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UTRECHT

DEPARTMENT
OF
MATHEMATICS

DEGREES, REDUCTIONS AND REPRESENTABILITY IN THE LAMBDA CALCULUS

PREPRINT
NR. 22

FEBRUARY 1976

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DEGREES, REDUCTIONS AND REPRESENTABILITY IN THE LAMBDA CALCULUS.

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The three chapters of this preprint can be read independently. Only some technical lemma's used in I are proved in II.

Acknowledgement

We would like to thank Ms. Breughel, Jan Willem Klop and Sofia van Sterkenburg for careful typing.

CHAPTER I

DEGREES OF SENSIBLE LAMBDA THEORIES.

INTRODUCTION.

A *λ-theory* T is a consistent set of equations between λ-terms closed under derivability. The *degree* of T is the degree of the set of Gödel numbers of its elements. A λ-theory T is *sensible* iff $T \vdash \mathcal{H}_0$ (= $\{M=N \mid M,N \text{ unsolvable}\}$).

In §1 it is proved that the theory $\mathcal{H}$ is Σ_2^0-complete. We present Wadsworth's proof that its unique maximal consistent extension $\mathcal{H}^*$ (= $Th(D_\infty)$) is Π_2^0-complete.

In §2 it is proved that $\mathcal{H}\eta$ (= λη-calculus + $\mathcal{H}$) is not closed under the ω-rule (see [1]).

In §3 arguments are given to conjecture that $\mathcal{H}\omega$ (= λ + $\mathcal{H}$ + ω-rule) is Π_1^1-complete. This is done by representing recursive sets of sequence numbers as λ-terms and by connecting well-foundedness of trees with provability in $\mathcal{H}\omega$.

In §4 a set of equations independent over $\mathcal{H}\eta$ will be constructed. From this it follows that there are $2^{\aleph_0}$ sensible theories T s.t $\mathcal{H} \subset T \subset \mathcal{H}^*$ and $2^{\aleph_0}$ sensible hard models of arbitrarily high degrees.

In §5 some non-provability results needed in §1,2 are established. For this purpose one uses the theory $\mathcal{H}\eta$ extended with a reduction relation for which the Church-Rosser theorem holds. The concept of Gross reduction is used in order to show that certain terms have no common reduct.

Familiarity with [2] is assumed.

§1. Degrees of $\mathcal{H}$, $\mathcal{H}^*$ and $\mathcal{H}\omega$.

The λ-theory $\mathcal{H}$ has a unique maximal consistent extension $\mathcal{H}^*$ ([2] §4). Let $\mathcal{H}\omega$ be the set of equations provable in $\lambda + \mathcal{H} + \omega$-rule. Then one has $\mathcal{H} \subseteq \mathcal{H}\eta \subseteq \mathcal{H}\omega \subseteq \mathcal{H}^*$. The first two inclusions are trivial, the last one follows from the fact that $\mathcal{H}^* = Th(D_\infty)$ and D_∞ satisfies the ω-rule, see [6]. Moreover the inclusions are proper. $\mathcal{H} \neq \mathcal{H}\eta$ follows from the C-R property for $\mathcal{H}\eta$. $\mathcal{H}\eta \neq \mathcal{H}\omega$ is proved in 1.9. $\mathcal{H}\omega \neq \mathcal{H}^*$ follows by an extension of the consistency proof in [1]: It can be proved that if $\mathcal{H}\omega \vdash M=I$, then $\lambda \vdash M\vec{I} = I$ where $\vec{I}$ is some sequence of I's. If $\mathcal{H}\omega = \mathcal{H}^*$, then $\mathcal{H}\omega \vdash J=I$, where J is Wadsworth's term $Y(\lambda jxy.x(jy))$, since J and I have equivalent Böhm trees [2], 6.7. So $\lambda \vdash J\vec{I}=I$, contradicting the C-R property for the λ-calculus.
It will be proved that $\mathcal{H}(\eta)$ is Σ^0_2-complete and that $\mathcal{H}^*$ is Π^0_2-complete. It is conjectured that $\mathcal{H}\omega$ is Π^1_1-complete.

Notation. Ω denotes the term $(\lambda x.xx)(\lambda x.xx)$.
If $\longrightarrow$ is a reduction relation, $\xrightarrow{*}$ denotes its transitive reflexive closure.
$\xrightarrow[\beta]{}$, $\xrightarrow[\eta]{}$ are one step β- resp. η-reduction.
$\xrightarrow[\beta\eta]{} = \xrightarrow[\beta]{} \cup \xrightarrow[\eta]{}$.

1.1 Lemma. Let $R(x)$ be an r.e. predicate (on ω). Then for some term F

$\mathcal{H} \vdash F\underline{n} = I$ if $R(n)$

$\mathcal{H} \vdash F\underline{n} = \Omega$ if $\neg R(n)$.

Proof.
Let $R(x) \Longleftrightarrow \exists y\, A(x,y)$ with A recursive. Define by the fixed point combinator
$Fx =$ If $A(x,y)$ then I else $F(x+1)$. Then F works, since if $\neg R(n)$, then $F\underline{n}$ is unsolvable, hence $= \Omega$ in $\mathcal{H}$. ⊠

1.2 Def. (i) Ordered tuples are represented as terms as follows:

$\langle M_0 \rangle = M_0$

$\langle M_0,\ldots,M_{n+1} \rangle = [M_0, \langle M_1,\ldots,M_{n+1} \rangle]$,

where [,] is some pairing with inverses $\lambda x.(x)_0, \lambda x.(x)_1$.

(ii) If M_i is a definable sequence of terms (i.e. for some M, $\vdash M\underline{i} = M_i$ for all i), then the infinite sequence $\langle M_i \rangle$ is represented as a term $A\underline{o}$, where A is such that $A\underline{n} \xrightarrow[\beta]{*} [M_n, A\underline{n+1}]$. A exists by the fixed point theorem.

1.3 Lemma. There is a term $\lambda ix.\pi_i x$ such that $\lambda \vdash \pi_{\underline{n}} \langle M_j \rangle = M_n$ (for definable sequences $\langle M_j \rangle$).

Proof.

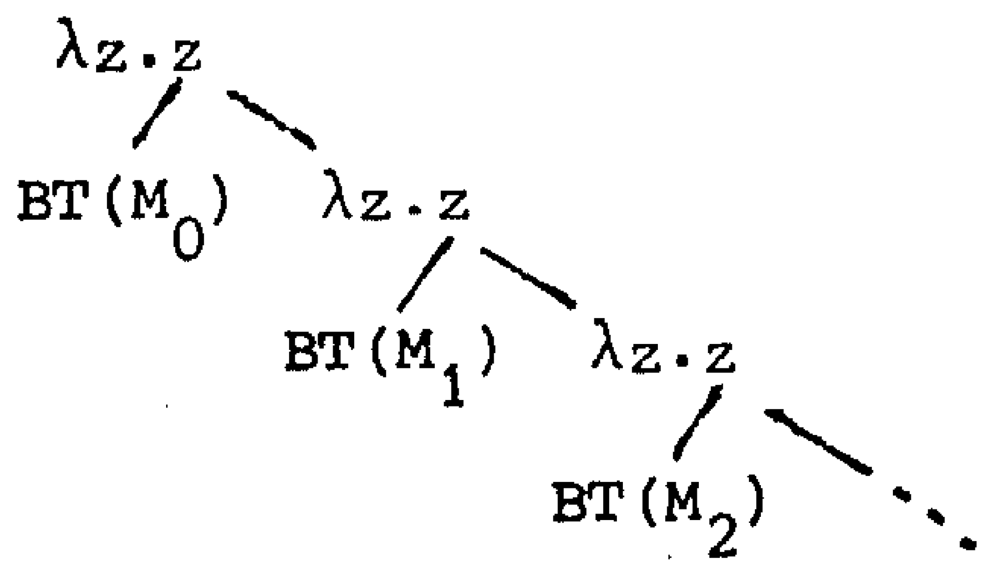
Define

$\pi_i x =$ If i=0 then $(x)_0$ else $\pi_{i-1}((x)_1)$. ⊠

1.4 Lemma. If $\langle M_i \rangle$, $\langle N_i \rangle$ are definable sequences then
$\forall i\ \mathcal{H}^* \vdash M_i = N_i \iff \mathcal{H}^* \vdash \langle M_i \rangle_{i\in\omega} = \langle N_i \rangle_{i\in\omega}$.

Proof.

⇒ : The Böhm tree of $\langle M_i \rangle$ is



and similarly for $\langle N_i \rangle$. By the theorem of Hyland and Wadsworth
$\mathcal{H}^* \vdash P=Q \iff BT(P) \sim_\eta BT(Q)$, see [2] 7.1.(i), it follows that the mentioned trees are equivalent and the result follows.

⇐ : By applying π_i of 1.3. ⊠

1.5 Theorem (Wadsworth [7]). $\mathcal{H}^*$ is Π^0_2-complete.

Proof.

(i) $\mathcal{H}^* \vdash M=N \iff M=N \in \mathcal{H}^* \iff \forall C[\ ]\,[C[M]$ is solvable $\iff C[N]$ is solvable$]$ (see [2], §5). The latter is clearly Π^0_2.

(ii) Let $\forall a\, \exists b\, A(a,b)$ be any Π^0_2 predicate; A is recursive (and has a not explicitly mentioned parameter c).

By 1.1 there is a term F such that

$\mathcal{H} \vdash F\underline{a} = I$ if $\exists b\, A(a,b)$

$\mathcal{H} \vdash F\underline{a} = \Omega$ else.

Let $H = \langle F\underline{o}, F\underline{1}, \ldots \rangle$, $H' = \langle I, I, \ldots \rangle$.

Now $\forall a\, \exists b\, A(a,b)$

$\iff \forall a\ \ \mathcal{H} \vdash F\underline{a} = I$

$\iff \forall a\ \ \mathcal{H}^* \vdash F\underline{a} = I$ since $\mathcal{H} \subset \mathcal{H}^*$ and $\mathcal{H}^* \nvdash I = \Omega$

$\iff \mathcal{H}^* \vdash H = H'$ by 1.4.

Therefore each Π^0_2 predicate can be reduced to provability in $\mathcal{H}^*$ (since the H,H' can be found uniformly in the parameter c in A). ⊠

1.6 Theorem. $\mathcal{H}(\eta)$ is Σ_2^0-complete.

Proof.

(i) The set of axioms $\mathcal{H} = \{M=N \mid M,N \text{ unsolvable}\}$ is clearly Π_1^0, therefore they generate a Σ_2^0 theory.

(ii) Let $\exists a\, \forall b\, A(a,b)$ be any Σ_2^0 predicate. By 1.1 there is a term F such that

$$\mathcal{H} \vdash F\underline{a} = \Omega \quad \text{if } \forall b\, A(a,b)$$
$$= I \quad \text{else.}$$

Let $H\, i\, \underline{a} \overset{*}{\underset{\beta}{\rightarrow}} [I,\, F\underline{a}(H\, i\, \underline{a+1})]$ by the fixed point operator.

Let x,y be different variables.

Claim: $\exists a\, \forall b\, A(a,b) \iff \mathcal{H}(\eta) \vdash H\, x\, \underline{o} = H\, y\, \underline{o}$.

⇒: If $\exists a\, \forall b\, A(a,b)$, then
$\mathcal{H} \vdash H\, x\, \underline{o} = [I, I, \ldots, \Omega] = H\, y\, \underline{o}$

⇐: If $\neg\exists a\, \forall b\, A(a,b)$, then
$\forall a \quad \mathcal{H} \vdash F\underline{a} = I$, so $H\, i\, \underline{n} \overset{*}{\underset{\beta}{\rightarrow}} [I, H\, i\, \underline{n+1}]$ and $H\, i\, \underline{o} \overset{*}{\underset{\beta}{\rightarrow}} [I, I, \ldots, H\, i\, \underline{n}]$.
Then $\mathcal{H}\eta \nvdash H\, x\, \underline{o} = H\, y\, \underline{o}$ as is proved in §5.

So each Σ_2^0 predicate can be reduced to provability in $\mathcal{H}(\eta)$. ⊠

§2. $\mathcal{H}\eta \nvdash \omega$.

2.1 Def. A λ-theory T is closed under the ω-rule, notation $T \vdash \omega$, if for all closed F,F'

$T \vdash FZ = F'Z$ for all closed Z $\Rightarrow$

$T \vdash F = F'$.

Note that $Th(\mathcal{M}) \vdash \omega$ iff $\mathcal{M}^0$ is extensional.

In [4] it is shown that $\lambda\eta \nvdash \omega$.
Now two proofs will be given that $\mathcal{H}\eta \nvdash \omega$.
In the first proof the terms constructed play a symmetric role. Not so in the alternative one. There a term A is constructed which in $\mathcal{H}\eta$ is constant on all closed terms, but not constant in general.
Also in [4] a pseudo-constant term is used to prove $\lambda\eta \nvdash \omega$. The construction is totally different however. See also [1].

2.2 Lemma. Let $FM^{\sim n} = F\underbrace{M\ldots M}_{n \text{ times}}$. Then

$\forall Z$ closed $\exists n$ $\mathcal{H} \vdash Z\Omega^{\sim n} = \Omega$.

Proof.
If Z is unsolvable, then $\mathcal{H} \vdash Z = \Omega$.
Otherwise Z has a head normal form ([2], 4.3) $\lambda x_1 \ldots x_n . x_i N_1 \ldots N_m$. Then $\mathcal{H} \vdash Z\Omega^{\sim n} = \Omega$. ⊠

2.3 Theorem. $\mathcal{H}\eta \nvdash \omega$.
First Proof.
Define a term O such that $O\, i\, \underline{n} \overset{*}{\underset{\beta}{\to}} \lambda y . y\Omega^{\sim n}\ (O\, i\ \underline{n+1}\, y)$
O can be constructed by the fixed point theorem and an F such that $\lambda \vdash F\,y\,\underline{n} = y\Omega^{\sim n}$. Take e.g. $F\,y\,n =$ <u>If</u> Zero n <u>then</u> y <u>else</u> $Fy(n-1)\Omega$.

Claim 1. $\forall Z$ closed $\mathcal{H} \vdash O\,x\,\underline{o}\,Z = O\,y\,\underline{o}\,Z$

2. $\mathcal{H}\eta \nvdash O\,x\,\underline{o} = O\,y\,\underline{o}$.

As to 1. $\mathcal{H} \vdash O\,x\,\underline{o}\,Z = Z\Omega(O\,x\,\underline{1}\,Z) = \ldots =$

$= Z\Omega(Z\Omega\Omega(\ldots(Z\Omega^{\sim n}(O\,x\,\underline{n+1}\,Z))\ldots)) = \ldots$

Hence by 2.1 there exists an n such that

$\mathcal{H} \vdash O\,x\,\underline{o}\,Z = Z\Omega(\ldots(Z\Omega^{\sim n-1})\ldots) = O\,y\,\underline{o}\,Z$.

As to 2. This is proved in §5.

By the claim $\mathcal{H}\eta \nvdash \omega$. ⊠

Alternative proof. Define a term A such that
$Az \overset{*}{\underset{\beta}{\to}} \lambda y . y(A(z\Omega))$. Then

$\mathcal{H} \vdash AI = \lambda y.y(A\Omega) = \lambda y.y(\lambda y.y(A\Omega)) = \ldots$

$= C_*^n(A\Omega) = \ldots,$ where $C_*a = \lambda y.ya$.

Claim: $\forall Z$ closed $\mathcal{H} \vdash AZ = AI$. Indeed,

$AZ \twoheadrightarrow_\beta \lambda y.y(A(Z\Omega)) \twoheadrightarrow_\beta C_*^2(A(Z\Omega^{\sim 2}))$

$\twoheadrightarrow_\beta C_*^n(A(Z\Omega^{\sim n})) \twoheadrightarrow_\beta C_*^n(A\Omega) \twoheadleftarrow_\beta AI$

for n large enough by 2.2.

Hence $\forall$ closed Z $\mathcal{H} \vdash AZ = K(AI)Z$.

But $\mathcal{H}\eta \nvdash A = K(AI)$ as is proved in §5. ⊠

§3. Conjecture: $\mathcal{H}\omega$ is Π_1^1-complete.

We will give a strong argument to conjecture that $\mathcal{H}\omega$, the λ-calculus extended by the axioms $\mathcal{H}$ and the ω-rule, is Π_1^1-complete.

Given a recursive set T of sequence numbers two terms B_o, B_1 can be defined.

It will be proved that: T is well-founded $\Rightarrow \mathcal{H}\omega \vdash B_o = B_1$. The converse is probably true.

3.1 Def. Let $\forall\alpha\, \exists n\, R(\overline{\alpha}(n))$, with R recursive, be any Π_1^1-predicate.

Define by 1.1 a term $\Box$ such that $\lambda\mathcal{H} \vdash \Box^{\underline{s}} = I$ if $\neg R(s)$

$= \Omega$ if $R(s)$.

As in the proof of 1.5 we do not exhibit explicitly the main parameter c; the whole construction is uniform in c.

Define by the double fixed point theorem, [1] 3.1, terms B,A such that

$$\lambda \vdash B_i^s \twoheadrightarrow_\beta \lambda y.\Box^s(A^s_{i,\underline{o}}\, y)$$

$$\lambda \vdash A^s_{i,\underline{n}} \twoheadrightarrow_\beta \lambda y.[B_i^{s*\langle \underline{n}\rangle}, y\Omega^{\sim n}(A^s_{i,\underline{n+1}}\, y)]$$

where $*$ is the representation of the concatenation function of sequence numbers.

Finally set $B_o = B_{\underline{o}}^{\langle\rangle}$, $B_1 = B_{\underline{1}}^{\langle\rangle}$. Note $B_i^{\langle\rangle} = [[[\ldots][\ldots]\ldots][[\ldots][\ldots]\ldots]\ldots]$.

3.2 Theorem. $\forall\alpha\, \exists n\, R(\overline{\alpha}(n)) \Rightarrow \lambda\mathcal{H}\omega \vdash B_o = B_1$.

Proof.

(i) $R(s) \Rightarrow \lambda\mathcal{H}\omega \vdash B_{\underline{o}}^{\underline{s}} = B_{\underline{1}}^{\underline{s}}$. Indeed $R(s) \Rightarrow \Box^{\underline{s}} = \Omega \Rightarrow B_{\underline{o}}^{\underline{s}} = \Omega = B_{\underline{1}}^{\underline{s}}$.

(ii) $\forall n\, [\lambda\mathcal{H}\omega \vdash B_{\underline{o}}^{\underline{s}*\langle \underline{n}\rangle} = B_{\underline{1}}^{\underline{s}*\langle \underline{n}\rangle}] \Rightarrow \lambda\mathcal{H}\omega \vdash B_{\underline{o}}^{\underline{s}} = B_{\underline{1}}^{\underline{s}}$.

Indeed the assumption implies as in the proof of 2.2 that

$$\lambda\mathcal{H}\omega \vdash B_{\underline{o}}^{\underline{s}}Z = B_{\underline{1}}^{\underline{s}}Z \quad \text{for all closed } Z.$$

Hence the conclusion follows by the ω-rule.

Now it follows by bar induction from (i), (ii) and the well-foundedness that $\lambda\mathcal{H}\omega \vdash B_{\underline{o}}^{\langle\rangle} = B_{\underline{1}}^{\langle\rangle}$, i.e. $\lambda\mathcal{H}\omega \vdash B_o = B_1$. ⊠

For the converse of 3.2, which establishes the conjecture that $\mathcal{H}\omega$ is Π_1^1-complete, a proof-theoretic analysis of $\mathcal{H}\omega$ is needed.

§4. $2^{\aleph_0}$ sensible hard models.

Let T be a λ-theory.

A set S of equations between λ-terms is *independent over T* if for $M=N \in S$
$T + S - \{M=N\} \nvdash M=N$.

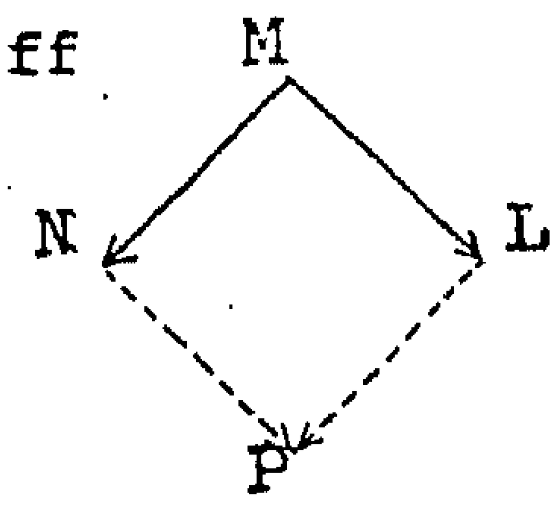
A set of terms X is independent over T if $S_X = \{M=N \mid M,N \in X, M\not\equiv N\}$ is independent over T.

We will construct a countable set of closed terms $\{B\underline{o}, B\underline{1}, \ldots\}$ independent over $\mathcal{H}\eta$. Hence the theories $T_A = \{B\underline{n} = B\underline{o} \mid n \in A\}$, with $A \subset \omega - \{o\}$, are all different.

Since an equation is provable in a λ-theory iff it is true in its term model, it follows that the closed term models of $\mathcal{H} + T_A$ are $2^{\aleph_0}$ sensible hard models. By taking the open term models of $\mathcal{H}\eta + T_A$, $2^{\aleph_0}$ sensible extensional models are obtained.

A relation $\longrightarrow$ between terms has the Church-Rosser (CR) property iff

i.e. $(M \longrightarrow N \ \& \ M \longrightarrow L) \Longrightarrow \exists P \ (N \longrightarrow P \ \& \ L \longrightarrow P)$.



4.1 Def. Let B be a term such that
$Bx \xrightarrow[\beta]{*} \lambda z.z(Bx)$. To be explicit
take $B = \omega\omega$ with $\omega = \lambda bxz.z(bbx)$.

It will be proved that $\{B\underline{o}, B\underline{1}, \ldots\}$ is an independent set over $\mathcal{H}\eta$.
In order to do this we introduce a reduction relation satisfying the Church-Rosser theorem, which generates the equality in the theory $\mathcal{H}\eta A = \mathcal{H}\eta + \{B\underline{n} = B\underline{o} \mid n \in A\}$ for $A \subset \omega - \{o\}$.

4.2 Def. (i) Ω-reduction $\xrightarrow[\Omega]{}$ is defined by

1. $H \xrightarrow[\Omega]{} \Omega$ for all unsolvables H
2. $M \xrightarrow[\Omega]{} N \Rightarrow MZ \xrightarrow[\Omega]{} NZ$, $ZM \xrightarrow[\Omega]{} ZN$, $\lambda x.M \xrightarrow[\Omega]{} \lambda x.N$, for all Z.
3. $M \xrightarrow[\Omega]{} M$.

(ii) $\xrightarrow[\beta\eta\Omega]{} \ = \ \xrightarrow[\beta\eta]{} \cup \xrightarrow[\Omega]{}$.

Clearly $\xrightarrow[\beta\eta\Omega]{}$ generates the equality in $\mathcal{H}\eta$.

4.3 Lemma. $\xrightarrow[\beta\eta\Omega]{*}$ has the CR property.

Proof:
See [3] §2.30. ☒

4.4 Def. (i) $\mathrm{Red}(Bx) = \{C(x) \mid Bx \xrightarrow[\beta]{*} C(x)\}$

(ii) The reduction relation $\xrightarrow[A]{}$ is defined by

1. $C(\underline{n}) \xrightarrow[A]{} C(\underline{o})$ for all $n \in A$ and $C(x) \in \mathrm{Red}(B)$

2. $M \xrightarrow[A]{} N \Rightarrow MZ \xrightarrow[A]{} NZ$, $ZM \xrightarrow[A]{} ZN$, $\lambda x.M \xrightarrow[A]{} \lambda x.N$ (all Z)

3. $M \xrightarrow[A]{} M$.

(iii) $\xrightarrow[\beta\eta\Omega A]{} \; = \; \xrightarrow[\beta\eta\Omega]{} \cup \xrightarrow[A]{}$.

Clearly $\xrightarrow[\beta\eta\Omega A]{}$ generates the equality of $\mathcal{H}\eta A$.

The following notation is used in order to facilitate the computation of the reduction tree of Bx.

4.5 Def. $\square := Bx \equiv \omega\omega x$. If Δ is a term, then
$1\Delta := ((\lambda xz.z\Delta)x)$ and $o\Delta := (\lambda z.z\Delta)$.

4.6 Lemma. $Bx \xrightarrow[\beta]{*} C(x) \iff C(x)$ has the form $i_1 \dots i_n \square$,
$i_1, \dots i_n \in \{o,1\}$ (*)

Proof.

Note that

(i) Each one step β-reduct of $o\Delta$ is $o\Delta'$ where Δ' is a one step β-reduct of Δ.

(ii) Each one step β-reduct of 1Δ is $o\Delta$ or $1\Delta'$ where Δ' is a one step β-reduct of Δ.

(iii) The only one step β-reduct of $\square$ is $1\square$.

From (i)-(iii) it follows that all possible β-reducts of $\square$ are of the form (*). Moreover all terms of the form (*) are reducts of $\square$. ⊠

4.7 Cor. Let $Bx \xrightarrow[\beta]{*} C(x)$. Then

(i) C(x) has no η or Ω redices.

(ii) The only free variable in C(x) occurs at the end.

(iii) $C(\underline{n}) \xrightarrow[\beta]{*} Z \Rightarrow Z = C'(\underline{n})$ with $Bx \xrightarrow[\beta]{*} C'(x)$.

(iv) $C(x) \equiv \lambda c.P \Rightarrow P \equiv cQ$ and $Bx \xrightarrow[\beta]{*} Q$.

Proof.

Immediate. ⊠

4.8 Lemma. $\xrightarrow[A]{}$ has the CR property.

Proof.

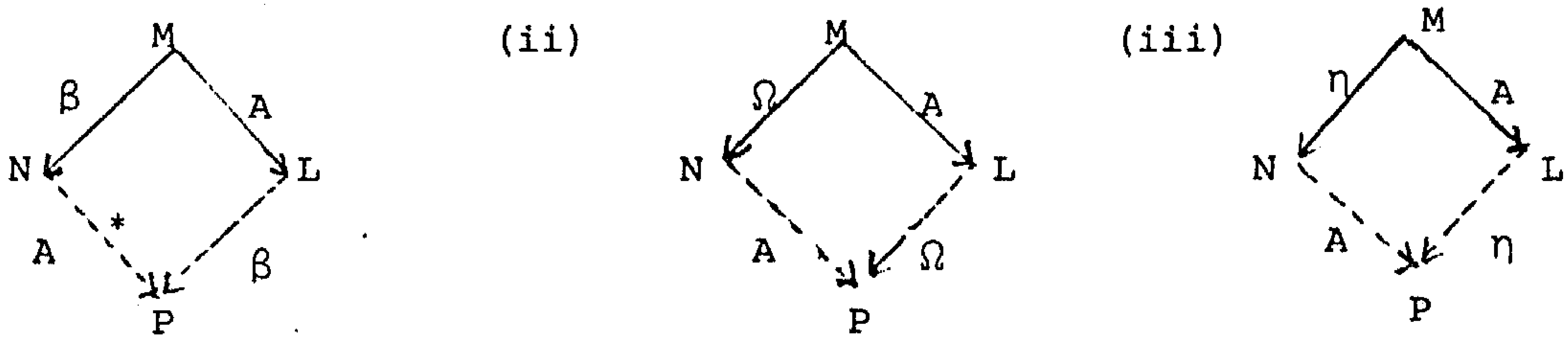
Let two terms be obtained from some term M by replacing some $\underline{n}$ by $\underline{o}$. Hence a common reduct P can be found by making both changes in M. ⊠

4.9 Lemma. $\xrightarrow[\beta\eta\Omega A]{*}$ is CR.

Proof.

By 4.8 and 4.3 $\xrightarrow[A]{}$ and $\xrightarrow[\beta\eta\Omega]{*}$ are CR. So by the lemma of Hindley-Rosen, [5] (1.2), it is sufficient to prove that they commute. For this it is sufficient to prove

(i) (ii) (iii)



(i) Let $R \equiv (\lambda z.V)W$ be the β-redex contracted in $M \xrightarrow[\beta]{} N$ and $C(\underline{n})$ the "A-redex" in $M \xrightarrow[A]{} L$.

