

## THE RUIJGROK-VAN HOVE MODEL OF FIELD THEORY IN TERMS OF "DRESSED" OPERATORS

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### Synopsis

A "dressing" transformation for the Ruijgrok-Van Hove model of field theory is given. By means of this transformation the Hamiltonian is expressed in terms of "dressed" operators only. It is shown that the Hamiltonian depends explicitly on the cut-off. The solution of the eigenvalue problem is given for one meson present in addition to the dressed nucleon. It also depends on the cut-off. A ghost state with negative norm as well as a bound state with positive norm are found, both belonging to the same energy value. The solution for one nucleon and several mesons is outlined. A detailed discussion of the metric in Hilbert space is given. It is shown that the "bare" particle representation is not adequate to describe the system considered. The most suitable representation, from physical and mathematical point of view, seems to be the asymptotic stationary states representation, introduced by Van Hove. A test as to the merits of these two representations is given in the case of indefinite metric; here the "bare" particle representation can no longer be used to describe the system, in contrast to the asymptotic stationary states representation which can be employed with success.

1. *Introduction.* In 1956 Ruijgrok and Van Hove<sup>1)</sup> have proposed a new model of field theory (hereafter quoted as R.-V.H.M.). This model was an extension of the Lee model: both kinds of heavy particles, i.e.  $V$ - and  $N$ -particles (we call them in the sequel nucleons and label them with the index  $q$ ;  $q=1$  denotes  $V$ ,  $q=2$  denotes  $N$ ), were treated in the same manner, as "dressed" particles. This model was investigated afterwards by several authors, e.g. by dell'Antonio, Duimio<sup>2)</sup>, Ruijgrok<sup>3)</sup>, Greenberg and Schweber<sup>4)</sup>.

Ruijgrok<sup>1)3)</sup> and Van Hove<sup>1)</sup> have performed for their model the mass renormalization as well as the determination of the renormalized coupling constants  $g_q$ ,  $q=1, 2$ ; they found also the eigenfunction corresponding to one nucleon and the norm of this function,  $N(q)$ ,  $q=1, 2$ . Further they showed that in their model, unlike in Lee's model, one can reach the point source keeping a non vanishing interaction and a definite metric in Hilbert space; if one keeps the unrenormalized coupling constants  $g_q^0$  constant with

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respect to the cut-off parameter  $L$  and regards  $g_1$  and  $g_2$  as functions of  $L$ , then in the infinite cut-off limit both renormalized coupling constants become always equal. They worked, however, with "bare" particle states and operators.

The goal of this note is:

- i) to find a suitable "dressing" transformation and express the Hamiltonian in terms of "dressed" operators only, instead of "bare" ones. We mention that to get rid entirely of the "bare" representation and establish a complete new mathematical scheme we are still obliged to introduce the metric.
- ii) to give a satisfactory discussion of the eigenvalue problem as well as of the properties of the Hamiltonian in terms of this new, "dressed" representation.

The natural way to introduce suitable "dressed" operators is to make use of the ingenious idea of Van Hove <sup>5)</sup> about the asymptotic stationary states (hereafter quoted as a.s.s.) representation. To build up this representation we are obliged to know the eigenstates of the Hamiltonian, which correspond to the presence of only one particle, nucleon or meson. There exist many transformations which yield the connection between the unperturbed one-particle states  $|\alpha\rangle$  and the corresponding physical one-particle states  $|A(\alpha)\rangle$

$$|A(\alpha)\rangle = D |\alpha\rangle \quad (1)$$

Among these transformations is at least one unitary  $U$  which transforms the complete orthonormal set  $|\alpha\rangle$  into a complete orthonormal set of eigenstates of the Hamiltonian. The point is, however, that we are not looking for  $U$ , because it is a forbidding task, but we are quite satisfied with any  $D$ -transformation. Loosely speaking the reason for this is that if one considers an eigenstate in an infinitely large box the outgoing or ingoing part of which corresponds to given numbers of particles, then for large spatial separation of these particles this eigenstate is well approximated by a product of a given number of one particle eigenstates; the difference arising from different  $D$ -transformations can be dropped in the asymptotic formulae (for more rigorous formulations see reference <sup>5)</sup>). Thus one can extend the definition (1) to all unperturbed states  $|\alpha\rangle$  and gets in this way a complete set of a.s.s.. This set is in general not orthogonal. In general only the one particle a.s.s. are eigenstates of the Hamiltonian.

We call  $D$  – the "dressing" transformation. This transformation can now be applied to any operator  $\Theta$

$$\Theta_{\text{dressed}} = D^{-1}\Theta D \quad (2)$$

In this way we get the "dressed" operators, in particular "dressed" creation and annihilation operators. They obey the same commutation rules as the

"bare" creation and annihilation operators. The "dressed" operators were already investigated by Greenberg and Schweber<sup>4)</sup>, and applied to R.-V.H.M. In this note we shall give a "dressing" transformation, slightly different from that given by Greenberg and Schweber. Our transformation secures not only the mass renormalization (as Greenberg and Schweber's does) but also yields the normalization of the "dressed" operators as well as the renormalization of the coupling constants.

With the help of (2) we are able to express our Hamiltonian in terms of the "dressed" operators only. If we express at the same time the unrenormalized quantities in renormalized ones, the new form of the Hamiltonian reveals the fact that the Hamiltonian depends *explicitly* on the cut-off parameter  $L$ . Thus we shall expect that both the eigenvalues as well as the eigenfunctions shall depend on the cut-off. That this must be so, follows to some extent from former work done by Ruijgrok<sup>1)3)</sup> and dell'Antonio and Duimio<sup>6)</sup>, particularly from the fact that the ratio of eigenfunction normalizations  $N(q)/N(q+1)$  becomes negative for certain values of  $L$ , as well as from the close connection with the Lee model (we get the Lee model by putting  $g_2^0 = 0$  in the R.-V.H.M.). In this note we establish this statement explicitly by investigating in detail the behaviour of the energy spectrum and eigenfunctions for nucleon-meson states (see § 5 and 7). Thus the appearance of  $L$  in the Hamiltonian is not a consequence of a non-appropriate choice of the  $D$ -transformation. As far as the energy spectrum is concerned a point of interest is that we find a "ghost state" (with negative norm), which appears for  $q = 1$  in the region of indefinite metric. It has a counterpart in a bound state (positive norm) for  $q = 2$  and in the region of definite metric. In general we are able to show that the energy spectrum of nucleon-meson states for  $q = 1$  and  $L = \epsilon + L_c$ , where  $L_c$  is the critical value of  $L$  (for the definition see below), is the same as for  $q = 2$  and  $L = -\epsilon + L_c$  and vice-versa, the eigenfunctions being, however, different in both cases. The reference just made to an indefinite metric will now be clarified. To make our investigation complete we must consider the metric in the Hilbert space of a.s.s.. We accept the metric given by Ruijgrok<sup>3)</sup>. This metric was established for  $L < L_c$ ; it depends implicitly on  $L$ . For  $L < L_c$  there is a one to one correspondence between  $|\alpha\rangle$  and  $|A(\alpha)\rangle$ . This correspondence breaks down when we enter the region  $L > L_c$ . The a.s.s. escape, together with the eigenfunctions of the Hamiltonian, into another Hilbert space with indefinite metric which only partly overlaps with the original space (see § 7). When we enter the region  $L > L_c$  our Hamiltonian expressed by "dressed" operators remains well defined, because it depends on  $L$  only through  $N(q+1)/N(q)$  (see below) and this ratio is well defined also for  $L > L_c$ , in contrast to  $N(q)$  which becomes meaningless in this region. We notice that we regard the renormalized coupling constants as given and independent of  $L$ ; thus the unrenormalized coupling constants are functions

of  $L$ . As the point source limit for given unrenormalized coupling constants gives  $g_1 = g_2$ , we expect to run into an indefinite metric taking the point source limit for each choice of  $g_a$  except  $g_1 = g_2$ . If  $g_1 \neq g_2$  we assume  $g_1 > g_2$  (to make possible the comparison with the Lee model). If we keep the commutation formulae for "dressed" operators as well as the expressions for the metric tensor unaltered for  $L > L_c$ , we get quite naturally the extension of our computation scheme from one region into the other. Our scheme is complete. We emphasize that using the language of a.s.s. we do not run into troubles like Källén and Pauli did by expressing the ghost state in terms of undisturbed states. We do not need to alter our metric in an arbitrary manner as was done by Källén and Pauli. The metric adopted from Ruijgrok's work alters by itself with  $L$ . Thus we can work with our a.s.s. formalism in both regions, of definite and indefinite metric, exactly in the same manner. The conclusion is that the a.s.s. representation is very appropriate for describing the behaviour of the system.

