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ON THE EXCESS PHOTON NOISE IN SINGLE-BEAM MEASUREMENTS WITH PHOTO-EMISSIVE AND PHOTO-CONDUCTIVE CELLS

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Synopsis

In this paper the so-called excess photon noise is theoretically considered with regard to noise power measurements with a single, illumined photo-emissive or photo-conductive cell. Starting from a modification of Mandel's stochastic association of the emission of photo-electrons with wave intensity, the frequency dependence of the excess noise power with photo-emissive cells is derived in Sect. 2. Use is made of formulas given by Rice for square-law detectors. In Sect. 3 a new hypothesis is introduced in order to calculate the excess photon noise effect with photo-conductive cells, in which the statistics of both the excitation and of the recombination have to be considered. It is shown that in the case of an ideal photo-conductor in thermodynamic equilibrium with the radiation field the result obtained is consistent with the expected Fermi-statistics for the electron gas. It is shown in Sect. 4 that under certain experimental conditions the excess photon-noise effect is detectable in noise-power measurements with a single photocell. This applies especially to the case of an infra-red sensitive photo-conductive or photo-emissive cell with a quantum-yield near to unity.

§ 1. *Introduction.* Ample attention has been given in the literature to the so-called excess photon-noise or wave-interaction noise, which occurs in addition to the well-known pure shot noise in the photo-current of phototubes (cf., e.g.,[1][2][3][4][5][6]). This excess noise has been explained by the Bose-Einstein statistics of photons (which give rise to the "bunching effect"), or classically, by the wave character of light (which gives rise to interference effects between the different frequency components). Its existence has been proved by measuring the coherence between the fluctuations in the emission currents of two phototubes illumined by (partly) coherent light. The chief application of such measurements is to the determination of the angular diameter of stars.

In this paper the case of a single, illumined photo-emissive or photo-conductive cell will be considered. In particular, the practical possibility of detecting the excess noise against a "background" of pure shot noise in noise measurements with a single photocell will be theoretically investigated. This possibility has been doubted in the past [1]). Counting techniques do not seem to be appropriate for that purpose, as good detectability of the excess noise requires very high counting rates (cf. [5]), which are not easy to register in practice. Therefore we shall here consider not the variance of the number of counts, but the noise power of the photo-current fluctuations, as measured, e.g., within a given frequency-interval by some square-law detector. Our analysis will also include the region of very high frequencies in order to show which kind of information may be gained from a frequency analysis of the noise power.

Under practical conditions the excess photon-noise effect is expected to be detectable especially with infra-red sensitive photocells with high quantum yield. Therefore the excess noise effect is here investigated theoretically also in the case of i.r. photo-conductive cells such as PbS-cells, etc. A theoretical investigation of this effect with photo-conductors has been made earlier by Van Vliet [7]. We have chosen, however, a different approach by introducing a stochastic hypothesis, in order to avoid the difficulties that have been discussed by Mandel [5] and Hanbury Brown and Twiss [1] with regard to the results obtained by Fellgett [8] and Clark Jones [9] from thermodynamic arguments.

In the case of a photo-emissive cell (to the neglect of transit-time effects) the noise power as a function of frequency might be derived from the general formula given by Mandel [5] for the variance $\overline{(\Delta n_T)^2}$, in which n_T represents the number of photo-electrons emitted in time-interval T. This might be done by making use of MacDonald's Inversion Theorem [10]. However, the incident radiation is usually specified with regard to its spectral distribution (and not with regard to its autocorrelation function). Moreover, we are interested here likewise in the "spectral" noise power of the photo-current fluctuations. So this method of derivation involving correlation functions would seem to be a detour. That is why we have chosen a different, more direct approach, which could also be applied in the case that transit-time effects are not negligible, and in the case of a photo-conductor.

§ 2. *Noise spectrum with photo-emissive cells.*

2a) The case of unresolved light-source, constant quantum-efficiency and no transit-time effects. We shall derive here the frequency spectrum of the photocurrent noise of a photo-emissive cell under rather restricted conditions, in order to make more clear the essential features of our analysis. These restrictions are the assumption of an unresolved light-source, of a quantum-efficiency that is independent of wavelength and of an infinitely short transit-time of the photoelectrons. In Section *2b* generalizations will be made which may be of practical interest.

Let us consider a plane, homogeneous photodetector, with quantum-efficiency η. The detector is illumined by plane, linear polarized waves of light from a Gaussian random source. The light-beam is assumed to be sufficiently homogeneous, i.e. the solid angle Θ under which the detector views the light-source should obey the condition $\Theta \ll \lambda_0^2/A$, with λ_0 = central wave-length of the light and A = detector area. This condition means that the light source essentially cannot be resolved by the detector. When the light-beam is, moreover, supposed to fall normally on the detector, we may denote the instantaneous amplitude of the light at the place of the detector by $y(t)$. $y(t)$ is a fluctuating quantity, which in the case of a spectral line with small spectral bandwidth, may be considered as a carrier sine wave (frequency ν_0) whose envelope fluctuates irregularly. The frequency range of the envelope fluctuations is of the order of the frequency bandwith of the spectral line. As usual in noise analysis, $y(t)$ may be expanded in a Fourier integral, and we may define $w_y(\nu)$ as the spectral noise power of $y(t)$. The frequency dependence of $w_y(\nu)$ is essentially the same as that of the spectral-energy distribution $\varphi(\nu)$ of the radiation concerned.