Case 1, $R \cap C(\underline{n}) = \emptyset$, is trivial.

Case 2, $R \subseteq C(\underline{n})$. By 4.7 (iii) we are done.

Case 3, $C(\underline{n}) \subset R$. 3.1: $C(\underline{n}) \subseteq W$, is easy. 3.2: $C(\underline{n}) \subseteq V$: since $C(\underline{n})$ is closed this case is trivial. 3.3: $C(\underline{n}) \equiv \lambda z.V$. By 4.7 (iv) $C(\underline{n}) \equiv \lambda z.zC'(\underline{n})$ where $C'(x) \in Red(Bx)$; hence $N \equiv \ldots WC'(\underline{n})\ldots$, $L \equiv \ldots C(\underline{o})W\ldots \equiv \ldots(\lambda z.zC'(\underline{o}))W\ldots$. Take $P \equiv \ldots WC'(\underline{o})\ldots$.

(ii) Let H be the Ω-redex and $C(\underline{n})$ the A-redex in M.

Case 1, $H \cap C(\underline{n}) = \emptyset$, is trivial.

Case 2, $H \subseteq C(\underline{n})$, does not occur, by 4.7 (i).

Case 3, $C(\underline{n}) \subset H$; $H \equiv H'[C(\underline{n})]$, $M \equiv \ldots H\ldots$, $N \equiv \ldots\Omega\ldots$, $L \equiv \ldots H'[C(\underline{o})]\ldots$.

Claim: $H'[C(\underline{o})]$ is unsolvable. So take $P \equiv N$ to complete the diagram.

Proof of claim (see [2] for the concepts of Böhm-tree and solvably equivalence). $C(\underline{n})$ and $C(\underline{o})$ have the same Böhm-tree, hence are solvably equivalent, i.e. for every context D[ ] we have:

$D[C(\underline{n})]$ is unsolvable $\Longleftrightarrow$ $D[C(\underline{o})]$ is unsolvable.

Now take $D[\ ] \equiv H'[\ ]$.

(iii) Let $E \equiv \lambda x.Fx$ be the η-redex and $C(\underline{n})$ be the A-redex in M.

Case 1, $E \cap C(\underline{n}) = \emptyset$, is trivial.

Case 2, $E \subseteq C(\underline{n})$, does not occur, by 4.7 (i).

Case 3, $C(\underline{n}) \subset E$; 3.1: $C(\underline{n}) \equiv Fx$ cannot occur by 4.7 (ii).
3.2: $C(\underline{n}) \subseteq F$: easy. ⊠

4.10 Lemma. For $n \notin A$, $n \neq o$, $\mathcal{H}\eta A \nvdash B\underline{n} = B\underline{o}$.

Proof.

If the equation were provable there would be a term Z s.t. $B\underline{n} \xrightarrow[\beta\eta\Omega A]{*} Z$ and $B\underline{o} \xrightarrow[\beta\eta\Omega A]{*} Z$. By 4.7 (ii) it would follow that $\underline{n}$ and $\underline{o}$ would occur at the same place in Z. ⊠

4.11 **Cor.** Let $o \notin A$, $A' \subset \omega$ and $A \neq A'$. Then
$\mathcal{H}\eta A \neq \mathcal{H}\eta A'$.

Proof.

Let $n \in A$ but $n \notin A'$, say. Then
$\mathcal{H}\eta A \vdash B\underline{n} = B\underline{o}$ and $\mathcal{H}\eta A' \nvdash B\underline{n} = B\underline{o}$ by 4.10. ⊠

4.12 **Theorem.** There are $2^{\aleph_0}$ theories between $\mathcal{H}\eta$ and $\mathcal{H}^*$.

Proof.

By 4.10 each $\mathcal{H}\eta A$ is consistent, hence $\subset \mathcal{H}^*$ by [2],4.8. The result follows from 4.11. ⊠

4.13 **Cor.** (i) There are $2^{\aleph_0}$ sensible hard models.
(ii) There are $2^{\aleph_0}$ sensible extensional models.

Proof.

Note that for λ-theories T,T' $\mathcal{M}^{(0)}(T) = \mathcal{M}^{(0)}(T') \iff T = T'$.
The results follow by taking closed respectively open term models. ⊠

§5. Applications of Gross-reduction.

In the preceding paragraphs we have postponed some technicalities, viz. the proofs of

1. $\mathcal{H}\eta \nvdash Hx\underline{o} = Hy\underline{o}$ where H is a term s.t. $Hx\underline{n} \xrightarrow[\beta]{*} [I, F\underline{n}(Hx\,\underline{n+1})]$ and $F\underline{n} \xrightarrow[\beta]{*} I$ for all n.
2. $\mathcal{H}\eta \nvdash Ox\underline{o} = Oy\underline{o}$ where O is s.t. $Ox\underline{n} \xrightarrow[\beta]{*} \lambda z.z\Omega^{\sim n}\,(Ox\,\underline{n+1})$
3. $\mathcal{H}\eta \nvdash Ax = AI$ where A is s.t. $Ax \xrightarrow[\beta]{*} \lambda z.z(A(x\Omega))$.

In all three cases the proof is similar: if an equation were provable, the terms would have a common reduct by the Church-Rosser theorem for $\mathcal{H}\eta$. In order to prove that this is impossible one wants to show that the first term has in each reduct the free variable x (and it is clear that for no reduct of the second term x occurs freely in it).
The verification of the last statement is still quite intricate, since the reduction trees of the terms involved are quite complicated due to many detour reductions. To overcome this difficulty we use the concept of a (deterministic) Gross-reduction chain which is cofinal in the reduction tree. This cofinality enables us to reduce properties of the whole reduction tree to the more easily computable Gross-reduction chain.

5.1 **Def.** The *Gross-contraction* of a term M, notation M^*, is the complete reduction of M w.r.t. all of its redices.

In [3] it is shown that this definition makes sense for $\mathcal{H}\eta$.

The Gross-reduction-chain of M is the sequence $(M)_o = M$, $(M)_{n+1} = (M)_n^*$.

5.2 Lemma. For $\mathcal{H}\eta$ the Gross-reduction-chain of M is cofinal in the reduction tree of M.

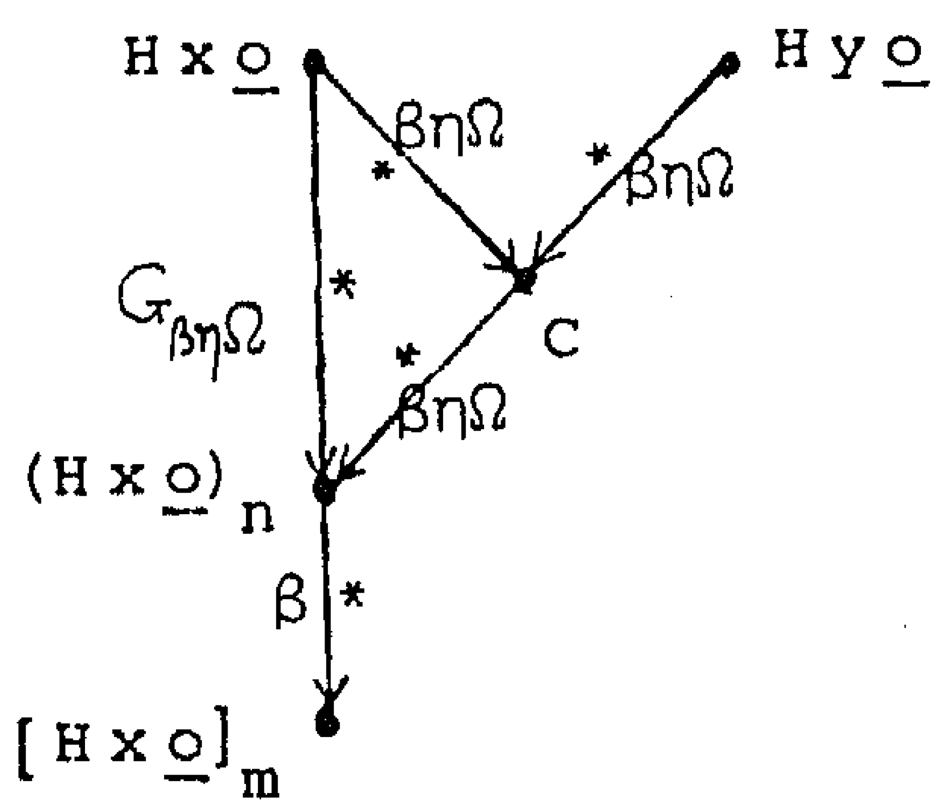
Proof.

See [3]. ☒

5.3 Proof of 1,2,3.

1. Define $[H\,x\,\underline{o}]_n = [\underbrace{I,[I,\ldots[I}_{n\text{ times}}, H\,x\,\underline{n}]\ldots]]$. Simple but tedious calculation shows: $(H\,x\,\underline{o})_n \xrightarrow[\beta]{*} [H\,x\,\underline{o}]_m$ for some m. (*)

Now $\lambda\beta\eta\Omega \nvdash H\,x\,\underline{o} = H\,y\,\underline{o}$. Suppose not, then by (5,2), (5.3), (*) we have



for some n,m. But $x \in FV([H\,x\,\underline{o}]_m)$, hence $x \in FV(H\,y\,\underline{o})$, contradiction.

2. Define $[O\,x\,\underline{o}]_n = \lambda y.y(y\Omega(y\Omega\Omega(\ldots(y\Omega^{\sim n}(O\,x\;\underline{n+1}))\ldots)))$. Then $(O\,x\,\underline{o})_n \xrightarrow[\beta]{*} [O\,x\,\underline{o}]_n$ as direct computation shows.

The rest of the proof is entirely analogous to that of 1.

3. $Ax \equiv \omega\omega x$, $\omega \equiv \lambda axz.z(aa(x\Omega))$.

Define $[Ax]_n = \underbrace{\lambda z.z(\lambda z.z(\ldots(\lambda z.z}_{n\text{ times}}(\omega\omega(x\Omega^{\sim n})))\ldots))$. A simple calculation shows $(Ax)_n \xrightarrow[\beta]{*} [Ax]_n$. The rest of the proof is (almost) analogous to that of 1. ☒

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CHAPTER II

SOME NOTES ON LAMBDA REDUCTION

Introduction. In sections 1 and 2 the Church-Rosser theorem is proved for the $\lambda\beta$ - respectively $\lambda\beta\eta(\Omega)$-calculus.

To add another proof of this theorem needs some motivation, especially since for the $\lambda\beta$-calculus there is a shorter one due to Tait and Martin-Löf (see [1]), which can be extended to the $\lambda\beta\eta$-calculus using the lemma of Hindley-Rosen, see [9].

For the $\lambda\beta$-calculus a proof of the Church-Rosser theorem is given in [7], via the finite developments (FD) theorem, which idea goes back to Curry, see also [6]. As a byproduct [6] shows the possibility of defining Gross-reduction and its cofinality.

We prove the FD theorem using a labeling of variables. This method applies also to the $\lambda\beta\eta(\Omega)$-calculus, which is the extension of the theory obtained by adding extensionally (and equating all unsolvable terms).

Now the motivation of the first three sections of this paper is that it gives a straightforward proof of the FD theorem which works for all theories considered. Secondly, we establish the cofinality of Gross-reduction for the $\lambda\beta\eta\Omega$-calculus, which result was used in [2].

In section 4 we consider reduction strategies; i.e. functions that assign to a term one of its reducts. We distinguish various kinds of strategies, and some known strategies are classified accordingly. Furthermore the (non)existence of certain kinds of (recursive) strategies is proved.

In section 5 we prove that there exists a recursive strategy that finds an infinite reduction sequence if it exists.

In section 6 we prove by one method two theorems in [4], [5], viz. the postponement of η-reductions and the fact that β- and $\beta\eta$-norma-normalizibility are the same.

Finally in section 7 some non-normalizing S-terms are constructed.

1. THE CHURCH-ROSSER THEOREM FOR THE $\lambda\beta$-CALCULUS.

1.0. The finite developments (FD) theorem states that for any set of redices in a term M, all developments of $\mathcal{R}$ are finite.
See [6] for terminology.
We formulate and prove FD using underlined and labeled terms.
The underlining specifies and keeps track of a set $\mathcal{R}$ of redices.
The labeling is used to prove that underlined terms strongly normalize.

1.1.Def. lab.$\underline{\lambda\beta}$ is the set of labeled and underlined λ-terms, defined by 1. $x^n \in$ lab.$\underline{\lambda\beta}$ for every variable x and every $n \geqslant 1$.

2. $M \in$ lab.$\underline{\lambda\beta} \Longrightarrow \lambda x.\ M \in$ lab.$\underline{\lambda\beta}$
3. M, $N \in$ lab.$\underline{\lambda\beta} \Longrightarrow MN \in$ lab.$\underline{\lambda\beta}$ and $(\underline{\lambda}x.\ M)N \in$ lab.$\underline{\lambda\beta}$

Remark that only variables not preceded by λ are labeled.

1.2.Def. $\xrightarrow[\text{lab.}\underline{\beta}]{}$ is one step underlined β-reduction between terms $\in$ lab.$\underline{\lambda\beta}$ defined by

$$C\left[(\underline{\lambda}x.\ M)N\right] \xrightarrow[\text{lab.}\underline{\beta}]{} C\left[\,[N|x]\ M\,\right], \text{ where}$$

N, $M \in$ lab.$\underline{\lambda\beta}$, $C[\ \ ] \in$ lab.$\underline{\lambda\beta}$ is a context with one hole, and $[\ |\ ]$ is the substitution-operator defined by $[N|x]\ x^n = N$ and the usual other rules.
Remark that β-redices whose head-λ is not underlined, are not allowed to contract.

1.3.Def. The same system without labels will be called $\underline{\lambda\beta}$, the corresponding reduction $\xrightarrow[\underline{\beta}]{}$.

1.4. Def. Let $M \in lab.\underline{\lambda\beta}$ and $N \subseteq M$ (N is subterm of M). Then:

$|N|$ = sum of the labels occurring in N

Remark that $|N| > 0$.

1.5. Def. Let $M \in lab.\underline{\lambda\beta}$. The labeling of M is called <u>decreasing</u> iff for every $\underline{\beta}$ -redex $(\underline{\lambda}x.\ P)Q$ in M we have $|x| > |Q|$ for all $x \in P$.

Example: $(\underline{\lambda}x.\ x^6x^7)(\lambda x.\ x^2x^3)$ is decreasingly labeled, but

$(\underline{\lambda}x.\ x^4x^7)(\lambda x.\ x^2x^3)$ is not.

1.6. Lemma. Let $M \in \underline{\lambda\beta}$. Then there is a decreasing labeling for M.

Proof: number the occurrences in M of variables from right to left, starting with 1. Give the n-th occurrence the label 2^n.

Example: if $M = xy((\underline{\lambda}z.z)x)$ the result is $x^{16}y^8((\underline{\lambda}z.z^4)x^2)$.

Obviously this is a decreasing labeling, since $2^n > 2^{n-1} + \ldots + 2$. ⊠

1.7. Lemma. Let $M \in lab.\underline{\lambda\beta}$, such that M's labeling is decreasing, and let $M \xrightarrow[lab.\underline{\beta}]{} N$. Then (i) $|M| > |N|$

(ii) N's labeling is again decreasing.

Proof of (i). Let $(\underline{\lambda}x.\ P)Q$ be the $\underline{\beta}$ -redex contracted in $M \longrightarrow N$. Each $x \in P$ is replaced by Q; since $|x| > |Q|$ this means that the sum of the labels in the contractum is getting less. Also if P contains no x this holds, since Q vanishes and $|Q| > 0$. $⊠_{(i)}$

Proof of (ii). Let $(\underline{\lambda}x'.\ P')Q'$ be the $\underline{\beta}$ -redex of whose residuals we must check that they satisfy the condition for a labeling to be decreasing. For the numbering of the cases, see the cases 11.. in the scheme of relative positions of redices.

Scheme of relative positions of redices.

	1 $R \equiv (\underline{\lambda}x.P)Q$	2 $E \equiv \underline{\lambda}y.Dy$	3 $\underset{\cdots}{H}$
1 $R' \equiv (\underline{\lambda}x'.P')Q'$	111 $R' \cap R = \emptyset$ 112 $R' \subset R$: 1121 $R' \subsetneq P$; 1122 $R' \subsetneq Q$. 113 $R' \equiv R$ 114 $R' \supset R$: 1141 $R \subsetneq P'$; 1142 $R \subsetneq Q'$	121 $R' \cap E = \emptyset$ 122 $R' \subset E$: 1221 $R' \subsetneq D$; 1222 $R' \equiv Dy$ 123 $R' \supset E$: 1231 $E \subsetneq P'$; 1232 $E \equiv \lambda x'.P'$; 1233 $E \subsetneq Q'$	131 $R' \cap H = \emptyset$ 132 $R' \subset H$ 133 $R' \equiv H$ 134 $R' \supset H$: 1341 $H \subsetneq P'$.; 1342 $H \equiv \lambda x'.P'$; 1343 $H \subsetneq Q'$
2 $E' \equiv \underline{\lambda}y'.D'y'$		221 $E' \cap E = \emptyset$ 222 $E' \subset E$ $(\Rightarrow E' \subsetneq D)$ 223 $E' \equiv E$ 224 $E' \supset E$ $(\Rightarrow E \subsetneq D')$	231 $E' \cap H = \emptyset$ 232 $E' \subset H$ 233 $E' \equiv H$ 234 $E' \supset H$: 2341 $H \subsetneq D'$; 2342 $H \equiv D'y'$
3 $\underset{\cdots}{H'}$			331 $H' \cap H = \emptyset$ 332 $H' \subset H$ 333 $H' \equiv H$ 334 $H' \supset H$

Of the cases 11.. (all the other cases are for use in section 2) only two are not trivial:

$$1121 \quad M = \ldots(\underline{\lambda}x.\ \boxed{\ldots x \ldots (\underline{\lambda}x'.\ P'(x))Q'(x)\ldots})Q \quad \ldots$$
$$\downarrow$$
$$N = \ldots \quad \boxed{\ldots Q \ldots (\underline{\lambda}x'.\ P'(Q))Q'(Q)\ldots} \quad \ldots$$

(Here $Q'(Q) = \left[Q \middle| x\right] Q'$)

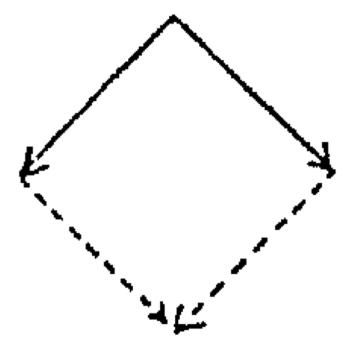
Since $|x| > |Q|$ for all $x \in P$, $|Q'(Q)| \leqslant |Q'|$. Also $|x'| > |Q'|$ for all $x' \in P'$. Hence $|x'| > |Q'(Q)|$ for all $x' \in P'$.

$$1142. \quad M = \ldots(\underline{\lambda}x'.\ P')\ \boxed{..(\underline{\lambda}x.\ P(x))Q\ ..} \ \ldots \text{ where } \square = Q'$$
$$\downarrow$$
$$N = \ldots(\underline{\lambda}x'.\ P')\ \boxed{..\quad P(Q)\quad ..} \ \ldots \text{ where } \square = Q''$$

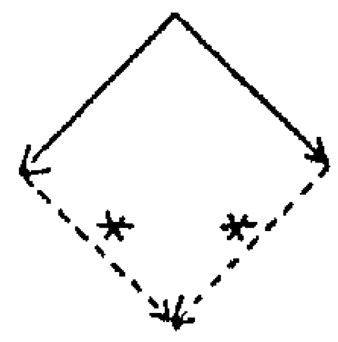
Now $|(\underline{\lambda}x.\ P(x))Q| > |P(Q)|$, so $|Q'| > |Q''|$, hence in N for all $x' \in P'$ we have $|x'| > |Q''|$. $\boxtimes$

1.8. Coroll. (Finite developments theorem: FD).
If $M \in \underline{\lambda\beta}$, then every $\underline{\beta}$-reduction sequence starting with M terminates.
Proof. By 1.6 and 1.7 since in a reduction labels can be taken along. ⊠

1.9. Def. Let $\longrightarrow$ be an arbitrary binary relation. Then
(i) $\xrightarrow{*}$ is the transitive and reflexive closure of $\longrightarrow$.
(ii) $\xrightarrow{*\neq 0}$ is the transitive closure of $\longrightarrow$.
(iii) $\longrightarrow$ has the Church-Rosser property (is CR) iff



(iv) $\longrightarrow$ has the weak Church-Rosser property (is weakly CR) iff



(v) if $M \longrightarrow N$, N will be called a successor of M.
(vi) an endpoint is a point without successors.
(vii) M has an endpoint N iff $M \xrightarrow{*} N$ where N is an endpoint.

1.10. Lemma. $\xrightarrow[\beta]{}$ is weakly CR.
Proof: A trivial analysis of a few cases, using

$$M \xrightarrow[\beta]{} M',\ N \xrightarrow[\beta]{} N' \implies [N|x]M \xrightarrow[\beta]{} [N'|x]M' \ . ⊠$$

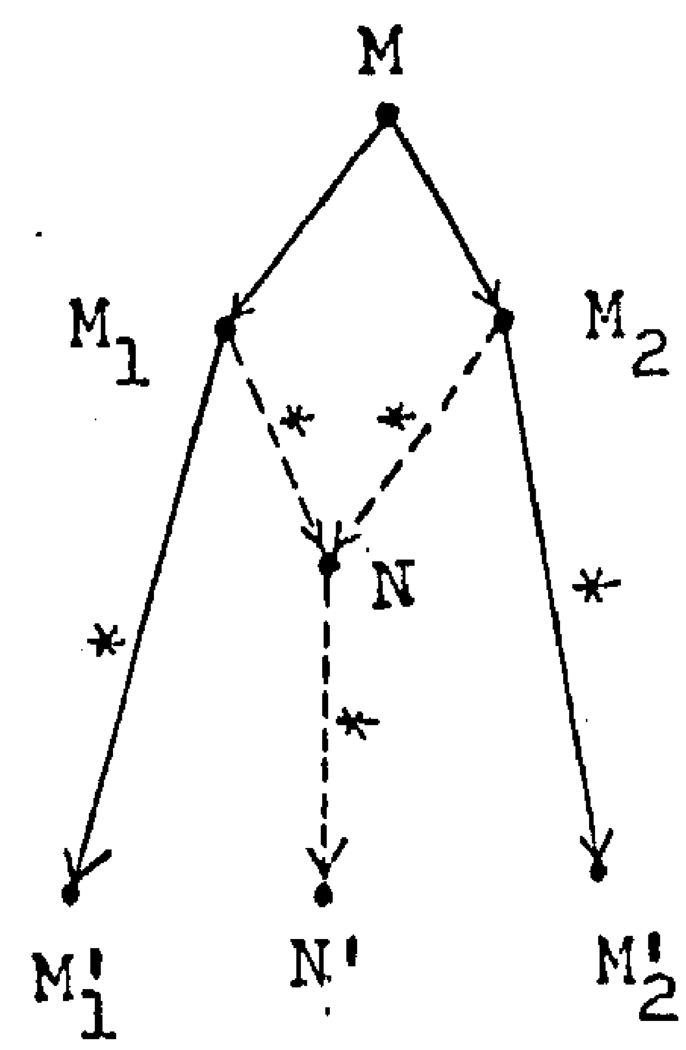
1.11. Lemma. Let $\longrightarrow$ be a reduction relation such that (1) every reduction sequence terminates and (2) $\longrightarrow$ is weakly CR. Then every term has a unique endpoint.

Proof. (Bar induction)

A term is bivalent if M has at least two different endpoints, otherwise univalent.

Claim: if M is bivalent, then M has a bivalent successor.

Indeed, let $M \longrightarrow M_1 \xrightarrow{*} M_1^!$ and $M \longrightarrow M_2 \xrightarrow{*} M_2^!$ be two reduction sequences terminating in different endpoints $M_1^!$, $M_2^!$, possibly with $M_1 = M_2$.
By the weak CR property there exists a term N such that $M_1 \longrightarrow N$, $M_2 \longrightarrow N$. Let N' be an endpoint of N. If, say, M_1 were univalent, then $M_1^! = N'$, therefor $M_2^! \neq N'$, and hence M_2 is bivalent, which proves the claim.

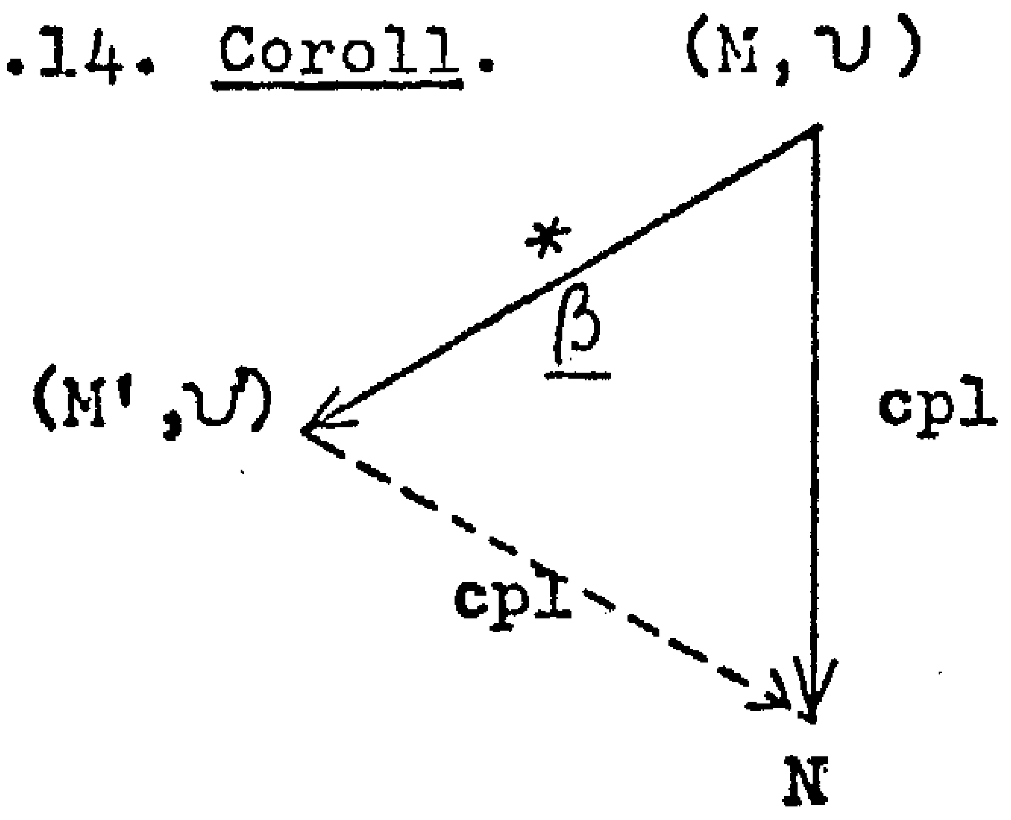


By the claim a bivalent term would yield a non-terminating reduction-sequence, contradicting (1). Hence each term is univalent. ⊠

1.12. <u>Cor.</u> (FD^+). Let $M \in \underline{\lambda\beta}$. Then every maximal $\underline{\beta}$-reduction sequence starting with M, terminates in a unique result N, which will be called the <u>complete reduct</u> of M.

1.13. <u>Notation</u>. (i) If $M' \in \underline{\lambda\beta}$ we will also write $M' = (M, \upsilon)$ where $M \in \lambda$ and υ is the underlining of M, i.e. the set of occurrences of $\underline{\lambda}$.
(ii) $(M, \emptyset)$ where $\emptyset$ is the empty underlining, will be identified with $M \in \lambda$. ($M \in \lambda$ means: M is a λ-term, without underlining.)
(iii) If $N = (N, \emptyset)$ is the complete reduct of $(M, \upsilon) \in \underline{\lambda\beta}$, we will write $(M, \upsilon) \xrightarrow[cpl]{} N$.

