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Positivity for perturbations of polyharmonic operators with Dirichlet boundary conditions in two dimensions.

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Abstract. Higher order elliptic partial differential equations with Dirichlet boundary conditions in general do not satisfy a maximum principle. Polyharmonic operators on balls are an exception. Here it is shown that in $\mathbb{R}^2$ small perturbations of polyharmonic operators and of the domain preserve the maximum principle. Hence the Green function for the clamped plate equation on an ellipse with small eccentricity is positive.

1. Historical comments and the main result

Maximum and comparison principles have proved to be a powerful tool in the theory of second order elliptic differential equations. So, for a better understanding of higher order elliptic differential equations it is an obvious step to investigate the question whether similar results do exist there. That is, if Ω is an appropriate bounded domain, does a function u that satisfies the higher order elliptic differential inequality $Lu \geq 0$, with zero Dirichlet boundary condition, have a fixed positive sign? As an example one may think of the clamped plate equation

$$\begin{cases} \Delta^2 u = f \text{ in } \Omega, \\ u_{|\partial\Omega} = \frac{\partial}{\partial n} u_{|\partial\Omega} = 0. \end{cases}$$

Here the question can be rephrased as:

For which shapes does upwards pushing imply upwards bending?

The pushing resp. bending is denoted by f, resp. u and $\Omega \subset \mathbb{R}^2$ is the shape.

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Throughout this paper only two dimensional domains $\Omega \subset \mathbb{R}^2$ will be considered, if nothing different is stated. We have to leave the question open, whether the results of the present paper can be generalized to higher dimensions.

Boggio [3] (1901) and Hadamard [13] (1908) conjectured that in arbitrary reasonable domains Ω, $f \geq 0$ implies $u \geq 0$. Boggio in [4] could show this in case of the ball ($\Omega = B =$ unit ball). Furthermore, for any polyharmonic operator, $(-\Delta)^m$, and balls $B \subset \mathbb{R}^n$, he calculated the Green function $G_{m,n}(\cdot,\cdot)$ for the Dirichlet problem and showed that $G_{m,n}(\cdot,\cdot) > 0$ in B^2. In 1909 Hadamard [14] already knew, that the positivity conjecture is false in annuli with small inner radius. But he, and also Boggio, as Hadamard mentioned, believed, that there is no serious doubt, that Green's function should be positive at least in convex domains.

Starting about 40 years later, numerous counterexamples [5], [6], [8], [9], [16], [17], [18], [22], [24] disprove the Boggio-Hadamard conjecture. The most striking examples have been found by Coffman, Duffin and Garabedian. For example, Garabedian ([9], see also [10, p. 275]) found that in an ellipse in $\mathbb{R}^2$, with the ratio of the half axes $\simeq 2$, the Green function for the biharmonic operator Δ^2 changes sign (for an elementary proof, see also [22]). Coffman and Duffin [6] could show the same result, i.e. change of sign of Green's function, in rectangles, including the square. That means that neither in arbitrarily smooth, uniformly convex domains nor in rather symmetric domains, we may expect Green's function to be positive. Things are still worse: even the first Dirichlet-eigenfunction of Δ^2 need not be unique and of one sign, see [5], [16].

In a recent paper [12] we could show that a polyharmonic operator in a ball which is slightly perturbed in the lower order terms still has the positivity preserving property mentioned above. In the present paper we shall show that in two dimensions one can even allow some small perturbation in the highest order term of the operator as well as in the shape of the domain and still has the positivity preserving property.

The higher order elliptic problem that we consider is

$$\text{(1.1)} \qquad \begin{cases} Lu & \geq & 0 & \text{in } \Omega, \\ D_m u & = & 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\text{(1.2)} \; L = \left(- \sum_{1 \leq i \leq j \leq 2} a_{ij}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \right)^m + \sum_{|\alpha| \leq 2m-1} b_\alpha(x) \left(\frac{\partial}{\partial x_1} \right)^{\alpha_1} \left(\frac{\partial}{\partial x_2} \right)^{\alpha_2},$$

$a_{ij} \in C^{2m-1,\gamma}(\overline{\Omega})$, $b_\alpha \in C^{0,\gamma}(\overline{\Omega})$ and $D_m u = 0$ on $\partial\Omega$ is the zero Dirichlet boundary condition:

$$\left(\frac{\partial}{\partial x} \right)^\alpha u_{|\partial\Omega} = 0 \text{ for all } \alpha \text{ with } |\alpha| \leq m-1.$$

For the sake of easy statement we define closeness of domains and operators.

Definition 1.1. Let $\varepsilon > 0$. We call Ω ε-*close* in $C^{k,\gamma}$-sense to Ω^*, if there exists a $C^{k,\gamma}$-mapping $g : \overline{\Omega^*} \to \overline{\Omega}$ such that $g\left(\overline{\Omega^*}\right) = \overline{\Omega}$ and

$$\|g - Id\|_{C^{k,\gamma}(\overline{\Omega^*})} \leq \varepsilon.$$

Remark 1. For convex Ω^*, $k \geq 1$ and ε small we easily find that g is bijective and even that $g^{-1} \in C^{k,\gamma}(\overline{\Omega})$, $||g^{-1} - Id||_{C^{k,\gamma}(\overline{\Omega})} = \mathcal{O}(\varepsilon)$. Local injectivity follows directly. Since Ω^* is assumed to be convex, we find

$$\begin{aligned} |g(x) - g(y)| &\geq |x-y| - |g(x) - x - (g(y) - y)| \geq \\ &\geq |x-y| - \left\|(g - Id)'\right\|_{C^0} |x-y| \geq (1-\varepsilon)|x-y|, \end{aligned}$$

implying that g is globally injective.