In the derivation of the noise spectrum we use the basic idea of Mandel[5]) in assuming a stochastic association of the photoelectrons ejected with the wave intensity at the place of the photodetector. In contrast to Mandel, however, we here identify the wave intensity directly and more simply by the envelope, or the low-frequency part of the stochastic variable $y^2(t)$. Thus we write:

$$p(t) \cdot \mathrm{d}t = \alpha\{y^2(t)\}_{\mathrm{l.f.}} \cdot \mathrm{d}t \tag{1}$$

in which $p(t) \cdot \mathrm{d}t$ is the probability that a photo-electron is emitted in time interval $t;\ t + \mathrm{d}t$, and $\{y^2(t)\}_{\mathrm{l.f.}}$ is the low-frequency part of $y^2(t)$. Here α is a constant which contains the quantum-efficiency and area of the detector. The assumption (1) seems reasonable in the case of radiation within a narrow frequency range $\Delta\nu_0(\ll \nu_0)$, in as much as the duration of the emission process is then certainly much smaller than the auto-correlation time of $\{y^2(t)\}_{\mathrm{l.f.}}$, which is of the order $(\Delta\nu_0)^{-1}$. According to Rice[12]) the spectral noise power of $\{y^2(t)\}_{\mathrm{l.f.}}$ is given by:

$$\int_{-\infty}^{\infty} w_y(x) \cdot w_y(f - x)\, \mathrm{d}x, \quad \text{with} \quad f \ll \nu_0\ ^{*)}$$

so that the spectral noise power of $p(t)$ is given by:

$$w_p(f) = \alpha^2 \int_{-\infty}^{\infty} w_y(x) \cdot w_y(f - x)\, \mathrm{d}x \quad (f \ll \nu_0) \tag{2}$$

We now have to find the spectral noise power $w_i(f)$ of a current caused by the emission of photo-electrons, the emission probability $p(t)$ itself being a stochastic function of time. We shall solve this problem by using a general result that has been derived elsewhere [11]) in connection with non-linear noise effects, where a stochastic probability function also occurs. If we expand the (fluctuating) photocurrent in a Fourier series during a very long time-

*) Here optical frequencies are denoted by ν and frequencies related to the fluctuating photo-current by f.

interval 0; T_0, it can be derived that:

$$w_i(f) = (2e^2/T_0) \cdot \overline{RR^*}^{\text{ens.}} \tag{3}$$

in which e = total electric charge that passes through the output leads of the detector when one photo-electron is emitted. R is a complex number defined by:

$$R \equiv \Sigma_{r=1}^{N} \exp[j2\pi n_0 t_r/T_0] \tag{4}$$

Here t_r denotes the moment, at which the r-th electron is emitted (total number of electrons emitted in time-interval T_0 is N); n_0 is defined by $n_0 = f \cdot T_0$. This formula holds for phototubes under saturation conditions, when the electrons emitted do not interfere with each other and when no additional noise effects occur. The transit-time is supposed to be infinitely short. In formula (3) the average is taken over an ensemble of macroscopically identical systems consisting of a photocell placed in a radiation field.

For each system the probability $K(\vartheta) \cdot \Delta\vartheta$ that a term $\exp[j2\pi n_0 t_r/T_0]$ in R occurs with argument between ϑ and $\vartheta + \Delta\vartheta$, is given by:

$$K(\vartheta) \cdot \Delta\vartheta = (T_0 \cdot \Delta\vartheta/2\pi n_0) \, \Sigma_{k=0}^{n_0-1} \, p(\vartheta \cdot T_0/2\pi n_0 + kT_0/n_0) \tag{5}$$

as we have the relation: $\vartheta = 2\pi n_0 t/T_0$ modulo 2π. Here $\Delta\vartheta$ is assumed to be infinitesimal in the sense that $p(t)$ is practically constant during any time-interval $\Delta t = (T_0/2\pi n_0) \cdot \Delta\vartheta$. Let the actual number n_i of terms $\exp[j2\pi n_0 tr/T_0]$ with argument between ϑ_i and $\vartheta_i + \Delta\vartheta$ for a given system of the ensemble be: $n_i \equiv K(\vartheta_i) \cdot \Delta\vartheta + \Delta n_i$ then R may be written for that system as:

$$R = \Sigma_{\vartheta_i=0}^{2\pi} \{K(\vartheta_i) \cdot \Delta\vartheta + \Delta n_i\} \, (\cos\vartheta_i + j \sin\vartheta_i)$$

We now consider a *sub*-ensemble of systems having all the same $p(t)$ for any t, thus also the same $K(t)$ for any t, but differing from each other with regard to Δn_i. The sub-ensemble average $\overline{RR^*}'$ may then be written as:

$$\overline{RR^*}' = \Sigma_{\vartheta_i} \Sigma_{\vartheta_j} (\cos\vartheta_i \cdot \cos\vartheta_j + \sin\vartheta_i \cdot \sin\vartheta_j) \overline{\{K(\vartheta_i)\Delta\vartheta + \Delta n_i\} \times \{K(\vartheta_j)\,\Delta\vartheta + \Delta n_j\}}'$$