We are assuming consistently our system to be enclosed in a finite box (discrete momentum spectrum). Many authors (e.g. Källén and Pauli<sup>8)</sup>, Glaser and Källén<sup>9)</sup>) mix often in their computations the cases of discrete and continuous spectra. It seems to us that these two cases have different regularities and therefore have to be treated separately (e.g. in the case of infinite volume a cut along the real axis appears for the resolvent, giving three sets of solutions (ingoing, outgoing, standing waves) instead of one, and in addition metastable states may appear due to complex poles on other sheets of the Riemann surface). We comment on the case of infinite volume at the end of § 5.

We are working throughout the note with the Schrödinger picture. Thus the procedure is not covariant.

In § 2 we define and discuss the "dressing" transformation as well as the a.s.s. representation.

In § 3 we give its "dressed form" to the Hamiltonian. From the integrals of motion we can infer about the general form of the eigenfunctions. In view of this form the Hamiltonian will be transformed in the most convenient way for further investigations.

In § 4 we discuss the dependence of the theory on the cut-off parameter  $L$ .

In § 5 we give the solution of the eigenvalue problem for one nucleon-one meson states. This paragraph in our view contains the most important new results.

In § 6 we outline the solution of the eigenvalue problem for two and more mesons and one nucleon.

Finally § 7 is devoted to the discussion of the metric. This seems to us of some importance not only for this particular model but from a general point of view.

2. *Dressing transformation.* The Hamiltonian of R.-V.H.M.<sup>1)3)</sup> reads

$$H = H_0 + V \quad (3a)$$

$$H_0 = \sum_{q=1}^2 \sum_p m \psi_q^\dagger(\mathbf{p}) \psi_q(\mathbf{p}) + \sum_k \omega(k) a^\dagger(\mathbf{k}) a(\mathbf{k}) \quad (3b)$$

$$V = - \sum_k \omega(k) X(k) [a(\mathbf{k}) T(\mathbf{k}) + a^\dagger(\mathbf{k}) T^\dagger(\mathbf{k})] - \sum_{q=1}^2 \sum_p \delta m_q \psi_q^\dagger(\mathbf{p}) \psi_q(\mathbf{p}) \quad (3c)$$

$$T^\dagger(\mathbf{k}) = \sum_{q=1}^2 \sum_p g_q^0 \psi_{q+1}^\dagger(\mathbf{p}) \psi_q(\mathbf{p} + \mathbf{k}) \quad (3d)$$

The notation is the same as in <sup>3)</sup>. The non vanishing (anti) commutators are

$$[a(\mathbf{k}), a^\dagger(\mathbf{k})]_- = \{\psi_q(\mathbf{p}), \psi_q^\dagger(\mathbf{p})\}_+ = 1 \quad (4)$$

As already mentioned the goal of this note, among others, is to eliminate the representation of bare particle states  $|\alpha\rangle$  from the formalism and to express all quantities – operators and vectors in Hilbert space – in terms of the a.s.s. introduced by Van Hove<sup>5)</sup>. This program was suggested by Van Hove and partly carried out by Ruijgrok<sup>3)</sup> and Greenberg and Schweber<sup>4)</sup>. The a.s.s. representation is the same as the dressing representation.

To obtain our dressing transformation we begin with the observation that the a.s.s. for one nucleon and several mesons as defined in <sup>3)</sup> can be written

$$|A(q, \mathbf{p}; \{m_k\}_n)\rangle = e^F N^\dagger \psi_q^\dagger(\mathbf{p}) \prod_k a^\dagger(\mathbf{k})^{m_k} (m_k!)^{-\frac{1}{2}} |0\rangle \quad (5)$$

with

$$F = \sum_k X(k) a^\dagger(\mathbf{k}) T^\dagger(\mathbf{k}) \quad (6)$$

and  $N^\dagger$  being an operator defined by

$$N^\dagger |\alpha\rangle = N^\dagger(\alpha) |\alpha\rangle \quad (7)$$

where  $N(\alpha) = \prod_i N(q_i)$ . We assume  $N(\alpha) \neq 0$  and  $N^\dagger |0\rangle = |0\rangle$ . Since  $N(\alpha)$ , and consequently  $N$ , will depend on the cut-off parameter  $L$  – as we shall see – this definition holds only for  $L < L_c$ . The cut-off parameter is defined by

$$L = \sum_k X^2(k) \quad (8)$$

The definition of  $L_c$  will be given later in § 4. Because

$$e^F N^\dagger |0\rangle = |0\rangle$$

we can write (5) in the form

$$|A(q, \mathbf{p}; \{m_k\}_n)\rangle = O_q^\dagger(\mathbf{p}) \prod_k a^\dagger(\mathbf{k})^{m_k} (m_k!)^{-\frac{1}{2}} |0\rangle \quad (9)$$

where

$$O_q^\dagger(\mathbf{p}) = e^F N^\dagger \psi_q^\dagger(\mathbf{p}) N^{-\frac{1}{2}} e^{-F} = N^\dagger(q) e^F \psi_q^\dagger(\mathbf{p}) e^{-F} \quad (10)$$

is the dressed creation operator of the nucleon of kind  $q$  with momentum  $\mathbf{p}$ ;  $O_q^\dagger(\mathbf{p})$  commutes with  $a^\dagger(\mathbf{k})$ . As  $F$  and  $N$  commute with  $a^\dagger(\mathbf{k})$  the bare and dressed creation operator for a meson of momentum  $\mathbf{k}$  coincide.

We adopt

$$e^{FN^\dagger} \quad (11)$$

as the dressing transformation. It is not unitary. As already mentioned this dressing operation is, of course, not unique. By means of this transformation and its inverse we can dress each bare operator following (2)

$$\Theta_{\text{dressed}} = e^{FN^\dagger} \Theta N^{-\dagger} e^{-F} \quad (12)$$

and each bare particle state

$$\begin{aligned} |A(\alpha)\rangle &= e^{FN^\dagger} |\alpha\rangle \\ \langle \bar{A}(\alpha) | &= \langle \alpha | N^{-\dagger} e^{-F} \end{aligned} \quad (13)$$

Equations (13) give us the a.s.s. and the states contragredient to them. In terms of (13) the unit operator reads

$$I = \sum_{\alpha} |A(\alpha)\rangle \langle \bar{A}(\alpha)|$$

It is of some interest that

$$\langle A(q, \mathbf{p}; \{0\}) | = N^{-\dagger}(q) \langle 0 | \psi_q(\mathbf{p})$$

which is the so-called renormalized bare particle state,  $N^{-\dagger}(q) \psi_q(\mathbf{p})$  being the renormalized field operator <sup>8)</sup>.

Transformation (11) is slightly different from the transformation given by Greenberg and Schweber <sup>4)</sup> which is just  $e^F$ . The difference is minor. Nevertheless our transformation has the advantage of yielding the renormalization of the coupling constants and the normalization of the dressed operators.

Making use of (12) the dressed annihilation operator of a nucleon of kind  $q$  and momentum  $\mathbf{p}$  is

$$\overline{O_q(\mathbf{p})} = N^{-\dagger}(q) e^F \psi_q(\mathbf{p}) e^{-F} \quad (14)$$

and the dressed annihilation operator of a meson with momentum  $\mathbf{k}$  is

$$\overline{a(\mathbf{k})} = a(\mathbf{k}) - X(\mathbf{k}) T^\dagger(\mathbf{k}) \quad (15)$$

To get the last formula we used

$$[a(\mathbf{k}), e^{-F}]_- = \partial e^{-F} / \partial a^\dagger(\mathbf{k})$$

From (10), (14) and (15) one finds easily the reciprocal relations between bare and dressed operators.

