

EXACTLY RENORMALIZABLE MODEL IN QUANTUM FIELD THEORY

III. RENORMALIZATION IN THE CASE OF TWO V-PARTICLES

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Synopsis

A simpler formulation of the renormalized Schrödinger equation is given for the model discussed previously ¹⁾. The renormalization is extended to the case of more than one heavy particle. The final results contain no divergences and are easily visualized in terms of diagrams which are calculated using only the renormalized coupling constants. It is conceivable that this formulation of the theory also applies to more realistic models.

1. *Introduction.* A simple field theory, which is an extension of the Lee model, was discussed in three papers ¹⁾, the last two of which will be quoted hereafter as I and II. It describes a system of what we called V-particles, interacting with a θ -particle field. In I we showed that the stationary V-particle states could be determined exactly in the bare particle representation. The renormalization of the mass of the V-particles and of the coupling constants was performed explicitly and it was proved that the results were in agreement with those of the usual S-matrix procedure. In II we discussed the asymptotic stationary (a.s.) states of Van Hove ²⁾ for our model. These states were written as a product of creation operators for the physical, i.e. dressed, particles acting on the vacuum state. We proved that the scalar products (s.p.) of a.s. states with one V-particle could be calculated, using only the renormalized coupling constants. This enabled us, by expanding the stationary states into a.s. states to give a completely renormalized form of the Schrödinger equation (eq. (3.6) of II). This renormalized Schrödinger equation, however, is given in the form of a complicated set of equations, which does not allow a clear physical interpretation. The first purpose of this paper is to put (II.3.6) into a very simple form by writing the hamiltonian with creation and absorption operators for dressed particles. The latter will be defined in section 2. The second purpose will be the renormalization of the Schrödinger equation for states with any number of V-particles which will be done in section 3. The result will be given in the form of simple rules for the calculation of diagrams, thereby using only renormalized

coupling constants. To check whether or not divergences appear in the intermediate steps of a calculation, we have determined the stationary state of two-V-particles in fixed positions and the corresponding energy, both to the first non trivial order in the renormalized coupling constants. The results are given in section 4 together with some remarks on the general validity of the method.

2. *Absorption operators for physical particles.* Throughout this paper we will use the same notation as in I and II. Therefore the creation operator of a physical V_q -particle of momentum \mathbf{p} is written as $O_q^*(\mathbf{p})$. An explicit expression for $O_q^*(\mathbf{p})$ in terms of bare particle creation operators is given in formula (II.2.2). The creation operator $a^*(\mathbf{k})$ for a physical θ -particle is the same as that for a bare particle.

We define the absorption operators for a physical V-particle as follows:

$$\Psi_q(\mathbf{p}) = N^{-\frac{1}{2}}(q) \left[\psi_q(\mathbf{p}) + \sum_{m=1}^{\infty} \sum (-1)^m \prod_{j=q-1}^{q-m} g_j^0 \cdot \psi_{q-m}(\mathbf{p} + \sum_{\mathbf{k}} l_{\mathbf{k}} \mathbf{k}) \cdot \prod_{\mathbf{k}} \frac{[X(\mathbf{k})a^*(\mathbf{k})]^{l_{\mathbf{k}}}}{l_{\mathbf{k}}!} \right], \quad (2.1)$$

where the second summation runs over all sets $\{l_{\mathbf{k}}\}_m$ of m θ -particles. This will always be understood when a summation sign without index occurs. The operator which absorbs a θ -particle from an a.s. state is

$$b(\mathbf{k}) = a(\mathbf{k}) - X(\mathbf{k})T^*(\mathbf{k}) \quad (2.2)$$

It can be shown that (II.2.2), (2.1) and (2.2) can be written in the following compact form *)

$$O_q^*(\mathbf{p}) = N^{\frac{1}{2}}(q)e^{B^*}\psi_q^*(\mathbf{p})e^{-B^*} \quad (2.3)$$

$$\Psi_q(\mathbf{p}) = N^{-\frac{1}{2}}(q)e^{B^*}\psi_q(\mathbf{p})e^{-B^*} \quad (2.4)$$