From the very definition of $K(\vartheta_i) \cdot \Delta\vartheta$ and Δn_i it follows that $\overline{\Delta n_i}' = 0$ for any i. As no mechanism is conceivable that would make one expect any correlation between the random deviations Δn_i and Δn_j $(i \neq j)$ (no interaction between the emitted photoelectrons was assumed), we also have:

$$\overline{\Delta n_i \cdot \Delta n_j}' = 0 \quad \text{for any } i \text{ and } j \neq i$$

So we get:

$$\overline{RR^*}' = \Sigma_i \Sigma_j (\cos\vartheta_i \cdot \cos\vartheta_j + \sin\vartheta_i \cdot \sin\vartheta_j) \cdot K(\vartheta_i) \cdot K(\vartheta_j) \cdot \Delta\vartheta^2 + \Sigma_i \overline{\Delta n_i^2}' \tag{6}$$

The first term in this expression may be written as $Q \cdot Q^*$ with:

$$Q \equiv \int_0^{2\pi} (\cos\vartheta + j\sin\vartheta)\cdot K(\vartheta)\cdot \mathrm{d}\vartheta$$

in which we have replaced the summation by an integration. We now expand $p(t)$, occurring in $K(\vartheta)$, in a Fourier series in the interval 0; T_0:

$$p(t) = \bar{p} + \Sigma_{n=1}^{\infty} p_n \cos(2\pi nt/T_0 + \psi_n) \tag{7}$$

In the expression for Q the contribution of $\bar{p}$ to $K(\vartheta)$ vanishes, so we get from (5) and (7):

$$Q = \int_0^{2\pi}(\cos\vartheta + j\sin\vartheta)(T_0/2\pi n_0)\{\Sigma_{n=1}^{\infty}\Sigma_{k=0}^{n_0-1} p_n\cdot\cos(\vartheta n/n_0 + 2\pi nk/n_0 + \psi_n)\}\,\mathrm{d}\vartheta$$

$$= \int_0^{2\pi}(\cos\vartheta + j\sin\vartheta)(T_0/2\pi n_0)\,\Sigma_{k=0}^{n_0-1} p_{n_0}\cdot\cos(\vartheta + \psi_{n_0})\cdot\mathrm{d}\vartheta$$

$$= \tfrac{1}{2}T_0\cdot p_{n_0}\cdot(\cos\psi_{n_0} - j\sin\psi_{n_0})$$

This gives us:

$$Q\cdot Q^* = \tfrac{1}{4}\cdot p_{n_0}{}^2\cdot T_0{}^2 \tag{8}$$

As a Poisson distribution may be expected for Δn_i in the expression (6) we may write: $\overline{\Delta n_i{}^2}' = \overline{n_i}' = K(\vartheta_i)\cdot\Delta\vartheta$. We then get, when passing to integrals:

$$\Sigma_i\,\overline{\Delta n_i{}^2}' = \int_0^{2\pi} K(\vartheta)\cdot\mathrm{d}\vartheta = \bar{p}\cdot T_0$$

after the Fourier expansion (7) for $p(t)$ has been substituted in the expression for $K(\vartheta)$.

From (6), (8) and this last result we find:

$$\overline{RR^*}' = \bar{p}\cdot T_0 + \tfrac{1}{4}\cdot p_{n_0}{}^2\cdot T_0{}^2$$

and by averaging over the *whole* ensemble:

$$\overline{RR^*}^{\text{ens.}} = \bar{p}\cdot T_0 + \tfrac{1}{4}\,\overline{p_{n_0}{}^2}\cdot T_0{}^2$$

From this expression and by replacing as usual $\overline{p_{n_0}{}^2}$ by $2\cdot w_p(f)/T_0$, with $f = n_0/T_0$, we get from (3):

$$w_i(f) = 2e^2\cdot\bar{p}\{1 + \tfrac{1}{2}w_p(f)/\bar{p}\} \tag{9a}$$

Or, by writing $e\bar{p} = \bar{\imath}$:

$$w_i(f) = 2e\cdot\bar{\imath}\{1 + \tfrac{1}{2}w_p(f)/\bar{p}\} \tag{9b}$$

The second term in the expression between brackets, which is essentially positive, describes the excess photon noise effect as a function of frequency. Writing (9*b*) as: $w_i(f) = 2e\bar{\imath} + e^2\cdot w_p(f)$, it appears more clearly that the total noise power is simply composed of a pure shot-noise and a pure "modulation" noise term. No mixed terms occur.