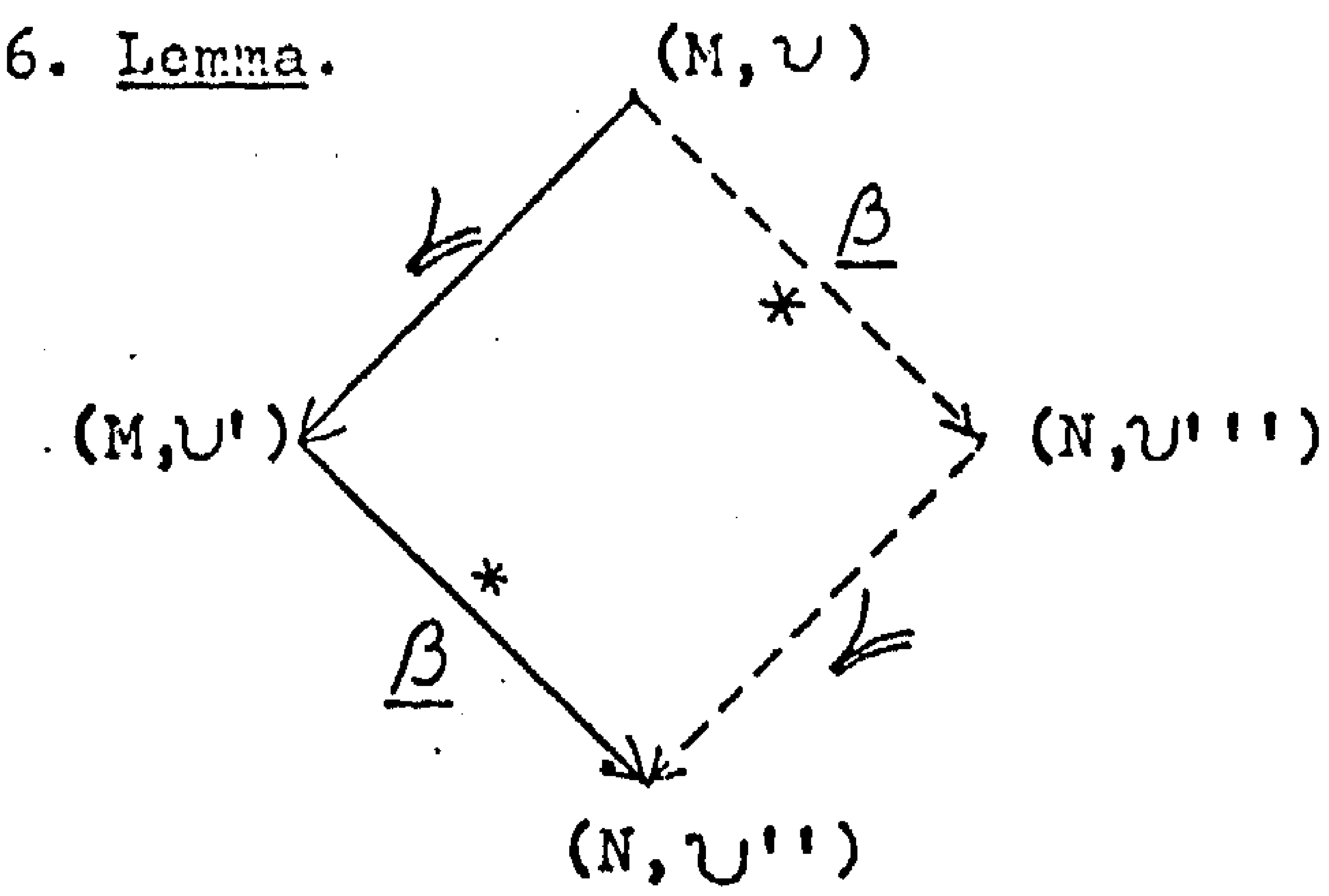
1.14. <u>Coroll.</u>



Proof: 1.12. ⊠

1.15. <u>Def</u>. (i) $(M, \upsilon) \succcurlyeq (M, \upsilon')$ iff $\upsilon' \subseteq \upsilon$.
(ii) $(M, \upsilon) + (M, \upsilon') = (M, \upsilon \cup \upsilon')$

1.16. <u>Lemma</u>.



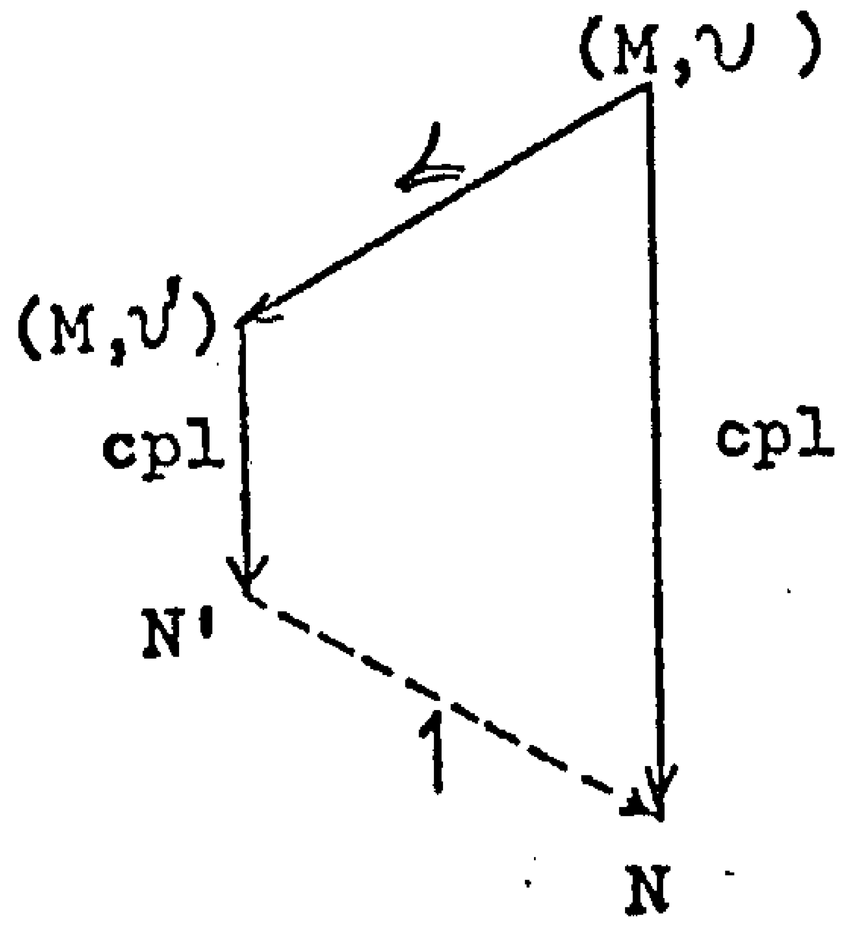
Proof: it is sufficiënt to consider the case where $(M, \upsilon') \xrightarrow[\underline{\beta}]{*} (N, \upsilon'')$ is one step; and for this case the lemma is trivial. ⊠

1.17. Def. Let $M, N \in \lambda$. Then

$$M \xrightarrow[1]{} N \iff (M,\upsilon) \xrightarrow[\text{cpl}]{} N \text{ for some underlining } \upsilon \text{ of } M.$$

Remark that $\xrightarrow[1]{} \supseteq \xrightarrow[\beta]{}$ (the usual one step β-reduction between λ-terms). This follows by considering the underlining of just one β-redex.

1.18. Lemma.



Proof.

a: lemma 1.16.

b: coroll. 1.14.

c: def. 1.17.

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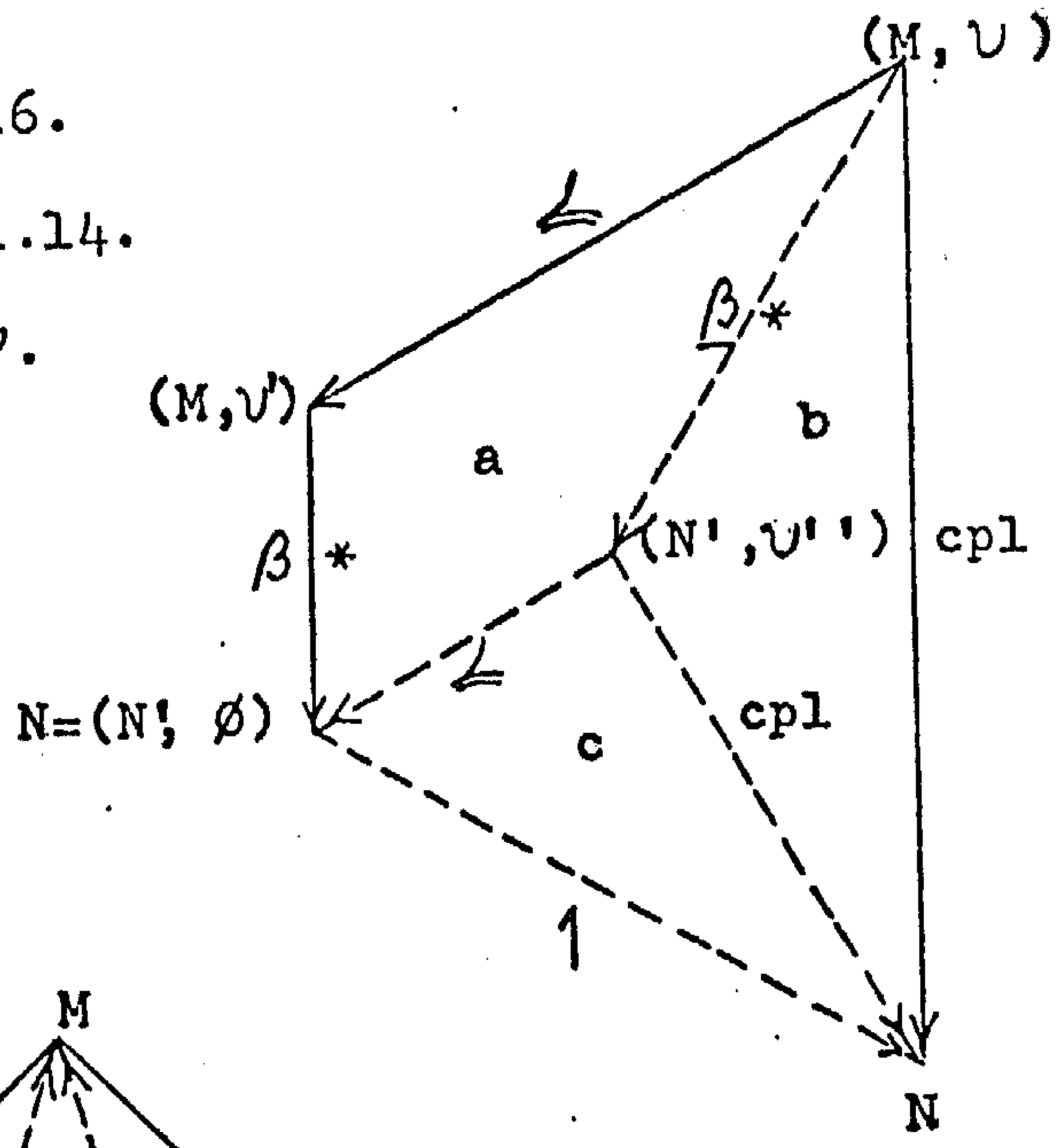

1.19. Lemma. $\xrightarrow[1]{}$ is CR.

Proof:

a, a': def. 1.17.

b: def. 1.15.

c, c': lemma 1. 18.

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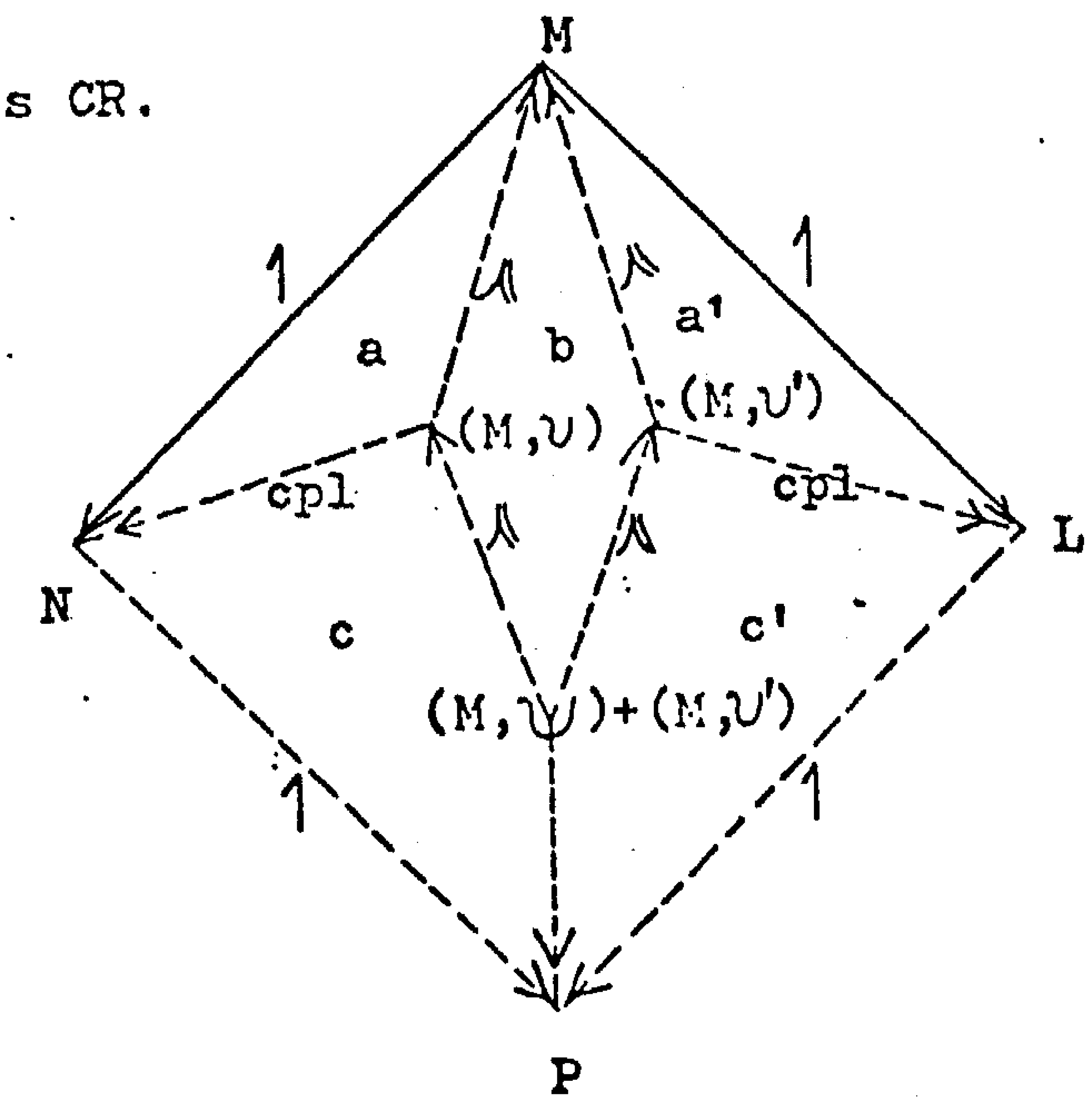


1.20. Theorem (CHURCH - ROSSER): $\xrightarrow[\beta]{*}$ is CR.

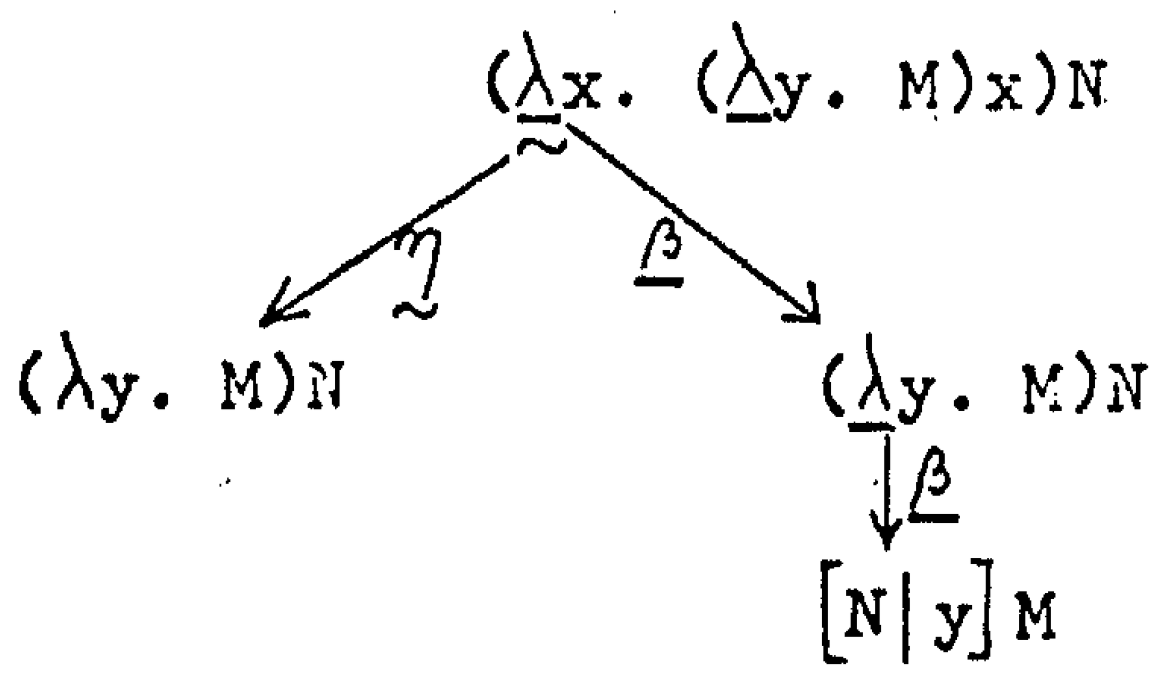
Proof. $\xrightarrow[\beta]{} \subseteq \xrightarrow[1]{}$ (see remark at 1.17) and $\xrightarrow[1]{} \subseteq \xrightarrow[\beta]{*}$,

hence $\xrightarrow[\beta]{*} = \xrightarrow[1]{*}$.

$\xrightarrow[1]{}$ is CR, hence $\xrightarrow[1]{*}$ is CR. ☒

2. THE CHURCH-ROSSER THEOREM FOR THE $\lambda\beta\eta(\Omega)$-CALCULUS

2.0. Introduction. The concept of underlining and underlined reduction in the $\lambda\beta$-calculus generalizes directly to the $\lambda\beta\eta$-calculus, so we want to use the method of section 1 for the $\lambda\beta\eta$-calculus. But then there is a problem when one tries to define the analogon of $\xrightarrow[1]{}$; using the method of the labels as in section 1, the FD theorem is still valid, but 1.10 fails, as is shown by the example in [4] p.119:



Here $\_$ denotes a specified β-redex and $\sim$ a specified η-redex, in our notation.
That FD is still valid can be easily verified, but we will not do this because by a simple restriction (on what sets of specified redices, i.e. what underlinings, are allowed) we have moreover FD^+ (see 1.12) for the $\lambda\beta\eta$-calculus.

$\mathcal{H}$ is the λ-theory in which all unsolvable terms are equated. As with ordinary conversion, it is useful to have a reduction relation satisfying the Church-Rosser theorem which generates equality in $\mathcal{H}$. This will be $\beta\eta\Omega$-reduction which is defined by adding Ω-contraction defined by

$$C[H] \xrightarrow[\Omega]{} C[\Omega] \quad ,$$

where $\Omega = (\lambda x.\ xx)(\lambda x.\ xx)$ and H is unsolvable.

The CR theorem is proved first for a restricted form of Ω-reduction (the Ω-redices have to be maximal w.r.t. inclusion of subterms; this is called Ω'-reduction) from which the theorem is proved for the general form. While β- and η-redices interfere in a nasty way, there is no interference of β-, η-redices and Ω'-redices.

The CR theorem for the $\lambda\beta\eta$-calculus can be proved more easily than below by using the lemma of Hindley-Rosen [9] 1.2. The benefit of the method here is that it gives an easy proof for the theorems in section 3.

2.1. <u>Def</u>. (i) To the language of the lab.$\underline{\lambda\beta}$ -terms we add an extra symbol $\sim$, which will be used to indicate η-redices, relative to which η-reduction is allowed. $\sim$ will be written under the head $-\lambda$ of an η-redex: $\underset{\sim}{\lambda} y.\ My$ $\quad (y \notin FV(M))$

Formally: extend def. 1.1 with the extra rule for term-formation:

$$4.\ M \in \text{lab.}\underline{\lambda\beta\eta} \Longrightarrow \underset{\sim}{\lambda} y.\ My \in \text{lab.}\underline{\lambda\beta\eta} \qquad (y \notin FV(M))$$

$\underline{\lambda\beta\eta}$ is the corresponding set of terms without labels.

<u>Remark</u>. In a (lab.)$\underline{\lambda\beta\eta}$-term a λ can be underlined with $\_$ if it is a β-redex λ and with $\sim$ if it is an η-redex λ, but no λ can be underlined by both $\_$ and $\sim$.

2.2. <u>Def</u>. Let $M, N \in \underline{\lambda\beta\eta}$. Then $M \xrightarrow[\beta]{} N$ is defined as follows.

Let $(\underline{\lambda} x.\ P)Q \subseteq M$ be the β-redex to reduce.

Case 1. $M \equiv C\left[\underset{\sim}{\lambda} y.\ (\underline{\lambda} x.\ P(x))y\right]$, i.e. $Q \equiv y$, $y \notin FV(P)$, and $C[\ ]$ is some context (with one hole). Then

$$M \xrightarrow[\beta]{} C\left[\lambda y. P(y)\right]$$

Case 2. If not case 1, then

$$M \equiv C\left[(\underline{\lambda} x.\ P(x))Q\right] \xrightarrow[\beta]{} C\left[P(Q)\right]$$

2.3. <u>Def</u>. Let $M, N \in \underline{\lambda\beta\eta}$. Then $M \xrightarrow[\eta]{} N$ is defined as follows.

Let $\underset{\sim}{\lambda} y.\ Dy \subseteq M$ be the $\underset{\sim}{\eta}$-redex to reduce.

Case 1. $M \equiv C\left[\underset{\sim}{\lambda} y.\ (\underline{\lambda} x.\ P)y\right]$, i.e. $Dy \equiv (\underline{\lambda} x.\ P)y$, $y \notin FV(P)$, and $C[\ ]$ some context. Then

$$M \xrightarrow[\eta]{} C\left[\lambda x.\ P\right]$$

Case 2. If not case 1, then

$$M \equiv C\left[\underset{\sim}{\lambda} y.\ Dy\right] \xrightarrow[\eta]{} C\left[D\right]$$

2.4. Def. $\xrightarrow[\text{lab.}\underline{\beta}]{}$ and $\xrightarrow[\text{lab.}\underline{\eta}]{}$ are the corresponding reduction relations in the presence of labels.

2.5. Remark. The underlinings and underlined reductions formalize the same concept of residual of β- and η-redices as in [4] page 117, 118.

2.6. Def. (i) Let $M \in \lambda$ and $N \subseteq M$. The subterm N is a maximal unsolvable subterm iff 1. N is unsolvable

and 2. (all L) $N \subseteq L \subseteq M$ & L unsolvable $\Rightarrow N = L$.

(ii) (lab.)$\underline{\lambda\beta\eta\Omega'}$ is (lab.)$\underline{\lambda\beta\eta}$ where maximal unsolvable subterms can be underlined with a dot-line .... . Such underlined maximal unsolvable subterms will be called $\underset{\cdots}{\Omega'}$-redices.

(iii) $\xrightarrow[\underset{\cdots}{\Omega'}]{}$, one step $\underset{\cdots}{\Omega'}$-reduction in $\underline{\lambda\beta\eta\Omega'}$, is defined by

$$C[\underset{\cdots}{H}] \xrightarrow[\underset{\cdots}{\Omega'}]{} C[\Omega]$$

where $\Omega = (\lambda x.\ xx)(\lambda x.\ xx)$ and $C[\ \ ]$ is some context with one hole .

$\xrightarrow[\text{lab.}\underset{\cdots}{\Omega'}]{}$ is the corresponding reduction in lab.$\underline{\lambda\beta\eta\Omega}$ defined by

$$C[\underset{\cdots}{H}] \xrightarrow[\text{lab.}\underset{\cdots}{\Omega'}]{} C[\Omega^1]$$

where $\Omega^1 \equiv (\lambda x.\ x^1 x^1)(\lambda x.\ x^1 x^1)$ and $C[\ \ ]$ is some context in lab.$\underline{\lambda\beta\eta\Omega}$.

2.7. Def. Extension of (i) $\xrightarrow[\text{(lab)}\underline{\beta}]{}$, (ii) $\xrightarrow[\text{(lab.)}\underline{\eta}]{}$ from (lab.)$\underline{\lambda\beta\eta}$ to (lab.)$\underline{\lambda\beta\eta\Omega'}$.

(i) Let $M' \in$ (lab.)$\underline{\lambda\beta\eta\Omega'}$, then M' can be considered as a pair (M, U) where $M \in$ (lab.)$\underline{\lambda\beta\eta}$ and U is the set of occurrences of $\underset{\cdots}{\Omega'}$-redices. Let $M \xrightarrow[\text{(lab.)}\underline{\beta}]{} N$, and let V be the set of descendants of the subterms in U. (Here the concept of descendant is in a natural way suggested by keeping track of underlinings during the reduction.)

Further, let W be the set of maximal unsolvable subterms of N generated by V as follows: $H \in W \iff$ 1. H is maximal unsolvable subterm

and 2. $\exists\, L \subseteq H$ such that $L \in V$.

Now define $\xrightarrow[(\text{lab.})\underline{\beta}]{}$ in $(\text{lab.})\underline{\lambda\beta\eta\Omega'}$ by

$$(M, U) \xrightarrow[(\text{lab.})\underline{\beta}]{} (N, W).$$

Example: $(\underline{\lambda} x.\ x(\underset{\cdots}{\Omega} x))\Omega \xrightarrow[\underline{\beta}]{} \underset{\cdots\cdots\cdot}{\Omega(\Omega\Omega)}$

(ii) Similar definition for $\xrightarrow[(\text{lab.})\underset{\sim}{\eta}]{}$ in $(\text{lab.})\underline{\lambda\beta\eta\Omega'}$.

(Remark: in this case V = W)

2.8. Def. $\xrightarrow[(\text{lab.})\underline{\beta\eta\Omega'}]{} \;=\; \xrightarrow[(\text{lab.})\underline{\beta}]{} \cup \xrightarrow[(\text{lab.})\underset{\sim}{\eta}]{} \cup \xrightarrow[(\text{lab.})\underset{\cdots}{\Omega'}]{}$

2.9. Def. Let $M \in \text{lab.}\underline{\lambda\beta\eta\Omega'}$. Again, M's labeling is called decreasing iff all its $\underline{\beta}$-redices $(\underline{\lambda} x.\ P)Q$ are decreasingly labeled, i.e. for all $x \in P$: $|x| > |Q|$.

2.10. Lemma. Let $M \in \underline{\lambda\beta\eta\Omega'}$. Then there is a decreasing labeling for M.

Proof: same as of lemma 1.6. ☒

2.11. Lemma. Let $M \in \text{lab.}\underline{\lambda\beta\eta\Omega'}$, such that M's labeling is decreasing, and let $M \xrightarrow[\text{lab.}\underline{\beta\eta\Omega'}]{} N$. Then (i) $|M| > |N|$

(ii) N's labeling is again decreasing.

Proof of (i). In $M \xrightarrow[\underline{\beta\eta\Omega'}]{} N$ is contracted 1) a $\underline{\beta}$-redex, 2) an $\underset{\sim}{\eta}$-redex, or 3) an $\underset{\cdots}{\Omega'}$-redex.

Case 1) was considered in the previous section.

2) Let the $\underset{\sim}{\eta}$-redex be $\underline{\lambda} y.Dy^k$, $y \notin FV(D)$. Then

$$|\underline{\lambda} y.Dy^k| = k + |D| > |D| \text{ since } k > 0.$$

3) Let H be this $\underset{\dots}{\Omega}'$ -redex. Then $\underset{\dots}{H} \xrightarrow[\omega\beta\eta\Omega]{} (\lambda x.\ x^1 x^1)(\lambda x.\ x^1 x^1)$.

A simple analysis shows that if H is unsolvable, then $H \equiv \Omega$, containing 4 variables, or H contains more than 4 variables. In the latter case, $|H| \geqslant 5$, since the labels are $\geqslant 1$; so indeed $|H| > |\text{contractum of } H|$.