Definition 1.2. Let $\varepsilon > 0$ and let L be as above. We call the operator L *ε-close* in $C^{k,\gamma}$-sense to $(-\Delta)^m$ on $\overline{\Omega}$, if additionally $a_{ij} \in C^{k,\gamma}(\overline{\Omega})$ and

$$\|a_{ij} - \delta_{ij}\|_{C^{k,\gamma}(\overline{\Omega})} \leq \varepsilon,$$
$$\|b_\alpha\|_{C^0(\overline{\Omega})} \leq \varepsilon.$$

Remark 2. For ε small, L is uniformly elliptic.

Our main result shows that it is the large deviation from the constant flexure of the ball that yields a change of sign for the Green function:

Theorem 1.3. *There exists $\varepsilon_0 = \varepsilon_0(m) > 0$ such that, for $\varepsilon \in [0, \varepsilon_0)$ we have: If $\partial\Omega \in C^{2m,\gamma}$, Ω is ε-close in C^{2m}-sense to B and L is ε-close in $C^{2m-1,\gamma}$-sense to $(-\Delta)^m$ on $\overline{\Omega}$, then every $u \not\equiv 0$ that satisfies* (1.1) *is strictly positive in Ω.*

Remark 3. In principle $\varepsilon_0 = \varepsilon_0(m)$ could be calculated explicitly. We expect $\varepsilon_0(m) \searrow 0$ as $m \to \infty$.

The theorem will be proven in two steps. First we will assume $a_{ij} = \delta_{ij}$ and consider "small" perturbations of the domain, see Section 2. This result and the theory of canonical forms will allow for perturbations of the leading coefficients of the operator too, see Section 3.

Corollary 1.4. *Let $E_\varepsilon = \left\{(x_1, x_2); x_1^2 + (1+\varepsilon)x_2^2 \leq 1\right\}$ denote ellipses close to the unit ball. There exists $\varepsilon_m > 0$ such that for all $|\varepsilon| \leq \varepsilon_m$ the Green function is positive for*

$$\begin{cases} (-\Delta)^m u = f & \text{in } E_\varepsilon, \\ D_m u = 0 & \text{on } \partial E_\varepsilon. \end{cases}$$

Remark 4. In 1951 Garabedian showed in [9] (see also [10, p. 275]) that the Green function on some eccentric ellipses (ratio of half axes ≈ 2 for $m = 2$) changes sign. Our Corollary answers the question whether the Green function changes sign on any ellipse that is not a ball.

2. Domain perturbations

Throughout the rest of this paper we may assume $m > 1$.

We take the crucial lemma from our previous paper [12, sect. 5], which has been proved by means of Green's function estimates.

Lemma 2.5. *There is a $\delta_0 = \delta_0(m) > 0$ such that the following holds. Let Ω be a simply connected, bounded smooth $C^{2m,\gamma}$-domain, L be as in Theorem 1.3 with $a_{ij} = \delta_{ij}$. Let $h : B \to \Omega$ be a biholomorphic mapping with $h \in C^{2m,\gamma}(\overline{B})$, $h^{-1} \in C^{2m,\gamma}(\overline{\Omega})$. If*

$$\begin{aligned} ||h - Id||_{C^{2m-1}(\overline{B})} &\leq \delta_0, \\ ||b_\alpha||_{C^0(\overline{\Omega})} &\leq \delta_0, \end{aligned}$$

for $|\alpha| < 2m$, then the Green function of L in Ω under homogeneous Dirichlet boundary conditions is positive.

Remark 1. The existence of holomorphic mappings with qualitative properties as in the lemma is ensured by theorems of Riemann and Warschawski, see [19].

Remark 2. An explicit conformal mapping from the unit ball to any ellipse is given in the note of H. A. Schwarz [21]. So one could think that Corollary 1.4 above could be proven by an explicit analysis of this mapping. But, since elliptic functions are involved, this seems to be at least very difficult.

This example shows the need for a notion of closeness of domains as in Definition 1.1. Such a condition may be checked more easily.

Theorem 1.3 with $a_{ij} = \delta_{ij}$ will be proven, if we can show that C^{2m}-closeness to the ball with respect to differentiable mappings implies also C^{2m-1}-closeness with respect to holomorphic mappings.

Proposition 2.6. *Let δ_0 be given. Then there is some $\varepsilon_0 = \varepsilon_0(\delta_0, m) > 0$, such that for $\varepsilon \in [0, \varepsilon_0)$ we have the following.*
If the $C^{2m,\gamma}$-domain Ω is ε-close in C^{2m}-sense to B, then there is a biholomorphic mapping $h : B \to \Omega$, $h \in C^{2m,\gamma}(\overline{B})$, $h^{-1} \in C^{2m,\gamma}(\overline{\Omega})$ with

$$||h - Id||_{C^{2m-1}(\overline{B})} \leq \delta_0.$$

Proof. Let $g : \overline{B} \to \overline{\Omega}$ be a mapping according to Definition 1.1 such that $||g - Id||_{C^{2m}(\overline{B})} < \varepsilon$. The number ε is assumed to be small enough. According to [7], cf. also [23, sect. 4.2], the holomorphic mapping $h^{-1} : \Omega \to B$, which has the desired qualitative properties, may be constructed in the following way. First set

$$w(x) := 2\pi G(x, 0).$$

Here G is the Green function for $-\Delta$ in Ω under homogeneous Dirichlet condition. Next define the conjugate harmonic function

$$w^*(x) := \int_{1/2}^{x} \left(-\frac{\partial}{\partial \xi_2} w(\xi)\, d\xi_1 + \frac{\partial}{\partial \xi_1} w(\xi)\, d\xi_2 \right),$$

where the integral is taken with respect to any curve from $\frac{1}{2}$ to x in $\Omega \setminus \{0\}$. w^* is well defined up to multiples of 2π. One finds that h^{-1} is well defined by

$$h^{-1}(x) := \exp\left(-w\left(x\right) - iw^*\left(x\right)\right) \text{ for } x \in \overline{\Omega},$$

where $\mathbb{R}^2$ and $\mathbb{C}$ are identified. The function h^{-1} maps 0 onto 0 and the point $\frac{1}{2}$ somewhere into the positive real half-axis. Moreover, for $x \in \partial\Omega$ we find that $\left|h^{-1}(x)\right| = \left|\exp\left(-iw^*\left(x\right)\right)\right| = 1$ and hence $h^{-1}\left(\partial\Omega\right) \subset \partial B$. For $x \in \Omega\setminus\{0\}$ we have $w\left(x\right) > 0$ and hence $\left|h^{-1}(x)\right| < 1$ implying $h^{-1}\left(\Omega\right) \subset B$.