This equation, together with eq. (2), leads to the noise spectrum of the photocurrent, if the spectral energy distribution of the radiation is known (cf. above). Here we can profit by the explicit formulas listed by Rice[12] for the case that a noise current of given spectral noise power (determined by the transmission characteristics of some typical electric filters) passes a square-law device. So we find immediately:

1. in the case of Doppler line shape with

$$w_y(\nu) = (\overline{y^2}/\sigma\sqrt{2\pi})\cdot\exp[-(\nu-\nu_0)^2/2\sigma^2] \text{ where } \sigma \ll \nu_0$$

one gets:

$$w_i(f) = 2e\bar{\imath}\cdot\{1 + (\bar{\imath}/2e\sigma\sqrt{\pi})\exp[-f^2/4\sigma^2]\} \tag{10a}$$

2. in the case of Lorentz line shape with

$$w_y(\nu) = \overline{y^2}B/\pi\{B^2 + (\nu-\nu_0)^2\} \text{ where } B \ll \nu_0$$

one gets:

$$w_i(f) = 2e\bar{\imath}[1 + \bar{\imath}/2\pi eB\{1 + (f/2B)^2\}] \tag{10b}$$

3. in the case of rectangular line shape with

$$w_y(\nu) = \overline{y^2}/B \text{ for } \nu_0 - B/2 \leqslant \nu \leqslant \nu_0 + B/2$$

and $w_y(\nu) = 0$ for all other values of ν,
one gets:

$$\begin{cases} w_i(f) = 2e\bar{\imath}\{1 + (\bar{\imath}/eB)\,(1 - f/B)\} \text{ for } 0 \leqslant f \leqslant B \\ w_i(f) = 2e\bar{\imath} \text{ for } f \geqslant B. \end{cases} \tag{10c}$$

It appears here that, inversely, information may be gained, in principle at least, about the shape, in particular the width of a spectral line from an *electric* frequency analysis of the photocurrent noise.

One can derive in the usual way from formulas (10) a formula for the variance of the number of counts in time-interval T. The results for $T \gg B^{-1}$ and $T \ll B^{-1}$ agree with those derived by Mandel[5]).

2*b*. Generalizations. Formula (9*b*) has been derived under rather restricted conditions. Here we consider some generalizations of practical interest.

From (10) it is seen that the frequency dependence of the excess noise effect becomes characteristic only as soon as frequencies are of the order of the frequency band-width of the radiation used, which in practical cases may amount to 6.10^8 cps. and higher[13]). However, at such high frequencies *transit-time effects* of the photo-electrons, which have been disregarded in Sect. 2*a*, will also influence the noise spectrum. Here we want to investigate this influence, by again expanding the photo current in a Fourier series in the interval 0; T_0. (With T_0 very large when compared to all correlation times involved). Assuming that the current pulses caused by the transitions of the individual photoelectrons have identical shape, we now write, instead of (3) (cf. [11]): $w_i(f) = 2T_0 \cdot A_f \cdot A_f^* \cdot \overline{RR^*}$ (3*a*), in which A_f denotes the (complex) amplitude of the Fourier component with frequency f of a current pulse starting at time $t = 0$. R has here again the same meaning as in (3). Comparing eq. (3*a*) with eq. (3) one derives immediately from eqs. (9):

$$w_i(f) = 2A(f)(\bar{\imath}/e)\{1 + w_p(f)/2\bar{p}\} \tag{9c}$$

with:

$$\bar{p} = \bar{\imath}/e \text{ and } A(f) \equiv \lim_{T_0 \to \infty} (A_f \cdot A_f^* \cdot T_0^2)$$

We see from (9*c*) that in the case of current pulses having all the same shape the *ratio* of excess photon noise to pure shot noise is not altered by transit-time effects (cf. Sect. 4). This will not hold in general, when the individual current pulses have different shapes, although their time-integrals are constant and equal to e. A proof thereof can be given, in a rather elaborate way, by dividing the shapes of the current-pulses in classes, denoted by affix s, and by repeating the derivation of Sect. 2*a* introducing now also summations over s.

If the radiation on the photocell is *unpolarized*, we simply have to add the noise spectra belonging to both polarization directions, as they are completely uncorrelated. Then the total noise spectrum is described again by formula (9*a*), if outside the brackets $\bar{p}$ is replaced by $\bar{\imath}_{\text{total}}/e$ and inside the brackets by $\frac{1}{2} \cdot \bar{\imath}_{\text{total}}/e$, when the photocell sensitivity is equal for both polarization directions.

As long as the dimensions of the photocell in the direction of the light-beam are small compared with c/f (c = light-velocity), the above formula for the noise spectrum will also hold for cells of *arbitrary shape*. It may be noted that c/f is the wavelength of the beat between two radiation components with $\nu_1 - \nu_2 = f$.

In the case that the *quantum yield* depends noticeably on frequency within the frequancy range of the incident radiation, the appropriate formula for the noise power of the photocurrent may be found by dividing this frequency range in small sub-intervals, for each of which formula (10*c*) holds with good approximation. For the limiting case $f \to 0$ only we may then derive the total noise power by integrating the noise power of all sub-intervalls. If $j(\nu)$ equals $e \cdot n(\nu) \cdot \eta(\nu)$, with $n(\nu)$ is the mean number of quanta per sec and per unit of bandwidth at frequency ν, and $\eta(\nu)$ is the quantum yield at frequency ν, we get:

$$w_i(0) = 2e \int_0^\infty j(\nu)\, d\nu \{1 + \int_0^\infty j(\nu)^2\, d\nu / e \cdot \int_0^\infty j(\nu)\, d\nu\} \tag{11}$$

This formula agrees with a result obtained by Hanbury Brown in the calculation of correlation effects [1]).