In the former case, trouble seems to arise when $\underset{\dots}{H} \equiv \underset{\dots}{(\lambda x.x^1x^1)(\lambda x.x^1x^1)}$ but this case does not occur since the labels in the initial labeling, i.e. the labeling in the preceding lemma, are $\geqslant 2$. Hence if a label of some x is 1, then x occurs in $(\lambda x.\ x^1 x^1)(\lambda x.x^1x^1)$ which can only be a contractum of an $\underset{\dots}{\Omega}'$ -redex and has therefore no underlining. ⊠(i)

Proof of (ii). We have to check that all residuals of the $\underline{\beta}$ -redices in M are again decreasingly labeled. To do this we use the first row of the scheme of relative positions of redices, i.e. the cases 1... Here R, E or H are the redices contracted in $M \xrightarrow[\beta\eta\Omega']{} N$, and R' is the $\underline{\beta}$ -redex in M whose residuals in N we have to check. Only the non-trivial cases will be mentioned.

Case 1222. Remark that R' has no underlined residual:

$$\underset{\sim}{\lambda} y.\ Dy \equiv \underset{\sim}{\lambda} y.\ (\underline{\lambda} x'.\ P')y \xrightarrow[\beta\eta\Omega']{} \lambda x'.\ P'$$

1232 does not occur, since _ and $\sim$ are not allowed to coincide.

1233. Here $Q' \equiv C[E]$ for some context $C[\ ]$. Now if $x' \in P'$, then $|x'| > |C[E]|$ by the hypothesis of the lemma, and $|E| > |D|$, hence in the residual of R': $|x'| > |C[D]|$, so this $\underline{\beta}$ -redex is again decreasingly labeled.

132, 133. R' has no underlined residual.

1342 does not occur, since $\lambda x'$, P' is not maximal unsolvable.

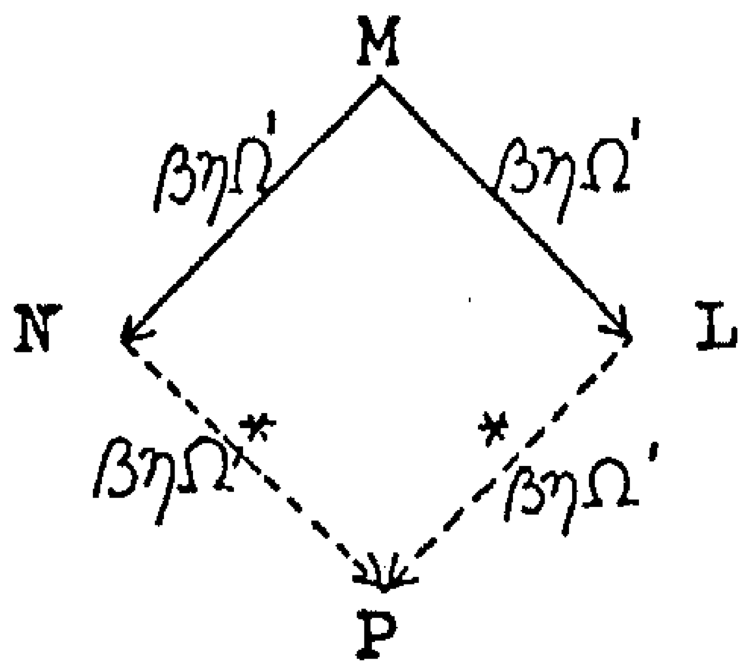
1343. Let $Q' \equiv C[\underset{\dots}{H}]$.Now if $x' \in P'$ then $|x'| > |C[\underset{\dots}{H}]| > |C[\Omega]|$.

⊠

2.12. Coroll. If $M \in \underline{\lambda\beta\eta\Omega'}$, then every $\underline{\beta\eta\Omega'}$-reduction sequence starting with M terminates.

Proof. By 2.10 and 2.11 since in a reduction labels can be taken along. ☒

2.13. Lemma. $\xrightarrow[\beta\eta\Omega']{}$ is weakly CR, i.e.:



Proof. See the scheme of relative positions of redices. Let R', E', H' be the redex contracted in $M \xrightarrow[\beta\eta\Omega']{} N$ and R, E, H the redex contracted in $M \xrightarrow[\beta\eta\Omega']{} L$. Only the non-trivial cases will be mentioned.

The cases 11.. are analogous to the corresponding cases in §1.

Case 1222. $M \equiv C[\lambda y.\ (\underline{\lambda} x'.\ P'(x'))y]$ where $C[\ ]$ is some context, $N \equiv C[\lambda y.\ P'(y)]$ and $L = C[\lambda x'.\ P'(x')]$. Take $P \equiv N \equiv L$.

1232. $M \equiv C[(\underset{\sim}{\underline{\lambda}} y.\ Dy)Q']$. However, _ and ~ are not allowed to coincide (see 2.1.(i)), so this case does not occur.

132, 133. Notice that the set of unsolvables is closed under β-reduction.

232, 233. Notice that the set of unsolvables is closed under η-reduction. 1341 and 2341 follow by 2.29(3).

1342, 2342, 332 and 334 do not occur because an $\underset{\cdots}{\Omega'}$-redex is defined as a maximal unsolvable subterm. ☒

2.14. Coroll. Let $M \in \underline{\lambda\beta\eta\Omega'}$. Then every maximal $\underline{\beta\eta\Omega'}$-reduction sequence starting with M, terminates in a unique result, which will be called the complete reduct of M.

Proof. 2.12, 2.13 and 1.11. ☒

2.15. Notation. Analogon of 1.13 for $\underline{\lambda\beta\eta\Omega'}$ and $\xrightarrow[\beta\eta\Omega']{}$ in stead of $\underline{\lambda\beta}$ and $\xrightarrow[\beta]{}$. $\mathcal{V}$ is now a triple consisting of a set of occurrences of $\underline{\lambda}$, a set of λ's, and a set of maximal unsolvable subterms.

2.16. Coroll. Analogous to coroll. 1.14.

2.17. Def. Let M, N $\in \lambda$. Then $M \xrightarrow[1]{} N \iff (M, \upsilon) \xrightarrow[cpl]{} N$ for some underlining υ of M.

2.18. Remark. Up to here we generalized the reduction $\xrightarrow[1]{}$ from the $\lambda\beta$- to the $\lambda\beta\eta\Omega'$-calculus. The next step, as in §1, is to prove that $\xrightarrow[1]{}$ is CR. In §1 this was done by taking the union of two underlinings of M. In the present case this would result in coïncidences of _ and ~ (i.e. $\underset{\approx}{\lambda}$) which is forbidden; hence the definition of a 'union' of two underlinings requires more consideration.
Suppose we are given a λ-term with an underlining in which two lines (_ or ~) occur. These two lines allow two reduction-steps; except in the following case: $\ldots \underset{\sim}{\lambda}y.\ (\underline{\lambda}x.\ M)y \ldots$
This motivates the definition of a chain and its energy.

2.19. Def. (i) Let λ_1, λ_2 be two occurrences of λ in M $\in \lambda$.
λ_1 and λ_2 are connected, written $\lambda_1 \sim\!\sim \lambda_2$, iff they occur in a context $\lambda_1 x.\ (\lambda_2 y.\ N)x$ for some N such that $x \notin FV(N)$.
(ii) A maximal sequence of connected λ's is called a chain.
The length of the chain is its number of λ's.
Example: $\lambda a.(\lambda b.(\lambda c.(\lambda d.N)c)b)a$

$\lambda \sim\!\sim \lambda \sim\!\sim \lambda \sim\!\sim \lambda$

(iii) A non-connected λ forms a chain on its own.

2.20. Notation. Let M $\in \lambda$. Then $(M)_0 \equiv M$

$$(M)_{n+1} \equiv \lambda x.(M)_n x \text{ where } x \notin FV((M)_n).$$

2.21. Remark. Sometimes we will identify a chain of length n+1 with its corresponding λ-term, which can be written as $(\lambda a.A)_n$, where $n \geq 0$, $(\lambda a.A)_n$ does not occur in a context $\lambda b.(\lambda a.A)_n b$, and $A \not\equiv (\lambda a'.A')a$.

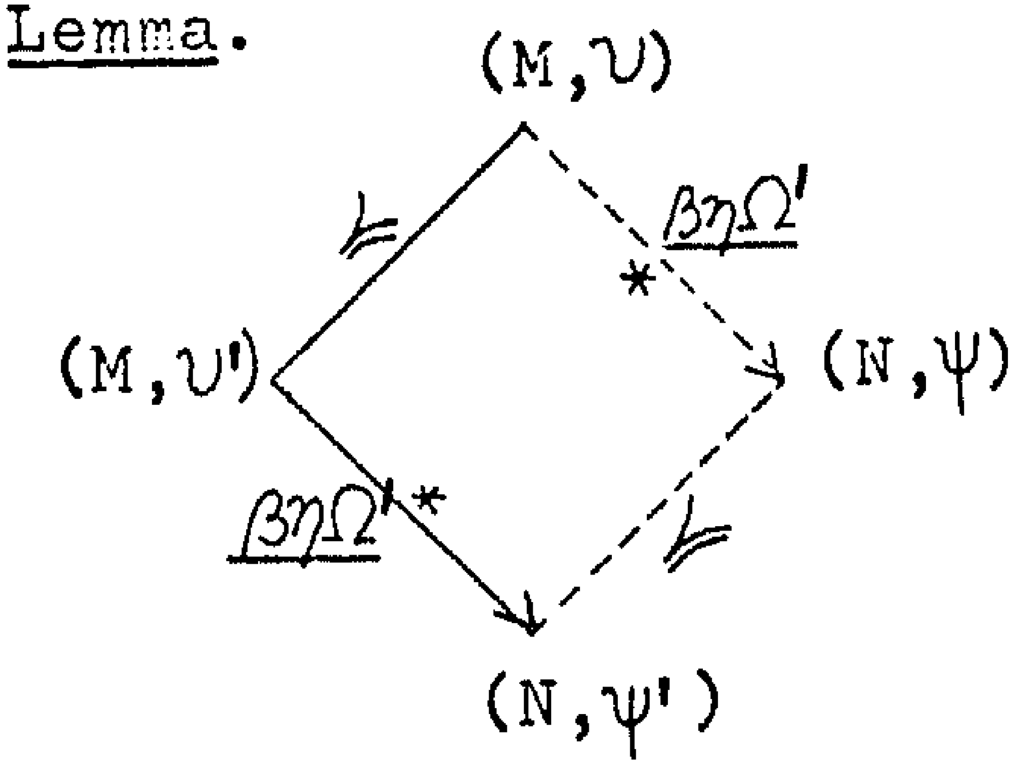
2.22. <u>Def</u>. Let (C,ν) be a chain with underlining ν. The <u>energy</u> of (C,ν), $\|(C,\nu)\|$, is the number of occurrences of $\_, \sim$ in (C, ν_{normal}) where ν_{normal} results from ν as follows: whenever $\underset{\sim}{\underline{\lambda}} \sim\sim \underline{\lambda}$ occurs in (C,ν) then it is replaced by $\lambda \sim\sim \underline{\lambda}$. (Remark: it is also possible to delete $\_$ in stead of $\sim$.)

2.23. <u>Def</u>. (i) $(M,\nu) \preccurlyeq (M, \nu') \iff$ <u>1</u>. $\|(C,\nu)\| \leqslant \|(C, \nu')\|$ for every chain C in M, and <u>2</u>. $\Omega'_{(M,\nu)} \subseteq \Omega'_{(M, \nu')}$ where $\Omega'_{(M,\nu)}$ is the set of occurrences of Ω'-redices in (M,ν).

(ii) $(M,\nu) + (M, \nu') = (M, \nu'')$ where ν'' is some underlining such that <u>1</u>. $\|(C, \nu'')\| = \max.(\|(C,\nu)\|, \|(C, \nu')\|)$ for every chain C in M, and <u>2</u>. $\Omega_{(M, \nu'')} = \Omega_{(M,\nu)} \cup \Omega_{(M, \nu')}$.

2.24. <u>Coroll</u>. $(M,\nu) + (M, \nu') \succcurlyeq (M,\nu), (M, \nu')$.

2.25. <u>Lemma</u>.



Proof. It is sufficient to consider the case that $(M,\nu') \xrightarrow[\beta\eta\Omega']{*} (N,\psi')$ is one step. There are 3 cases: <u>1</u>. a $\underline{\beta}$-redex, <u>2</u>. an $\underset{\sim}{\eta}$-redex, or <u>3</u>. an Ω'-redex is contracted.

Case <u>1</u>,<u>2</u>. Consider the chain C in (M,ν') of which the head-λ of the contracted $\underline{\beta}$- or $\underset{\sim}{\eta}$-redex is a part. We will distinguish two subcases: <u>a</u>. the lenght of C is >1, <u>b</u>. the length of C is 1.

<u>a</u>. Let $C \equiv \lambda a_n.(\lambda a_{n-1}. \ldots(\lambda a_1.(\lambda a.A)a_1)\ldots)a_n \equiv (\lambda a.A)_n$, $n \geqslant 1$ and let $\|(C,\nu')\| = m' \geqslant 1$. By def. 2.23, $\|(C,\nu)\| = m \geqslant m' \geqslant 1$, hence (C,ν) has an underlined λ ($\underline{\lambda}$ or $\underset{\sim}{\lambda}$). After contraction of such a $\underline{\lambda}$ or $\underset{\sim}{\lambda}$ (it does not matter which underlined λ of the chain is contracted),

the descendant of C in (N,ψ) is clearly a chain C' such that $\|(C',\psi)\| = m-1$. Because $\|(C',\psi')\| = m'-1$ we have indeed $\|(C',\psi)\| \geqslant \|(C',\psi')\|$. The other chains in M are clearly not affected by the contraction.

b. In this case there is the problem that new chains can be created, by concatenation of two (or in one case even three) chains.

Example. Let $C_1 \equiv (\lambda a.za)_n$ and $C_2 \equiv (\lambda b.B)_m$; then

$$(\underline{\lambda}z.C_1)C_2 \xrightarrow[\beta]{} (\lambda b.B)_{m+1+n} \;.$$

The problem is as follows: suppose that $C_1 \equiv \ldots\sim\sim\underset{\sim}{\lambda}$ and $C_2 \equiv \underline{\lambda}\sim\sim\ldots$ concatenate to $\ldots\sim\sim\underset{\sim}{\lambda}\sim\sim\underline{\lambda}\sim\sim\ldots$, then there would be a 'loss of energy' of 1, and the lemma would fail. This situation cannot happen, however. For suppose $C_1 \equiv (\underset{\sim}{\lambda}a.Aa)_n$ where $a \notin FV(A)$ and $A \not\equiv \lambda a'.\,A'$ (Φ), and $C_2 \equiv \underline{\lambda}b.B$. Then C_2 must occur in (C_2D) for some D. Let $(\underline{\lambda}b.B')D'$ be the residual of C_2D after the contraction that causes the concatenation. The concatenated chains must have the form $(\underset{\sim}{\lambda}a.(\underline{\lambda}b.B')a)_n$. Hence $D' \equiv a$, but this can only be the case if $C_1 \equiv (\underset{\sim}{\lambda}a.(\underline{\lambda}b.B)a)_n$, in contradiction with (Φ).

Case 3. Suppose $H \subseteq (M,\upsilon')$ is the contracted $\underset{\dots}{\Omega}'$-redex. Let (N,ψ) be the result of the contraction of the homologous $H \subseteq (M,\upsilon)$.
Let $C \subseteq N$ be a chain and let us compare $\|(C,\psi)\|$ and $\|(C,\psi')\|$.
By the maximality of $\underset{\dots}{\Omega}'$-redices there are only the following two possibilities: i) $C \subseteq N$ is the descendant of the 'same' chain in (M,υ) (although the corresponding subterms can be different),

ii) C is the result of concatenation of some chains in (M,υ).

In case i) there is no problem. In case ii) we prove by the same argument as above that there is no loss of energy. ⊠

2.26. Lemma. Analogous to lemma 1.18.

2.27. Lemma. $\xrightarrow[1]{}$ is CR.

Proof: analogous to the proof of 1.19. ⊠

2.28. Lemma. $\xrightarrow[\beta\eta\Omega']{*}$ is CR.

Proof: analogous to the proof of 1.20. ⊠

Now we will prove that $\xrightarrow[\beta\eta\Omega]{*}$ as defined in 2.0. is CR.

2.29. Lemma.

(1) $\lambda\beta\eta\Omega' \vdash N_1 = N_2 \Rightarrow \exists L \quad N_1 \xrightarrow[\beta\eta\Omega']{*} L \xleftarrow[\beta\eta\Omega']{*} N_2$

(2) $\xrightarrow[\beta\eta\Omega']{} \subset \xrightarrow[\beta\eta\Omega]{}$

(3) $M \xrightarrow[\Omega]{} M' \Rightarrow \exists L \quad M \xrightarrow[\Omega']{} L \xleftarrow[\Omega']{} M'$

Proof: (1) is the standard reformulation of the Church-Rosser theorem 2.28. (2) follows directly from the definitions.

(3). Let H' be the Ω-redex in M and let H be the maximal unsolvable subterm of M containing H'. Then contraction of H gives the required L. ⊠

2.30. Theorem. $\xrightarrow[\beta\eta\Omega]{*}$ is CR.

Proof. Suppose that $N_1 \xleftarrow[\beta\eta\Omega']{*} N_0 \xrightarrow[\beta\eta\Omega']{*} N_2$.

Then by 2.29 (3) $\lambda\beta\eta\Omega' \vdash N_1 = N_2$ and hence by 2.29 (1)

N_1, N_2 have a common $\xrightarrow[\beta\eta\Omega']{*}$-reduct N_3.

By 2.29 (2) N_3 is also a common $\xrightarrow[\beta\eta\Omega]{*}$-reduct of N_1, N_2.

Hence $\xrightarrow[\beta\eta\Omega]{*}$ is CR. ⊠

3. THE COFINALITY OF GROSS - REDUCTION

3.0. For the $\lambda\beta$-calculus the reduction strategy introduced in this section and its cofinality were communicated to us by professor Gross, see also [6]. Due to the result in the previous section we can introduce a similar strategy for the $\lambda\beta\eta(\Omega)$-calculus.

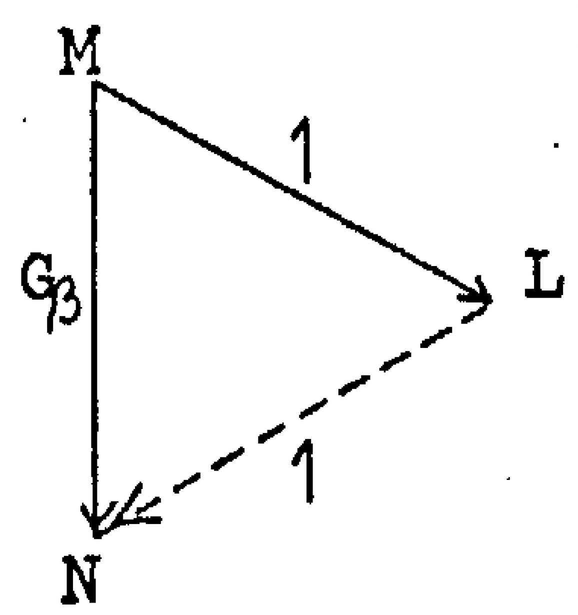
3.1. Gross-reduction in the $\lambda\beta$-calculus.

3.1.1. Def. $\xrightarrow[G_\beta]{}$, one step Gross-reduction, is defined by:

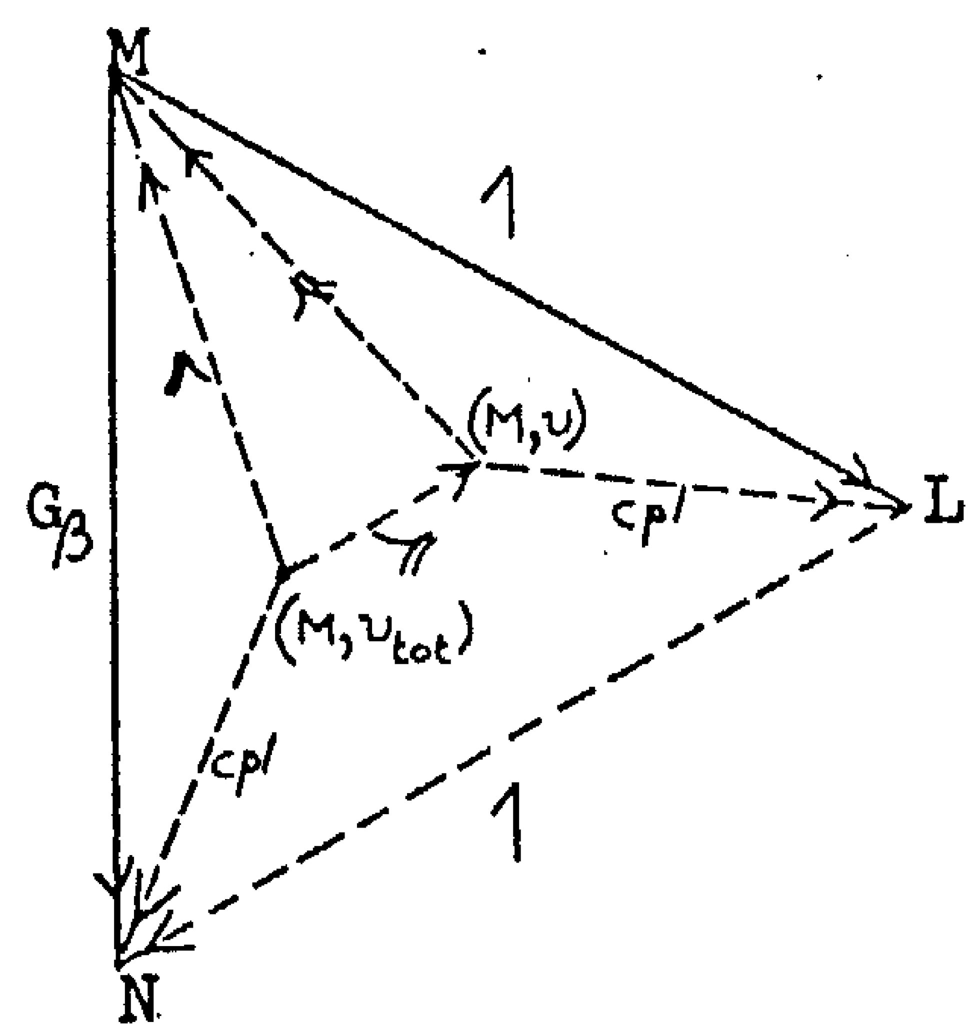
$$M \xrightarrow[G_\beta]{} N \iff (M, \upsilon_{tot}) \xrightarrow[cpl]{} N$$

where υ_{tot} is the total underlining of all β-redices in M.

3.1.2. Lemma.

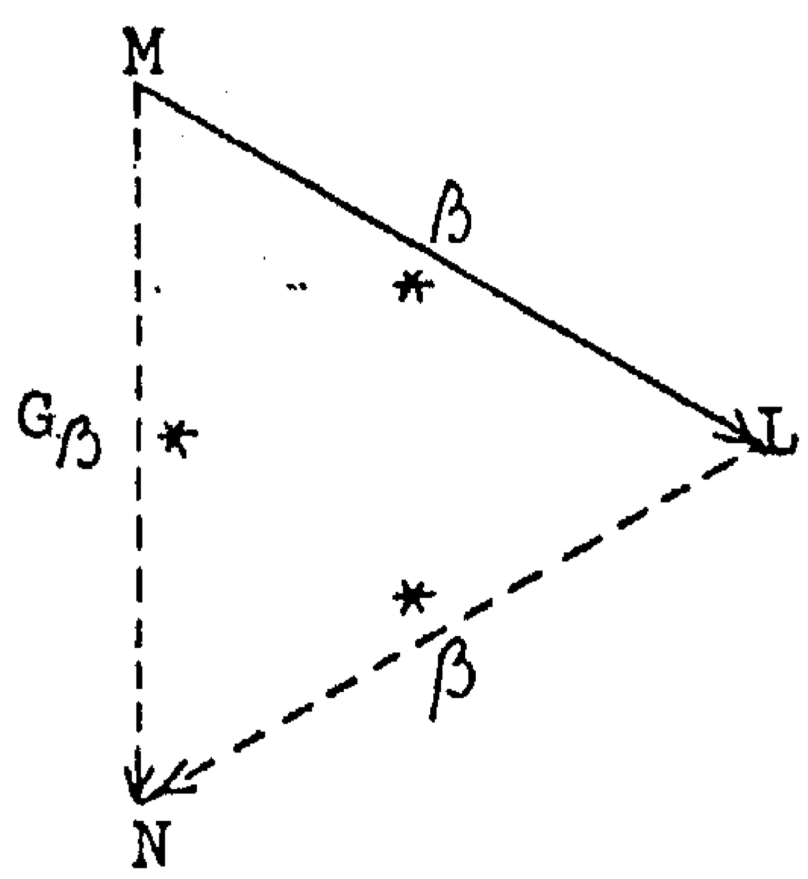


Proof:



The proof is a direct consequence of def. 3.1.1 and 1.17, and lemma 1.18. ⊠

3.1.3. Theorem

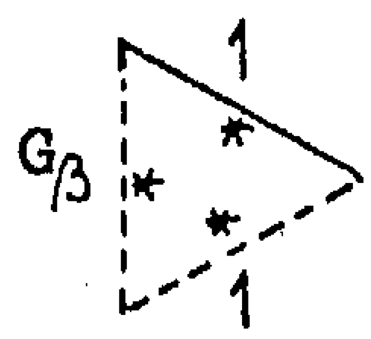


Proof: $\xrightarrow[\beta]{} \subseteq \xrightarrow[1]{}$, so $\xrightarrow[\beta]{*} \subseteq \xrightarrow[1]{*}$

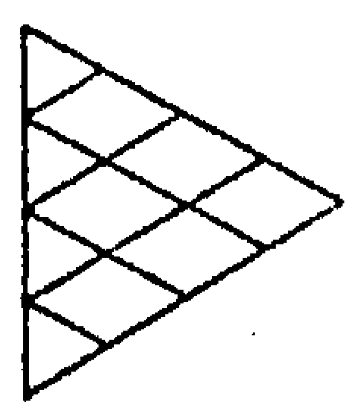
$\xrightarrow[1]{} \subseteq \xrightarrow[\beta]{*}$, so $\xrightarrow[1]{*} \subseteq \xrightarrow[\beta]{*}$;

hence $\xrightarrow[\beta]{*} = \xrightarrow[1]{*}$.

So we have to prove:



and this follows immediately from 3.1.2, 1.19 and a simple diagram-chasing argument as suggested by the figure:



⊠

3.2. Gross - reduction in the $\lambda\beta\eta(\Omega)$ - calculus.

3.2.1. Def. (i) $\xrightarrow[G_{\beta\eta\Omega}]{}$, one step Gross-reduction, is defined by:

$$M \xrightarrow[G_{\beta\eta\Omega}]{} N \iff (M, \upsilon_{tot}) \xrightarrow[cpl]{} N \text{ , where}$$

(ii) υ_{tot}, the total underlining, is defined as follows:

a. Underline all maximal unsolvable subterms of M with $\dots$.

b. Underline all non-connected η-redex λ's with $\sim$.

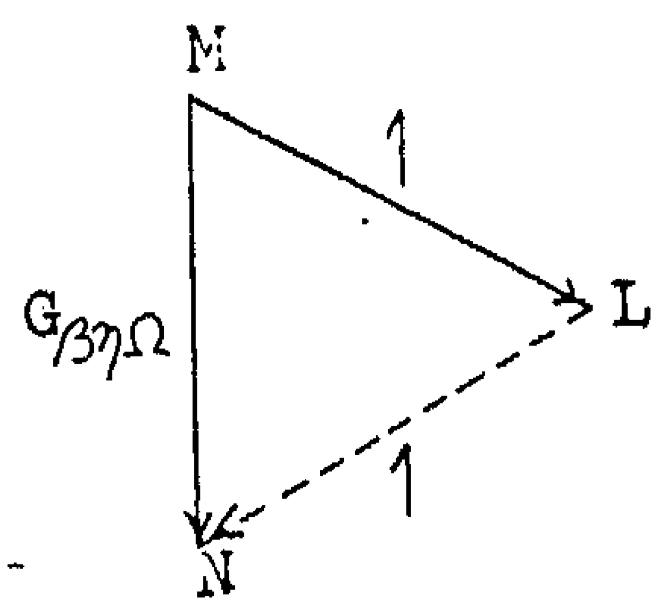
c. Underline all non-connected β-redex λ's with _, unless such a is allready underlined by b.

d. If $C \equiv (\lambda a.A)_n$ is a chain of length $\geqslant 2$, we distinguish three cases.