3. *The dressed Hamiltonian.* The general form of the solution of the eigenvalue problem. The Hamiltonian expressed in terms of the dressed operators reads

$$H = H_{as} + V_{as} \quad (16a)$$

$$H_{as} = m \sum_{q=1}^2 \sum_p O_q^\dagger(\mathbf{p}) \overline{O_q(\mathbf{p})} + \sum_k \omega(k) a^\dagger(\mathbf{k}) \overline{a(\mathbf{k})} \quad (16b)$$

$$V_{as} = -$$

$$- \sum_k \omega(k) X(k) \sum_{q=1}^2 \sum_p g_q (N(q+1)/N(q)) [e^{-F} O_q^\dagger(\mathbf{p}+\mathbf{k}) \overline{O_{q+1}(\mathbf{p})} e^F \overline{a(\mathbf{k})}] \quad (16c)$$

$$- e^{-F} \sum_{q'=1}^2 \sum_{p'} g_{q'} X(k) O_q^\dagger(\mathbf{p}+\mathbf{k}) O_{q'+1}^\dagger(\mathbf{p}') \overline{O_{q+1}(\mathbf{p})} \overline{O_{q'}(\mathbf{p}'+\mathbf{k})} e^F \quad (16d)$$

where  $F$  is again given by (6), but with  $T^\dagger(k)$  written as

$$T^\dagger(k) = \sum_q \sum_p g_q O_{q+1}^\dagger(\mathbf{p}) \overline{O_q(\mathbf{p}+\mathbf{k})} \quad (17)$$

We recall that

$$g_q = g_q^0 (N(q)/N(q+1))^\dagger \quad (18)$$

The calculations are indicated in Appendix I. As stated in this Appendix we get the mass renormalization from the equation

$$\begin{aligned} H_{as} |O_q^\dagger(\mathbf{p}) |0\rangle &= m O_q^\dagger(\mathbf{p}) |0\rangle \\ V_{as} |O_q^\dagger(\mathbf{p}) |0\rangle &= 0 \end{aligned} \quad (19)$$

If we arrange (16) in normal products we get the Hamiltonian in form of an infinite series of terms each of which describes a certain physical process. E.g. the first few terms give the absorption of a meson by a nucleon, the scattering of a meson on a nucleon, scattering of a meson on a nucleon connected with production of additional mesons, scattering of two nucleons with or without production of mesons.

Formula (16c, d) reveals one interesting feature of our Hamiltonian. Although expressed in terms of  $m$  and  $g$ 's it depends on the ratio  $N(q+1)/N(q)$ , which depends in turn on  $L$ . We shall see later that also the eigenvalues and eigenfunctions depend very strongly on  $L$ .

For R.-V.H.M. there are two constants of motion which can be easily found (see 2))

$$Q_1 \equiv \sum_{q=1}^2 \sum_p \psi_q^\dagger(\mathbf{p}) \psi_q(\mathbf{p}) \quad (20a)$$

$$Q_2 \equiv \sum_k a^\dagger(\mathbf{k}) a(\mathbf{k}) - \sum_{q=-\infty}^{+\infty} q \sum_p \psi_q^\dagger(\mathbf{p}) \psi_q(\mathbf{p}) \quad (20b)$$

In the second constant it is understood that in contrast to what is done in other equations  $q$  cannot be taken modulo 2 (one can also interpret  $Q_2$  in a different manner as a constant of motion modulo 2).

$Q_1$  divides the Hilbert space into independent subspaces corresponding to fixed numbers of nucleons.

The eigenfunctions belonging to the zero nucleon subspace coincide with the corresponding  $|\alpha\rangle$ -states. Thus this subspace is of little interest to us.

The one nucleon subspace is the most interesting for our purpose. The

main aim of our note (see § 5) is to find solutions belonging to it. The general form of such solutions must be

$$|\Psi_\lambda\rangle = G_\lambda O_q^\dagger(\mathbf{p}) |0\rangle \quad (21a)$$

with

$$G_\lambda = \sum_{l_1} \sum_{l_2} \dots h(l_1, l_2, \dots) \prod_{\mathbf{k}} (l_{\mathbf{k}}!)^{-1} [a^\dagger(\mathbf{k}) T^\dagger(\mathbf{k})]^{l_{\mathbf{k}}} \quad (21b)$$

Here we took into account (20b), which for (21) has the value  $q$ . We attached the index  $\lambda$  to express that  $G$  depends also on the eigenvalue.

The eigenvalue problem reads

$$H |\Psi_\lambda\rangle = (m - \lambda) |\Psi_\lambda\rangle \quad (22)$$

Making use of (21) we can cast (22) into a form, suitable for further investigation in the subspace of one nucleon

$$\begin{aligned} & \sum_{\mathbf{k}} \{ \sum_j a(\mathbf{k}, j) z(j) + b(\mathbf{k}) \} [\partial G_\lambda / \partial z(\mathbf{k})] O_q^\dagger(\mathbf{p}) |0\rangle + \\ & + \sum_{\mathbf{k}} \sum_j c(\mathbf{k}) z(j) [\partial^2 G_\lambda / \partial z(j) \partial z(\mathbf{k})] O_q^\dagger(\mathbf{p}) |0\rangle + \lambda G_\lambda O_q^\dagger(\mathbf{p}) |0\rangle = 0 \end{aligned} \quad (23)$$

where

$$a(\mathbf{k}, j) \equiv -\gamma_q^2 \omega(k) X(k) X(j) + \omega(k) \delta_{kj} \quad (24a)$$

$$b(\mathbf{k}) \equiv -g_q^2 (N(q+1)/N(q)) \omega(k) X(k) \quad (24b)$$

$$c(\mathbf{k}) \equiv -\gamma_q^2 \omega(k) X(k)$$

$$z(\mathbf{k}) \equiv a^\dagger(\mathbf{k}) T^\dagger(\mathbf{k}) \quad (24c)$$

and

$$\begin{aligned} \gamma_q^2 & \equiv g_{q+1}^2 (N(q)/N(q+1)) - g_q^2 (N(q+1)/N(q)) = \\ & = (-)^q 2a [\sinh\{2a(L_c - L)\}]^{-1} \end{aligned} \quad (25)$$

The derivation of (23) is outlined in Appendix II. The quantities  $L_c$  and  $a$  will be defined presently.

4. *The dependence of  $N(q+1)/N(q)$  on  $L$ .* Using formulae given by Ruijgrok and Van Hove<sup>1)3)</sup> we can write  $N(1)/N(2)$  in a simple form

$$N(1)/N(2) = \tanh a(L_c - L) / \tanh aL_c \quad (26)$$

with  $a = g_1 g_2$ , and

$$g_2/g_1 = \tanh aL_c, \quad (g_1 > g_2) \quad (27)$$

Formula (27) gives us the definition of the critical value of the cut-off parameter. Because  $0 < N(i) < 1$  due to the positive definite metric of the Hilbert space of  $|\alpha\rangle$ -states (see 7)), the only possible values of  $L$  for given  $g_1$  and  $g_2$  are  $L < L_c$ . From (27) follows that only in the case of equal renormalized coupling constants can  $L_c$  become infinite. Otherwise  $L_c$  is always finite.



If we enter the region  $L > L_c$  the problem becomes obscure. If (26) becomes negative no mathematical meaning can be attached to  $N(q)$  and  $(N(q))^\dagger$ . Consequently the relation between the  $|\alpha\rangle$  and  $|A(\alpha)\rangle$ -states, given by (13) becomes meaningless (except for the no nucleon states). There is no longer a connection between these two sets. Moreover  $g_q^0$  lose their meaning too, because from (18) and (26) follows that  $g_q^{02} < 0$ .\*). Thus the only way of reconciling these facts with mathematical consistency is to state that for  $L > L_c$  the  $g_q^0$  do not exist.