Finally we have to discuss the case that the light-source is *resolved* by the detector, which means that Θ can no longer be considered as small compared with λ_0^2/A (compare Sect. 2*a*). We now only assume that the mean radiation intensity per unit of solid angle is constant within the total solid angle under which the source is seen by the detector. This case may lead to complicated calculations (cf. [2]). Here we have used a result obtained by Hanbury Brown and Twiss[2]) for the limiting case that the light-source is completely resolved by the detector ($\Theta \cdot A/\lambda_0^2 \gg 1$), which is of great practical interest (cf. below). This result leads immediately to:

$$w_i(0) = 2e\bar{\imath}\{1 + (\lambda_0^2/\Theta \cdot A)(\bar{\imath}/e \cdot B)\} \tag{12}$$

in the case that the radiation intensity is constant within a small fractional frequency-band (width B), centered about λ_0, and is zero everywhere else. This formula replaces (10*c*) for $f = 0$, and will be further discussed in Sect. 4.

§ 3. *Noise spectrum with photoconductive cells.* The essential difference between photo-emissive and photoconductive cells, with regard to noise effects, is the fact that the total electric charge transported through the outer leads by each excited carrier is invariably equal to e in the former case, whereas it is a stochastic variable in the latter case. We want to discuss here photo-conductive cells, in which the mean life-time of the free carriers is short when compared with the drift-time. We shall then have to consider not only the statistics of the excitation process (as in the preceding Section), but also the statistics of the recombination process. Since photons or phonons, obeying Bose-Einstein statistics, may be emitted in the recombination process, we could expect that this might have some bearing upon the statistics of the recombining electrons.

First an ideal case of thermodynamic *equilibrium* is considered, for which we expect a priori that the electrons in the photoconductor follow Fermi statistics. Suppose that the photo-conductor has only two energy states, a ground state and an excited, conducting state, and that transitions between these states can be induced through absorption and emission of radiation only. The energy levels may be broadened to some extent, in the sense that the absorption coefficient may be assumed to have a positive, frequency-independent value within a narrow frequency-band ν_0; $\nu_0 + B$ (with $B \ll \nu_0$), but is zero everywhere outside this band. We now assume that this photoconductor is placed in a radiation field, the density of which in the frequency band ν_0; $\nu_0 + B$ is given by Planck's Law at (effective) temperature T_r. Let the instantaneous value of the carrier occupancy in the conducting state be given by $N_0 + \Delta n$, in which N_0 is the equilibrium value at temperature T_r. For the stochastic, time-dependent variable Δn we may now write in first approximation the Langevin equation (cf. also [7]):

$$\frac{\mathrm{d}(\Delta n)}{\mathrm{d}t} = -\Delta n/\tau^* + \Delta F_e(t) - \Delta F_r(t) \tag{13}$$

Here τ^* is the radiative life-time, whereas $\Delta F_e(t)$ and $\Delta F_r(t)$ represent the "random forces" which describe here the fluctuations in the excitation, and in the recombination process, respectively. Between these "random forces" a certain degree of coherence might exist (see below). In order to calculate the spectral noise power of Δn from eq. (13), we first have to compute the spectral noise power of $\Delta F_e(t) - \Delta F_r(t)$ by making a new hypothesis on the stochastic association of light-wave intensity and excitation and recombination probabilities, respectively. To simplify matters we again restrict ourselves to the one-dimensional case, characterized by the condition: $\Theta \cdot A \ll \lambda_0^2$ (compare Sect. 2*a*). Here Θ refers to both the incident and the emitted radiation.

For the probability $p_e(t)$ per sec of an excitation process through light-absorption at any moment t, we again put forward the hypothesis made with photo-emissive cells (Sect. 2*a*):

$$p_e(t) = \alpha_e \cdot \{y_1{}^2(t)\}_{\text{l.f.}} \tag{14a}$$

Here $y_1(t)$ is the part of the Fourier expansion of $y(t)$ with frequencies between ν_0 and $\nu_0 + B$.

Consider now a hypothetical case in which the photoconductor does not receive any radiation (surroundings being at zero temperature), but only emits radiation. It then seems reasonable to assume a similar stochastic association between the probability $p_r(t)$ per sec of a recombination process (followed by light-emission), and the intensity of the radiation field. Hence:

$$p_r(t) \cdot \mathrm{d}t = \alpha_r \{y_1{}^2(t)\}_{\text{l.f.}} \cdot \mathrm{d}t \tag{14b}$$

This formula could be understood to express that the probability of the emission of a photon in any time-interval t; $t + \mathrm{d}t$ is proportional to the probability of finding an (emitted) photon at the place of the photoconductor during that interval. The latter probability, in turn, may be considered as related to the intensity of the radiation field.