1. C is active, i.e. occurs in a context (CD) for some D. Then the underlining will be $\underline{\lambda}\sim\sim\underline{\lambda}\sim\sim\ldots\sim\sim\underline{\lambda}\sim\sim\underline{\lambda}$.
2. C is not active, and $A \equiv A'a$ for some A' s.t. $a \notin FV(A')$. Then the underlining will be $\underset{\sim}{\lambda}\sim\sim\underset{\sim}{\lambda}\sim\sim\ldots\sim\sim\underset{\sim}{\lambda}\sim\sim\underset{\sim}{\lambda}$
3. Neither 1. nor 2. Then the underlining will be $\underset{\sim}{\lambda}\sim\sim\underset{\sim}{\lambda}\sim\sim\ldots\sim\sim\underset{\sim}{\lambda}\sim\sim\underline{\lambda}$.

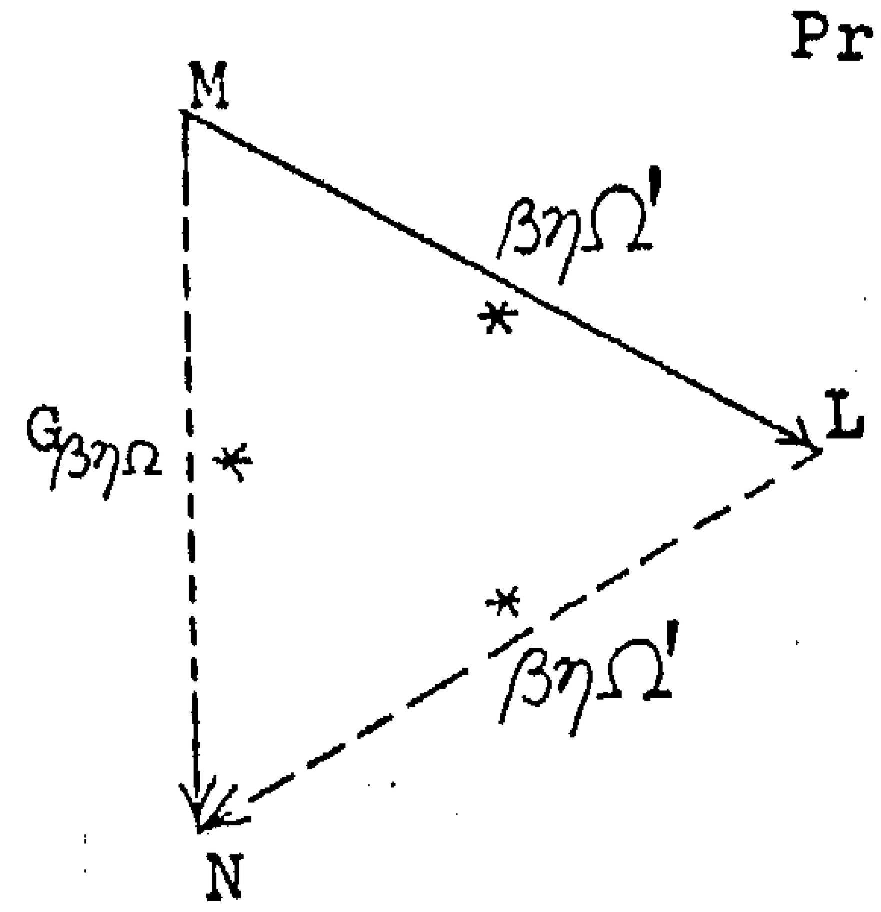
In this way we have given each chain in M a maximal amount of energy. Clearly $(M, \upsilon_{tot}) \succcurlyeq (M,\upsilon)$ for all υ.

3.2.2. Lemma.



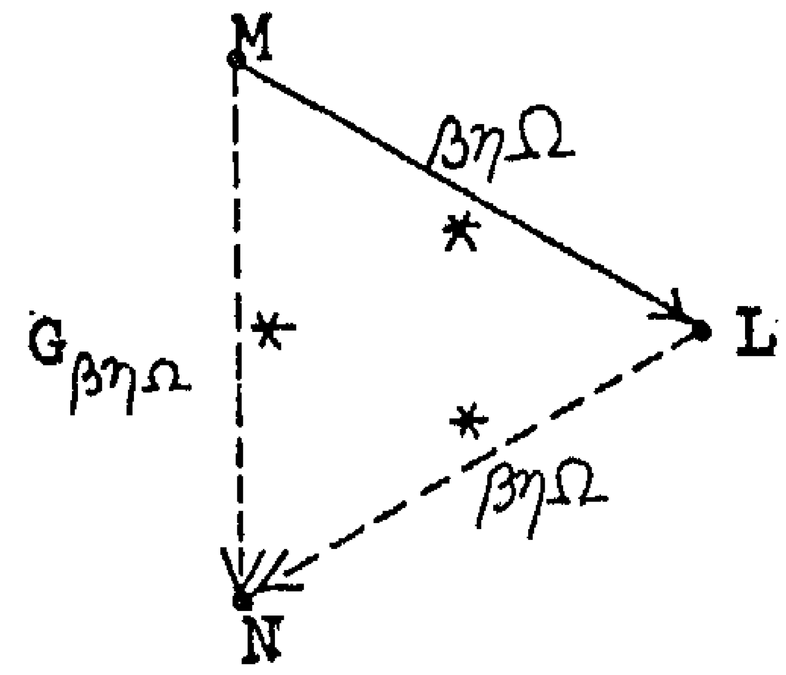
Proof: similar to that of 3.1.2. ☒

3.2.3. Lemma.

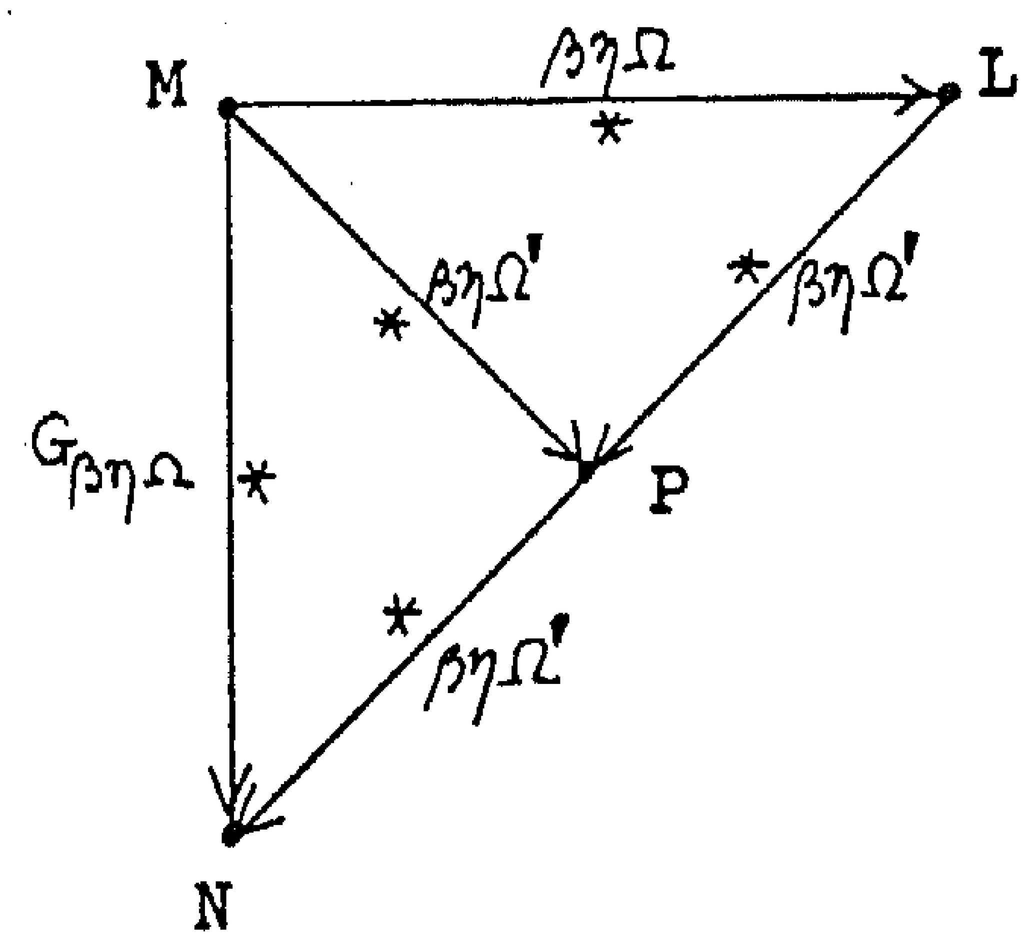


Proof: similar to that of 3.1.3. ☒

3.2.4. <u>Theorem</u>.



Proof. $M \xrightarrow[\beta\eta\Omega]{*} L \implies \lambda\beta\eta\Omega' \vdash M = L$ by 2.29.(3). By 2.29.(1) M and L have a common $\xrightarrow[\beta\eta\Omega']{*}$ -reduct P. So by 3.2.3. there is an N such that .



By 2.29(2) N is as required. ⊠

4. REDUCTION STRATEGIES

4.1. Def. Let λ be the set of all λ-terms.

A (reduction) strategy is a map $F : \lambda \longrightarrow \lambda \cup \{\otimes\}$ such that

(i) $M \xrightarrow[\beta]{*} F(M)$ if M is not in normal form.

(ii) $F(M) = \otimes \Longleftrightarrow$ M is in normal form (nf).

A strategy is a 1-strategy (or one step strategy) if for all M not in nf $M \xrightarrow[\beta]{} F(M)$. In contrast to standard use $\xrightarrow[\beta]{}$ is not reflexive.

A strategy is recursive if it is recursive after coding of the terms.

4.2. Examples of recursive 1-strategies are

(i) Gross-reduction, defined in 3.1.1.

(ii) Normal reduction (see [4] p.140).

Note that both strategies only depend on the skeleton of terms. (The skeleton of e.g. $y(\lambda x.\ xx)$ is $\Box(\lambda\Box\,.\,\Box\Box)$.)

4.3. Def. Let F be a strategy.

(i) F is normalizing if for all M having a nf, $\exists n\ \ F^n(M) = \otimes$.

(ii) F is cofinal if for all M and N such that $M \xrightarrow[\beta]{*} N$, $\exists n\ \ N \xrightarrow[\beta]{*} F^n(M)$.

(iii) F is perpetual if for all M: M has an infinite reduction sequence $\Longrightarrow \forall n\ \ F^n(M) \neq \otimes$.

Remarks. (i) Let Δ have no nf. Then $KI\Delta$ is a term with a normal form which also has an infinite reduction sequence ($KI\Delta \xrightarrow[\beta]{} KI\Delta' \xrightarrow[\beta]{} \ \ldots$). Thus in order to show that a term has no nf it is not sufficient to show that a term M has an infinite reduction sequence. Therefore a normalizing strategy F is useful, since it shows that a term M has no nf if F does not terminate on M.
(ii) There are even terms M such that each subterm has a nf, but M does have an infinite reduction sequence, e.g. PP, with $P \equiv \lambda z.(\lambda xy.y)(zz)$.
(iii) In 5.18 it is proved that a perpetual strategy cannot depend only on the skeleton of a term.

4.4. Proposition. (i) Any cofinal strategy is normalizing. (ii) Gross-reduction is a recursive cofinal strategy. (iii) Normal reduction is a recursive one step strategy.
Proof. (i) Obvious. (ii) See 3.1.3. (iii) See [4] p 142. ⊠

In the next section a recursive perpetual strategy will be constructed.

4.5. Definition. Let F be a strategy. Then

$L_F(M) = \mu n.(F^n(M) = \otimes)$

$B_F(M) = \max.\{\text{length}(F^n(M)) \mid n\in\omega \ \&\ k \leqslant n \Rightarrow F^k(M) \neq \otimes\}.$

$L_F(M)$, $B_F(M)$ are possibly ∞.

4.6. Def. Let F and G be normalizing strategies.

$F \leqslant_L G \Leftrightarrow \forall M\ [M \text{ has a nf} \Rightarrow L_F(M) \leqslant L_G(M)]$

$F \leqslant_B G \Leftrightarrow \forall M\ [M \text{ has a nf} \Rightarrow B_F(M) \leqslant B_G(M)]$

F is L(B)-better than G if $F \leqslant_{L(B)} G$ and not $G \leqslant_{L(B)} F$.

F is L(B)-optimal if no strategy is L(B)-better than F.

F is L(B)-1-optimal if F is a 1-strategy and no 1-strategy is L(B)-better than F.

4.7. Proposition. There exists

i. an L-optimal strategy

ii. a B-optimal strategy

iii. no recursive L-optimal strategy

iv. no recursive B-optimal strategy

v. an L-1-optimal strategy

vi. a B-1-optimal strategy.

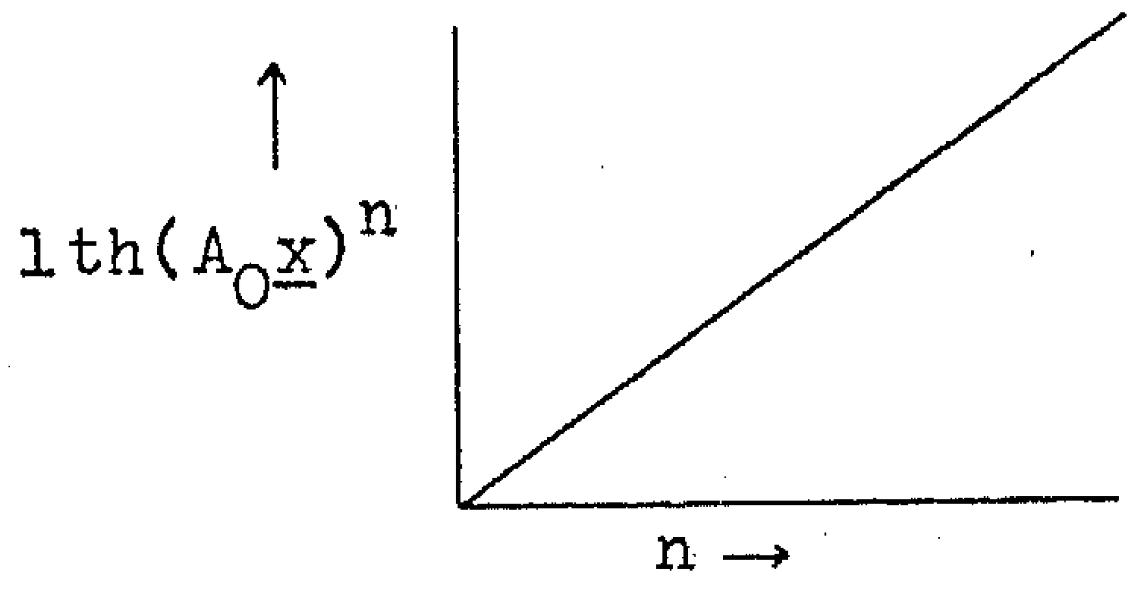
Proof. i,ii are trivial. iii: let F be a recursive L-optimal strategy. Then for all M having a nf F(M) is a nf or $\otimes$, hence M has a nf iff F(M) is a nf or $\otimes$. This makes 'M has a nf' decidable which is impossible. iv: similar to the proof of thm. 4.8. v and vi are trivial. ☒

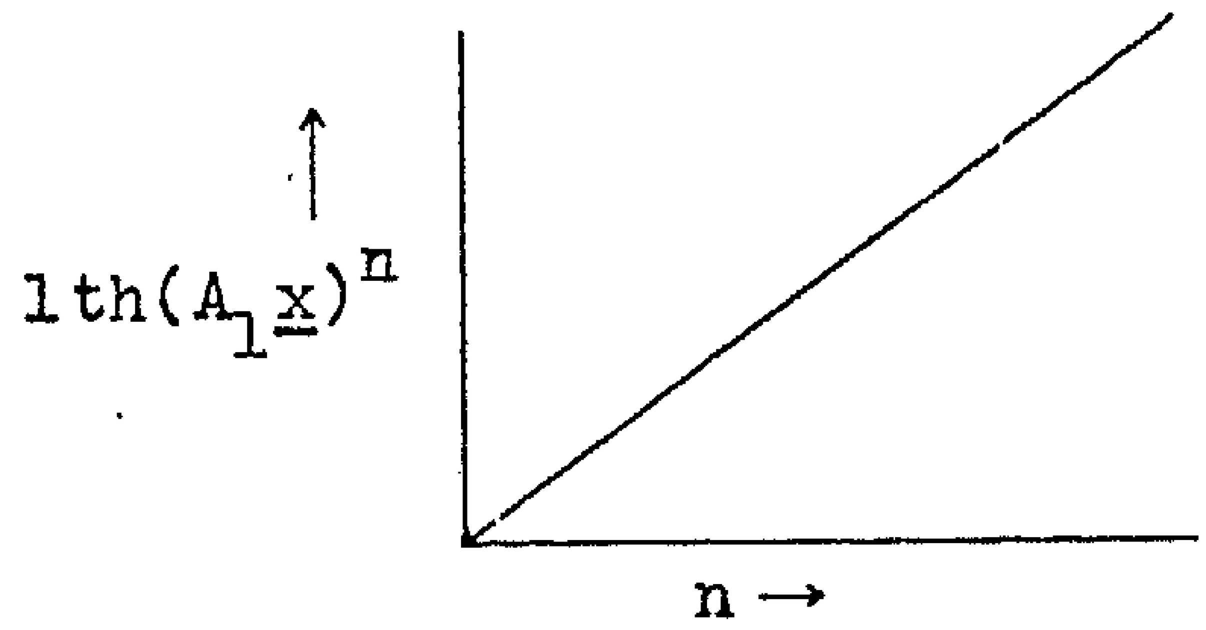
4.8. Theorem. There is no recursive B-1-optimal strategy.

Proof. Let φ be a partial recursive function with index e such that the $W_i = \{x \mid \varphi(x) = i\}$ for i = 0,1 are recursively inseparable. We can find terms A_1 and A_2 which have the following properties (which are stated in a very informal way):

i) The terms $A_i\underline{x}$ can reduce in at most one way. Their reducts have the same property and so on. Let $(A_i x)$ be the n-th reduct (i.e. n times one step reduction) of $A_i\underline{x}$ if it exists.

ii) If $\varphi(x) \neq 0$ and $\varphi(x) \neq 1$ the length of the $(A_i\underline{x})^n$ depends as follows on n:

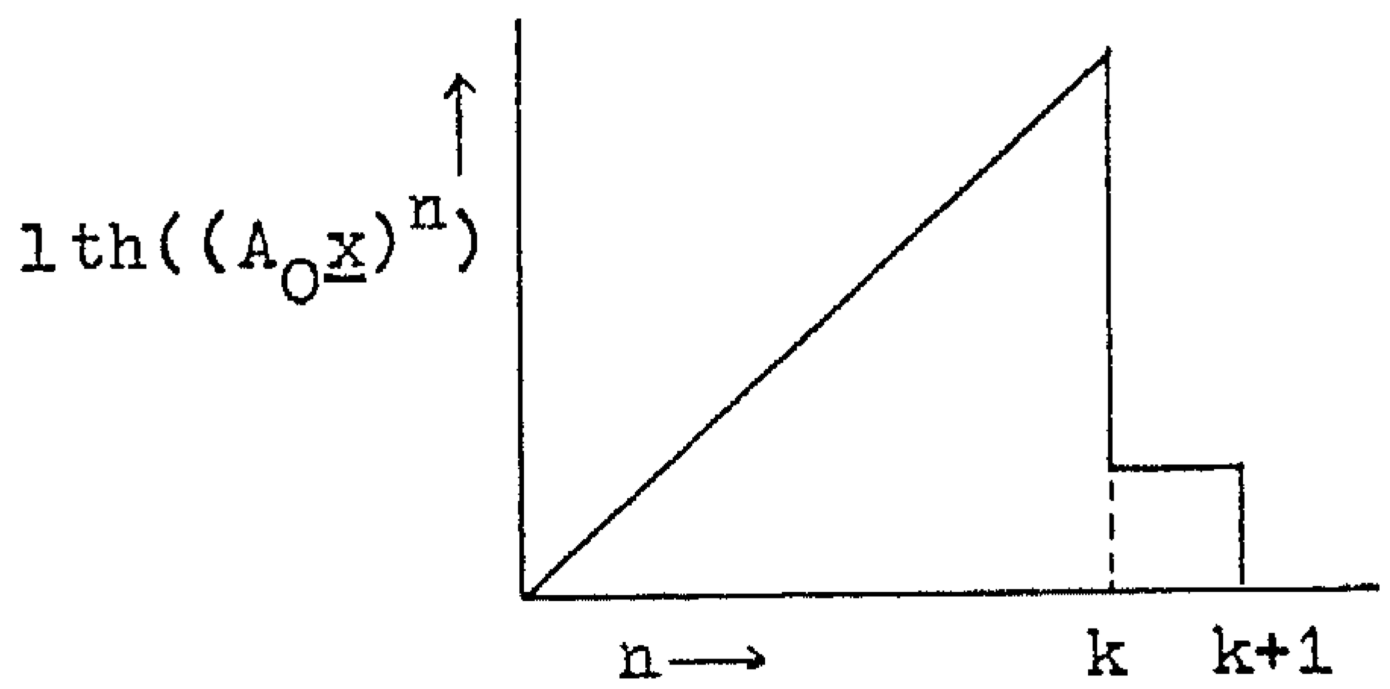


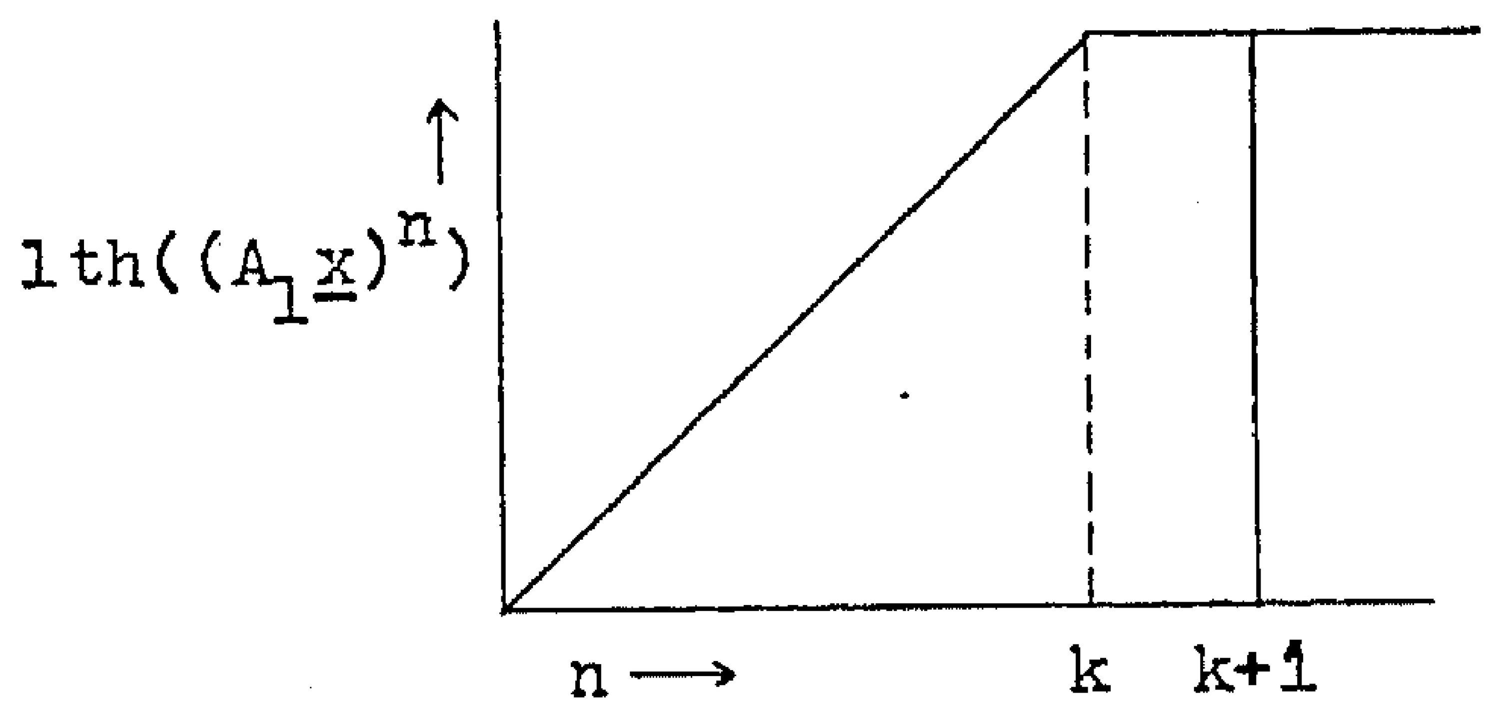


In this case both $A_0\underline{x}$ and $A_1\underline{x}$ have no normal form.

iii) If $\varphi(x) = 0$ then the dependence is as follows.

(Let k be the length of the computation $\{e\}(x)$.)





Further $(A_0\underline{x})^{k+1}$ and $(A_1\underline{x})^{k+1}$ are in normal form.

iv) Finally if $\varphi(x) = 1$, then the dependence of lth$((A_i\underline{x})^n)$ on n is as in iii) but with the pictures interchanged.

Let $C \equiv \lambda y.\ y(A_0\underline{x})(A_1\underline{x})$. Normalizations of C consist just of mixtures of normalizations of $A_0\underline{x}$ and $A_1\underline{x}$. Suppose F is a recursive strategy which minimalizes breadth. Obviously there exists a recursive f such that $f(x) = 0$ if F says that first $A_0\underline{x}$ has to be reduced

and $\qquad f(x) = 1$ if the reduction has to start with $A_1\underline{x}$.

We claim that f gives a recursive separation of W_0 and W_1.

Suppose $x \in W_1$, then $\varphi(x) = 1$. In this case for some n, $(A_1\underline{x})^{n+1}$ is in nf and lth$((A_1\underline{x})^{n+1}) = 1$. Further lth$((A_0\underline{x})^k)$ 'stabilizes' on a high level. It is clear that the smallest breadth is reached if we first reduce $A_1\underline{x}$ to a normal form (of length 1) and then $A_0\underline{x}$. So $f(x) = 1$.

Similarly $x \in W_0$ implies $f(x) = 0$. ⊠

4.9. Theorem. There is no recursive L-1-optimal 1-strategy.

Proof. In the same spirit as the proof of the previous theorem, again using a pair of recursively inseparable r.e. sets. ⊠

4.10. Theorem. There is no one step strategy which is both L-1 and B-1 optimal.

Proof. Consider the term $Z = \underbrace{(\lambda xy.pxx(yI))\underbrace{((\lambda x.\ pxx)A)}_{1}}_{2}I$

where A is in nf and very long. We show that if $L_F(Z)$ is minimal, F(Z) must result from Z by a reduction of redex 1, whereas minimalization of $B_F(Z)$ requires first to reduce redex 2. The first fact is obvious as reducing redex 2 first results in duplication of redex 1. On the other hand if we just compare the breadth of normalizations starting with a reduction of 1 and 2 we see that starting with redex 2 minimalizes the breadth. Reduction of 2 yields:

$$
\begin{aligned}
&(\lambda y.p((\lambda x.pxx)A)((\lambda x.pxx)A)(yI))I \xrightarrow[\beta]{} \\
&\quad p((\lambda x.pxx)A)((\lambda x.pxx)A)(II) \xrightarrow[\beta]{} \\
&\quad p((\lambda x.pxx)A)((\lambda x.pxx)A)I \xrightarrow[\beta]{} \\
&\quad p((\lambda x.pxx)A)(pAA)I \xrightarrow[\beta]{} \\
&\quad p(pAA)(pAA)I\ .
\end{aligned}
$$

First reduction of 1 yields:

$$
\begin{aligned}
&(\lambda xy.pxx(yI))(pAA)I \xrightarrow[\beta]{} \\
&(\lambda y.p(pAA)(pAA)(yI))I\ .
\end{aligned}
$$

This term is longer than the final result in the previous reduction. ⊠

5. A RECURSIVE PERPETUAL STRATEGY

5.0. Introduction. In this section we will construct a recursive perpetual one step strategy F_∞. As an application of F_∞ we show that the contraction of a redex $(\lambda x.P)Q$ where $x \in FV(P)$ in a term with an infinite reduction seqence yields a similar term.