$$b(\mathbf{k}) = e^{B^*}a(\mathbf{k})e^{-B^*} \quad (2.5)$$

with

$$B = \sum_{\mathbf{k}} X(\mathbf{k})a(\mathbf{k})T(\mathbf{k}). \quad (2.6)$$

Almost the same equations have been derived by Greenberg and Schwember³⁾. From (2.3) and (2.4) it follows immediately that $\Psi_q(\mathbf{p})$ and $O_{q'}^*(\mathbf{p}')$ obey the anticommutation relation

$$\{\Psi_q(\mathbf{p}), O_{q'}^*(\mathbf{p}')\} = \delta_{qq'} \delta_{\mathbf{p}\mathbf{p}'}. \quad (2.7)$$

Because $[\Psi_q(\mathbf{p}), a^*(\mathbf{k})] = 0$ it follows, as stated before, that $\Psi_q(\mathbf{p})$ absorbs a V_q -particle of momentum \mathbf{p} when acting on an a.s. state containing that particle. Similarly one sees that

$$[b(\mathbf{k}), a^*(\mathbf{k}')] = \delta_{\mathbf{k}\mathbf{k}'} \quad \text{and} \quad [b(\mathbf{k}), O_q^*(\mathbf{p})] = 0,$$

*) I am extremely indebted to Dr. J. Lopuszański for providing me with these formulae.

so that e.g.

$$b(\mathbf{k})(a^*(\mathbf{k}))^n O_q^*(\mathbf{p})|0\rangle = n(a^*(\mathbf{k}))^{n-1} O_q^*(\mathbf{p})|0\rangle,$$

i.e. $b(\mathbf{k})$ absorbs a θ -particle of momentum k from an a.s. state. Formulae (2.3) and (2.4) can be used to derive a number of useful equations, one of which is

$$e^{B^*} T^*(\mathbf{k}) e^{-B^*} = T^*(\mathbf{k}) = \sum_{qp} g_q O_{q+1}^*(\mathbf{p}) \Psi_q(\mathbf{p} + \mathbf{k}), \quad (2.8)$$

i.e. a generalization of (II.3.2).

Applying the operators given in (2.1), (2.2) and (2.3), together with $b^*(\mathbf{k})$, the hermitian conjugate of $b(\mathbf{k})$, we can bring the hamiltonian of the model, given by (I.2.4) and (I.2.5) into the following form:

$$H = \sum_{qp} m O_q^*(\mathbf{p}) \Psi_q(\mathbf{p}) + \sum_k \omega(k) b^*(\mathbf{k}) b(\mathbf{k}) - \sum_k \omega(k) U(\mathbf{k}), \quad (2.9)$$

with

$$U(\mathbf{k}) = -X^2(k) \sum_{qq'\mathbf{p}\mathbf{p}'} g_q \circ g_{q'} \circ \psi_q^*(\mathbf{p} + \mathbf{k}) \psi_{q'+1}(\mathbf{p}') \psi_{q+1}(\mathbf{p}) \psi_{q'}(\mathbf{p}' + \mathbf{k}) \quad (2.10)$$

Thus the δm_q -terms no longer occur explicitly in the hamiltonian. However, (2.9) is not yet in a completely renormalized form because the operator $U(\mathbf{k})$ still contains unrenormalized coupling constants and bare particle creation and absorption operators.

In II we only considered states with one V-particle. In this case the hamiltonian is

$$H = m + \sum_k \omega(k) b^*(\mathbf{k}) b(\mathbf{k}) \quad (2.11)$$

The Schrödinger equation for the eigenstate $|\psi\rangle$ of energy $m + \omega_0$ can then be written as the following set of equations

$${}_{\text{as}}\langle\alpha| \sum_k \omega(k) b^*(\mathbf{k}) b(\mathbf{k}) |\psi\rangle = \omega_0 {}_{\text{as}}\langle\alpha|\psi\rangle, \quad (2.12)$$

where ${}_{\text{as}}\langle\alpha|$ may be any a.s. state with one V-particle. When $|\psi\rangle$ is expanded into a.s. states, it follows that (2.12) is equivalent to (II.3.6), when keeping in mind that $b^*(\mathbf{k})$ acts to the left as an absorption operator for a θ -particle. In II we showed that (2.12) is in a completely renormalized form by proving that the s.p. ${}_{\text{as}}\langle\alpha|\alpha'\rangle_{\text{as}}$ for states with one V-particle contained only the renormalized coupling constants and that in the calculations no infinities showed up. In the next section we will do the same for states with two V-particles.