The crucial step in our analysis of the more general case, in which the photo-conductor emits as well as absorbs radiation, is the introduction of the combined hypotheses:

$$\left.\begin{aligned} p_e(t) &= \alpha_e \cdot \{y_1{}^2(t)\}_{\text{l.f.}} \\ p_r(t) &= \alpha_r \cdot \{y_1{}^2(t)\}_{\text{l.f.}} \end{aligned}\right\} \tag{14c}$$

Here again $y_1(t)$ denotes the instantaneous amplitude of the radiation field within the frequency-band B at the place of the photo-conductor, which is now composed of both the incident and the emitted radiation. It should be noted that α_r contains not only the contribution of the spontaneous emission but also of the "stimulated emission". So in general α_r still depends on the radiation density, and thus on the temperature T_r, which was introduced above. In assuming that α_r is a constant, depending only on the *mean* radiation density (i.e. on T_r), a first order approximation has been made that does not seem to be very essential.

The derivation of the noise spectrum $w_F(f)$ of the fluctuation in the excitation and recombination rates $\{\Delta F_e(t) - \Delta F_r(t)\}$ may proceed along similar lines as in Sect. 2*a*. Allowance should here be made, however, for a certain degree of coherence between $\Delta F_e(t)$ and $\Delta F_r(t)$ on account of (14*c*), and for the fact that a recombination process has just the opposite effect on the number of free carriers as an excitation process. Formally, the situation here may be compared with a diode consisting of two hot cathodes, which is connected to a condenser. $\Delta F_e(t)$ and $\Delta F_r(t)$ are then to be compared with the fluctuations in the two opposite electron streams respectively, between which a certain degree of coherence is to be assumed. The number of free carriers in the photoconductor then compares with the charge on the condenser, which should be counted in units of 1 electron charge.

We then may write:

$$w_F(f) = (2/T_0)\cdot\overline{RR^*}^{\,\text{ens.}}$$

by replacing the electron charge e in (3) by unity. Here R is defined by:

$$R \equiv \Sigma_{n=1}^{N} \exp[j2\pi n_0 t_r'/T_0]$$

in which t_r' equals time t_r at which the r-th transition took place in the interval 0; T_0, if this transition was an excitation, and $t_r' = t_r \pm 1/(2f)$, if the r-th transition was an recombination. This definition of t_r' accounts for the opposite effects of an excitation and a recombination process on the number of free carriers.

In analogy to Sect. 2*a* we now have for the probability $K(\vartheta)\cdot\Delta\vartheta$ that a term in R has an argument between ϑ and $\vartheta + \Delta\vartheta$:

$$K(\vartheta)\cdot\Delta\vartheta = (K_e + K_r)\,\Delta\vartheta$$

with:

$$K_e(\vartheta) = (T_0/2\pi n_0)\ \Sigma_{k=0}^{n_0-1} p_e(\vartheta T_0/2\pi n_0 + kT_0/n_0)$$

and:

$$K_r(\vartheta) = (T_0/2\pi n_0)\ \Sigma_{k=0}^{n_0-1}\ p_r\{\vartheta T_0/2\pi n_0 + (k + \tfrac{1}{2})\ T_0/n_0\}$$

The number of terms in R with argument between ϑ and $\vartheta + \Delta\vartheta$ for an actual system in the interval 0; T_0 being given by:

$$n_i \equiv K_e\cdot\Delta\vartheta + (\Delta n_i)_e + K_r\cdot\Delta\vartheta + (\Delta n_i)_r$$

we may write for that system:

$$R = \Sigma_{\vartheta_i=0}^{2\pi}\ \{K_e\Delta\vartheta + K_r\Delta\vartheta + (\Delta n_i)_e + (\Delta n_i)_r\}(\cos\vartheta_i + j\sin\vartheta_i)$$

Introducing again a sub-ensemble as defined in Sect. 2*a*, we now find for the sub-ensemble average of RR^*:

$$\overline{RR^*}' = QQ^* + \Sigma_i\{\overline{(\Delta n_i)_e{}^2} + \overline{(\Delta n_i)_r{}^2}\}$$

with:

$$Q \equiv \int_0^{2\pi} (K_e + K_r)(\cos\vartheta + j\sin\vartheta)\,d\vartheta$$

Writing:

$$p_r(t) = (\alpha_r/\alpha_e)\cdot p_e(t)$$

on account of (14*c*) and expanding $p_e(t)$ in a Fourier series (compare Sect. 2*a*), we now find:

$$Q = \tfrac{1}{2}T_0(p_{n_0})_e\,(1 - \alpha_r/\alpha_e)(\cos\psi_{n_0} - j\sin\psi_{n_0})$$

in which $(p_{n_0})_e$ is the amplitude of the n_0-th term in the Fourier expansion of $p_e(t)$.