5.1. Definition. (i) Let $M \in \lambda$. Then the predicate ∞ is defined by

$$\infty(M) \Longleftrightarrow M \text{ has an infinite reduction sequence.}$$

(ii) Let $R = (\lambda x.P)Q$. If $x \in FV(P)$ we call R an I-redex, otherwise R is called a K-redex.

(iii) If M is not in nf, the left-most redex of M is the redex of which the head-λ is not preceded by the head-λ of any other redex.

5.2. Definition. Let the reduction strategy F_∞ be defined as follows by induction on the length of the terms:

$$F_\infty(M) = \begin{cases} \otimes \text{ if } M \text{ is in nf.} \\ \text{Otherwise, let } M \equiv C[(\lambda x.P)Q] \text{ where } R \equiv (\lambda x.P)Q \\ \text{is the left-most redex of } M. \text{ Then:} \\ \begin{cases} C[[Q|x]P] \text{ if } R \text{ is an I-redex.} \\ \text{Otherwise(if } R \text{ is a K-redex):} \\ \begin{cases} C[P] \text{ if } Q \text{ is in nf.} \\ C[(\lambda x.P)(F_\infty(Q))] \text{ if } Q \text{ is not in nf.} \end{cases} \end{cases} \end{cases}$$

5.3. Definition. (i) Let R be the redex $(\lambda x.P)Q$. The re of R is $(\lambda x.P)$ and the dex of R is Q.

(ii) Let M be a term not in nf. The derived term of M, notation M^+, is the dex of the left-most redex in M.

(iii) let M be a term. Its derived sequence $M^0, M^1, \ldots, M^n$ is defined by $M^0 \equiv M$, $M^{k+1} \equiv (M^k)^+$, as long as M^k is not in nf, otherwise M^{k+1} is not defined. Clearly each derived sequence is finite.

(iv) R^i, the left-most redex of M^i, is called the special redex of order i of M ($0 \leqslant i < n$). See figure:

$$
\begin{array}{l}
M \equiv M^0 \\
\qquad \cup\!| \\
R^0 \equiv (\lambda x_0.P_0)M^1 \\
\qquad\qquad \cup\!| \\
\qquad R^1 \equiv (\lambda x_1.P_1)M^2 \\
\qquad\qquad\qquad \cup\!| \\
\qquad\qquad\qquad\quad \ddots \\
\qquad\qquad\qquad\qquad R^{n-1} \equiv (\lambda x_{n-1}.P_{n-1})M^n \text{ where } M^n \text{ is in nf.}
\end{array}
$$

5.4. Remark. As can be seen from def. 5.2 and 5.3. F_∞ contracts the first I-redex in the sequence $R^0, R^1, \ldots, R^{n-1}$ if there is such a redex, otherwise F_∞ contracts R^{n-1}.

5.5. Lemma. Let $M \equiv C[(\lambda x.P)Q]$ where $R \equiv (\lambda x.P)Q$ is the left-most redex. Suppose that

(i) R is a K-redex, (ii) $\infty(M)$ and (iii) not $\infty(Q)$.

Then $\infty(C[P])$.

Proof. Let $M \equiv M_0 \xrightarrow{\beta} M_1 \xrightarrow{\beta} M_2 \xrightarrow{\beta} \ldots$ be an infinite reduction sequence. There are two cases.

1. $(\lambda x.P)Q$ is never contracted in the reduction sequence. Then for all i, $M_i \equiv C_i[(\lambda x.P_i)Q_i]$ where $P_i \xrightarrow{\beta} P_{i+1}$, $Q_i \xrightarrow{\beta} Q_{i+1}$ and $C_i[\ ]$ are contexts (with one empty place) such that $C_i[z] \xrightarrow{\beta} C_{i+1}[z]$ (z is a fresh variable). That the reductions of M are separated in this way, follows because $(\lambda x.P)Q$ is left-most, hence nothing can be substituted in P or Q. Moreover $(\lambda x.P_i)Q_i$ stays left-most in M_i for the same reason.

Now because not $\infty(Q)$, there is an m such that $Q_{m'} \equiv Q_m$ for all $m' \geqslant m$, i.e. for all n in the reduction $M_{m+n} \xrightarrow{\beta} M_{m+n+1}$ is a redex outside Q_{m+n} contracted. Hence for some $f \in \omega^\omega$ there is an infinite reduction sequence $\{C_{f(i)}[P_{f(i)}] \mid i \in \omega\}$ where $C_{f(i)}[P_{f(i)}] \xrightarrow{\beta} C_{f(i+1)}[P_{f(i+1)}]$.

2. In the reduction sequence $\{M_i\}$ the redex $(\lambda x.P)Q$ is contracted: by the same argument as in case 1. we have

$M_0 \xrightarrow{\beta} \ldots \xrightarrow{\beta} M_n \equiv C_n[(\lambda x.P_n)Q_n] \xrightarrow{\beta} M_{n+1} \equiv C_n[P_n] \xrightarrow{\beta} \ldots$ for some n.

Hence for some $g \in \omega^\omega$ there is an infinite reduction sequence $\{C_{g(i)}[P_{g(i)}] \mid i \in \omega\}$. ⊠

5.6. Theorem. F_∞ is a recursive perpetual one step strategy.

Proof. From def. 5.2. it is clear that F_∞ is a recursive one step strategy. Now we prove that F_∞ is perpetual. Let $M \equiv C[(\lambda x.P)Q]$ where $R \equiv (\lambda x.P)Q$ is the left-most redex. Let $\infty(M)$ and let $M \equiv M_0 \xrightarrow{\beta} M_1 \xrightarrow{\beta} M_2 \xrightarrow{\beta} \ldots$ be an infinite reduction sequence. The proof that $\infty(F_\infty(M))$ uses induction on the length of M (in the case 2.2.2).

1. R is an I-redex.

1.1. There is an n such that in $M_n \xrightarrow{\beta} M_{n+1}$ R_n is contracted, where $R_n \subseteq M_n$ is the residual of R. Because R is left-most it is evident that $M_i \equiv C_i[(\lambda x.P_i)Q_i]$ for $i \leqslant n$ where $R_i \equiv (\lambda x.P_i)Q_i$ is the residual of R and R_i is left-most in M_i. Moreover $P_i \xrightarrow{\beta} P_{i+1}$, $Q_i \xrightarrow{\beta} Q_{i+1}$ and $C_i[z] \xrightarrow{\beta} C_{i+1}[z]$ $(i+1 \leqslant n)$.
Now $\infty(F_\infty(M))$ because $F_\infty(M) \equiv C[[Q|x]P] \xrightarrow[\beta]{*} C_n[[Q_n|x]P_n] \equiv M_{n+1}$.

1.2. Here for all i, $M_i \equiv C_i[(\lambda x.P_i)Q_i]$ where $P_i \xrightarrow{\beta} P_{i+1}$, $Q_i \xrightarrow{\beta} Q_{i+1}$ and $C_i[z] \xrightarrow{\beta} C_{i+1}[z]$. This gives the infinite reduction sequence $\{C_i[[Q_i|x]P_i] \mid i \in \omega\}$. That this sequence is indeed infinite, follows because $x \in P_i$.

2. R is a K-redex.

2.1. Q is in nf. Then $F_\infty(M) = C[P]$ and because not $\infty(Q)$ we have by 5.5 $\infty(C[P])$.

2.2. Q is not in nf. Then $F_\infty(M) = C[(\lambda x.P)(F_\infty(Q))]$.

2.2.1. not $\infty(Q)$. By 5.5 $\infty(C[P])$, hence $\infty(F_\infty(M))$ because $F_\infty(M) = C[(\lambda x.P)(F_\infty(Q))] \xrightarrow{\beta} C[P]$.

2.2.2. $\infty(Q)$. By induction hypothesis $\infty(F_\infty(Q))$, hence $\infty(F_\infty(M))$.

5.7. Remark. The proof of 5.6 is non-constructive; however realizing the explicit action of F_∞ (see 5.4) a constructive proof of 5.6 can be given.

As an application of the perpetuity of F_∞ we will prove the following theorem.

5.8. Theorem. If $M \xrightarrow{\beta} M'$ by contracting an I-redex in M, then
$\infty(M) \Rightarrow \infty(M')$.

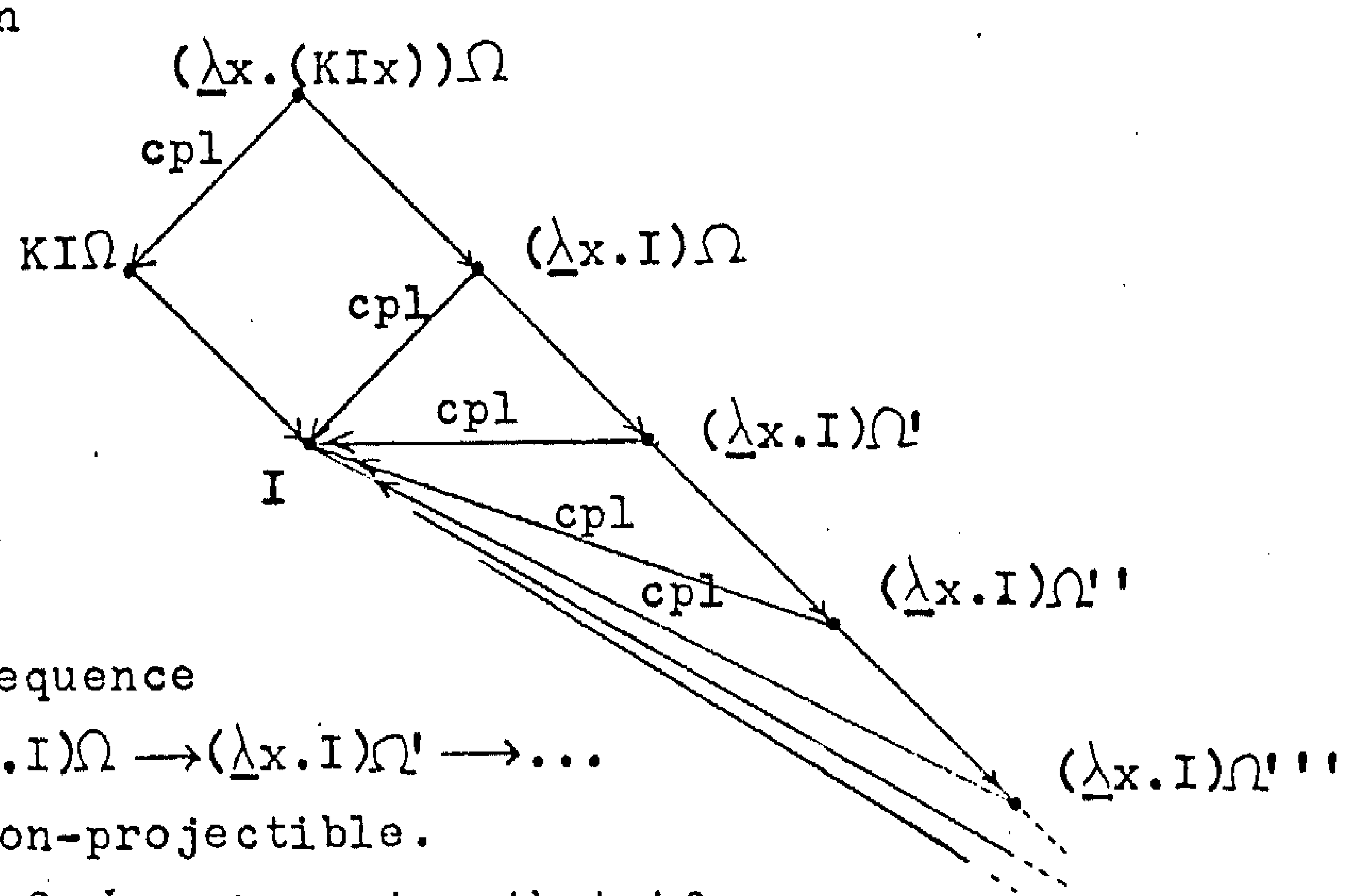
As a corollary we obtain two known results of the λI-calculus.

5.9. Corollary. In the λI-calculus one has
(i) If M has a nf, then M strongly normalizes, i.e. each reduction sequence of M terminates.
(ii) If M has a nf, then all subterms of M have a nf.

Proof. (i) immediate by 5.8.
(ii) If M had a subterm without nf, then M would have an infinite reduction sequence and hence by (i) no nf. ⊠

The proof of 5.8 for the λK-calculus is more complicated than that of 5.9 for the λI-calculus. The latter proof runs as follows:

Let $M \equiv C[(\lambda x.P)Q]$ and $M' \equiv C[[Q|x]P]$. Let $M \equiv M_0 \xrightarrow[\beta]{} M_1 \xrightarrow[\beta]{} \cdots$ be an infinite reduction sequence of M. Underline in M as follows $C[(\underline{\lambda}x.P)Q]$. Then each term M_i in the sequence can be provided with lines which indicate the residuals of $(\lambda x.P)Q$ in M_i. By taking the complete developments of the resulting underlined sequence an infinite reduction sequence is obtained, which will be called the projection of the reduction sequence $\{M_i\}$. The following example shows that this method of proof is false for the λK-calculus:
Let $\infty(\Omega)$, then



Therefore the sequence
$(\underline{\lambda}x.KIx)\Omega \longrightarrow (\underline{\lambda}x.I)\Omega \longrightarrow (\underline{\lambda}x.I)\Omega' \longrightarrow \ldots$
is said to be non-projectible.
We will prove 5.8 by observing that if $\infty(M)$, then $F^{i}_{\infty}(M)$ is an infinite reduction of M, which is projectible and hence $\infty(M')$. The proof occupies 5.10 - 5.16.

5.10. Definition. (i) $M \in I\underline{\lambda\beta}$ iff $M \in \underline{\lambda\beta}$, see 1.2, and only I-redices in M are underlined.
(ii) Let $M \in \underline{\lambda\beta}$. The special redices of M are defined analogously to 5.3.

5.11. Definition. The reduction relation $\xrightarrow[\underline{\beta}]{}$ contracts only underlined redices. Another reduction relation $\xrightarrow[-\beta]{}$ for terms in $\underline{\lambda\beta}$ is defined as follows:

$$C\left[(\lambda x.P)Q\right] \xrightarrow[-\beta]{} C\left[[Q|x]P\right]$$

where $C[\ ]$ is a context with one hole.

5.12. Lemma. A special redex R of M (being underlined or not) is not part of the re of any redex in M.

Proof. Induction on the order i of R.

i = O. Then R is the left-most redex of M. If R would be in the re of some redex of M, then R would not be the left-most redex of M.

i+1. Then R is the special redex of order i of M^+. By the induction hypothesis, R is not part of the re in a redex in M^+. Hence if R were part of some re in M, then also M^+ is part of this re. But then R^O (see 5.3) would not be the left-most redex in M. ⊠

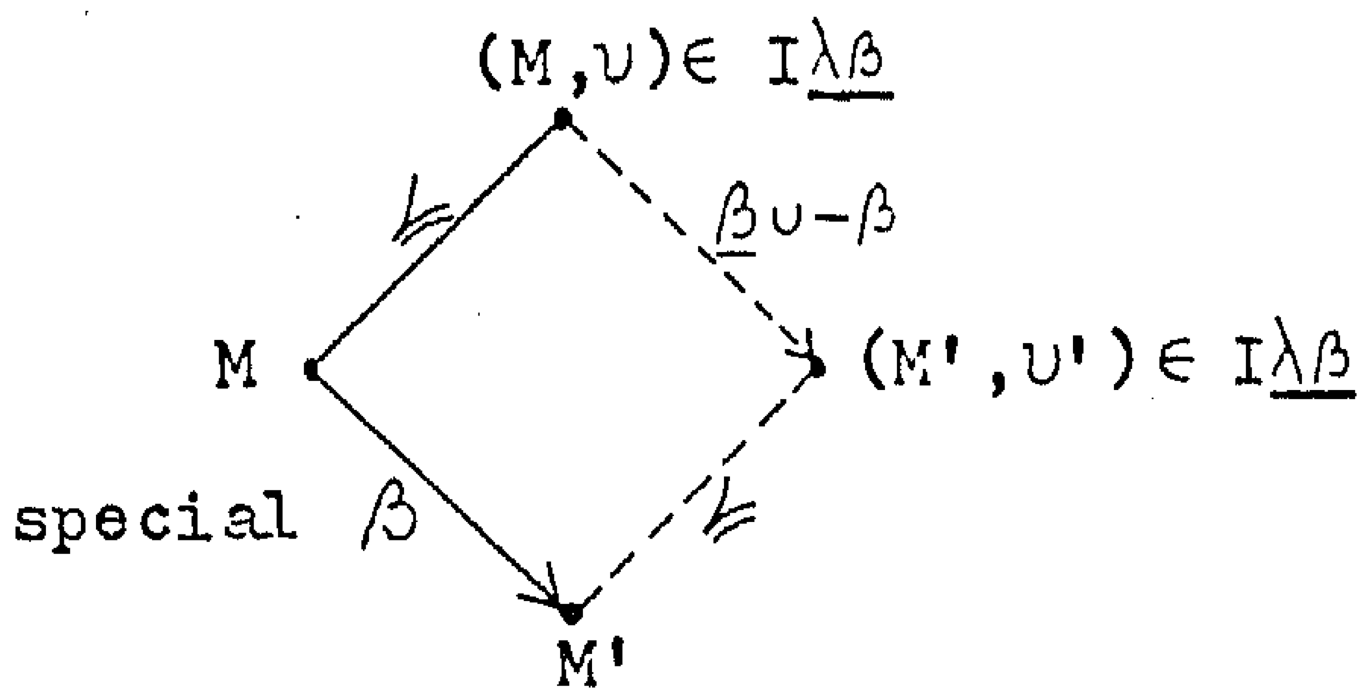
5.13. Lemma. Let M, N ∈ $\underline{\lambda\beta}$. If (i) M $\xrightarrow[\underline{\beta}]{}$ N or (ii) M $\xrightarrow[-\beta]{}$ N by contracting a special redex, then M ∈ I$\underline{\lambda\beta}$ $\Longrightarrow$ N ∈ I$\underline{\lambda\beta}$.

Proof. An underlined I-redex $(\underline{\lambda}x.P)Q$ can degenerate to a K-redex only if (*) inside P all free occurrences of x are erased.

In case (i) (*) cannot happen since the contracted redex is an I-redex.

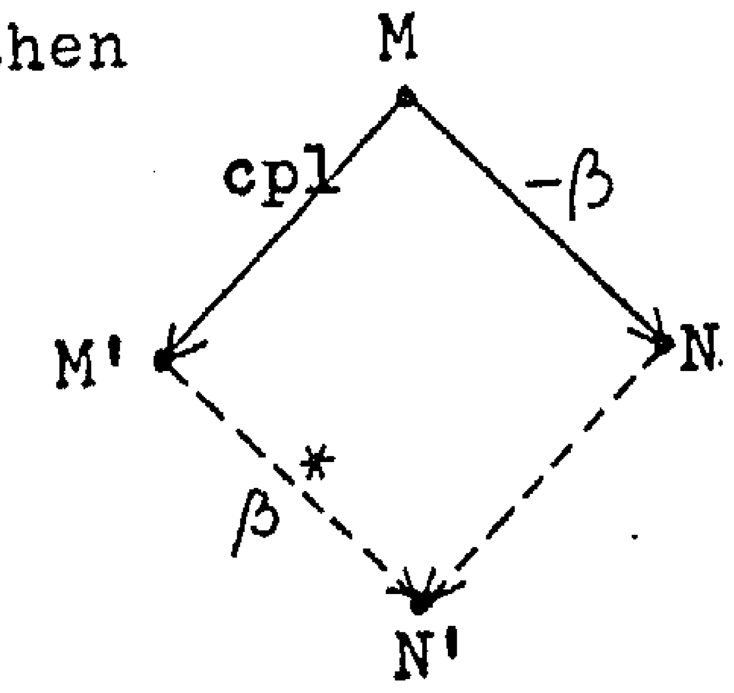
In case (ii) (*) cannot happen since no special redex is part of P, by 5.11. ⊠

5.14. Lemma. Let M, N ∈ λ and let M $\xrightarrow[\beta]{}$ M' by contraction of a special redex. Let (M,υ) ∈ I$\underline{\lambda\beta}$. Then there exists a (M',υ') ∈ I$\underline{\lambda\beta}$ such that (M,υ) $\xrightarrow[\underline{\beta}\cup-\beta]{}$ (M',υ'), where $\xrightarrow[\underline{\beta}\cup-\beta]{}$ is $\xrightarrow[\underline{\beta}]{}$ or $\xrightarrow[-\beta]{}$.



Proof. The underlining υ' for M' follows from that of M by making the contraction in (M,υ) homologous to that in M. That (M',υ') ∈ I$\underline{\lambda\beta}$ follows by 5.3. ⊠

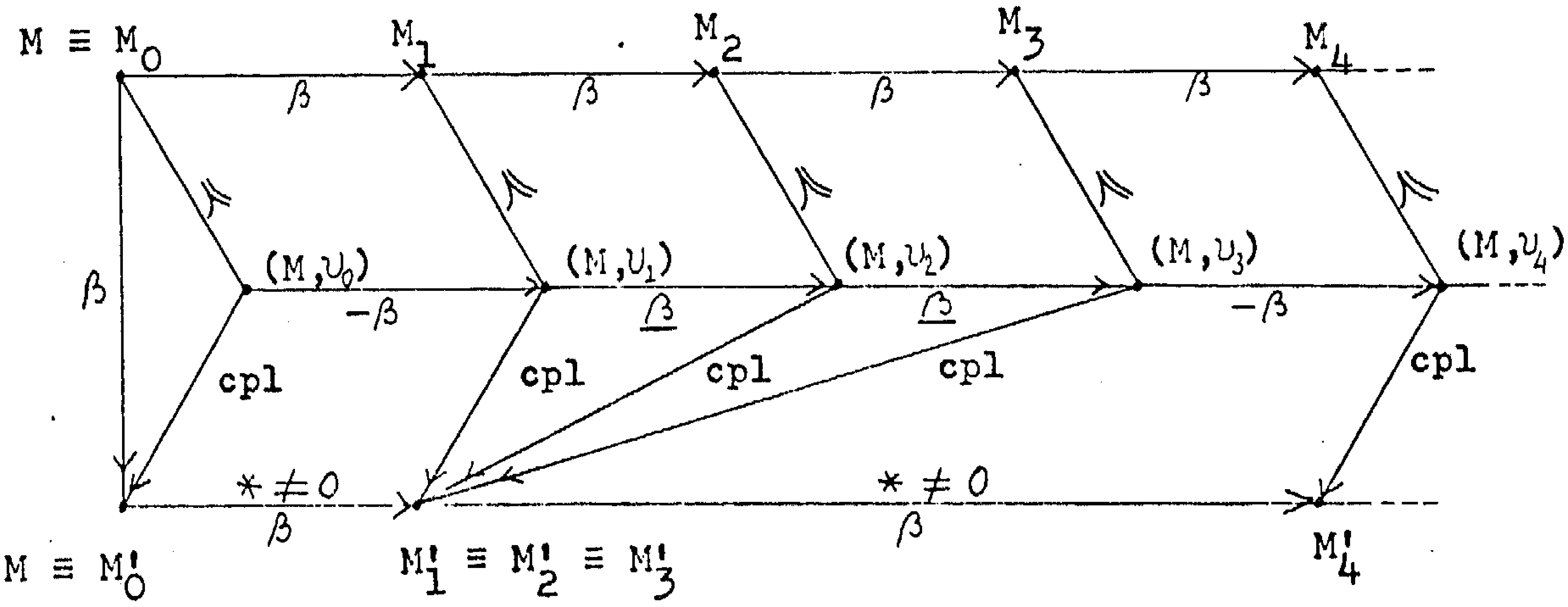
5.15. Lemma. Let $M, N \in \underline{\lambda\beta}$, then



Moreover if $M \in I\underline{\lambda\beta}$, then $M' \xrightarrow[\beta]{* \neq 0} N'$ in the diagram.

Proof. Let $(\lambda x.P)Q$ be the redex contracted in $M \xrightarrow[-\beta]{} N$. Underline this redex as follows: $(\underline{\lambda} x.P)Q$.
$M \xrightarrow[-\beta]{} N$ is then the complete reduction of M as $\underline{\lambda\beta}$-term and hence $M \xrightarrow[-\beta]{} N \xrightarrow[\text{cpl}]{} N'$ is a complete reduction of M as ---- and _ underlined term. By FD^+, 1.12, it follows that $M' \xrightarrow[\beta]{*} N'$ which is in fact a complete reduction w.r.t. the underlining ----.
Now suppose $M \in I\underline{\lambda\beta}$. Then by 5.13(i) each term in the _-complete reduction of M is an $I\underline{\lambda\beta}$-term. Hence each of these terms contains a $\underline{\lambda\beta}$-redex. Therefore $M' \xrightarrow[\beta]{* \neq 0} N'$. ⊠

5.16. Proof of 5.8.
Suppose $\infty(M)$ and $M \equiv C[(\lambda x.P)Q] \xrightarrow[\beta]{} M' \equiv C[[Q|x]P]$ is the contraction of an I-redex. We have to show $\infty(M')$. Let $M_n \equiv F_\infty^n(M)$. By the perpetuity of F_∞, $M \equiv M_0 \xrightarrow[\beta]{} M_1 \xrightarrow[\beta]{} \dots$ By 5.4, $M_i \xrightarrow[\beta]{} M_{i+1}$ is the result of contracting a special redex.
Let $(M, \mathcal{V}_0)$ be $C[(\underline{\lambda}x.P)Q] \in I\underline{\lambda\beta}$. By 5.14 there are $\mathcal{V}_i$ such that $M_i \preccurlyeq (M_i, \mathcal{V}_i) \in I\underline{\lambda\beta}$ and $(M_i, \mathcal{V}_i) \xrightarrow[\underline{\beta} \cup -\beta]{} (M_{i+1}, \mathcal{V}_{i+1})$.
Let M_i' be the result of a complete reduction of $(M_i, \mathcal{V}_i)$.
If $(M_i, \mathcal{V}_i) \xrightarrow[-\beta]{} (M_{i+1}, \mathcal{V}_{i+1})$ then by 5.15 $M_i' \xrightarrow[\beta]{* \neq 0} M_{i+1}'$.
If $(M_i, \mathcal{V}_i) \xrightarrow[\underline{\beta}]{} (M_{i+1}, \mathcal{V}_{i+1})$ then by 1.14 $M_i' \equiv M_{i+1}'$.
Hence we have



By FD a sequence of consecutive $\xrightarrow[\beta]{}$ is always finite. Therefore a subsequence of the M_i' is an infinite reduction for M', i.e. ∞(M').

5.17. Remark. Note that by 1.18 in the figure in the proof of 5.15 one has $M_i \xrightarrow[\beta]{*} M_i'$. This observation is the essence of the proof of the Church-Rosser theorem in [4].

5.18. Remark. A perpetual strategy F cannot depend only on the skeleton of a term.

Proof. Let $M \equiv (\underline{\lambda}x.x(\lambda a.aa))(\lambda y.(\underline{\lambda}pq.qq)(\lambda z.vv))$ and

$$M' \equiv (\underline{\lambda}x.v(\lambda a.aa))(\lambda y.(\underline{\lambda}pq.pp)(\lambda z.zz)).$$

Both terms have the same skeleton and only two redices (the underlined ones). To obtain an infinite reduction sequence F must contract in M the first and in M' the second redex.

6. β- VERSUS $\beta\eta$- REDUCTION.

In [5] p.124 it is proved that: M has a $\beta\eta$-normal form $\Leftrightarrow$ M has a β-normal form. The implication $\Leftarrow$ is trivial, but $\Longrightarrow$ gives some problems. The authors remark that while it seems as if the proof should be trivial, they did not know a shorter proof than the one they give there.