The Green function $G(x,0)$ is defined by

$$G(x,0) = -\frac{1}{2\pi}\left(\log|x| - r(x)\right), \quad x \in \overline{\Omega},$$

where

$$\begin{cases} \Delta r = & 0 & \text{in } \Omega, \\ r\left(x\right) = & \varphi(x) := \log|x| & \text{for } x \in \partial\Omega. \end{cases}$$

Since

$$h^{-1}(x) = x\exp\left(-r\left(x\right) - ir^*\left(x\right)\right) \text{ for } x \in \overline{\Omega},$$

again identifying $\mathbb{R}^2$ and $\mathbb{C}$, one finds that

$$||r||_{C^{2m-1}(\overline{\Omega})} = \mathcal{O}(\varepsilon) \tag{2.3}$$

will imply $||h^{-1} - Id||_{C^{2m-1}(\overline{\Omega})} = \mathcal{O}(\varepsilon)$ and consequently $||h - Id||_{C^{2m-1}(\overline{B})} = \mathcal{O}(\varepsilon)$. The estimate in (2.3) will follow from the extension of the boundary data $\varphi_{|\partial\Omega}$ to some $\hat{\varphi}$ on $\overline{\Omega}$ with

$$||\hat{\varphi}||_{C^{2m}(\overline{\Omega})} = \mathcal{O}(\varepsilon). \tag{2.4}$$

Indeed, the estimate for $||r||_{C^0(\overline{\Omega})}$ is immediate by the maximum principle. Furthermore, by means of elliptic estimates for second order equations (see [1, Theorem 7.3], [11, chapt. 6.4]) we find $||r||_{C^{2m-1,\gamma}(\overline{\Omega})} = \mathcal{O}(\varepsilon)$. Note that due to the closeness of Ω to B in C^{2m}-sense, according to Definition 1.1, the constants in these estimates may be chosen independently of Ω.

It remains to show the existence of some $\hat{\varphi}$ that satisfies (2.4). This is done as follows. Since Ω is ε-close to B in C^{2m}-sense, one can show that $(\varphi \circ g)_{|\partial B}$ may be extended to φ_g on $\overline{B}$ with $||\varphi_g||_{C^{2m}(\overline{B})} = \mathcal{O}(\varepsilon)$. That means we have to estimate the "tangential derivatives" of $\varphi \circ g_{|\partial B}$ only.

Set $\psi\left(t\right) := \varphi\left(g\left(\cos t, \sin t\right)\right)$. We are done, if we have shown that

$$\max_{j=0,\ldots,2m} \max_{t\in[0,2\pi]} \left|\left(\frac{d}{dt}\right)^j \psi\right| = \mathcal{O}(\varepsilon).$$

We observe that $\psi(t) = \mathcal{O}(\varepsilon)$, since $\log|g(\cos t, \sin t)| = \log\left(1 + \mathcal{O}(\varepsilon)\right) = \mathcal{O}(\varepsilon)$. Let us denote $\tilde{g}(t) := g(\cos t, \sin t)$. For $j \geq 1$ a tedious application of chain and product rule yields:

$$\left(\tfrac{d}{dt}\right)^j \psi = \left(\tfrac{d}{dt}\right)^j (\varphi \circ \tilde{g}) =$$

$$= \sum_{|\alpha|=1}^{j} ((D^\alpha \varphi) \circ \tilde{g}) \left(\sum_{\substack{p_1 + \ldots + p_{|\alpha|} = j \\ 1 \le p_l}} d_{j,\alpha,\vec{p}} \prod_{l=1}^{|\alpha|} \left(\tfrac{d}{dt}\right)^{p_l} \tilde{g}^{(\beta_l)} \right)$$

with some suitable coefficients $d_{j,\alpha,\vec{p}}$, $\beta_l = 1$ for $l = 1, \ldots, \alpha_1$ and $\beta_l = 2$ for $l = \alpha_1 + 1, \ldots, |\alpha|$. We want to compare this with the corresponding expression with g replaced by Id. Denote $\tilde{g}_0(t) = Id \circ (\cos t, \sin t)$.

$$\left(\tfrac{d}{dt}\right)^j \psi = \sum_{|\alpha|=1}^{j} \left(\left((D^\alpha \varphi) \circ \tilde{g} - (D^\alpha \varphi) \circ \tilde{g}_0 \right) + (D^\alpha \varphi) \circ \tilde{g}_0 \right) \times$$

$$\times \left(\sum_{\substack{p_1 + \ldots + p_{|\alpha|} = j \\ 1 \le p_l}} d_{j,\alpha,\vec{p}} \prod_{l=1}^{|\alpha|} \left(\left(\left(\tfrac{d}{dt}\right)^{p_l} \tilde{g}^{(\beta_l)} - \left(\tfrac{d}{dt}\right)^{p_l} \tilde{g}_0^{(\beta_l)} \right) + \left(\tfrac{d}{dt}\right)^{p_l} \tilde{g}_0^{(\beta_l)} \right) \right).$$