3. *The renormalization for states with two V-particles.* In this section we will prove that for a.s. states $|\alpha\rangle_{\text{as}}$ and $|\alpha'\rangle_{\text{as}}$ with two V-particles the s.p. ${}_{\text{as}}\langle\alpha|\alpha'\rangle_{\text{as}}$ and the matrix elements (m.e.) ${}_{\text{as}}\langle\alpha|U(\mathbf{k})|\alpha'\rangle_{\text{as}}$ can be expressed in terms of the renormalized coupling constants only. Once this is done we can expand the state vector into a.s. states and write the time in-

dependent Schrödinger equation as a set of equations, involving only these renormalised *s.p.* and *m.e.* The solution to these equations must then be free of divergences and this was, indeed, established for some examples.

To find the general rules for the calculation of *s.p.* we will restrict ourselves to the investigation of the following example

$$\langle 0 | O_{q_1}(\mathbf{p}_1) O_{q_2}(\mathbf{p}_2) O_{q_3}^*(\mathbf{p}_3) O_{q_4}^*(\mathbf{p}_4) | 0 \rangle = \prod_{i=1}^4 N^\dagger(q_i) \langle 0 | \psi_{q_1}(\mathbf{p}_1) \psi_{q_2}(\mathbf{p}_2) e^{B e^{B^*} \psi_{q_3}^*(\mathbf{p}_3) \psi_{q_4}^*(\mathbf{p}_4)} | 0 \rangle \quad (3.1)$$

The r.h.s. of (3.1) is derived by using (2.3). Expanding e^B and e^{B^*} leads to

$$\prod_{i=1}^4 N^\dagger(q_i) \sum_{m=0}^{\infty} \sum_k \frac{[X^2(k)]^{l_k}}{l_k!} \langle 0 | \psi_{q_1}(\mathbf{p}_1) \psi_{q_2}(\mathbf{p}_2) T(\mathbf{k}_1) \dots T(\mathbf{k}_m) T^*(\mathbf{k}_m) \dots T^*(\mathbf{k}_1) \psi_{q_3}^*(\mathbf{p}_3) \psi_{q_4}^*(\mathbf{p}_4) | 0 \rangle. \quad (3.2)$$

The operators $T(\mathbf{k})$ commute among themselves, i.e. $[T(\mathbf{k}), T(\mathbf{k}')] = 0$. We further remark that

$$T^*(\mathbf{k}) \psi_{q'}^*(\mathbf{p}) \psi_{q'}^*(\mathbf{p}') | 0 \rangle = g_{q'}^0 \psi_{q+1}^*(\mathbf{p} - \mathbf{k}) \psi_{q'}^*(\mathbf{p}') | 0 \rangle + g_{q'}^0 \psi_{q'}^*(\mathbf{p}) \psi_{q'+1}^*(\mathbf{p}' - \mathbf{k}) | 0 \rangle \quad (3.3)$$

$$T(\mathbf{k}) \psi_q^*(\mathbf{p}) \psi_{q'}^*(\mathbf{p}') | 0 \rangle = g_{q-1}^0 \psi_{q-1}^*(\mathbf{p} + \mathbf{k}) \psi_{q'}^*(\mathbf{p}') | 0 \rangle + g_{q'-1}^0 \psi_q^*(\mathbf{p}) \psi_{q'-1}^*(\mathbf{p}' + \mathbf{k}) | 0 \rangle. \quad (3.4)$$

Taking now for each T^* and T in (3.2) either the first or the second term in the r.h.s. of (3.3) or (3.4), respectively, we can write the *m.e.* in (3.2) as a sum of terms. We represent each term by a diagram consisting of two horizontal full lines for the two V-particles, directed from right to left. These lines we call V-lines. For each T and for each T^* in (3.2) we put a dot on one of the V-lines. The dots for $T^*(\mathbf{k})$ and $T(\mathbf{k})$ are located on the upper (lower) line, when in the r.h.s. of (3.3) and (3.4) the first (second) term is chosen. The T -dots are all on the left of the T^* -dots. For each \mathbf{k}_j the dots associated with $T^*(\mathbf{k}_j)$ and $T(\mathbf{k}_j)$ are connected by a dotted line representing a θ -