The proof that we present here is more straightforward and has the advantage of proving something more: viz. the theorem of Postponement of η-reductions [4] p.132, Thm.2. This is useful because the proof presented there contains an error, as noted by Nederpelt in [8] p.65. He also gives a proof of this theorem of which we will give a brief sketch. There is a simple connection between his proof and ours.

6.1. Notation. Let M be a λ-term. Then $(M)_o = M$, $(M)_{n+1} = \lambda x. (M)_n x$ where $x \notin FV((M)_n)$.

6.2. Remark. The M_n are η-expansions of M and one easily verifies:

(i) $((M)_n)_m = (M)_{n+m}$

(ii) $(M)_n \xrightarrow[\beta]{*} (M)_1$

(iii) $(M)_n N \xrightarrow[\beta]{*} MN$

6.3. Definition. A labeling L of a λ-term M is a map which assigns a natural number to each occurrence of a subterm of M.

Remark: the labelings in this paragraph have nothing to do with those in §1,2 and 3.

6.4. Notation. (i) If M is labeled by L we write M^L. Also we write the labels as superscripts; example: $M^L = (x^1(\lambda y.(y^2y^0)^1)^2)^0$.

Sometimes a self-explaining notation like $(M^L N^{L'})^k$ is used.

(ii) $M^L \xrightarrow[\approx]{} M$ means that M is the result of omitting the labels in M^L.

6.5. Definition. Let φ be the mapping which changes superscripts into subscripts. By our notation φ maps labeled λ-terms to λ-terms.

Example. Let M^L be as in 6.4. Then $\varphi(M^L) = M_L = x_1(\lambda y.(y_2 y_0)_1)_2 =$ $(\lambda f.xf)(\lambda e.(\lambda d.(\lambda y.(\lambda c.((\lambda b.(\lambda a.ya)b)y)c))d)e)$

6.6. Notation. In stead of $A = \varphi(B^L)$ we will write $B^L \xrightarrow[\varphi]{} A$.

6.7. Lemma. (i) $A \xleftarrow[\varphi]{} B^L \Rightarrow A \xrightarrow[\eta]{*} B$

(ii) $((\lambda x.\ P_p)_r\ Q_q)_s \xrightarrow[\beta]{*} [Q_q \mid x]\ P_{p+s}$

Proof of (i) is evident.

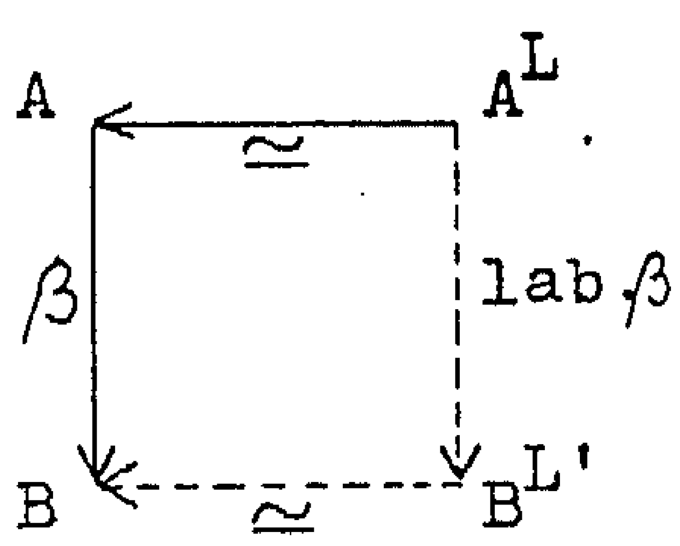
(ii) $((\lambda x.\ P_p)_r\ Q_q)_s \xrightarrow[\beta^*]{6.2.(iii)} ((\lambda x.\ P_p)\ Q_q)_s \xrightarrow[\beta]{}$ $([Q_q|x]P_p)_s = [Q_q|x]P_{p+s}$.

6.8. Definition of labeled β-reduction $\xrightarrow[lab.\beta]{}$:

(i) $((\lambda x.\ P^p)^r\ Q^q)^s \xrightarrow[lab.\beta]{} [Q^q|\ x]\ P^{p+s}$

(ii) If $M^L \xrightarrow[lab.\beta]{} N^{L'}$ then $C[M^L] \xrightarrow[lab.\beta]{} C[N^{L'}]$ for every labeled context with one empty place. Here $[\ |\ ]$ is the usual substitution operator, plus the extra rule: $[Q^q|\,x]\ x^n = Q^{q+n}$.

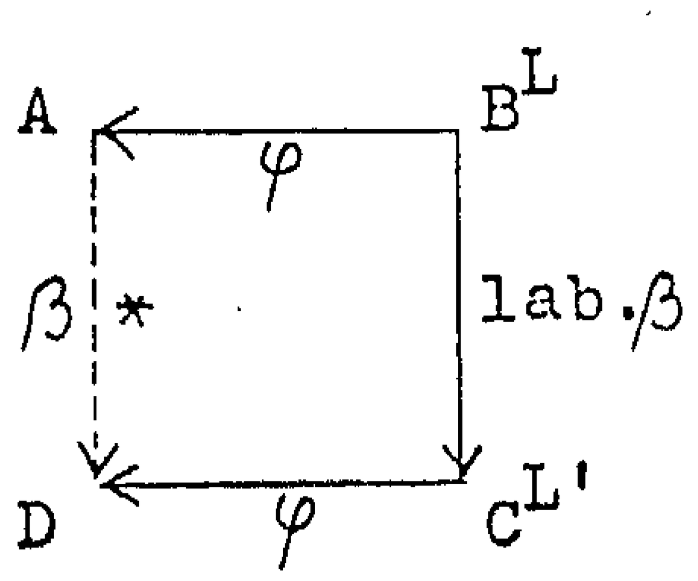
6.9. Lemma.



i.e. if $A \xrightarrow[\beta]{} B$ and L is a labeling of A, then there is a labeling L' of B such that $A^L \xrightarrow[lab.\beta]{} B^{L'}$.

Proof. Clear. ⊠

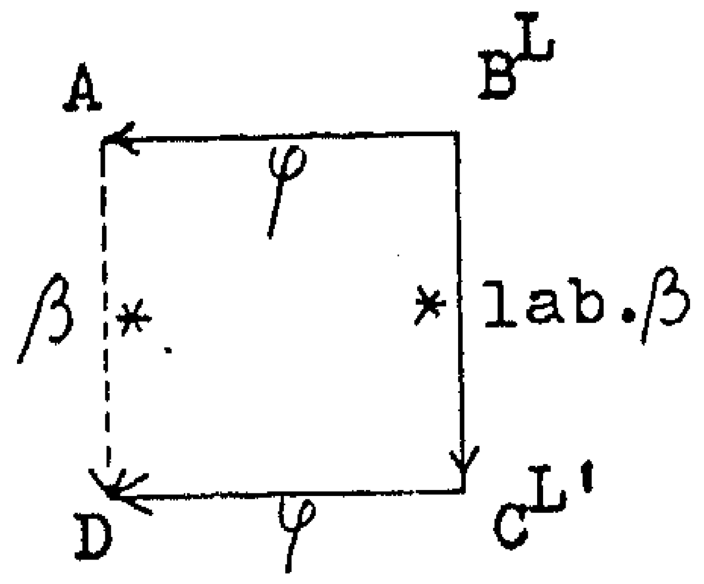
6.10. Lemma.



i.e. $B^L \xrightarrow[lab.\beta]{} C^{L'} \Rightarrow \varphi(B^L) \xrightarrow[\beta]{*} \varphi(C^{L'})$

Proof. Immediate from 6.5, 6.8 and 6.7(ii). ⊠

6.11. Main lemma.



Proof. Immediate from 6.10. ⊠

As a first corollary we prove: if M has a $\beta\eta$-normal form, then M has a β-normal form. We need two lemma's to do this:

6.12. Lemma. M is a β-nf. $\Longrightarrow$ $\varphi(M^L)$ has a β-nf. (for all L).

Proof. The class BNF of β-nf's can be inductively defined by:

i) $x \in$ BNF

ii) $M_1, \ldots, M_n \in \text{BNF} \Longrightarrow xM_1 \ldots M_n \in \text{BNF}$

iii) $M \in \text{BNF} \Longrightarrow \lambda x.M \in \text{BNF}$.

Now we apply induction on this definition.

Case i): $M \equiv x$, $\varphi(x^n) = (x)_n \xrightarrow[\beta]{*} (x)_1 \equiv \lambda y.\ xy$ by 6.2(ii), hence $\varphi(x^n)$ has a β-nf.

Case ii): for simplicity suppose $M \equiv xAB$. Then $\varphi(M^L) \equiv \varphi((xAB)^L) \equiv \varphi(((x^{\ell_1}A^{L_1})^{\ell_2}B^{L_2})^{\ell_3}) = ((x_{\ell_1}A_{L_1})_{\ell_2}B_{L_2})_{\ell_3}$. By 6.2(iii), $\varphi(M^L) \xrightarrow[\beta]{*}$ $(xA_{L_1}B_{L_2})_{\ell_3} \xrightarrow[\beta]{*} (xA_{L_1}B_{L_2})_1$, the last reduction if $l_3 > 0$.

By induction hypothesis A_{L_1}, B_{L_2} have a β-nf., hence $xA_{L_1}B_{L_2}$ and $(xA_{L_1}B_{L_2})_1$ also.

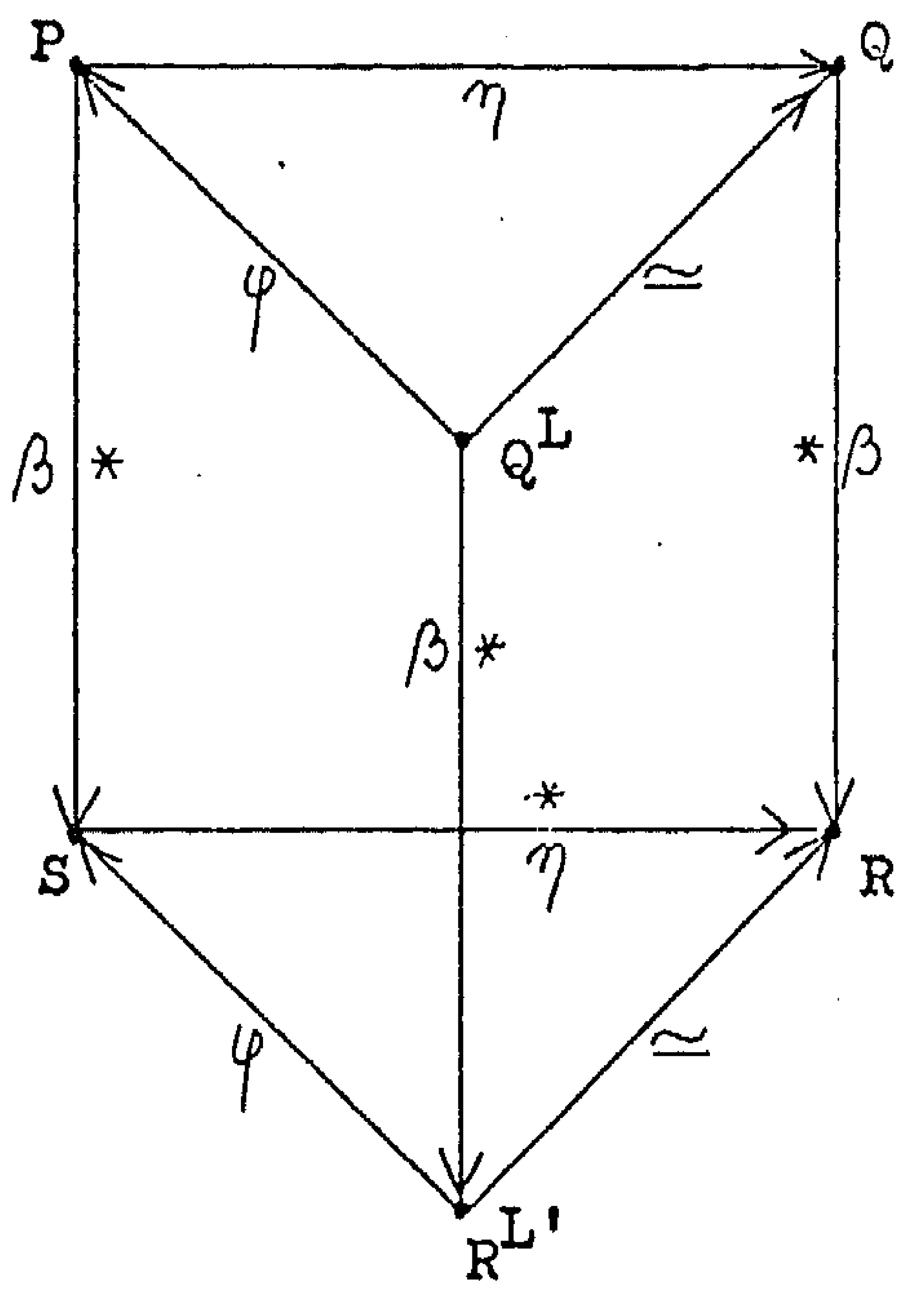
Case iii): $M \equiv \lambda x.\ N.$ $\varphi(M^L) \equiv \varphi((\lambda x.\ N^{L'})^{\ell}) \equiv (\lambda x.\ N_{L'})_{\ell} \xrightarrow[\beta^*]{\text{if } \ell > 0}$ $(\lambda x.\ N_{L'})_1 \equiv \lambda y.\ (\lambda x.\ N_{L'})y \xrightarrow[\beta]{} \lambda y.\ [y|x]N_{L'} \equiv \lambda x.\ N_{L'}$.

By induction hypothesis $N_{L'}$ has a β-nf., hence also $\lambda x.\ N_{L'}$ and $\varphi(M^L)$. ⊠

6.13. Lemma. If $P \xrightarrow[\eta]{} Q$ and Q has a β-nf., then P has a β-nf.

Proof. (See diagram) Let $Q \xrightarrow[\beta]{*} R$. Let the η-redex contracted in $P \xrightarrow[\eta]{} Q$ be $\lambda x.\ Mx$. Then we label Q by giving the resulting M label 1 and every other subterm (occurrence) label 0. This is Q^L. Evidently

$Q^L \xrightarrow[\varphi]{} P$. By 6.9 an $R^{L'}$ such that $Q^L \xrightarrow[\text{lab.}\beta]{*} R^{L'} \simeq R$ can be found.



By the main lemma we have $P \xrightarrow[\beta]{*} \varphi(R^{L'}) \equiv S$ and by 6.7(i) $S \xrightarrow[\eta]{*} R$. Suppose moreover that R is the β-nf. of Q. Then by 6.12 S has a β-nf., hence P has a β-nf. ⊠

6.14. <u>Corollary</u>. M has a β-nf. $\Longleftrightarrow$ M has a $\beta\eta$-nf.

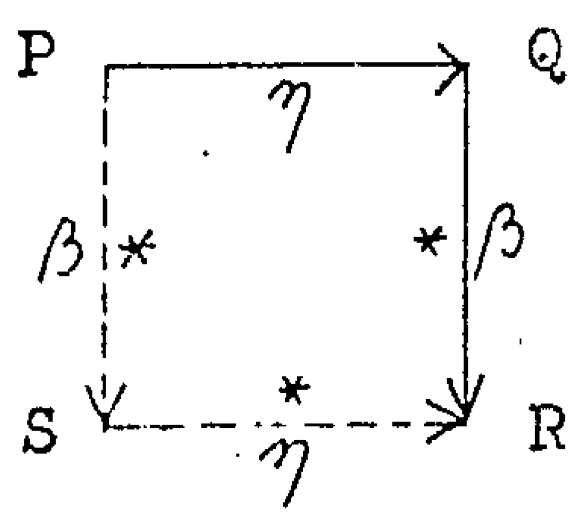
Proof. ($\Longleftarrow$) Induction on the numbers of steps necessary to reduce M to $\beta\eta$-nf., using 6.13. ⊠

($\Longrightarrow$) Trivial, since η-contractions of a β-nf. do not create new β-redices. ⊠

6.15. <u>Corollary</u>. (Postponement of η-reductions)

$$M \xrightarrow[\beta\eta]{*} N \Longrightarrow \exists L \quad M \xrightarrow[\beta]{*} L \xrightarrow[\eta]{*} N$$

Proof. In the first part of the proof of 6.13 we proved



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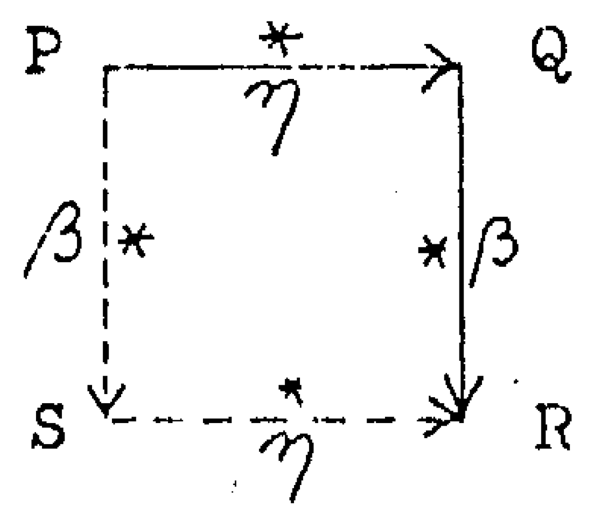


directly follows.

The rest of the proof is routine. ⊠

Now we compare our method of proof of 6.14 with that of [7]7.28. Nederpelt defines a new reduction $\xrightarrow[K]{}$ as follows:

I. $A \xrightarrow[K]{} A$

II. $\dfrac{A \xrightarrow[K]{} B}{\lambda x.Ax \xrightarrow[K]{} B}$ $(x \notin FV(A))$

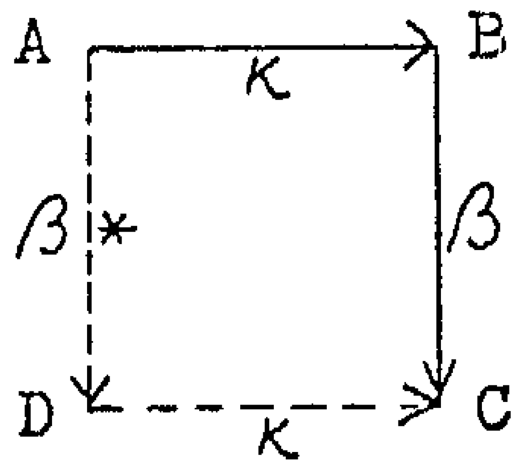
III. $\dfrac{A \xrightarrow[K]{} B \qquad C \xrightarrow[K]{} D}{AC \xrightarrow[K]{} BD}$

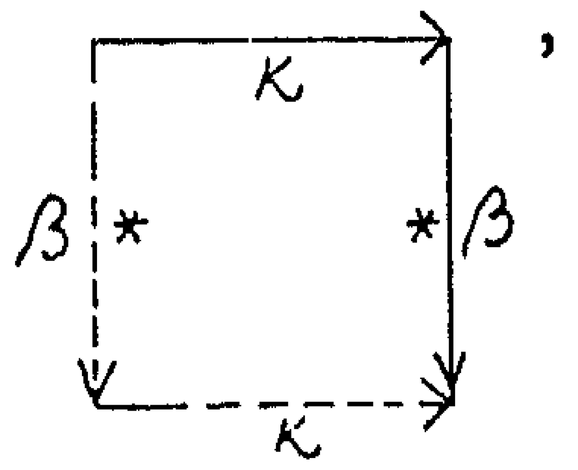
IV. $\dfrac{A \xrightarrow[K]{} B}{\lambda x.A \xrightarrow[K]{} \lambda x.B}$

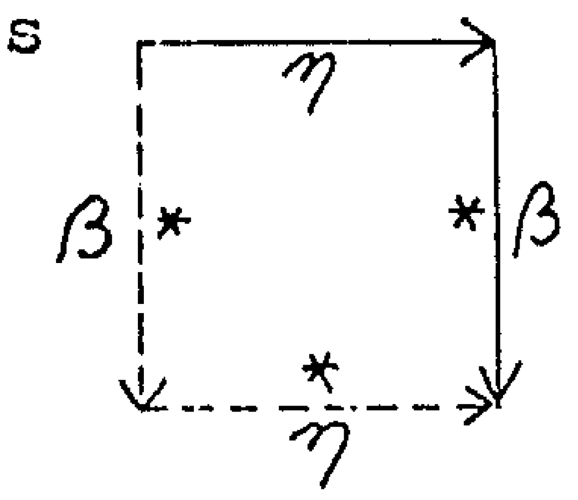
$\xrightarrow[K]{}$ has the following properties:

i) $A \xrightarrow[K]{} B \implies A \xrightarrow[\eta]{*} B$

ii) $A \xrightarrow[\eta]{} B \implies A \xrightarrow[K]{} B$

iii) 

This gives directly 

, which gives 

,

from which the theorem of Postponement of η-reductions follows.

Remark that $\xrightarrow[K]{}$ is not transitive; example:

$\lambda x.\ A(\lambda y.\ xy) \xrightarrow[K]{} \lambda x.\ Ax \xrightarrow[K]{} A$ but not

$\lambda x.\ A(\lambda y.\ xy) \xrightarrow[K]{} A.$

Now there is a simple connection between Nederpelts and our method:

$A \xrightarrow[K]{} B \iff A \xleftarrow[\varphi]{} B^L$ for some L.

Properties i) - iii) about $\xrightarrow[K]{}$ follow from properties of $\xrightarrow[\varphi]{}$.

7. NON-NORMALIZING S-TERMS

An S-term is an applicative combination of S's.
At the Rome conference on λ-calculus (March 1975) the question was raised whether there are S-terms without a normal form (nf).
Several people, including ourselves, provided independently solutions.
We will treat three examples. In each case the proof that the term has no nf is rather different.
The interest in the examples is that they provide terms with a rather unusual reduction pattern.

The length of an S-term is the number of its S's. If a_n is the number of S-terms with length n, then by the formula of Catalan (cf. [3] p. 64)

$$a_n = \frac{1}{2n-1}\binom{2n-1}{n}.$$

The first values of a_n are indicated in fig. 1.
Let b_n be the number of S-terms of length n without a nf.
Mr. Duboué has calculated by computer upper bounds for b_n, for $n < 10$, see fig. 1.

n	1	2	3	4	5	6	7	8	9	10
a_n	1	1	2	5	14	42	132	429	1430	4862
b_n	0	0	0	0	0	0	2	$\leqslant 39$	$\leqslant 231$	

(fig. 1)

The bounds are not exact, since the computer only reduced a term a (large) finite number of times in order to conclude that it might be non-normal. For n = 7, theorem 6.4 proves that the bound is exact.

7.1. Notations. C[ ] is a context containing one or more holes.
$F^0X = X$; $F^{n+1} = F(F^nX)$.
$M \xrightarrow{\circledast} N \iff CL \vdash M \longrightarrow C[N]$ for some context C[ ], and $M \not\equiv C\,N$.

7.2. Lemma. If M is an S-term having an infinite $\xrightarrow{\circledast}$ reduction path, then M has no nf (in combinatory logic, nor its translation M_λ in the λ-calculus).
Proof. Since S-terms are λI-terms this is a well-known property, cf. 5.9.(i). ⊠

7.3. <u>Theorem</u>. (Petorossi) Let $A = SSS$, $\omega = SAA$, $M = \omega\omega$.
Then M has no nf.

<u>Proof</u>. Note that $Axy \xrightarrow{\bullet} xy$, hence by induction $A^n xy \xrightarrow{\bullet} xy$.
Claim: $A^n\omega(A^n\omega) \xrightarrow{\bullet} A^{n+1}\omega(A^{n+1}\omega)$.
Indeed, $A^n\omega(A^n\omega) \xrightarrow{\bullet} \omega(A^n\omega) = SAA(A^n\omega) \longrightarrow A^{n+1}\omega(A^{n+1}\omega)$.
By the claim M has an infinite $\xrightarrow{\bullet}$ path, hence no nf. ⊠

7.4. <u>Theorem</u>. The shortest S-term without nf is of length 7.
In fact there are exactly two such terms: $X_1 = S(SS)SSSS$ and
$X_2 = SSS(SS)SS$.

<u>Proof</u>. (A different proof has been given by Monique Baron.)
Mr. Duboué has shown by computer that all S-terms of length $\leqslant 6$ are normalizable, as well as all other S-terms of length 7.
Now we will prove that $X_{1,2}$ have no nf.

Let $B = S(SS)$, $C = S(BS)S$ and $Y = BC$.

Then $Bxy \longrightarrow S(xy)(y(xy))$, $Bxyz \xrightarrow{\bullet} y(xy)z$,

$Cx \xrightarrow{\bullet} x(Sx)(Sx)$, and

(1) $Yx \xrightarrow{\bullet} x(Sx)(Sx)$.

Now

(2) $X_1 \longrightarrow X_2 \longrightarrow BSSC \longrightarrow BBC \xrightarrow{\bullet} C(BC) \xrightarrow{\bullet} Y(SY)(SY)$.

Def. $\mathcal{A}$ is the set of S-terms inductively defined by

$SY \in \mathcal{A}$

$M \in \mathcal{A} \implies SM \in \mathcal{A}$

$M, N \in \mathcal{A} \implies MN \in \mathcal{A}$

(3) Lemma. For all $M \in \mathcal{A}$: $Mxy \xrightarrow{\bullet} Y\, C_1^M[x,y]\; C_2^M[x,y]$,
where C_1^M, C_2^M are contexts such that (after reduction) $C_{1,2}^M[P,Q] \in \mathcal{A}$ for all $P, Q \in \mathcal{A}$.

Proof. Induction on the structure of M ($\in \mathcal{A}$).

$M \equiv SY$: $SYxy \longrightarrow Yy(xy)$

$M \equiv SN$: $SNxy \longrightarrow Ny(xy) \xrightarrow{\bullet} Y\, C_1^N[y,xy]\; C_2^N[y,xy]$ by the induction hypothesis.

$M \equiv PQ$: $PQxy \xrightarrow{\bullet} PQx \xrightarrow{\bullet} Y\, C_1^P[Q,x]\; C_2^P[Q,x]$.

(4) Cor. $\forall M_1, M_2 \in \mathcal{A}\ \exists M_1', M_2'\quad Y M_1 M_2 \xrightarrow{\bullet} Y M_1' M_2'$.

Proof. Let $M_{1,2} \in \mathcal{A}$. Then by (1) and (3)

$$Y M_1 M_2 \xrightarrow{\bullet} M_1 (SM_1)(SM_1) \xrightarrow{\bullet} Y\, C_1^{M_1}[SM_1, SM_1]\ C_2^{M_1}[SM_1, SM_1] \equiv$$

$Y M_1' M_2'$. ⊠

Now it follows by (2) and (4) that $X_{1,2}$ have an infinite $\xrightarrow{\bullet}$ path. Therefore by 7.2 $X_{1,2}$ have no nf. ⊠

Now we present a third method of proving that an S-term has no nf.

7.5. Theorem. Let A = SSS. Then AAA has no nf.

Proof. (den Hartog)

1. Def. Let SA be the calculus with terms built up by application from constants S, A.

2. Fact. Each SA-term M is of the form $SM_1 \ldots M_n$ or $AM_1 \ldots M_n$.
   The M_i are called the i^{th} component of M.

3. Def. Reduction in SA is defined by
   $$A\vec{M} \longrightarrow SSS\vec{M}$$
   $$SPQR\vec{M} \longrightarrow PR(QR)\vec{M}$$

4. Lemma. Let M be a subterm of an SA-reduct M' of AAA. Then the components of M all end with the letter A except possibly the 1^{st} and 2^{nd} components, in which case they are S.
   Proof. By induction on the SA-reduction sequence $AAA \longrightarrow M'$. ⊠

5. Def. If $M \longrightarrow N$ is an one step SA-reduction, then M is a predecessor of N.

6. Theorem. AAA has in SA an infinite reduction path.
   Proof. The SA-reduction of AAA only can terminate in a term of the form S, SP or SPQ.
   Clearly S and SP have no predecessors. The only possible predecessor of SPQ is SSYP. The only possible predecessor of SSYP is SSXSP. But this term does not satisfy the condition of lemma 4. ⊠

7. Cor. The S-term AAA has no nf.
   Proof. Since AAA has an infinite reduction chain, so has AAA.

⊠

7.6. Fact. Let $A = SSS$, $B = S(SS)$. Then the following terms have no nf:

SAA(SAA)

BSSSS

AAA

SAAA

SBBB

AA(SS).