Since $\varphi(\tilde{g}_0(t)) = \log|(\cos t, \sin t)| \equiv 0$, all expressions containing $\tilde{g}_0$ only (and not a difference), sum up to zero. In the remaining sum, every term contains at least one factor of the form

$$(D^\alpha \varphi) \circ \tilde{g} - (D^\alpha \varphi) \circ \tilde{g}_0 \qquad \text{or} \qquad \left(\tfrac{d}{dt}\right)^{p_l} \left(\tilde{g}^{(\beta_l)} - \tilde{g}_0^{(\beta_l)} \right).$$

For ε small, each of these factors is at most $\mathcal{O}(\varepsilon)$. The other factors remain uniformly bounded with respect to $\varepsilon \in [0, \varepsilon_0)$, ε_0 chosen appropriately. We come up with

$$\max_{j=0,\ldots,2m} \max_{t \in [0,2\pi]} \left| \left(\tfrac{d}{dt}\right)^j \psi \right| = \mathcal{O}(\varepsilon),$$

as required. □

3. Operator perturbations

Denote

$$L_0 = - \sum_{1 \le i \le j \le 2} a_{ij}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}. \tag{3.5}$$

First we assume that $a_{ij} \in C^3(\overline{\Omega})$, that Ω is convex and that $\partial\Omega \in C^3$. In order to simplify the notation we will use in the major part of this section (x, y) instead of (x_1, x_2) and set $a := a_{11}$, $b := \frac{1}{2} a_{12}$ and $c := a_{22}$.

Second order linear equations in two variables can be reduced to a canonical form. See e.g. [10, p. 66-68]. For equations of elliptic type it means that there exist quasiconformal transformations $(x, y) \mapsto (\xi, \eta)$ such that with $v(x, y) = V(\xi, \eta)$ we have

$$(L_0 v)(x, y) = A(\xi, \eta) \left(- \Delta V(\xi, \eta) - w(\xi, \eta) \cdot \nabla V(\xi, \eta) \right). \tag{3.6}$$

For references about reduction to canonical form see e.g. [2], [10], [20]. Here we have to deal with geometrically very simple domains. In this case some global properties of the mapping $(x,y) \mapsto (\xi,\eta)$ can be shown rather directly. For the reader's convenience we include these proofs in the following.

For the transformation that we use we fix the boundary condition of the second component η to equal the second component of the identity.

We find that η satisfies the second order elliptic boundary value problem

$$\begin{cases} M\eta = 0 \text{ in } \Omega, \\ \eta = y \text{ on } \partial\Omega, \end{cases} \tag{3.7}$$

with M defined by

$$M\phi = \frac{\partial}{\partial x}\frac{a\phi_x + b\phi_y}{\sqrt{ac-b^2}} + \frac{\partial}{\partial y}\frac{b\phi_x + c\phi_y}{\sqrt{ac-b^2}}. \tag{3.8}$$

Lemma 3.7. *Let Ω be as above. Then (3.7) has a unique solution $\eta \in C^{2,\gamma}(\overline{\Omega})$ and $\nabla\eta \neq 0$ in $\overline{\Omega}$. Moreover, every level line $\ell_{\eta,t}(\overline{\Omega})$, defined by*

$$\ell_{\eta,t}(\overline{\Omega}) = \left\{(x,y) \in \overline{\Omega}; \eta(x,y) = t\right\},$$

with $\min\{y;(x,y) \in \overline{\Omega}\} < t < \max\{y;(x,y) \in \overline{\Omega}\}$, consists of one C^1-arc connecting two boundary points.

Proof. Existence and regularity for η follows from standard elliptic theory. We have to show that $\nabla\eta \neq 0$. First we fix $\hat{p} \in \partial\Omega$. If the tangential direction $\tau(\hat{p})$ at $\hat{p}$ is not parallel $(1,0)$, then $\frac{\partial\eta}{\partial\tau} = \frac{\partial y}{\partial\tau} = \tau_2(\hat{p}) \neq 0$ and $\nabla\eta$ is not parallel with $n(\hat{p})$, where $n(\hat{p})$ is the normal direction. If $\tau(\hat{p})$ and $(1,0)$ are parallel, then, since Ω is convex either $\eta \geq \eta(\hat{p})$ or $\eta \leq \eta(\hat{p})$ on $\partial\Omega$ and not identical 0, the maximum principle and Hopf's boundary point Lemma imply $\frac{\partial\eta}{\partial y}(\hat{p}) > 0$. It shows that $\nabla\eta \neq 0$ on $\partial\Omega$ and hence that the Brouwer degree $\deg(\nabla\eta,\Omega,0)$ is well defined. Moreover, we will show that the homotopy $[0,1]\times\partial\Omega \to \mathbb{R}^2$, $(t,p) \mapsto t(0,1)+(1-t)\nabla\eta(p)$ is admissible. This is obvious for the two points p with $\tau(p)$ and $(1,0)$ parallel. If $\tau(p)$ and $(1,0)$ are not parallel, $t(0,1)+(1-t)\nabla\eta(p) = 0$ for $t \in [0,1]$, $p \in \partial\Omega$ would give: $t \neq 1$, $\eta_x(p) = 0$, $\eta_y(p) = t/(t-1)$, hence $0 \neq \tau_2(p) = \frac{\partial\eta}{\partial\tau}(p) = \eta_x(p)\tau_1(p)+\eta_y(p)\tau_2(p) = \tau_2(p)\cdot t/(t-1)$, a contradiction. Consequently we have

$$\deg(\nabla\eta,\Omega,0) = \deg((0,1),\Omega,0) = 0.$$

Now suppose that $\nabla\eta(\hat{p}) = 0$ for some $\hat{p} \in \Omega$. Then the Carleman-Hartman-Wintner Theorem (see [20] and the appendix) implies that $\hat{p}$ is an isolated zero of $\nabla\eta$ and moreover, that the local degree of $\nabla\eta$ at $\hat{p}$ satisfies $\deg(\nabla\eta, B_\varepsilon(\hat{p}),0) < 0$ (for some $\varepsilon > 0$ small). By the additivity property of the degree we obtain a contradiction with $\deg(\nabla\eta,\Omega,0) = 0$. Hence $\nabla\eta \neq 0$ in $\overline{\Omega}$.