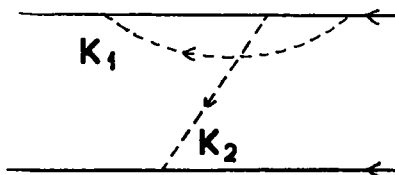


Fig. 1. Example of a diagram for the calculation of a *s.p.*

particle which is emitted in the T^* -dot and absorbed in the T -dot. Such a line will be called a θ -line. Let us for a moment attach an index 1 or 2 to the $T^*(\mathbf{k})$ and $T(\mathbf{k})$ to indicate whether the first or the second term in (3.3) and (3.4) has been taken. The choice $T_1(\mathbf{k}_1) T_2(\mathbf{k}_2) T_1^*(\mathbf{k}_2) T_1^*(\mathbf{k}_1)$ in the term of (3.2) with $m = 2$ is then represented by the diagram of figure 1.

θ -lines connecting a V-line to itself, such as \mathbf{k}_1 in fig. 1, will be called θ -

lines of the first kind; those which connect the two V-lines, such as \mathbf{k}_2 in fig. 1, will be called θ -lines of the second kind. It is possible to sum the contributions of all diagrams which differ only through θ -lines of the first kind. This gives a factor $N^{-1}(q)$ for both V-lines in the diagram. Using the formula

$$g_q = N^{\frac{1}{2}}(q)N^{-\frac{1}{2}}(q+1)g_q^0 \quad (\text{I.3.3})$$

it then follows that these two factors, together with $\prod_{i=1}^4 N^{\frac{1}{2}}(q_i)$, are just enough to renormalize the coupling constants occurring in (3.2). Thus in calculating s.p. we can restrict ourselves to diagrams without θ -lines of the first kind. The factor to be accounted for each vertex is now simply $g_q X(k)$, where g_q is renormalized.

The calculation of a m.e. of $U(\mathbf{k})$ between a.s. states amounts to considering

$$\langle 0 | \psi_{q_1}(\mathbf{p}_1) \psi_{q_2}(\mathbf{p}_2) e^{BU(\mathbf{k})} e^{B^* \psi_{q_3}^*(\mathbf{p}_3) \psi_{q_4}^*(\mathbf{p}_4)} | 0 \rangle \quad (3.5)$$

In (3.5) the exponentials are again expanded and as before a diagram is associated with each term. Observing that

$$U(\mathbf{k}) \psi_{q'}^*(\mathbf{p}) \psi_{q''}^*(\mathbf{p}') | 0 \rangle = X^2(k) [g_q^0 g_{q'-1}^0 \psi_{q+1}^*(\mathbf{p} - \mathbf{k}) \psi_{q'-1}^*(\mathbf{p}' + \mathbf{k}) + g_{q-1}^0 g_{q'}^0 \psi_{q-1}^*(\mathbf{p} + \mathbf{k}) \psi_{q'+1}^*(\mathbf{p}' - \mathbf{k})] | 0 \rangle, \quad (3.6)$$

we can account for the action of the operator $U(\mathbf{k})$ by adding a double θ -line to the diagram, connecting a dot on the upper V-line with one on the lower V-line. These two dots lie to the left of the T^* -dots and to the right of the T -dots. When we choose the first (second) term in the r.h.s. of (3.6) this double θ -line is directed from the upper (lower) to the lower (upper) V-line. Again we can perform a partial summation, which exactly renormalizes all coupling constants g_q^0 , so that only diagrams with θ -lines of the second kind must be considered. Figure 2 is an example of a diagram which now appears. For the calculation of these diagrams we must substitute again for each vertex a factor of the form $g_q X(k)$.

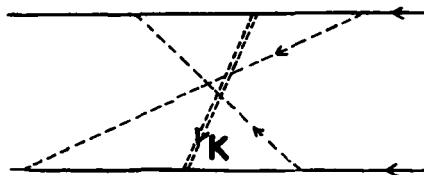


Fig. 2. Diagram for m.e. of $U(k)$.