The first three were treated above. Proofs of the non-normalization of the other terms were given by Hindley and Gerd and Aleid Mitschke. Other examples were provided by Duboué, and Börger and Carstens.

7.7. Question. 1. Is convertibility between S-terms decidable?

2. Is the set of S-terms having a nf decidable?

7.8. Exercise. Prove that S(SS)(SS)(SS)SS has a nf.

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CHAPTER III

REPRESENTABILITY IN LAMBDA ALGEBRAS

Acknowledgement. The authors would like to thank the referee for carefully reading the manuscript and making several improvements in the text.

Introduction. Let $\mathcal{M} = \langle M, \cdot \rangle$ be a λ-algebra (i.e. a model of the λ-calculus). Elements of M are thought of as functions. Arbitrary $f: M \to M$ are called external functions. Such a function is representable (by an element $a \in M$) if $\forall b \in M\ f(b) = a \cdot b$. f is definable in $\mathcal{M}$ if f is representable by $[\![F]\!]^{\mathcal{M}}$ for some closed term F. Here $[\![F]\!]^{\mathcal{M}}$ denotes the value of F in the model $\mathcal{M}$.

Other notations:

x, y, ... denote variables of the λ-calculus.

a, b, ... denote variables ranging over the elements of a λ-algebra.

F, G, ... denote λ-terms.

The numerals $\underline{0}$, $\underline{1}$, $\underline{2}$, ... denote some adequate representation of the natural numbers as λ-terms e.g. those of Church:

$\underline{n} = \lambda fx.\ f^n(x)$.

If T is a consistent extension of the λ-calculus, $\mathcal{M}^{(o)}(T)$ is the closed) term-model of T, i.e. all (closed) λ-terms modulo provable equality in T.

A λ-algebra $\mathcal{M}$ is hard if its domain consists exactly of the images of closed terms. In such an $\mathcal{M}$ a function is representable iff it is definable.

For other terminology see Barendregt [1976].

The three sections of the paper treat different aspects of the notion of representability.

In §1 attention is restricted to the standard extensional term model $\mathcal{M} = \mathcal{M}(\lambda\eta)$.

Church's δ is an external function satisfying

(*) $\delta MM = \underline{0}$ if M is in normal form (nf)

$\delta MM' = \underline{1}$ if M,M' are different nf's.

In Böhm [1972] it is proved that $\forall N_1 \ldots N_n$ different $\beta\eta$ nf's $\exists F \vdash FN_i = \underline{i}$. As a consequence it follows that for every finite set of nf's there is a term δ satisfying (*).

At the Orléans logic conference (1972) the question was raised whether the general Church's δ is definable as a λ-term. We will give a negative answer which was already established in Barendregt [1972] and independently in Wadsworth [1972] (see also Hindley and Mitschke [1975]). All three proofs of the non-existence of δ are different.

Furthermore it is proved that the only definable functions from the terms into the numerals are the constant functions.

In §2 it will be proved that definable functions in various λ-algebras have a range of cardinality 1 or $\aleph_0$. For representable functions this is not true in D_∞ and $P\omega$.

Two external functions f and g on $\mathcal{M}$ are <u>dual</u>, notation $f \sim_{\mathcal{M}} g$, if $f(a).b = g(b).a$ for all $a, b \in \mathcal{M}$.

A model $\mathcal{M}$ is <u>rich</u> if for all f, g:

$$f \sim_{\mathcal{M}} g \implies f \text{ and } g \text{ are representable in } \mathcal{M}.$$

The results of §3 are: D_∞ and $\mathcal{M}(\lambda\eta)$ are rich; rich models are extensional; hard sensible models (e.g. the interior of D_∞) are not rich.

We would like to draw the proof of 3.6 to the readers attention. There variables of the λ-calculus are not just used in the usual way, but also serve as separate entities.

§1. Non definability results.

The main tool in this section is the "Böhm out" technique 1.4. This result is also of use in §2.

1.1 Def. Let BT(M) be the Böhm tree of M, see Barendregt [1976], §5. $x \in BT(M)$ iff $x \in FV(M)$ and occurs as a head variable in some label at a node of BT(M).

1.2 Def. (i) A selector is a term of the form $U \equiv \lambda x_1 \ldots x_n . x_i$. A permutator is a term of the form $C \equiv \lambda x_1 \ldots x_n \cdot x_{\pi(1)} \cdots x_{\pi(n)}$ for some permutation π.

(ii) Simple terms are inductively defined by: Any variable, selector or permutation is a simple term. If P,Q are simple terms, so is PQ.

1.3 Lemma. Simple terms have a normal form (nf).

Proof. Realizing that each simple term is of the form $x\vec{P}$, $U\vec{P}$, $C\vec{P}$ with $\vec{P}$ simple U a selector and C a permutator, it can be shown by induction on the term length that they have a nf. ⊠

1.4 Theorem. Let $FV(M) = \{x\}$ and $x \in BT(M)$. Then

(i) For some $\vec{P},\vec{Q}$, with $x \notin FV(\vec{P})$, $\lambda \vdash M\vec{P} = x\vec{Q}$ ("x is Böhmed out").

(ii) Moreover $\vec{P}$ can be chosen as a sequence of simple terms.

Proof. Let x occur in BT(M) at depth $k > 0$. By a similar construction as in Barendregt [1976] 6.14, 6.15 for some Böhm-transformation π, x occurs in $BT(M^\pi)$ at depth $k-1$. Iterating this leads to $M^\pi = \lambda\vec{y} \cdot x\vec{Q}$, hence $M^\pi\vec{y} = x\vec{Q}$.

Checking the details of the construction of π one verifies that $M^\pi y \equiv M \ldots x_i \ldots [x_j/Cx_j] \ldots [x_k/Ux_k] \ldots \vec{y} \equiv M\vec{P}$ for some simple terms $\vec{P}$ with $x \notin FV(\vec{P})$ (where C is a permutator and U a selector). ⊠

1.5 Lemma. If F is not constant, i.e. $\nvdash FX = FX_1$ for some X_1, X_2, and for some M, FM has a nf., then $x \in BT(Fx)$ for all x.

Proof. Note that if P, P' have equal finite Ω-free Böhm-trees, then $\lambda \vdash P = P'$. Now suppose $x \notin BT(Fx)$ for some $x \notin FV(M)$. Then BT(Fx) = BT(FM). But since FM is in nf, BT(FM) is finite and Ω-free and hence $\vdash Fx = FM$, i.e. F is constant. This contradiction shows $x \notin FV(M) \Longrightarrow x \in BT(Fx)$. But then by substitution $x \in BT(Fx)$ for all x. ⊠

1.6 Def. $\underline{\underline{0}} = I$, $\underline{\underline{n+1}} = K\ \underline{\underline{n}}$.

1.7 Lemma. The function sg is not λ-definable with respect to $\{\underline{\underline{n}} | n \in \omega\}$, i.e. for no λ-term F $\vdash F\ \underline{\underline{0}} = \underline{\underline{0}}$, $\vdash F\ \underline{\underline{n+1}} = \underline{\underline{1}}$.

Proof. Suppose F exists. Then by 1.5 $x \in BT(Fx)$. Hence by 1.4 $Fx\vec{P} = x\vec{Q}$ for some $\vec{P}, \vec{Q} = Q_1 \ldots Q_m$. But then for all $n > m$, $\vdash \underline{\underline{1}}\vec{P} = F\ \underline{\underline{n}}\ \vec{P} = \underline{\underline{n}}\ Q_1 \ldots Q_m = \underline{\underline{n-m}}$ contradicting the Church-Rosser theorem since the $\underline{\underline{k}}$ are different nf's. ⊠

1.8 Def. A system of terms $\{M_n | n \in \omega\}$ is an *adequate system of numerals* iff

(i) Each M_n has a nf.

(ii) Each recursive function can be λ-defined with respect to the M_n.

In Barendregt [197∞] is shown that the second condition can be replaced by (ii'): The successor, predecessor and sg functions can be λ-defined with respect to the M_n.

The following corollary was proved independently by Barendregt [1972] and Wadsworth [1972].

1.9 Cor. (Barendregt). (i) $\{\underline{\underline{n}} | n \in \omega\}$ is not an adequate system of numerals. (ii) Church's δ is not λ-definable.

Proof. (i) Immediate. (ii) If δ were λ-definable, then so would be F in 1.7 since $F = \lambda x \cdot\ \delta x\ \underline{\underline{0}}\ \underline{\underline{0}}\ \underline{\underline{1}}$. ⊠

1.10 Let $\underline{\omega} = \{\underline{n} | n \in \omega\}$ be an adequate system of numerals and let f be a map into $\underline{\omega}$ definable by F. Then f is constant.

Proof. First assume $\underline{\omega}$ is Church's system of numerals, i.e. $\underline{n} = \lambda fx \cdot f^n(x)$ $(= \lambda fx \cdot \underbrace{f \ldots (f}_{n \text{ times}} x))$. Suppose F is not constant, then by 1.5 $x \in BT(Fx)$. Hence for some simple $\vec{P}$ and $\vec{Q}$, $\lambda \vdash Fx\vec{P} = x\vec{Q}$. Hence $\lambda \vdash FM\vec{P} = M\vec{Q}$ for all M. But $M\vec{Q}$ can take arbitrary values and not $FM\vec{P}$, since $\underline{n}\ \vec{P} = P_1^n(P_2)P_3 \ldots P_k$ is always in nf by 1.3. Now let $\underset{\sim}{\omega}$ be an arbitrary system of numerals. It is wellknown how to define a term G such that $G\underset{\sim}{n} = \underline{n}$.
Suppose a non constant f : terms $\rightarrow \underset{\sim}{\omega}$ would be definable, then $G \circ f$ were a definable non constant mapping into $\underline{\omega}$. ⊠

First alternative proof (due to the referee).
Suppose F is not constant, i.e. let $n_1 \neq n_2 \in Ra(F)$. Define G as the λ-defining term of the recursive function $g(x) = 0$ if $x = n_1$, and $g(x) = 1$ else. Then the range of $G \circ F$ is $\{\underline{0}, \underline{1}\}$ contrary to 2.3.

Second alternative proof. By Barendregts lemma in de Boer [1975] it follows that if Ω is unsolvable and N a nf, then

$$F\Omega = N \Longrightarrow Fx = N \text{ for all } x.$$

(General genericity lemma) Now if the values of F are numerals it follows that $P\Omega$ has a nf, i.e. F is constant. ⊠

1.11 Cor. There is no F such that

$FM = \underline{0}$ if M is a numeral (i.e. $\vdash M = \underline{n}$ for some n)

$\quad\quad \underline{1}$ else

for any adequate system.

1.12 Question: Is there a term F such that

FM has a nf (is solvable) if M is a numeral

has no nf (is unsolvable) else.

§2. The range property

2.1 Def. Let $\mathcal{M} = \langle M, \cdot \rangle$ be a λ-algebra. For each $f \in M$, we define $Ra^{\mathcal{M}}(f)$, the range of f in $\mathcal{M}$, as follows:

$$Ra^{\mathcal{M}}(f) = \{f \cdot x \mid x \in M\}.$$

Notation. $Ra^{\mathcal{M}}(F) = Ra^{\mathcal{M}}([\![F]\!]^{\mathcal{M}})$ for terms F.

When possible, the superscript $\mathcal{M}$ will be dropped in $Ra^{\mathcal{M}}$.

2.2 Def. A λ-algebra $\mathcal{M}$ satisfies the range property if for all $f \in M$, the cardinality of $Ra^{\mathcal{M}}(f)$ is 1 or $\aleph_0$.

2.3 Range theorem: (Barendregt; Myhill). Let T be a r.e. λ-theory. Then $\mathcal{M}(T)$ (and also $\mathcal{M}^{0}(T)$) has the range property.

Proof. Suppose $f \in M$ and $Ra(f) = \{m_0, \ldots, m_k\}, k > 0$. Define $N_i = \{x \mid f \cdot x = m_i\} \subset M$. Every such N_i is r.e. Therefore $N = \bigcup_1^k N_i$, the complement of N_0 is also r.e.. Hence N_0 is recursive.

On the other hand N_0 is non-trivial and closed under equality, which contradicts Scott's theorem, (Barendregt [1976] 2.21).

The proof for $\mathcal{M}^{o}(T)$ is the same. ⊠

2.4 Cor. $\mathcal{M}^{(o)}(\lambda(\eta))$ has the range property.

The range property, however, is not satisfied in every λ-algebra.

2.5 Theorem. $P\omega$ and D_∞ do not satisfy the range property.

Proof. Since the proof is similar in both cases, let $\mathcal{S} = (S, \leqslant)$ denote either $(P_\infty, \subseteq)$ or $(D_\infty, \sqsubseteq)$. We define the following function $f: S \longrightarrow S$ by $f(x) = \top$ if $x \neq \bot$ else $\bot$ ($\top$ and $\bot$ are the largest respectively smallest element of S.)

Claim f is continuous. Then by Scott [1972], [1975] f is representable and since f has range of cardinality two we are done.

For open O in S one has: $x \in O$ and $x \leq y \implies y \in O$.
See Scott [1972], [1975] for definition of the topologies involved.
Hence for open O, $\bot \in O \implies O = S$, and $O \neq \emptyset \implies \top \in O$.

Now for every open set O, $f^{-1}(O)$ is open:

Case 1. $\bot \in O$. Then $O = S$ so $f^{-1}(S) = S$ which is open.

Case 2. $\bot \notin O$. If $O = \emptyset$, then we are done. Else $\top \in O$ and hence

$$f^{-1}(O) = S - \{\bot\} = \{x \mid x \not\leq \bot\} \underset{def}{=} U_\bot .$$

$U_\bot$ is open in D_∞, see e.g. Barendregt [1976] 4.2.

$U_\bot$ is open in $P\omega$: Let $O_k = \{x \mid e_k \subseteq x\}$. Note $e_o = \emptyset = \bot$ and that the O_k form a base for the topology on $P\omega$.

Now: $x \in U_\bot \iff x \notin O \iff \exists k_{k \neq 0}\, e_k \subseteq x \iff x \in \bigcup_{k \neq 0} O_k$

which is, as a union of elements of a base, indeed open. ⊠

The following theorem was announced in Wadsworth [1973] for the D_∞ case.

2.6. Theorem. Let $\mathcal{S}$ be D^0_∞ or $P^0\omega$. Then $\mathcal{S}$ satisfies the range property.

Proof. Let F be a closed term. Consider BT(Fx).

Case 1. $x \notin$ BT(Fx). Then BT(FM) = BT(FM') for all M,M'. Since terms with equal Böhm trees are equal in $\mathcal{S}$, see Hyland [1975], Barendregt [1976], it follows that $Ra^{\mathcal{S}}(F)$ has cardinality 1.

Case 2. $x \in$ BT(Fx). Then by 1.4 $\lambda \vdash Fx\vec{P} = x\vec{Q}$.

Since $[\![N\vec{Q}]\!]^{\mathcal{S}}$ can take arbitrary values in $\mathcal{S}$ when N ranges over the closed terms, $Ra^{\mathcal{S}}(F)$ is infinite. ⊠

2.6 Conjecture. $\mathcal{M}(\mathcal{H})$ satisfies the range property.

2.7 Question. Does every hard λ-algebra $\mathcal{M}$ (i.e. $\mathcal{M} = \mathcal{M}^0$) satisfy the range theorem?

§3. Duality

3.1 Def. Let f,g be two external functions on a λ-algebra $\mathcal{M} = \langle M, \cdot \rangle$

f,g are dual iff $\forall a,b \in M$: $f(a)\cdot b = g(b)\cdot a$. Notation $f \sim_{\mathcal{M}} g$, or simply $f \sim g$.

Remarks. (i) Let f be an external function on $\mathcal{M}$. f is locally representable iff for each $b \in M$ the function h defined by $h(a) = f(a).b$ is representable. Then f is locally representable iff f has a dual. A model is rich iff all locally representable functions are representable.

(ii) If f is representable (by $f_0 \in M$, say), then f has a dual g which is also representable (by $g_0 = \lambda ab.f_0 ba$).

(iii) Let $\mathcal{M}$ be extensional. Then f has at most one dual. Hence if $f \sim_{\mathcal{M}} g$ and f is representable, then by (ii) g is representable.

3.2 Def. $\mathcal{M}$ is rich iff all dual functions on $\mathcal{M}$ are representable in $\mathcal{M}$.

3.3 Theorem. If $\mathcal{M}$ is rich, then $\mathcal{M}$ is extensional.

Proof. Suppose $\mathcal{M}$ is not extensional. Then there exist $b,b' \in M$ such that for all $c \in M$ $b\cdot c = b'\cdot c$ and $b \neq b'$.

Define $f(a) = \begin{cases} b' & \text{if } a = b \\ b & \text{else.} \end{cases}$

and $g = [\![\lambda y.K(by)]\!]^{\mathcal{M}}$,

then for all $a, a' \in M$: $f(a).a' = b.a' = g(a').a$, hence $f \sim g$.
But f cannot be representable since it has no fixed point. Thus $\mathcal{M}$ is not rich. ⊠

3.4 Cor. The following λ-algebras are not rich:

$P\omega$; $P^0\omega$; $\mathcal{M}(\lambda)$; $\mathcal{M}^0(\lambda)$; $\mathcal{M}^0(\lambda\eta)$.

Proof.

1. $P\omega$ is not extensional:

Take for example $a = \{(0,0)\}$ and

$b = \{(0,0),(1,0)\}$

Then $\forall c \in P\omega\ a \cdot c = b \cdot c$ but $a \neq b$.

2. $P^0\omega$ is not extensional: Let $1 = \lambda xy \cdot xy$, then $P^0\omega \models Ixy = 1xy$, but $P^0\omega \not\models I = 1$ for otherwise $P\omega \models I = 1$, so $P\omega \models \forall xy\ x = \lambda y \cdot xy$ which implies that $P\omega$ were extensional.

3. By the Church Rosser property $\lambda \not\vdash I = 1$. So $\mathcal{M}(\lambda)$, $\mathcal{M}^0(\lambda)$ are not extensional.

4. $\mathcal{M}^0(\lambda\eta)$ is not extensional because the λ-calculus is ω-incomplete, see Plotkin [1974].

3.5 Theorem. D_∞ is rich.

Proof. Suppose that f,g are dual i.e.:

$$\forall a,b \in D_\infty\colon f(a) \cdot b = g(b) \cdot a.$$

We have to show that f,g are representable.

It is sufficient to show that f,g are continuous. Take a directed $X \subset D_\infty$. For all $b \in D_\infty$ $f(\sqcup X) \cdot b = g(b) \cdot \sqcup X = \sqcup\{g(b) \cdot a | a \in X\} = \sqcup\{f(a) \cdot b | a \in X\} = \sqcup\{f(a) | a \in X\} \cdot b$ by the duality condition and the continuity of application.

Thus by extensionality in D_∞: for all directed X $f(\sqcup X) = \sqcup\{f(a) | a \in X\}$ i.e. f is continuous. The proof for g is dual. ⊠

3.6 Theorem. $\mathcal{M}(\lambda\eta)$ is rich.

Proof. Define $M =_{\lambda\eta} N$ iff $\lambda\eta \vdash M = N$ and $x \in_{\lambda\eta} M$ iff for all $M' =_{\lambda\eta} M$ one has $x \in FV(M')$.

Let f,g be dual functions on $\mathcal{M}(\lambda\eta)$.

3.6.1 Lemma. (i) $x \in_{\lambda\eta} \lambda y \cdot P \Rightarrow x \in_{\lambda\eta} P$. (ii) Let $x \not\equiv y$, then $x \in_{\lambda\eta} M \Longleftrightarrow x \in_{\lambda\eta} My$.

Proof. (i) Let $x \in_{\lambda\eta} \lambda y \cdot P$. Suppose $N =_{\lambda\eta} P$, then $\lambda y \cdot N =_{\lambda\eta} \lambda y \cdot P$. Therefore $x \in FV(\lambda y \cdot N) \subset FV(N)$.

(ii) $\Rightarrow$: Let $x \in_{\lambda\eta} M$. Case 1. $M =_{\lambda\eta} \lambda y \cdot P$. Then $x \in_{\lambda\eta} \lambda y \cdot P$, so by (i) $x \in_{\lambda\eta} P =_{\lambda\eta} My$. Case 2. $M =_{\lambda\eta} X \Rightarrow X$ is not of the form $\lambda y.P$. Suppose $N = My$. By the Church-Rosser theorem there is a Z such that $\lambda\eta \vdash N \to Z$, $My \to Z$. Then $Z \equiv M'y$ and $M' =_{\lambda\eta} M$. Therefore $x \in FV(M') \subset FV(Z) \subset FV(N)$.

$\Leftarrow$: Let $x \in_{\lambda\eta} My$. Suppose $N =_{\lambda\eta} M$. Then $Ny =_{\lambda\eta} My$. Therefore $x \in FV(Ny)$ and hence $(x \not\equiv y)$ $x \in FV(N)$. $\boxtimes_{3.6.1}$

3.6.2 Lemma. If $\exists y \not\equiv x \; x \in_{\lambda\eta} f(y)$, then $\forall y \not\equiv x \; x \in_{\lambda\eta} g(y)$ (and hence $\forall y \not\equiv x \; x \in_{\lambda\eta} f(y)$).

Proof. Suppose $x \in_{\lambda\eta} f(y)$, $y \not\equiv x$. Let $y' \not\equiv x$. Then by 3.6.1 (ii) $x \in_{\lambda\eta} f(y) \cdot y' =_{\lambda\eta} g(y') \cdot y$. Hence, 3.6.1 (ii), $x \in_{\lambda\eta} g(y')$. (The rest follows by applying the statement to $x \in_{\lambda\eta} g(y)$). $\boxtimes_{3.6.2}$

3.6.3 Main lemma. There is a variable x such that for all terms M: $f(x)[x/M] = f(M)$.

Proof. Let v be any variable. Choose $x \not\equiv v$ such that $x \notin_{\lambda\eta} f(v)$. Then $x \notin_{\lambda\eta} g(z)$ for all $z \not\equiv x$, by the dual of 3.6.2.

Given M, one can find a y such that $y \notin_{\lambda\eta} M, f(M), x, f(x)$. Hence $x \notin_{\lambda\eta} g(y)$. Now since $y \not\equiv x$ and $x \notin_{\lambda\eta} g(y)$, $(f(x)[x/M])y = (f(x) \cdot y)[x/M] = (g(y) \cdot x)[x/M] = g(y) \cdot M = f(M) \cdot y$.

Since $y \notin f(x), M, f(M)$, extensionality yields $f(x)[x/M] = f(M)$. $\boxtimes_{3.6.3}$

Now it follows by 3.6.3 that f can be represented by the term $\lambda x \cdot f(x)$ and similary for g. $\boxtimes$

The following construction is needed for the proof of 3.10.

3.7 <u>Def</u>. Let $\#$ be a Gödel numbering of terms. $\ulcorner M \urcorner$ is the numeral $\underline{\#M}$. A sequence of terms M_n is recursive if $\lambda n.\#M_n$ is a recursive function.

3.8 <u>Lemma</u>. (Coding of infinite sequences). Let $\{M_n\}$ be a recursive sequence of terms such that $FV(M_n) \subseteq \{x\}$ for all n. Then there exists a term X such that $p\underline{i}X = M_i$, for all i, where p is some closed term. Par abus de langage we write $\langle M_n \rangle_{n\in\omega}$ for X.

<u>Proof</u>.

As in Curry et al. [1972], 13 B3 there is a term E which enumerates all terms with x as only free variable:

$$E(\#M) = M, \text{ for } M \text{ with } FV(M) = \{x\}.$$

Let $[M,N]$ be a pairing of terms defined by $\lambda z.zMN$. Then $[M,N]K = M$ and $[M,N](KI) = N$. Define ordered tuples as follows: $[M] = M$, $[M_1, \ldots, M_{n+1}] = [M_1,[M_2, \ldots, M_{n+1}]]$.

Let M_n with $FV(M_n) \subseteq \{x\}$ be a recursive sequence of terms, i.e. $f = \lambda n.\#M_n$ is recursive. We want to code the sequence M_n as a λ-term. Let S^+ be such that $S^+\underline{n} \xrightarrow{*}_{\beta} \underline{n+1}$ and let $b \equiv \lambda xy.[E(Fy),(x(S^+y))]$, where f λ-defines f, and $B \equiv FP\ b$. Then

$B\underline{n} \xrightarrow{*}_{\beta} bB\underline{n} \xrightarrow{*}_{\beta} [E(F\underline{n}),B\underline{n+1}] \xrightarrow{*}_{\beta} [M_n,B\underline{n+1}]$. So $B\underline{o} = [M_o,B_1] = [M_o,M_1,B_2] = \ldots$. Hence by setting $\langle M_n \rangle_{n\in\omega} = B\underline{o}$ we have a coding for infinite sequences of terms with one fixed free variable.

It is easy to construct a term p such that $p\underline{m}\langle M_n \rangle_{n\in\omega} = M_m$, (take e.g. pxa = if zero x then aK else $p(x-1)(a(KI))$, using the fixed point theorem). ⊠

3.9 <u>Lemma</u>. For all closed Z there is an n such that $Z\Omega^n =_{\mathcal{H}} \Omega$.

($Z\Omega^n$ is short for $Z\underbrace{\Omega\Omega\ldots\Omega}_{n \text{ times}}$)

<u>Proof</u>.

Case 1. Z is unsolvable; then $Z =_{\mathcal{H}} \Omega$, so $n = 0$.

Case 2. Z is solvable; then Z has a HNF, $Z = \lambda\vec{x}.x_iA_1\ldots A_m$ $(x_i \in \vec{x})$.

Take $n = i$, so $Z\Omega^i = \lambda\vec{x}'.\Omega A_1\ldots A_m =_{\mathcal{H}} \Omega$. ⊠

3.10 <u>Theorem</u>. If $\mathcal{M}$ is hard and sensible, then $\mathcal{M}$ is not rich.

<u>Proof</u>. If $\mathcal{M}$ is hard, then $\mathcal{M}$ is isomorphic to $\mathcal{M}^o(T)$, where $T = Th(\mathcal{M})$. We reason in $\mathcal{M}^o(T)$. Since $\mathcal{M}$ is sensible, $\mathcal{H} \subseteq T$.

Let $h: \omega \to \omega$ be a function not definable in $\mathcal{M}$. h exists since a hard model is countable.

Let $A_n(x,y)$ be the term $x\Omega^n(y\Omega^n(h\underline{n}))$, $n \in \omega$. For closed M the sequence $A_o(M,y)$, $A_1(M,y)$, ... is by 3.9

$My(h\underline{o})$, $M\Omega(y\Omega(h\underline{1}))$, ..., $M\Omega^n(y\Omega^n(h\underline{n}))$, Ω, Ω, ... ,

where n is such that $M\Omega^{n+1} = \Omega$. Thus $\lambda n.A_n(M,y)$ is a recursive sequence containing one fixed free variable and hence representable as a term. Define $f(M) = \lambda y. \langle A_n(M,y)\rangle_{n\in\omega}$. Similarly for closed N $\lambda n. A_n(x,N)$ is recursive and it is possible to define $g(N) = \lambda x. \langle A_n(x,N)\rangle_{n\in\omega}$. Then for all closed M, N: f(M) and g(N) are well defined and $f(M).N = g(N).M = \langle A_n(M,N)\rangle_{n\in\omega}$ by construction. So f and g are dual.

Suppose now that $\mathcal{M}$ is rich, i.e. f were representable by some closed F. Then for all closed M,N: $FMN = f(M)N = \langle A_nM,N\rangle_{n\in\omega}$. But then $\underline{p}\underline{n}(F(K^nI)(K^nI)) = \underline{p}\underline{n}\langle h(\underline{n})\rangle_{n\in\omega} = h(\underline{n})$, hence h were definable, contradiction. Thus $\mathcal{M}$ is not rich. ⊠

3.11 Corollary. D^0_∞ and $\mathcal{M}^0(T)$ for $T \supset \mathcal{H}$ are poor.

3.12 Questions. (i). Is every extensional term model $\mathcal{M}(T)$ rich?
(ii). Is $\mathcal{M}^0(\lambda\omega)$ rich?

Here $\lambda\omega$ is the λ-theory obtained by adding the ω-rule to the theory, see Barendregt [1974].

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