Since $\nabla\eta \neq 0$ in $\overline{\Omega}$ the level lines are C^1. Note that the maximum principle implies that every component of a level set $\overline{\Omega}^+_{\eta,t} = \{p \in \overline{\Omega}; \eta(p) > t\}$ or $\overline{\Omega}^-_{\eta,t} = \{p \in \overline{\Omega}; \eta(p) < t\}$ has to intersect the boundary. Since Ω is convex and because of the boundary condition for η the boundary level sets $(\partial\Omega)^+_{\eta,t}$ and $(\partial\Omega)^-_{\eta,t}$ are connected and hence

the level sets $\overline{\Omega}^+_{\eta,t}$ and $\overline{\Omega}^-_{\eta,t}$ are (simply) connected. The last claim of the Lemma follows from the fact that

$$\ell_{\eta,t}(\overline{\Omega}) = \partial(\overline{\Omega}^+_{\eta,t}) \cap \partial(\overline{\Omega}^-_{\eta,t}).$$

□

The first component ξ is defined up to a constant by the Beltrami equations

$$\begin{cases} \xi_x = \frac{b\eta_x + c\eta_y}{\sqrt{ac-b^2}} \text{ in } \Omega, \\ \xi_y = -\frac{a\eta_x + b\eta_y}{\sqrt{ac-b^2}} \text{ in } \Omega. \end{cases} \tag{3.9}$$

We fix ξ by $\xi(0,0) = 0$. Since the differential equation in (3.7) implies that $\xi_{xy} = \xi_{yx}$ and since Ω is simply connected, we find that ξ is well defined by (3.9). Denote

$$F(x,y) := (\xi(x,y), \eta(x,y)) \tag{3.10}$$

and $\Omega^* = F(\Omega)$.

Lemma 3.8. *Let Ω, η and ξ be as above. Then the transformation $F : \overline{\Omega} \to \overline{\Omega^*}$ is bijective and its Jacobian satisfies*

$$\mathcal{J}_F > 0 \text{ on } \overline{\Omega}.$$

Proof. Since F is continuous up to the boundary $F(\overline{\Omega}) = \overline{F(\Omega)}$. For the bijectivity note that $\nabla\eta \neq 0$ in $\overline{\Omega}$ implies $\nabla\xi \neq 0$ in $\overline{\Omega}$ and $\mathcal{J}_F > 0$ on $\overline{\Omega}$. Indeed

$$\mathcal{J}_F = \det(\nabla\xi \ \nabla\eta) = \frac{1}{\sqrt{ac-b^2}}\left(a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2\right) > 0 \tag{3.11}$$

implies $\nabla\xi \neq 0$ in $\overline{\Omega}$. The inequality in (3.11) also implies that ξ is strictly monotone on any level line $\ell_{\eta,t}(\overline{\Omega})$. Since we know from the previous lemma that every level line $\ell_{\eta,t}(\overline{\Omega})$ consists of one component, the strict monotonicity of ξ implies that the transformation $(x,y) \to (\xi,\eta)$ is globally bijective. Notice that (3.11) in itself would only imply that the transformation is locally bijective. □

It remains to show that this transformation is close to the identity whenever L_0 is close to $-\Delta$.

Lemma 3.9. *Now suppose additionally that $\partial\Omega$ is $C^{2m,\gamma}$. Let A, w, ξ and η be as in (3.6). For all $\delta > 0$ there is $\varepsilon > 0$ such that if L_0 is ε-close in $C^{2m-1,\gamma}$-sense to $-\Delta$ on $\overline{\Omega}$, then*

$$\|(\xi(\cdot), \eta(\cdot)) - Id\|_{C^{2m,\gamma}} \leq \delta,$$

$$\|A(\xi,\eta) - 1\|_{C^{2m-1,\gamma}} \leq \delta,$$

$$\|w(\xi,\eta)\|_{C^{2m-2,\gamma}} \leq \delta.$$

Proof. First we consider η. Note that $\phi(x,y) = \eta(x,y) - y$ satisfies

$$\begin{cases} M\phi = -\frac{\partial}{\partial x}\left(\frac{b}{\sqrt{ac-b^2}}\right) - \frac{\partial}{\partial y}\left(\frac{c}{\sqrt{ac-b^2}}\right) \text{ in } \Omega, \\ \phi = 0 \text{ on } \partial\Omega, \end{cases}$$

with M as in (3.8). Since L_0 is ε-close in $C^{2m-1,\gamma}$-sense to $-\Delta$ we find that

$$\left\| \frac{\partial}{\partial x}\left(\frac{b}{\sqrt{ac-b^2}}\right) - \frac{\partial}{\partial y}\left(\frac{c}{\sqrt{ac-b^2}}\right) \right\|_{C^{2m-2,\gamma}} = \mathcal{O}(\varepsilon)$$