The total ensemble average of RR^* is finally found in a way analogous to that in Sect. 2*a*:

$$\overline{RR^*}^{\text{ens.}} = \bar{p}_e(1 + \alpha_r/\alpha_e)T_0 + \tfrac{1}{2}T_0 w_{pe}(f)(1 - \alpha_r/\alpha_e)^2$$

and for $w_F(f)$ we then have:

$$w_F(f) = 2\bar{p}_e(1 + \alpha_r/\alpha_e)[1 + \tfrac{1}{2}w_{p_e}(f)(1 - \alpha_r/\alpha_e)^2/\bar{p}_e(1 + \alpha_r/\alpha_e)] \qquad (15)$$

In thermodynamic equilibrium $\overline{p_e(t)}$ equals $\overline{p_r(t)}$, so: $\alpha_r = \alpha_e$. Then $w_F(f)$ reduces to $4p_e$ and from (13) we find:

$$w_{\Delta n}(f) = 4\bar{p}_e\cdot\tau^{*2}/(1 + 4\pi^2 f^2\tau^{*2}) \qquad (16)$$

Because of the equality of α_r and α_e, which is essential in thermodynamic equilibrium no excess photon noise term is found in this formula. As expected, the electrons appear, indeed, to obey Fermi statistics in the equilibrium case, which may support the validity of the general hypothesis (14*c*) made.

In the general *non-equilibrium* case, however, $w_F(f)$ and, consequently, $w_{\Delta n}(f)$ may contain an additional positive term, which depends on $w_{p_e}(f)$ and should be considered therefore as representing the excess photon noise. This case may occur, e.g., with a cooled photo-conductor, in which a strong interaction exists between the electrons and the lattice vibrations. Here recombination of free carriers is mainly accompanied by the emission of a phonon instead of a photon, whereas the excitation mainly occurs through absorption of a photon. For the photon-induced transitions (mainly excitations) we derive from (15*a*), by making use of $\alpha_r/\alpha_e \ll 1$

$$w_{F_1}(f) = 2\bar{p}_e\{1 + \tfrac{1}{2}w_{p_e}(f)/\bar{p}_e\} \qquad (17a)$$

Also the phonons should obey Bose-Einstein statistics, so that, in principle, an excess *phonon* noise effect could also be expected in the statistics of the recombination process. For the recombining transitions, accompanied by phonon emission, we thus have to write generally:

$$w_{F_2}(f) = 2\bar{p}_e(1 + \Omega) \qquad (17b)$$

as the mean number of recombinations per sec. equals, of course, the mean number $\bar{p}_e$ of excitations. Since the pure shot fluctuations in the excitation and recombination processes are, of course, not correlated, and since also the excess photon and excess phonon noise effects are uncorrelated (because the fluctuations in the photon and the photon fields are), we have:

$$w_F(f) = w_{F_1}(f) + w_{F_2}(f) = 4\bar{p}_e(1 + \tfrac{1}{4}w_{pe}(f)/\bar{p}_e + \tfrac{1}{2}\Omega) \qquad (17c)$$

Here $w_F(f)$ denotes the noise spectrum of the combined "random forces" in the Langevin-equation, in which the radiative lifetime τ^* has now to be replaced by τ_c, the collisional lifetime. We then get instead of (16):

$$w_{\Delta n}(f) = 4\bar{p}_e(1 + \tfrac{1}{4}w_{pe}(f)/\bar{p}_e + \tfrac{1}{2}\Omega)\tau_c^2/(1 + 4\pi^2 f^2\tau_c^2) \qquad (18)$$

We do not want to go into the details of the calculation of Ω, which is connected with the electron-lattice interaction. It may be remarked here only, that by choosing sufficiently low temperatures for the photoconductor and sufficiently high radiation densities. the term $\frac{1}{2}\Omega$ may vanish in comparison with the excess photon noise term. The magnitude of these excess terms is connected with the mean number of phonons and photons respectively in one Bose-cell (cf.[5]). By lowering the conductor temperature the density of the phonon field and consequently the mean number of phonons in one Bose cell is reduced. In that case formula (18) is reduced to:

$$w_{\Delta n}(f) = 4\bar{p}_e(1 + \tfrac{1}{4}w_p(f)/\bar{p}_e)\tau_c^2/(1 + 4\pi^2 f^2\tau_c^2) \qquad (18a)$$

Comparison of (18*a*) with (9*a*) tells us that for a given mean number of absorbed effective quanta per sec, the ratio of excess photon noise to pure shot noise is exactly *half* as large with a cooled photoconductor as with a photo-emissive cell. For $w_{p_e}(f)$ occurring in (18*a*) the same calculations and generalizations are applicable as treated in Sect. 2 for $w_p(f)$. So, if we restrict ourselves to small radiation bandwidths B with frequency-independent density and to noise frequencies f which are small compared with B we get from (18*a*) and (12)

$$w_{\Delta n}(f) = 4\bar{p}_e\{1 + \tfrac{1}{2}(\lambda_0^2/\Theta\cdot A)(\bar{p}_e/B)\}\cdot\tau_c^2/(1 + 4\pi^2 f^2\tau_c^2) \text{ with } f \ll B \qquad (18b)$$

§ 4. *Conclusion.* To conclude we shall consider here the detectability of the excess photon noise under experimental conditions. Usually this excess effect will largely be masked by pure shot noise, and the spectral noise power will be found practically to equal $2\cdot e\cdot \bar{\imath}$ in the case of a phototube. But the expressions found in the preceding Sections may guide us to choose more suitable experimental conditions, such as intensity of light-source, wavelength, kind of photocell, etc. in order to detect the excess noise.