Using the formula

$$a(\mathbf{k}) O_{q_1}^*(\mathbf{p}_1) O_{q_2}^*(\mathbf{p}_2) = g_q X(k) \{ O_{q_1+1}^*(\mathbf{p}_1 - \mathbf{k}) O_{q_2}^*(\mathbf{p}_2) + O_{q_1}^*(\mathbf{p}_1) O_{q_2+1}^*(\mathbf{p}_2 - \mathbf{k}) \} + O_{q_1}^*(\mathbf{p}_1) O_{q_2}^*(\mathbf{p}_2) a(\mathbf{k}), \quad (3.7)$$

it is easy to show how the diagram method for the calculation of s.p. and

m.e. must be extended to a.s. states, which also contain θ -particles. Examples of more general diagrams which then appear are drawn in figure 3. The rules for the evaluation of a diagram remain the same.

4. *Application and discussion.* The form of the theory as given in the last section may be called 'completely renormalized' if we can prove that no divergences will occur when perturbation theory is applied, using the renormalized coupling constants as expansion parameters. In II it was established for θ -V-scattering. We now have also considered the stationary state $|\psi^{(l)}(\mathbf{x}, \mathbf{x}')\rangle$ of two V-particles which are located in \mathbf{x} and \mathbf{x}' , and proved, using the diagram method, that in this case too no divergences arise. We made the following Ansatz:

$$|\psi^{(l)}(\mathbf{x}, \mathbf{x}')\rangle = [\sum_{s=1}^{n_0} \sum_{\mathbf{p}\mathbf{p}'} \alpha_{ls} e^{-i(\mathbf{p}\mathbf{x} + \mathbf{p}'\mathbf{x}')} O_s^*(\mathbf{p}) O_{l-s}^*(\mathbf{p}') + \sum_{j\mathbf{p}''\mathbf{p}'''\mathbf{k}} d_{lj}(\mathbf{k}, \mathbf{x}, \mathbf{x}') e^{-i(\mathbf{p}''\mathbf{x} + \mathbf{p}'''\mathbf{x}')} a^*(\mathbf{k}) O_j^*(\mathbf{p}'') O_{l-j+1}(\mathbf{p}''') + \dots] |0\rangle, \quad (l = 1, 2, \dots, n_0) \quad (4.1)$$

and determined α_{ls} and $d_{lj}(\mathbf{k}, \mathbf{x}, \mathbf{x}')$ to lowest order in the renormalized coupling constants g_1 and g_2 (we took the case of only two different kinds of V-particles, i.e. $n_0 = 2$). We found

$$d_{lj}(\mathbf{k}, \mathbf{x}, \mathbf{x}') = W(|\mathbf{x} - \mathbf{x}'|) \omega^{-1}(\mathbf{k}) X(k) [g_{l-j}(g_j^2 - g_{j-1}^2) \alpha_{lj} e^{-i\mathbf{k}\mathbf{x}} + g_{j-1}(g_{l-j+1}^2 - g_{l-j}^2) \alpha_{l,j-1} e^{-i\mathbf{k}\mathbf{x}'}], \quad (4.2)$$

with

$$W(r) = \sum_{\mathbf{k}} \omega(k) X^2(k) e^{i\mathbf{k}\mathbf{r}} = e^{-\mu r} / 8\pi r \quad (\mu \text{ is mass of } \theta\text{-particle}). \quad (4.3)$$

The last equality in (4.3) only holds for infinite cut-off. The energy of the state $|\psi^{(l)}(\mathbf{x}, \mathbf{x}')\rangle$ was written as $2m + U^{(l)}(|\mathbf{x} - \mathbf{x}'|)$ and calculated to fourth order in the coupling constants. We found

$$U^{(l)}(r) = -W(r) [A_l - B_l V(r)] \quad (4.4)$$

with

$$V(r) = \sum_{\mathbf{k}} X^2(k) e^{i\mathbf{k}\mathbf{r}} = K_0(\mu r) / 8\pi^2. \quad (4.5)$$