Since $\partial\Omega$ is $C^{2m,\gamma}$ it follows again by elliptic estimates for second order equations (see [1, Theorem 7.3], [11, chapt. 6.4]) that

$$\|\phi\|_{C^{2m,\gamma}} = \mathcal{O}(\varepsilon). \tag{3.12}$$

Hence $\|\eta_x\|_{C^{2m-1,\gamma}} = \mathcal{O}(\varepsilon)$ and $\|\eta_y - 1\|_{C^{2m-1,\gamma}} = \mathcal{O}(\varepsilon)$. Moreover, using (3.9) we find that $\psi(x,y) = \xi(x,y) - x$ satisfies

$$\begin{aligned} \psi_x &= \tfrac{b}{\sqrt{ac-b^2}}\eta_x + \left(\tfrac{c}{\sqrt{ac-b^2}} - 1\right)\eta_y + (\eta_y - 1), \\ \psi_y &= -\tfrac{a}{\sqrt{ac-b^2}}\eta_x - \tfrac{b}{\sqrt{ac-b^2}}\eta_y \end{aligned}$$

Since

$$\begin{aligned} \left\|\tfrac{a}{\sqrt{ac-b^2}} - 1\right\|_{C^{2m-1,\gamma}} &= \mathcal{O}(\varepsilon), \\ \left\|\tfrac{b}{\sqrt{ac-b^2}} - 0\right\|_{C^{2m-1,\gamma}} &= \mathcal{O}(\varepsilon), \\ \left\|\tfrac{c}{\sqrt{ac-b^2}} - 1\right\|_{C^{2m-1,\gamma}} &= \mathcal{O}(\varepsilon), \\ \|\nabla\eta - (0,1)\|_{C^{2m-1,\gamma}} &= \mathcal{O}(\varepsilon), \end{aligned}$$

we find $\|\nabla\psi\|_{C^{2m-1,\gamma}} = \mathcal{O}(\varepsilon)$. From $\psi(0,0) = 0$ it follows that

$$\|\psi\|_{C^{2m,\gamma}} = \mathcal{O}(\varepsilon). \tag{3.13}$$

From (3.12) and (3.13) we conclude that $\|(\xi(\cdot), \eta(\cdot)) - Id\|_{C^{2m,\gamma}} = \mathcal{O}(\varepsilon)$.

A and w remain to be estimated. With [10] we calculate that

$$\begin{aligned} A &= a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2, \\ w &= \tfrac{1}{A}\begin{pmatrix} a\xi_{xx} + 2b\xi_{xy} + c\xi_{yy} \\ a\eta_{xx} + 2b\eta_{xy} + c\eta_{yy} \end{pmatrix}^{\mathrm{T}}. \end{aligned}$$

Since $\|a-1\|_{C^{2m-1,\gamma}}$, $\|b\|_{C^{2m-1,\gamma}}$, $\|c-1\|_{C^{2m-1,\gamma}}$, $\|\xi_x - 1\|_{C^{2m-1,\gamma}}$, $\|\xi_y\|_{C^{2m-1,\gamma}}$ and hence $\|\xi_{xx}\|_{C^{2m-2,\gamma}}$, $\|\xi_{xy}\|_{C^{2m-2,\gamma}}$, $\|\xi_{yy}\|_{C^{2m-2,\gamma}}$ are all of order $\mathcal{O}(\varepsilon)$ we finally have

$$\begin{aligned} \|A-1\|_{C^{2m-1,\gamma}} &= \mathcal{O}(\varepsilon), \\ \|w\|_{C^{2m-2,\gamma}} &= \mathcal{O}(\varepsilon). \end{aligned}$$

□

Now we switch back to the original notation: $a_{11} = a$, $a_{12} = 2b$, $a_{22} = c$, (x_1, x_2) instead of (x, y).

Corollary 3.10. *Let L be as in* (1.2) *and let $\partial\Omega \in C^{2m,\gamma}$. For all $\delta > 0$ there is $\varepsilon > 0$ such that the following holds. Suppose that the operator L_0 as in* (3.5) *is ε-close in $C^{2m-1,\gamma}$-sense to $-\Delta$. Then there is $F : \bar{\Omega} \to \mathbb{R}^2$ such that*

1. *$F(\Omega)$ is δ-close in $C^{2m,\gamma}$-sense to Ω,*
2. *if u satisfies*
$$\begin{cases} Lu & \geq & 0 & \text{in } \Omega, \\ D_m u & = & 0 & \text{on } \partial\Omega, \end{cases}$$
then $U(F(x)) := u(x)$ satisfies
$$\begin{cases} \tilde{L}U & \geq & 0 & \text{in } F(\Omega), \\ D_m U & = & 0 & \text{on } \partial F(\Omega), \end{cases} \tag{3.14}$$
with
$$\tilde{L} = (-\Delta)^m + \sum_{|\alpha| \leq 2m-1} \tilde{b}_\alpha(x) \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}. \tag{3.15}$$
and
$$\sum_{|\alpha| \leq 2m-1} \left\| b_\alpha - \tilde{b}_\alpha \circ F \right\|_{C^0} < \delta. \tag{3.16}$$

Proof. The transformation F is defined above. The property in *1.* follows from the previous Lemma. The results in *2.* follow from tedious but straigthforward calculus. Indeed we find

$$\left(\sum_{1 \leq i \leq j \leq 2} -a_{ij} \tfrac{\partial}{\partial x_i} \tfrac{\partial}{\partial x_j} \right)^m U(F(x)) =$$

$$= \left(\sum_{1 \leq i \leq j \leq 2} -a_{ij} \tfrac{\partial}{\partial x_i} \tfrac{\partial}{\partial x_j} \right)^{m-1} A \left((-\Delta U) \circ F(x) - w \cdot \left((\nabla U) \circ F(x) \right) \right) =$$

$$= A^m \left(\left((-\Delta)^m U \right) \circ F(x) \right) + \sum_{|\alpha| \leq 2m-1} g_\alpha(\cdot) \left(\left(\tfrac{\partial}{\partial x} \right)^\alpha U \right) \circ F(x).$$

The coefficients g_α have at least one derivative of A of order between 1 and $2m-2$ as a factor. Furthermore, in the g_α appear derivatives up to order $2m-2$ of w, a_{ij} and F. Since $\left\| \left(\frac{\partial}{\partial x}\right)^\beta A \right\|_{C^0} = \mathcal{O}(\varepsilon)$ for all β with $1 \leq |\beta| \leq 2m-2$ the claim in (3.16) follows with help of Lemma 3.9. □

Proof of Theorem 1.3. Combine the results of Lemma 2.5, Proposition 2.6 and Corollary 3.10. □

A Appendix; on a result by Carleman, Hartman and Wintner

The next theorem is a corollary of the Carleman-Hartman-Wintner Theorem (see [20]). Their result has no direct extension to higher dimensional domains. For the sake of completeness we include a proof.