Notwithstanding this we are able to perform the computations in a consistent manner also for the region  $L > L_c$ . As mentioned in the introduction the a.s.s. representation enables us to extend our computation scheme from  $L < L_c$  to  $L > L_c$ , without any trouble. Indeed, the Hamiltonian is expressed in terms of dressed operators and renormalized quantities except for the factor  $N(q+1)/N(q)$ ; this factor, however, does not lose its meaning for  $L > L_c$ , it merely becomes negative. We know how to handle the dressed operators, we know their commutation rules. The only thing which we need to make our mathematical scheme complete is the metric\*\*). This was given for the subspaces of zero and one nucleon by Ruijgrok<sup>3)</sup>

$$\begin{aligned} \langle 0 | O_{q'}(\mathbf{p}') \prod_{\mathbf{k}'} (l_{\mathbf{k}'}!)^{-\frac{1}{2}} a(\mathbf{k}')^{l_{\mathbf{k}'}} \prod_{\mathbf{k}} (l_{\mathbf{k}}!)^{-\frac{1}{2}} a^\dagger(\mathbf{k})^{l_{\mathbf{k}}} O_q^\dagger(\mathbf{p}) | 0 \rangle = \\ = \delta_{q'-\sum_{\mathbf{k}'} l_{\mathbf{k}'}, q - \sum_{\mathbf{k}} l_{\mathbf{k}}} \delta_{\mathbf{p}' + \sum_{\mathbf{k}'} \mathbf{k}' l_{\mathbf{k}'}, \mathbf{p} + \sum_{\mathbf{k}} \mathbf{k} l_{\mathbf{k}}} \varphi_{q - \sum_{\mathbf{k}} l_{\mathbf{k}}}(\{l_{\mathbf{k}'}\}, \{l_{\mathbf{k}}\}) \end{aligned} \quad (28a)$$

the first few  $\varphi$ 's are

$$\varphi_q(0; 0) = 1; \varphi_q(\mathbf{k}'; 0) = g_q X(\mathbf{k}'); \varphi_{q-1}(\mathbf{k}'; \mathbf{k}) = \delta_{\mathbf{k}'\mathbf{k}} + g_q^2 X(\mathbf{k}') X(\mathbf{k}) \quad (28b)$$

It is important to know that all scalar products are positive and individually independent of  $L$ . We shall return to the subject of metric in § 7. Now we confine ourselves to the statement that the tensor formed of all scalar products (28a) depends on  $L$  and defines for  $L > L_c$  an indefinite metric. We adopt this tensor as suitable for our aims.

Thus we are in the position to forget all about the bare particle states which were our starting representation. We have at our disposal a complete and well defined description of the system in terms of a.s.s. We just extend our complete formalism from  $L < L_c$  to  $L > L_c$ . We shall see that working with the a.s.s. representation in the region  $L > L_c$  we get reasonable results, in contrast to the case when we work with the unperturbed state representation, which leads to mathematically meaningless formulae (see e.g. in Källén and Pauli's work<sup>8)</sup> formulae (32) and (37) for  $|V\rangle$  and  $|N, \Theta\rangle$ , resp., where  $N$  appears), unless a new metric is introduced ad hoc.

Let us say a few words about the special cases  $L = L_c$  and  $L = \infty$ .

The case  $L = L_c$  plays a singular role in the theory of R.-V.H.M. Many

\*) The situation will not be saved by introducing complex  $g_q^0$ ; the reason is that if we had worked from the very beginning with complex  $g_q^0$  we would have in all our formulae the absolute values of  $g_q^0$

\*\*) I am very indebted to Professor Van Hove for calling this to my attention.

quantities regarded as functions of  $L$  are symmetrical or antisymmetrical in the difference  $L - L_c$ . Thus e.g.  $N(1)/N(2)$  is antisymmetrical. This in turn causes some antisymmetrical effects in the interaction Hamiltonian  $V_{as}$ . To see it distinctly let us consider the neighbourhood of  $L_c$ . Since  $N(1)/N(2)$  vanishes at  $L = L_c$  one part of the interaction becomes infinite while the other vanishes. This corresponds to the unphysical situation in which we are left with only one unrenormalized coupling constant which becomes infinite\*). Thus it is to be expected that the energy values for certain eigenfunctions tend to infinity with  $L \rightarrow L_c$ . E.g. computations performed for the two nucleon problem yield for the energy

$$-Y/(L_c - L) 2g_1 + O(1)$$

where  $Y$  is the Yukawa potential

$$2 \sum_{\mathbf{k}} \omega(k) X^2(k) \exp[i\mathbf{k} \cdot (\mathbf{x}_1 - \mathbf{x}_2)] \quad (29)$$

$\mathbf{x}_1, \mathbf{x}_2$  are the position vectors of the nucleons. This result holds for positive as well as for negative values of  $L - L_c$ . The energy makes a jump at  $L = L_c$ . As we shall see in § 5 the energy spectrum of the nucleon-meson system shows a symmetry with respect to  $L_c$  between the system labelled by  $q = 1$  and  $L = L_c + \varepsilon$  and the system with  $q = 2$  and  $L = L_c - \varepsilon$ . There is, however, lack of symmetry in the eigenfunctions belonging to the same energy level.

The case  $L = \infty$  is certainly a pathological one. The limiting process is not uniform; e.g. if  $g_1 = g_2$ ,  $N(1)/N(2)$  tends to  $(+1)$  otherwise it tends to  $(-g_1/g_2)$  (we recall that we assume  $g_1 < g_2$ ). For  $L \rightarrow \infty$  we are able to write the Hamiltonian (16) in a much simpler form

$$m \sum_q \sum_p O_q^\dagger(\mathbf{p}) \overline{O_q(\mathbf{p})} + \sum_{\mathbf{k}} \omega(k) a^\dagger(\mathbf{k}) \overline{a(\mathbf{k})} + \quad (30a)$$

$$+ \sum_{\mathbf{k}} \omega(k) X(k) \sum_q \sum_p g_{q+1} O_q^\dagger(\mathbf{p} + \mathbf{k}) \overline{O_{q+1}(\mathbf{p})} \overline{a(\mathbf{k})} + \quad (30b)$$

$$- \sum_{\mathbf{k}} \omega(k) X^2(k) \sum_q \sum_p g_{q+1} \sum_{q'} \sum_{p'} g_{q'} O_q^\dagger(\mathbf{p} + \mathbf{k}) O_{q'+1}^\dagger(\mathbf{p}') \cdot \overline{O_{q+1}(\mathbf{p})} \overline{O_{q'}(\mathbf{p}' + \mathbf{k})} \quad (30c)$$

The physical contents of the theory in this limit become rather poor. We are able to write down easily exact solutions for the case of two interacting nucleons as a superposition of plane waves of a.s.s.; the energy is

$$2m \pm g_1 g_2 Y$$

with  $Y$  given by (29). The exact solution of a meson interacting with a nucleon yields no meson-scattering term; only absorption of the meson can take place. This inclination to absorption is, to some extent, a general feature of the a.s.s. representation; we make use of it in later paragraphs.

\*) I thank Dr. Ruijgrok for calling this point to my attention.

The energy is simply

$$m + \omega(k) \quad (31)$$

as follows from the general discussion in § 5.

