The Plancherel theorem for a reductive symmetric space

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§0. Introduction

1. In these notes, G is a reductive Lie group, i.e., its Lie algebra $\mathfrak{g}$ is a real reductive Lie algebra. At a later stage we will impose the restrictive condition that G belongs to a certain class $\mathcal{R}$, see appendix.

We assume σ to be an involution of G, i.e., $\sigma \in \mathrm{Aut}(G)$ and $\sigma^2 = I$. Moreover, H is an open subgroup of G^σ, or, equivalently, H is a subgroup with

$$(G^\sigma)_e \subset H \subset G^\sigma .$$

The homogeneous space $X := G/H$ is called a reductive symmetric space.

2. The reason for this terminology is the following. Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ also denote the derivative of σ at the identity element e. Then σ is an involution of the Lie algebra $\mathfrak{g}$, which therefore decomposes as the direct sum

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}\,, \tag{2-1}$$

where $\mathfrak{h}, \mathfrak{q}$ are the $+1$ and -1 eigenspaces for σ. We note that $\mathfrak{h} = \mathrm{Lie}(H)$ and that (2-1) is $\mathrm{Ad}(H)$-invariant.

It can be shown that there exists a non-degenerate indefinite inner product β_e on $\mathfrak{q}$, which is H-invariant. Indeed, if $\mathfrak{g}$ is semisimple then the restriction of the Killing form has this property; in general one may take β_e to be a suitable extension of the Killing form from $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{q}$ to $\mathfrak{q}$. From $T_{eH}(G/H) \simeq \mathfrak{g}/\mathfrak{h} \simeq \mathfrak{q}$ and the H-invariance of β_e it follows that β_e induces a G-invariant pseudo-Riemannian metric on G/H by the formula

$$\beta_{gH} := (\ell_g^{-1})^* \beta_e \qquad (g \in G)\,.$$

The natural map $\overline{\sigma} : G/H \to G/H$, $gH \mapsto \sigma(g)H$ is the geodesic reflection in the origin eH for the metric β. By homogeneity it follows that the (locally defined) geodesic reflection S_x at any point $x \in X$ extends to a global isometry. A space with this property is called symmetric. For a more general definition of symmetric space we refer to p. 98 of [S].

3. The following are motivating and guiding examples

Example 3.1 $\sigma = \theta$ a Cartan involution, then $H = K$ is maximally compact and $X = G/K$ is a *Riemannian symmetric space*. See [S], p. 101, Ex. 1.8.

Example 3.2 Let ${}^{\backprime}G$ be a reductive group, then $G = {}^{\backprime}G \times {}^{\backprime}G$ is reductive as well. G acts transitively on ${}^{\backprime}G$ by the left times right action given by $(g_1, g_2) \cdot x = g_1 x g_2^{-1}$. The stabilizer of ${}^{\backprime}e$ in G equals $H = \text{diagonal}\,({}^{\backprime}G \times {}^{\backprime}G)$. Hence

$${}^{\backprime}G \simeq G/H\,.$$

Moreover, $H = G^\sigma$, where σ is the involution of G defined by $(g_1, g_2) \mapsto (g_2, g_1)$.

See also [S], Example 1.5, p. 99.

Example 3.3 **The real hyperbolic spaces**. Let $p, q \geq 1$ be integers, and put $n = p + q$. We agree to write $x = (x', x'')$ according to $\mathbb{R}^n \simeq \mathbb{R}^p \times \mathbb{R}^q$. Let $(\cdot\,,\cdot)$ denote the standard inner products on $\mathbb{R}^p$ and $\mathbb{R}^q$ and define the indefinite inner product β on $\mathbb{R}^n$ by

$$\beta(x, y) = (x', y') - (x'', y'')\,.$$

We define $X_{p,q} := \{x \in \mathbb{R}^n \mid \beta(x,x) = 1\}$. Thus $X_{p,q}$ consists of $x \in \mathbb{R}^n$ with:

$$x_1^2 + \cdots x_p^2 - x_{p+1}^2 - \cdots - x_n^2 = 1 \,.$$

Moreover, if $p = 1$, we impose the additional condition $x_1 > 0$ to ensure that $X_{p,q}$ is connected. In the latter case we may visualize $X_{p,q}$ as follows