Theorem A.11. *Let Ω be a domain in $\mathbb{R}^2$ and let*

$$L = \sum_{1\le i\le j\le 2} a_{ij}(.)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{1\le j\le 2} b_j(.)\frac{\partial}{\partial x_j}$$

be uniformly elliptic on $\overline{\Omega}$ with $a_{ij}, b_j \in C^2(\overline{\Omega})$. Suppose that $\phi \in C^2(\overline{\Omega})$ satisfies $L\phi = 0$ in Ω and is non-constant. If $\hat{p} \in \Omega$ is such that $\nabla\phi(\hat{p}) = 0$, then $\hat{p}$ is an isolated singularity:

$$\text{there exists } r > 0 \text{ such that } \overline{B_r(\hat{p})} \subset \Omega \text{ and } \nabla\phi \neq 0 \text{ on } \overline{B_r(\hat{p})} \setminus \{\hat{p}\},$$

and moreover

$$\deg(\nabla\phi, B_r(\hat{p}), 0) < 0.$$

Proof. From the uniform ellipticity of L it follows that there exist $\lambda_1, \lambda_2 > 0$ and an orthogonal matrix Q, with $\det Q = 1$, such that

$$Q^{-1}\begin{pmatrix} a_{11}(\hat{p}) & \frac{1}{2}a_{12}(\hat{p}) \\ \frac{1}{2}a_{12}(\hat{p}) & a_{22}(\hat{p}) \end{pmatrix} Q = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

With the transformation $U : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$U(z_1, z_2) = \left(Q \begin{pmatrix} (\lambda_1)^{\frac{1}{2}} & 0 \\ 0 & (\lambda_2)^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \end{pmatrix} \right)^T$$

we find that $\varphi(z) := \phi(Uz) - \phi(\hat{p})$ satisfies a uniformly elliptic equation $\hat{L}\varphi = 0$ on $U^{inv}\Omega$ with the operator $\hat{L}$ satisfying $\hat{a}_{11}(0) = \hat{a}_{22}(0) = 1$, $\hat{a}_{12}(0) = 0$. Moreover $\varphi(0) = 0$ and $\nabla\varphi(0) = 0$. Hence we may use the Carleman-Hartman-Wintner Theorem (see [20, Th. 7.2.1]). We will also use the result [20, Th. 7.2.4]. Since $\varphi(z) = o(|z|)$ as $|z| \to 0$ it follows that either $\varphi(z) \equiv 0$ on $U^{inv}\Omega$, or there exists $m \in \mathbb{N}^+$ with

$$\lim_{|z|\to 0} \frac{\varphi_{z_1} - i\varphi_{z_2}}{(z_1 + iz_2)^m} = \alpha \in \mathbb{C} \setminus \{0\}. \tag{1.17}$$

If $\varphi(z) \equiv 0$ on $U^{inv}\Omega$ then $\phi(x) \equiv \phi(\hat{p})$ on Ω. Now suppose that $\varphi(z) \not\equiv 0$. Then there is $r^* > 0$ with $\overline{B_{r^*}(0)} \subset U^{inv}\Omega$ and $\nabla\varphi(z) \neq 0$ for $z \in \overline{B_{r^*}(0)} \setminus \{0\}$. Hence 0 is an isolated zero of $\nabla\varphi$ and a homotopy argument shows that

$$\begin{aligned} &\deg(\nabla\varphi, B_{r^*}(0), 0) = \\ &= \deg((\mathrm{Re}(\alpha(z_1 + iz_2)^m), -\mathrm{Im}(\alpha(z_1 + iz_2)^m)), B_{r^*}(0), 0), \end{aligned}$$

implying that $\deg\left(\nabla\varphi, B_{r^*}(0), 0\right) = -m < 0$. Now take a ball $B_r(\hat{p})$ such that $B_r(\hat{p}) \subset UB_{r^*}(0)$. Since $\nabla\varphi \neq 0$ on $\overline{B_{r^*}(0)} \setminus \left(U^{inv} B_r(\hat{p})\right)$ we have

$$\deg\left(\nabla\varphi, B_{r^*}(0), 0\right) = \deg\left(\nabla\varphi, U^{inv}\left(B_r(\hat{p})\right), 0\right).$$

According to Heinz [15, Lemma 7], this degree may be defined as follows. For some $\varepsilon > 0$ we have $|\nabla\varphi| \geq \varepsilon$ on $\partial\left(U^{inv}\left(B_r(\hat{p})\right)\right)$. Any $\chi \in C^0(\mathbb{R}^2, \mathbb{R})$ with $\mathrm{supp}\chi \subset B_{c\varepsilon}(0)$, $c \in (0,1)$ to be chosen below, and $\int_{\mathbb{R}^2} \chi(z)\, dz = 1$ is a normalized admissible testing function for $\nabla\varphi$. We have