5. *The solution of the nucleon - one meson problem.* Taking into account the remark at the end of § 4 we make the ansatz

$$G = \sum_k \alpha(k) z(k) + \beta. \quad (32)$$

We show in this paragraph that this ansatz yields an exact solution of the nucleon-meson problem. We dropped the index  $\lambda$  for convenience. We substitute (32) into (23). As the vectors

$$\prod_k z(k)^{l_k} O_q^\dagger(p) |0\rangle$$

are independent we must have

$$\sum_j a(j, k) \alpha(j) + \lambda \alpha(k) = 0 \quad (33a)$$

$$\sum_k b(k) \alpha(k) + \lambda \beta = 0 \quad (33b)$$

The trivial solution of (33) is  $\alpha(j) = 0$  and  $\lambda = 0$ , corresponding to the case of one nucleon only (see (19)). To get non trivial solutions we investigate the characteristic equation

$$\text{Det}(a(j, k) + \lambda \delta_{jk}) = 0 \quad (34)$$

Let us now assume for the present that the meson spectrum is bounded from above; it shall consist of  $n$  levels  $\omega(1) \dots \omega(n) < \infty$ . This assumption will help us to get better insight into the structure of the true energy spectrum. Thus, using (24), the determinant can be written in the form

$$\prod_j (\omega(j) + \lambda) [1 - \gamma_q^2 \sum_{k=1}^n \omega(k) X^2(k) \{\omega(k) + \lambda\}^{-1}] \quad (35)$$

For simplicity we shall leave out of consideration the case when the true energy level coincides with the unperturbed one as well as the case of degeneracy of the unperturbed spectrum (which, in fact, takes place, because we are dealing with a cubic lattice in momentum space); thus we confine ourselves to investigate

$$1 - (\gamma_q^2/2v) \sum_{k=1}^n (f(k)/\omega(k))^2 (\omega(k) + \lambda)^{-1} = 0 \quad (36)$$

with, following Ruijgrok's notation <sup>3)</sup>,

$$X(k) = (2v)^{-1/2} f(k) \omega(k)^{-3/2} \quad (37)$$

$f(k)$  is the cut-off factor. Equation (36) resembles the characteristic equation used by Källén and Pauli <sup>8)</sup>. Before discussing in detail the energy spectrum given by (36), let us compare our results with the case of the Lee

model (see 8)). For  $g_2 \rightarrow 0$  and  $q = 1$  we get

$$\gamma_1^2 = -(L_c - L)^{-1}$$

In the notation used by Källén and Pauli <sup>8)</sup> we have for Lee's model

$$g^2 = L_c^{-1}$$

$$N^2 = 1 - g^2 L$$

so

$$g^2/N^2 = -\gamma_1^2$$

and we get from (36)

$$1 + (g^2/2vN^2) \sum_k (f(k)/\omega(k))^2 (\omega(k) + \lambda)^{-1} = 0$$

a relation equivalent to relation (36) of the paper of Källén and Pauli <sup>8)</sup>.

We return now to the R.-V.H.M. We exclude from our consideration the case  $L = L_c$ . So we have from (36) and (25)

$$h_q(\lambda) = \sinh[2a(L_c - L)] \pm (a/v) \sum_k (f(k)/\omega(k))^2 (\omega(k) + \lambda)^{-1} \quad (38)$$

the upper sign refers to  $q = 1$ , the lower to  $q = 2$ .

$h_1$  and  $h_2$  considered as functions of  $\lambda$  show the following behaviour

$$\begin{aligned} h_1(-\infty) = h_1(+\infty) = h_2(-\infty) = h_2(+\infty) &= \sinh[2a(L_c - L)] \\ h_1(-\omega(1) + 0) &> 0; \quad h_2(-\omega(1) + 0) < 0 \end{aligned} \quad (39)$$

$$h_1(-\omega(n) - 0) < 0; \quad h_2(-\omega(n) - 0) > 0 \quad (39)$$

$\omega(1) = \mu \equiv$  meson mass.

Both  $h_1$  and  $h_2$  have  $n$  poles at  $-\omega(1), -\omega(2), \dots, -\omega(n)$ ; if the derivative exists we find

$$dh_1(\lambda)/d\lambda < 0 \quad dh_2(\lambda)/d\lambda > 0; \quad (40)$$

hence between  $-\omega(1)$  and  $-\omega(n)$  lie exactly  $(n-1)$  roots for both,  $h_1$  and  $h_2$ . The position of the  $n^{\text{th}}$  root depends on the sign of  $\sinh[2a(L_c - L)]$  (see figure 1).

For  $L < L_c$  (definite metric),  $\sinh[2a(L_c - L)] > 0$  and the  $n^{\text{th}}$  root of  $h_1$  falls into the interval  $(-\infty, -\omega(n))$ . For  $n \rightarrow \infty$  and  $\lim_{n \rightarrow \infty} \omega(n) = \infty$  this root will be pushed toward  $\infty$ ; it has no practical significance. For  $L > L_c$  (indefinite metric), the  $n^{\text{th}}$  root of  $h_1$  lies between  $-\omega(1)$  and  $+\infty$  (see the figure).

For  $v \rightarrow \infty$  the zero between  $-\omega(1)$  and  $-\omega(n)$  as well as the poles at  $-\omega(k)$  melt into a cut along the real axis in the complex  $\lambda$ -plane, extending from  $-\omega(1)$  to  $-\omega(n)$ . As  $h_1(\lambda)$  is no longer a one valued function we get three sets of meson-nucleon scattering states, instead of one: the first when we reach the cut from above (outgoing waves), the second when we reach it from below (ingoing waves), the third by taking the principal value (standing

waves). The  $n^{\text{th}}$  root becomes a bound state level: for  $L < L_c$  it has no significance, but for  $L > L_c$  it is the ghost. It is obvious that  $\lambda_1(1)$  can be larger than zero, e.g. for  $L$  sufficiently close to  $L_c$ .

There is throughout a one to one correspondence between the undisturbed energy levels and true eigenvalues; only the energy shifts are qualitatively different for  $L < L_c$  and  $L > L_c$ . One realizes it distinctly from the approximate evaluation of the true eigenvalues. If  $\lambda$  is sufficiently close to  $-\omega(r)$  one can write

$$\lambda_1(r) \cong -\omega(r) - (a/v)(f(r)/\omega(r))^2 [\sinh\{2a(L - L_c)\}]^{-1}$$

For  $L < L_c$  the shift is negative, for  $L > L_c$  it is positive.

For  $h_2$  the  $n^{\text{th}}$  root is located between  $-\omega(1)$  and  $+\infty$  if  $L < L_c$ , and between  $-\infty$  and  $-\omega(r)$  if  $L > L_c$ . Because the graph of  $h_2(\lambda)$  for  $L < L_c$  is a mirror image of the graph of  $h_1(\lambda)$  for  $L > L_c$ , (see the figure) the roots as well as the poles are identical in both cases. Thus  $\lambda_2(1)$  corresponding to  $\omega(1)$  is identical with the ghost level, mentioned above. As this level  $\lambda_2(1)$  can be larger than zero, it follows that for  $q = 2$  (we recall that  $g_2$  is the smaller coupling constant) there exist in the limit  $v \rightarrow \infty$  a bound state due to one nucleon and one meson with energy  $m - \lambda_2(1)$ , which can be smaller than  $m$ . Also  $h_1$  for  $L < L_c$  is a mirror image of  $h_2$  for  $L > L_c$ .

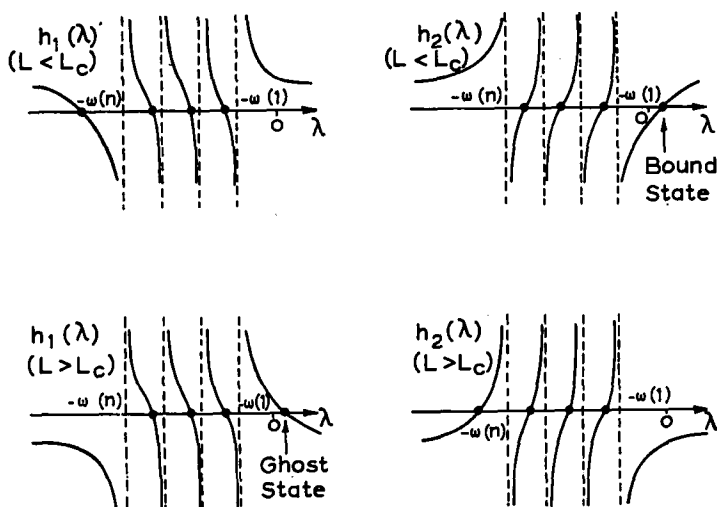


Fig. 1. The graphs of  $h_1(\lambda)$  and  $h_2(\lambda)$  versus  $\lambda$ .

For  $L \rightarrow L_c$   $h_q(-\infty) = h_q(+\infty) \rightarrow 0$ , i.e. the high energy as well as the low energy bound state levels (including the ghost) tend to  $\pm \infty$ , resp. For  $L \rightarrow \infty$  one has  $\gamma_q^2 = 0$  and from (35) follows (31).