The last equality in (4.5) again holds only in the limit of infinite cut-off. For the Besselfunction $K_0(\mu r)$ the following asymptotic expressions exist:

$$K_0(\mu r) \simeq \sqrt{(\pi/2\mu r)} e^{-\mu r} \text{ for } \mu r \gg 1,$$

$$K_0(\mu r) \simeq -\log \mu r \text{ for } \mu r \ll 1, \quad K_0(\mu r) > 0 \text{ for all } r.$$

The quantities A_l and B_l for the different lowest order solutions of α_{ls} are given below

$$l = 1 \quad \alpha_{11} = \pm \alpha_{12}; \quad A_1 = \pm(g_1^2 + g_2^2); \quad B_1 = (g_1^2 - g_2^2)^2$$

$$l = 2 \quad \alpha_{21} = \pm \alpha_{22}; \quad A_2 = \pm 2g_1 g_2; \quad B_2 = -(g_1^2 - g_2^2)^2.$$

Even for arbitrarily small coupling constants, it is possible to find a separation of the particles such that the fourth order term in the energy is more

important than the second order (Yukawa) term. It seems therefore, that the interaction between the two V-particles can have a repulsive core. However, it is easy to see that the term $B_I V(r)$ in (4.4) can only dominate the term A_I for very large cut-off. In fact the cut-off has to be so large that, according to (I.3.3), the unrenormalized coupling constants g_q^0 become imaginary. In the case of the Lee-model Källén and Pauli⁴⁾ had to introduce an in-

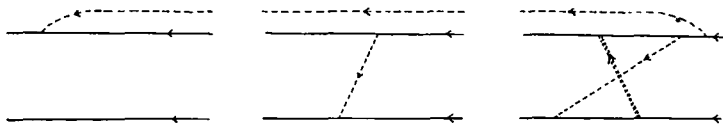


Fig. 3. More general diagrams.

definite metric to make sure that the physical states still had a positive norm. The norm of the ghost state which then appears, however, became negative and this made the S -matrix non-unitary. To check whether a similar feature occurs in our model we also have to introduce an indefinite metric. An appropriate definition of η (cf. ⁴⁾ formula (4.6)) is $\eta = (-1)^Q$ with

$$Q = \begin{cases} \sum_{qp} q\psi_q^*(\mathbf{p})\psi_q(\mathbf{p}) & \text{if } N(q=1) < 0 \\ \sum_{qp} q\psi_q^*(\mathbf{p})\psi_q(\mathbf{p}) + 1 & \text{if } N(q=1) > 0. \end{cases}$$

With this metric the norm of the states $|\phi_q(\mathbf{p})\rangle = O_q^*(\mathbf{p})|0\rangle$ is again positive. It could not yet be decided whether such a theory is consistent for a cut-off which makes g_q^0 imaginary and the hard core interaction between two heavy particles may therefore be meaningless.

A more important result is that all integrals, like e.g.

$$\sum_k X(k)d_{ij}(\mathbf{k}, \mathbf{x}, \mathbf{x}')e^{ikx},$$

which occur in the calculations, converge for infinite cut-off. This proves that the theory is in completely renormalized form. We further remark that $d_{ij}(\mathbf{k}, \mathbf{x}, \mathbf{x}')$ is of third order in the coupling constants and not of first order, as would be expected on first sight. The same feature was encountered in II when we calculated the θ -V scattering. It must probably be ascribed to the fact that we used expansions in a.s. states. The derivations given in this paper can easily be extended to the case of three and more V-particles.

We conclude that the most important result of our considerations is the possibility of calculating all physical quantities in terms of the renormalized coupling constants only, without ever running into divergences. For this result to be valid it is fundamental that our diagrams do not contain θ -lines of the first kind. This leads to the conjecture that it will be impossible to write the hamiltonian in renormalized form and to demand simultaneously that no divergences appear in the intermediate steps of a perturbation calculation.

The validity of the method developed here does not seem to depend very critically on the exact solubility of the V-particle states and it may therefore be worth while to apply the same idea to other theories, like e.g. the Chew-Low model. The problem of bound states can perhaps also be treated in a more satisfactory way than is done in the Bethe-Salpeter approximation.

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