$$\begin{aligned}
-m &= \deg\left(\nabla\varphi, U^{inv}\left(B_r(\hat{p})\right), 0\right) = \\
&= \int\limits_{U^{inv}(B_r(\hat{p}))} \chi\left(\nabla\varphi(z)\right) \det\begin{pmatrix} \varphi_{11}(z) & \varphi_{12}(z) \\ \varphi_{21}(z) & \varphi_{22}(z) \end{pmatrix} dz = \\
&= \int\limits_{U^{inv}(B_r(\hat{p}))} \chi\left(\nabla\phi(Uz)\ U'\right) \det\left((U')^{\mathrm{T}} \begin{pmatrix} \phi_{11}(Uz) & \phi_{12}(Uz) \\ \phi_{21}(Uz) & \phi_{22}(Uz) \end{pmatrix} U'\right) dz = \\
&= \int\limits_{B_r(\hat{p})} \chi\left(\nabla\left(\phi(p)\right)\ U'\right) \det\begin{pmatrix} \phi_{11}(p) & \phi_{12}(p) \\ \phi_{21}(p) & \phi_{22}(p) \end{pmatrix} \left|\det(U')\right|\ dp.
\end{aligned}$$

Here

$$U' = Q\begin{pmatrix} \lambda_1^{\frac{1}{2}} & 0 \\ 0 & \lambda_2^{\frac{1}{2}} \end{pmatrix}.$$

We note that $\det U' > 0$ and introduce

$$\tilde{\chi} \in C^0(\mathbb{R}^2, \mathbb{R}),\ \tilde{\chi}(p) := \det(U')\ \chi(pU').$$

As

$$\int_{\mathbb{R}^2} \tilde{\chi}(p)\, dp = \int_{\mathbb{R}^2} \det(U')\ \chi(pU')\ dp = \int_{\mathbb{R}^2} \chi(z)\, dz,$$

$\tilde{\chi}$ is a normalized admissible testing function for $\nabla\phi$, provided that c above has been chosen sufficiently small. We conclude by

$$0 > -m = \int_{B_r(\hat{p})} \tilde{\chi}\left(\nabla\phi(p)\right) \det\begin{pmatrix} \phi_{11}(p) & \phi_{12}(p) \\ \phi_{21}(p) & \phi_{22}(p) \end{pmatrix} dp = \deg\left(\nabla\phi, B_r(\hat{p}), 0\right).$$

□

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References

[1] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I, Commun. Pure Appl. Math. **12** (1959), 623-727.

[2] L. V. Ahlfors, Conformality with respect to Riemannian metrics, Ann. Acad. Sci. Fenn., Ser. AI **206** (1955), 1-22.

[3] T. Boggio, Sull'equilibrio delle piastre elastiche incastrate, Rend. Acc. Lincei **10** (1901), 197-205.

[4] ———, Sulle funzioni di Green d'ordine m, Rend. Circ. Mat. Palermo **20** (1905), 97-135.

[5] C. V. Coffman, On the structure of solutions to $\Delta^2 u = \lambda u$ which satisfy the clamped plate condition on a right angle, SIAM J. Math. Anal. **13**, 746-757 (1982).

[6] C. V. Coffman and R. J. Duffin, On the structure of biharmonic functions satisfying the clamped plate conditions on a right angle, Adv. Appl. Math. **1** (1980), 373-389.

[7] R. Courant, Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces, Interscience, New York, 1950.

[8] R. J. Duffin, On a question of Hadamard concerning super-biharmonic functions, J. Math. Phys. **27** (1949), 253-258.

[9] P. R. Garabedian, A partial differential equation arising in conformal mapping, Pacific J. Math. **1** (1951), 485-524.

[10] ———, Partial Differential Equations, second edition, Chelsea, New York, 1986.

[11] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd edition, Springer-Verlag, Berlin etc., 1983.

[12] H.-Ch. Grunau and G. Sweers, Positivity for equations involving polyharmonic operators with Dirichlet boundary conditions, to appear in Math. Ann.

[13] J. Hadamard, Mémoire sur le problème d'analyse relatif à l'équilibre des plaques élastiques incastrées, Mémoires présentés par divers savants a l'Académie des Sciences, Vol. 33 (1908), 1-128.

[14] ———, Sur certains cas intéressants du problème biharmonique, Atti IVe Congr. Intern. Mat. Rome (1909), 12-14.

[15] E. Heinz, An elementary analytic theory of the degree of mapping in n-dimensional space, J. Math. Mech. **8** (1959), 231-247.

[16] V. A. Kozlov, V. A. Kondrat'ev and V. G. Maz'ya, On sign variation and the absence of "strong" zeros of solutions of elliptic equations, Math. USSR Izvestiya **34** (1990), 337-353.

[17] Ch. Loewner, On generation of solutions of the biharmonic equation in the plane by conformal mappings, Pacific J. Math. **3** (1953), 417-436.

[18] S. Osher, On Green's function for the biharmonic equation in a right angle wedge, J. Math. Anal. Appl. **43** (1973), 705-716.

[19] Ch. Pommerenke, Boundary behaviour of conformal maps, Springer-Verlag, Berlin etc., 1992.

[20] F. Schulz, Regularity theory for quasilinear elliptic systems and Monge-Ampère equations in two dimensions, Lecture Notes in Math. 1445, Springer-Verlag, Berlin etc., 1990.

[21] H. A. Schwarz, Notizia sulla rappresentazione conforme di un' area ellittica sopra un' area circolare, Gesammelte Abhandlungen, 1890; reprint of Annali di Matematica pura ed applicata II^{a} serie, tomo III°, 1869-1870, pp 166-170.

[22] H. S. Shapiro and M. Tegmark, An elementary proof that the biharmonic Green function of an eccentric ellipse changes sign, SIAM Rev. **36** (1994), 99-101.

[23] G. Sweers, Positivity for a strongly coupled elliptic system by Green function estimates, J. Geometric Analysis **4** (1994), 121-142.

[24] G. Szegö, Remarks on the preceeding paper of Charles Loewner, Pacific J. Math. **3** (1953), 437-446.

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