We have still to discuss the norm of the eigenstates belonging to the energy levels discussed above, especially its sign. In particular the norm of

the bound or ghost states is of interest to us. Looking at equation (24), (25) and (33) one sees that simultaneous change of  $q$  to  $q + 1$  and  $L = L_c + \varepsilon$  to  $L = L_c - \varepsilon$  will cause no changes in (33a), consequently the eigenvalues are not influenced by this substitution. On the contrary equation (33b) will be modified:  $b(k)$  changes sign and form. The most important thing is the change of sign. Thus the eigenfunction belonging to the bound state ( $q = 2$ ,  $L = L_c - \varepsilon$ ) will have the same coefficients  $\alpha(j)$ 's as the eigenfunction belonging to the ghost ( $q = 1$ ,  $L = L_c + \varepsilon$ ), but different  $\beta$ 's; these  $\beta$ 's will have opposite sign. The norm written by means of the metric tensor (28) reads

$$\begin{aligned} \langle \psi_\lambda | \psi_\lambda \rangle = & \lambda^{-2} [\lambda^2 g_q^2 \sum_k \sum_j \alpha^*(j) \alpha(k) \{\delta_{jk} + g_{q+1}^2 X(k) X(j)\} + \\ & + \lambda g_q^4 (N(q+1)/N(q)) \sum_k \sum_j \alpha^*(j) \alpha(k) X(k) X(j) \{\omega(k) + \omega(j)\} + \\ & + g_q^4 (N(q+1)/N(q)^2) \sum_k \sum_j \alpha^*(j) \alpha(k) \omega(j) \omega(k) X(j) X(k)] \end{aligned} \quad (41)$$

For the two cases mentioned above the second term in (41) has different sign and for  $L = L_c + \varepsilon$  it is negative. One can show that this term makes the norm of the ghost state negative. The same conclusion can be drawn from (40) (the derivatives of  $h_1$  and  $h_2$  have different signs).

6. *On the solution of the nucleon-several mesons problem.* We make the two-meson ansatz

$$\begin{aligned} G = & \sum_i \sum_j \alpha(i, j) z(i) z(j) + \sum_j \beta(j) z(j) + \delta \\ \alpha(i, j) = & \alpha(j, i) \end{aligned} \quad (42)$$

We substitute (42) into (23), obtaining the set of equations

$$\begin{aligned} \sum_k (a(k, j) \{\alpha(k, i) + \alpha(i, k)\} + a(k, i) \{\alpha(k, j) + \alpha(j, k)\}) + \\ + \lambda [\alpha(i, j) + \alpha(j, i)] = 0 \end{aligned} \quad (43a)$$

$$\sum_k a(k, j) \beta(k) + \lambda \beta(j) + \sum_k (b(k) + c(k)) (\alpha(k, j) + \alpha(j, k)) = 0 \quad (43b)$$

$$\sum_k b(k) \beta(k) + \lambda \delta = 0 \quad (43c)$$

Barring exceptional cases (e.g. that the  $\lambda$ -value computed from (43a) coincides with one of the eigenvalues of (33a)), we can solve the problem as follows: we regard the double indices  $(i, j)$  as labelling the state and rewrite (43a) in the form

$$\sum_i \sum_j A_{i, j; k, l} \alpha(k, l) + \lambda \alpha(i, j) = 0 \quad (44a)$$

with

$$A_{i, j; k, l} = a(k, j) \delta_{il} + a(k, i) \delta_{jl} \quad (44b)$$

As we are interested in nontrivial solutions we require

$$\text{Det}(A_{ij; kl} + \lambda \delta_{ik} \delta_{jl}) = 0 \quad (45)$$

This characteristic equation yields eigenvalues which are also labelled with double indices. However, not all eigenvalues will be suitable. We take into account, only such  $\lambda_{ij}$  for which the symmetry requirement

$$\alpha(k, l) = \alpha(l, k) \quad (46)$$

is fulfilled.

If  $i, j, k, l$ , run again for the moment from 1 to  $n$  ( $n$  finite integer) we find that  $\lambda = 0$  is a  $n(n-1)/2$  fold root. Further we find that the eigenvectors belonging to the other  $n(n+1)/2$ -roots which are different from zero, have the required property (46), while the solutions for  $\lambda = 0$  must not satisfy (46). We were not able to investigate further the general case. We will, however, present a detailed discussion for  $n = 2$  which reveals some interesting features. It is to be expected that these regularities will be of more general validity.

The characteristic determinant can be written

$$\lambda(\lambda + a(1, 1) + a(2, 2)) \begin{vmatrix} a(1, 1) + \lambda/2 & a(1, 2) \\ a(2, 1) & a(2, 2) + \lambda/2 \end{vmatrix} = 0 \quad (47)$$

The root  $\lambda = 0$  yields the solution

$$\alpha(i, k) = \begin{pmatrix} -a(2, 1) & -a(2, 2) \\ a(1, 1) & a(1, 2) \end{pmatrix}$$

which does not fulfil (46). The root  $\lambda_{12} = -a(1, 1) - a(2, 2)$  is the trace of the matrix  $(-a(i, k))$  and therefore is equal to the sum of roots of (34), viz.

$$\lambda_{12} = \lambda_1 + \lambda_2$$

The last two roots differ from the roots of (34) by a factor 2,

$$\begin{aligned} \lambda_{11} &= 2\lambda_1 \\ \lambda_{22} &= 2\lambda_2 \end{aligned}$$

Thus we see at least for  $n = 2$ , that the energies are composed by adding energies of one meson states.

Turning to the eigensolutions, for  $\lambda \neq 0$  they can always be written in the form

$$\alpha(i, k) = (v_i w_k + v_k w_i)/2 \quad (48)$$

and in the case  $\lambda_{11}$  and  $\lambda_{22}$  even as direct products of solutions for the nucleon - one meson problem.

We are inclined to believe that the additiveness of energy as well as the property (48) will hold also for an arbitrary  $n$ . We therefore infer that also in case of two mesons at least one ghost state and one bound state should appear. The computation scheme for three and more mesons is the same as in the case of two mesons. For three mesons we make the ansatz that the

eigenstate involves at most three meson a.s.s. We label all coefficients as well as the eigenvalues by a group of three indices.

7. *On the metric.* From the results of § 5, especially from (38), as well from other facts mentioned earlier in this note, one realizes that the eigenvalues will depend on  $L$ . This implies that it is not possible to cast the Hamiltonian of R.-V.H.M. in an exactly renormalized form, neither by the a.s.s. representation nor by any other.

The most interesting thing is that if we enter the region  $L > L_c$  the metric becomes indefinite; that this is really so one notices on the norm of the ghost state as well as on the negative value of  $N(1)/N(2)$  (see 6)). Thus, as pointed out by Van Hove (private communication), the metric shows in the neighbourhood of  $L = L_c$  a singular behaviour. We shall give now a brief discussion of this important and interesting question.

As already mentioned in § 4 we accept the metric tensor (28) not only for the region  $L < L_c$  but also for  $L > L_c$ . Although the individual elements of this tensor do not depend on  $L$  explicitly, the tensor as a whole in fact depends on it implicitly. This can be checked by investigating the principal minors of the metric tensor; although all tensor elements are positive, some of these minors can become negative for certain values of  $g$ 's and  $X$ 's. We can show that for  $L > L_c$  the metric becomes indefinite.

Usually one starts with the bare particle states  $|\alpha\rangle$ . All states with positive norm, particularly all eigenfunctions of the system for  $L < L_c$  can be expressed as a linear combination of  $|\alpha\rangle$ ,

$$|\psi\rangle = \sum_{\alpha} c_{\alpha} |\alpha\rangle$$

Of course, each linear combination of  $|\alpha\rangle$  has a positive norm. For  $L > L_c$  difficulties occur with some of the eigenfunctions of the system, viz. with the ghost state. Although Källén and Pauli write the ghost state for the Lee model as a linear combination of  $|\alpha\rangle$ 's

$$|V_{-\lambda}\rangle = \hbar[|N| \psi_1^{\dagger}(\mathbf{p}) |0\rangle + (2v)^{-\frac{1}{2}} g \sum_{\mathbf{k}} f(\omega) \omega^{-\frac{1}{2}} (\omega + \lambda)^{-1} \psi_2^{\dagger}(\mathbf{p} - \mathbf{k}) a^{\dagger}(\mathbf{k}) |0\rangle] \quad (49)$$

this formula is in fact meaningless, because the coefficients are no longer numbers (since  $|N|^2$  must be negative). They circumvent this difficulty by introducing a new metric invented ad hoc.