As a measure for the detectability of the excess noise, we consider the "fractional excess noise", i.e., the ratio of excess noise power to pure shot noise power. As a practical case we suppose to have a light-beam, the spectral density of which is constant within a small frequency band ν_0; $\nu_0 + B$ ($B \ll \nu_0$) and is zero outside this band. This may be approximately realised with the aid of a monochromator. Within this band the quantum efficiency η of the detector is assumed to be constant. Furthermore the sensitive area of the detector and the solid angle under which the detector views the light-source are assumed to be sufficiently large to resolve the light-source completely (compare Sect. 2). From (10*c*) we see that the fractional excess noise is maximum for $f \to 0$, so we restrict ourselves to frequencies small compared to B and for which, moreover, transit-time effects are not noticeable. From the discussion in Sect. 2*b* it appears that the fractional excess noise is the same for the unpolarized radiation as for the two polarization directions separately. So the detectability of the excess noise is not ameliorated by polarizing the radiation.

Let T_r be the radiation temperature of the light-source, i.e., the temperature of a black body when the intensity of its radiation at ν_0, per unit of frequency band, per square cm and per solid angle, is the same as that of the light-source concerned. Let ζ be the factor that accounts for the radiation losses between light-source and detector, owing to absorption in lenses, prism, etc. Replacing $\bar{\imath} = \bar{p}\cdot e$ and writing $\bar{p} = n(\nu_0)\cdot\eta\cdot B$, where $n(\nu_0)$ is the mean number of photons arriving per sec and per unit of bandwidth at frequency ν_0, we find from formulas (12) and (18*b*), respectively, and from Planck's radiation law:

$$\frac{\text{excess photon noise power}}{\text{pure shot noise power}} = k\cdot\eta\cdot\zeta/(\exp[1{,}438/\lambda_0 T_r) - 1) \tag{19}$$

in which λ_0 is expressed in cm and T_r in °K $\cdot$ k equals 1 in the case of a photo-emissive cell (under saturation conditions) and k equals $\frac{1}{2}$ in the case of a cooled photo-conductive cell. It may be noted that the spectral radiation bandwidth B and the solid angle Θ do not occur in (19). So B and Θ may be chosen sufficiently large (with the restriction that B should still be small compared to ν_0) to make the radiation-induced noise large in comparison with the background noise (at dark cell). When choosing a small solid angle with $\Theta\cdot A/\lambda_0^2 \ll 1$, everything else being kept the same, the fractional excess noise is lowered, as may be proved by comparing (19) with (10*c*).

We take for example a vacuum photo-emissive tube with $\eta = 0{,}3$, illumined by the green spectral line at $\lambda_0 = 0{,}546\cdot 10^{-4}$ cm of a mercury discharge lamp with radiation temperature $T_r = 6750$°K. The fractional excess noise is now calculated to be as low as 1%, if no losses occur ($\zeta = 1$). Here the spectral line shape was assumed to be rectangular, which might be a reasonable approximation at high self-absorption. This is about the best to be expected with this kind of cell, as higher radiation temperatures are not easily realized and as at larger wavelengths these cells will have much lower quantum yields.

Better results may be expected with solid state detectors which have a quantum yield near to unity, also in the infra-red region. We here have to distinguish p–n junction cells with reverse bias, that behave as photoemissive cells, and photoconductive cells, for which $k = 1$ and $\frac{1}{2}$, respectively. For a typical germanium p–n junction cell illumined by a light-source with $T_r = 2500°K$ (e.g. a flame,) 3800°K (e.g. the positive crater of a carbon-arc) and 6000°K (the sun) at about $\lambda_0 = 1{,}8 \cdot 10^{-4}$ cm the fractional excess noise is calculated to be 7, 15 and 30%, respectively, if $\eta = 1$ and $\zeta = 1$. For a photoconductor illumined at wavelength $\lambda_0 = 4 \cdot 10^{-4}$ cm, with a quantum yield $\eta = 1$ at that wavelength, the fractional excess noise amounts to 15%, with $T_r = 2500°K$, $\zeta = 1$ and $k = \frac{1}{2}$.

It is rather difficult to obtain an accuracy of, say, 1% in an absolute measurement of noise power. Moreover, additional effects may cause (slight) deviations from the noise formulas given here. These difficulties in detecting the excess photon noise may be overcome by making use of *relative* noise measurements. In one (relative) measurement pure shot noise and excess noise are measured together; in another measurement only pure shot noise is measured at the *same* value of $\bar{\imath}$ (or $\bar{p}_e$). The latter measurement may be realized by inserting a light-attenuator between light-source and detector. One should then at the same time enlarge the solid angle under which the detector views the source, to such an extent that $\bar{\imath}$ (or $\bar{p}_e$) is again the same as in the first noise measurement. The fractional excess noise, which is influenced by the light-attenuation (represented by ζ in form. (19)) and not by the enlarged solid angle, can be made negligibly small in this way. By comparing the results of the two noise measurements (all other conditions being the same) the fractional excess photon noise is directly found.

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