Following ideas of Van Hove we employed throughout this note the representation of a.s.s. In this representation the transition from the region  $L < L_c$  into the region  $L > L_c$  involves no ad hoc modification. We just have extended our mathematical scheme formally into this new region. With the help of this representation we are able to express all eigenstates both with positive and negative norm, as shown in § 5. We emphasize that the coefficients of the a.s.s. are ordinary numbers. To emphasize this point



we compare (49) with the expression of the same ghost state of the Lee model given in terms of a.s.s.

$$|V_{-\lambda}\rangle = k[O_1^\dagger(\mathbf{p})|0\rangle + (2v)^{-\frac{1}{2}}g \sum_{\mathbf{k}} \omega^{-\frac{1}{2}}f(\omega)\{(\omega + \lambda)^{-1} - \omega^{-1}\} O_2^\dagger(\mathbf{p} - \mathbf{k}) a^\dagger(\mathbf{k})|0\rangle] \quad (50)$$

Using the metric tensor (28) this gives automatically a negative norm.

We conclude that the a.s.s. show a different behaviour from that of the unperturbed states. The linear combination of a.s.s.

$$|\psi\rangle = \sum_{\alpha} d_{\alpha} |A(\alpha)\rangle$$

must not necessarily have positive norm. As long as  $L < L_c$  holds there is a one to one correspondence between the sets  $|\alpha\rangle$  and  $|A(\alpha)\rangle$ ; it is given by (13) and depends on the actual value of  $L$ ; both sets are complete in the same space. For  $L < L_c$  such connection no longer exists, the a.s.s. escape partly into another space with indefinite metric; while of course the spaces of  $|\alpha\rangle$  and of  $|A(\alpha)\rangle$  overlap at least in the subspace without nucleons. Simultaneously with the a.s.s. also the eigenstates escape into this new space, spanned by a.s.s.; thus the bare states  $|\alpha\rangle$  are no longer adjusted to describe the physical states for  $L > L_c$ , except in the trivial case of the zero nucleon states.

The author is greatly indebted to Professor L. Van Hove for his most valuable and constructive criticism, discussions and constant encouragement. The author is very grateful to Dr. Th. W. Ruijgrok for many fruitful discussions. He also wishes to express his cordial thanks to Professor Van Hove for the kind hospitality extended to him at the Instituut voor Theoretische Fysica in Utrecht as well as to the Polish Ministry of High Education for the award of a grant.

*Appendix I.* As  $F$  commutes with the total number of bare nucleons, we are able to write the first term in (3b)

$$m \sum_q \sum_p \psi_q^\dagger(\mathbf{p}) \psi_q(\mathbf{p}) = m \sum_q \sum_p O_q^\dagger(\mathbf{p}) \overline{O_q(\mathbf{p})} \quad (A1)$$

If we combine the second term in (3b) with the second term of (3c) we get simply

$$\sum_{\mathbf{k}} \omega(\mathbf{k}) a^\dagger(\mathbf{k}) [a(\mathbf{k}) - X(\mathbf{k}) T^\dagger(\mathbf{k})] = \sum_{\mathbf{k}} \omega(\mathbf{k}) a^\dagger(\mathbf{k}) \overline{a(\mathbf{k})} \quad (A2)$$

in virtue of (15). The terms (A1) and (A2) give the unperturbed asymptotic Hamiltonian  $H_{as}$ . This Hamiltonian of free dressed particles is non-hermitian. The rest of the Hamiltonian, i.e. the first and third term in (3c), form the asymptotic interaction Hamiltonian,  $V_{as}$ , which describes the transient

interaction between dressed particles. We have

$$V_{as} \equiv - \sum_{\mathbf{k}} \omega(k) X(k) a(\mathbf{k}) T(\mathbf{k}) - \sum_q \sum_p \delta m_q \psi_q^\dagger(\mathbf{p}) \psi_q(\mathbf{p}) =$$

$$= - \sum_{\mathbf{k}} \omega(k) X(k) \sum_q \sum_p g_q^0 (N(q+1)/N(q))^\dagger [\overline{a(\mathbf{k})} e^{-F} O_q^\dagger(\mathbf{p} + \mathbf{k}) \overline{O_{q+1}(\mathbf{p})} e^F + \quad (\text{A3})$$

$$+ X(k) T^\dagger(\mathbf{k}) e^{-F} O_q^\dagger(\mathbf{p} + \mathbf{k}) \overline{O_{q+1}(\mathbf{p})} e^F + \quad (\text{A4})$$

$$+ (N(q)/N(q+1))^\dagger \delta m_q X(k) \{g_q^0 \sum_{\mathbf{k}'} \omega(k') X^2(k')\}^{-1} e^F O_q^\dagger(\mathbf{p}) \overline{O_q(\mathbf{p})} e^F] \quad (\text{A5})$$

We notice that by applying the dressing operation,  $T^\dagger(\mathbf{k})$  can be put into the form

$$T^\dagger(\mathbf{k}) = \sum_q \sum_p g_q O_{q+1}^\dagger(\mathbf{p}) \overline{O_q(\mathbf{p} + \mathbf{k})} \quad (\text{17})$$

Let us write (A3) in the form

$$- X(k) T^\dagger(\mathbf{k}) e^{-F} O_q^\dagger(\mathbf{p} + \mathbf{k}) \overline{O_{q+1}(\mathbf{p})} e^F + \quad (\text{A6})$$

$$+ e^{-F} O_q^\dagger(\mathbf{p} + \mathbf{k}) \overline{O_{q+1}(\mathbf{p})} X(k) T^\dagger(\mathbf{k}) e^F + \quad (\text{A7})$$

$$+ e^{-F} O_q^\dagger(\mathbf{p} + \mathbf{k}) \overline{O_{q+1}(\mathbf{p})} e^F \overline{a(\mathbf{k})} \quad (\text{A8})$$

and rearrange the creation and annihilation operators in (A7) in such a way that they form normal products. If we insert (A6–A8) in this form into (A3–A5) the expression in the paranthesis becomes

$$- X(k) T^\dagger(\mathbf{k}) e^{-F} O_q^\dagger(\mathbf{p} + \mathbf{k}) \overline{O_{q+1}(\mathbf{p})} e^F + \quad (\text{A9})$$

$$+ g_q X(k) e^{-F} O_q^\dagger(\mathbf{p} + \mathbf{k}) \overline{O_q(\mathbf{p} + \mathbf{k})} e^F + \quad (\text{A10})$$

$$+ e^{-F} \sum_{q'} \sum_{p'} g_{q'} X(k) O_q^\dagger(\mathbf{p} + \mathbf{k}) O_{q'+1}^\dagger(\mathbf{p}') \overline{O_{q+1}(\mathbf{p})} \overline{O_{q'}(\mathbf{p}' + \mathbf{k})} e^F + \quad (\text{A11})$$

$$+ e^{-F} O_q^\dagger(\mathbf{p} + \mathbf{k}) \overline{O_{q+1}(\mathbf{p})} e^F \overline{a(\mathbf{k})} + \quad (\text{A12})$$

$$+ X(k) T^\dagger(\mathbf{k}) e^{-F} O_q^\dagger(\mathbf{p} + \mathbf{k}) \overline{O_{q+1}(\mathbf{p})} e^F + \quad (\text{A13})$$

$$- (N(q)/N(q+1))^\dagger \delta m_q X(k) \{g_q^0 \sum_{\mathbf{k}'} \omega(k') X^2(k')\}^{-1} e^{-F} O_q^\dagger(\mathbf{p}) \overline{O_q(\mathbf{p})} e^F \quad (\text{A14})$$

Now we require that  $m$  shall be an eigenvalue of the Hamiltonian belonging to one nucleon eigenstate  $O_q^\dagger(\mathbf{p}) |0\rangle$ ; then the term (A14) must cancel against the term (A10) and we arrive at

$$\delta m_q = (g_q^0)^2 \sum_{\mathbf{k}} \omega(k) X^2(k) \quad (\text{A15})$$

in agreement with <sup>1)3)</sup>.

*Appendix II.* We start with the eigenvalue problem (22) written in the form

$$H_\vartheta G_\lambda O_q^\dagger(\mathbf{p}) |0\rangle = -\lambda G_\lambda O_q^\dagger(\mathbf{p}) |0\rangle \quad (\text{B1})$$

with

$$H_\vartheta = \sum_{\mathbf{k}} \omega(k) a^\dagger(\mathbf{k}) \overline{a(\mathbf{k})} - \sum_{\mathbf{k}} \omega(k) X(k) \sum_q \sum_p g_q (N(q+1)/N(q)) e^{-F}$$

$$O_q^\dagger(\mathbf{p} + \mathbf{k}) \overline{O_{q+1}(\mathbf{p})} e^F \overline{a(\mathbf{k})} \quad (\text{B2})$$

Here it was taken into account that the total number of nucleons is conserved as well as that we are dealing with states with only one nucleon. We drop in the following the index  $\lambda$  for simplicity. We recall that  $G$  depends only on  $z(\mathbf{k}) \equiv a^\dagger(\mathbf{k}) T^\dagger(\mathbf{k})$ . Taking into account

$$[\overline{a(\mathbf{k})}, G]_- = T^\dagger(\mathbf{k})(\partial G/\partial z(\mathbf{k})) = (\partial G/\partial z(\mathbf{k})) T^\dagger(\mathbf{k}) \quad (\text{B3})$$

we can write the first term on the left hand side of (B1)

$$\begin{aligned} \sum_{\mathbf{k}} \omega(\mathbf{k}) a^\dagger(\mathbf{k}) \overline{a(\mathbf{k})} G O_q^\dagger(\mathbf{p}) |0\rangle &= \\ &= \sum_{\mathbf{k}} \omega(\mathbf{k}) a^\dagger(\mathbf{k}) T^\dagger(\mathbf{k})(\partial G/\partial z(\mathbf{k})) O_q^\dagger(\mathbf{p}) |0\rangle \end{aligned} \quad (\text{B4})$$

For the second term on the left hand side we find

$$\begin{aligned} - \sum_{\mathbf{k}} \omega(\mathbf{k}) X(\mathbf{k}) \sum_{q'} \sum_{\mathbf{p}'} g_{q'}(N(q'+1)/N(q')) e^{-F} O_{q'}^\dagger(\mathbf{p}'+\mathbf{k}) \overline{O_{q'+1}(\mathbf{p}') e^F a(\mathbf{k})} \\ G O_q^\dagger(\mathbf{p}) |0\rangle &= \\ &= - \sum_{\mathbf{k}} \omega(\mathbf{k}) X(\mathbf{k}) e^{-F} S e^F (\partial G/\partial z(\mathbf{k})) O_q^\dagger(\mathbf{p}) |0\rangle \end{aligned} \quad (\text{B5})$$

with

$$S = \sum_{q'} g_{q'}^2 (N(q'+1)/N(q')) \sum_{\mathbf{p}'} O_{q'}^\dagger(\mathbf{p}') \overline{O_{q'}(\mathbf{p}')} \quad (\text{B6})$$

The last step was accomplished by means of (B3) and

$$[\overline{O_{q'+1}(\mathbf{p}')}, T^\dagger(\mathbf{k})]_- = \sum_{q''} \sum_{\mathbf{p}''} g_{q''} \delta_{q'q''} \delta_{\mathbf{p}'\mathbf{p}''} \overline{O_{q''}(\mathbf{p}''+\mathbf{k})} +$$

+ terms with two annihilation operators for nucleon at the right end.

$S$  is the only operator in  $(H_{\mathcal{S}} G)$  which does not commute with  $z(\mathbf{k})$  or  $T^\dagger(\mathbf{k})$ . We now get rid of it. Let us consider the identity

$$S e^F (\partial G/\partial z(\mathbf{k})) = [S, e^F (\partial G/\partial z(\mathbf{k}))]_- + e^F (\partial G/\partial z(\mathbf{k})) S \quad (\text{B7})$$

Our task is to transform the right hand side of (B7) in such a way that it can be expressed in terms of  $z(\mathbf{k})$  only if acting on  $O_q^\dagger(\mathbf{p}) |0\rangle$ . To this end we investigate the commutator; we have

$$[S, T^\dagger(\mathbf{k})]_- = \tau^\dagger(\mathbf{k}) \quad (\text{B8})$$

where

$$\begin{aligned} \tau^\dagger(\mathbf{k}) &= \sum_{q'} [g_{q'}^2 (N(q'+1)/N(q')) - g_{q'+1}^2 (N(q')/N(q'+1))] g_{q'+1} \sum_{\mathbf{p}'} O_{q'}^\dagger(\mathbf{p}') \\ &\quad \overline{O_{q'+1}(\mathbf{p}'+\mathbf{k})} \end{aligned} \quad (\text{B10})$$

We have, however,

$$[\tau^\dagger(\mathbf{k}), T^\dagger(\mathbf{j})]_- = 0 \quad (\text{B10})$$

Thus  $S$  does not commute with  $z(\mathbf{k})$  but its commutator with  $z(\mathbf{k})$  does. We state without proof the following lemma:

If the operator  $A$  and  $B$  do not commute, viz.

$$[A, B]_- = C$$

but  $C$  does commute with  $B$ , viz.

$$[B, C]_- = 0$$

then for each  $\varphi(B)$  which can be expanded in powers of  $B$  one has

$$[A, \varphi(B)]_- = C(\partial\varphi/\partial B)$$

Using this lemma we get

$$[S, e^F]_- = \sum_{\mathbf{k}} e^F X(\mathbf{k}) a^\dagger(\mathbf{k}) \tau^\dagger(\mathbf{k}) \quad (\text{B11})$$

and

$$[S, (\partial G/\partial z(\mathbf{k}))]_- = \sum_{\mathbf{k}'} a^\dagger(\mathbf{k}') \tau^\dagger(\mathbf{k}') (\partial^2 G/\partial z(\mathbf{k}) \partial z(\mathbf{k}')) \quad (\text{B12})$$

Equations (B11) and (B12) as well as

$$SO_q^\dagger(\mathbf{p}) |0\rangle = g_q^2(N(q+1)/N(q)) O_q^\dagger(\mathbf{p}) |0\rangle \quad (\text{B13})$$

enable us to write the equation (B1-2) in the form

$$\begin{aligned} \sum_{\mathbf{k}} \omega(\mathbf{k}) [z(\mathbf{k}) - X(\mathbf{k}) \sum_{\mathbf{k}'} X(\mathbf{k}') a^\dagger(\mathbf{k}') \tau^\dagger(\mathbf{k}') - X(\mathbf{k}) g_q^2(N(q+1)/N(q)) \\ (\partial G/\partial z(\mathbf{k})) O_q^\dagger(\mathbf{p}) |0\rangle - \\ - \sum_{\mathbf{k}} \omega(\mathbf{k}) X(\mathbf{k}) \sum_{\mathbf{k}'} a^\dagger(\mathbf{k}') \tau^\dagger(\mathbf{k}') (\partial^2 G/\partial z(\mathbf{k}) \partial z(\mathbf{k}')) O_q^\dagger(\mathbf{p}) |0\rangle + \\ + \lambda G O_q^\dagger(\mathbf{p}) |0\rangle = 0 \end{aligned} \quad (\text{B14})$$

We observe that

$$a^\dagger(\mathbf{k}) \tau^\dagger(\mathbf{k}) O_q^\dagger(\mathbf{p}) |0\rangle = \gamma_q^2 z(\mathbf{k}) O_q^\dagger(\mathbf{p}) |0\rangle \quad (\text{B15})$$

where

$$\gamma_q^2 = g_{q+1}^2(N(q)/N(q+1)) - g_q^2(N(q+1)/N(q)) \quad (25)$$

Thus (B14) can be rewritten with the help of (24) as (23).

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