In case $p > 1$ we may visualize $X_{p,q}$ as follows

The stabilizer of β in $\mathrm{SL}(n,\mathbb{R})$ is denoted by $\mathrm{SO}(p,q)$. Its identity component $\mathrm{SO}_e(p,q)$ acts transitively on $X_{p,q}$. Moreover, the stabilizer of $e_1 = (1,0,\ldots,0)$ equals $\mathrm{SO}_e(p-1,q)$, so that

$$X_{p,q} \simeq \mathrm{SO}_e(p,q)/\mathrm{SO}_e(p-1,q) \,.$$

We define a pseudo-Riemannian structure $\overline{\beta}$ on $X_{p,q}$ by

$$\overline{\beta}_x = \beta \,\Big|\, T_x X_{p,q} \,.$$

Clearly, $\overline{\beta}$ is $\mathrm{SO}_e(p,q)$-invariant. Moreover, from

$$T_{e_1} X_{p,q} \simeq \mathbb{R}^{p-1} \times \mathbb{R}^q$$

we read off that $\overline{\beta}$ has signature $(p-1,q)$. Thus, if $p=1$ then $X_{p,q}$ is Riemannian; if $p>1$ then $X_{p,q}$ is pseudo-Riemannian, and one can show that $X_{p,q}$ lies outside the range of the examples 3.1 and 3.2 except case $p=q=2$. See also [S], Example 1.6.

4. Exercise. Determine all geodesics on $X_{p,q}$. Determine $\overline{\sigma}$, the geodesic reflection in e_1.

5. Invariant measure. The groups G and H are unimodular, since $|\det \mathrm{Ad}(\cdot)| = 1$ on G and on H. It follows that $X = G/H$ carries a G-invariant measure, which we denote by dx. The associated space of square integrable functions on X is denoted by $L^2(X) = L^2(X, dx)$. The left regular representation L of G in $L^2(X)$, given by $L_g f(x) = f(g^{-1}x)$, for $f \in L^2(X)$, $x \in X$, $g \in G$, is unitary, because dx is G-invariant.

6. Scope of these lectures. The Plancherel theorem for X describes the decomposition of $(L, L^2(X))$ as a direct integral of unitary representations

$$(L, L^2(X)) \simeq \int_{\widehat{G}}^{\oplus} m_\pi \, \pi \, d\mu(\pi) \, .$$

In the next section we will describe the meaning of the above formula in more detail.

From the examples 3.1 and 3.2 it follows that the Plancherel theorem for reductive symmetric spaces includes both the Plancherel theorem for Riemannian symmetric spaces and the Plancherel theorem for real reductive groups. In the Riemannian case the Plancherel theorem was obtained by Harish–Chandra and Helgason ([32] and [40]) and in the group case it was obtained by Harish-Chandra [34–36]. Here we use the reference numbers from [BS].

For the hyperbolic spaces the Plancherel theorem was obtained by Molchanov, see Ref. 147 in [S]. In other special cases the Plancherel formula was obtained by Faraut, van Dijk, Harinck, Bopp. We refer to [S] for precise references.

In these notes I will discuss an explicit as possible Plancherel theorem for reductive symmetric spaces formulated in terms of their general structure theory. Such a result was announced by Oshima in the 80's, but not published. The most continuous part of the Plancherel theorem was described in joint work of Henrik Schlichtkrull and myself (B & S), [15]. In the fall of 1995, during the special year at the Mittag-Leffler Institute near Stockholm, Sweden, Patrick Delorme and B & S independently announced to have found a proof for the general Plancherel theorem. At the same time B & S announced a proof of the Paley–Wiener theorem as well. In the original announcement B & S needed Delorme's result on the so-called Maass–Selberg relations. In the meantime they have found an independent proof. The two now existing proofs of the Plancherel theorem are very different. In these notes we will mainly focus on the proof of B & S, but we will also hint at aspects of Delorme's proof. Delorme's proof has appeared in '98, see [29]. A Fourier inversion theorem by B & S, established in [17], will be the basis of their proofs of both the Plancherel and the Paley–Wiener theorem, which still have to appear, see [19]. For more details on the history, see §1 in [BS].

7. Outline. In the next section we will describe the structure of the Plancherel theorem in terms of general functional analysis, based on L. Schwartz' notion of Hilbert subspace.

We then proceed by discussing structure theory for reductive symmetric spaces, such as the Cartan decomposition, invariant differential operators, parabolic subgroups, induced representations. After this we introduce the explicit Fourier transform, and we are then able to state the Plancherel theorem (§6, n^o 25).

We finish these lectures by discussing some important concepts and ideas of the proof, such as Eisenstein integrals, Maass–Selberg relations and the spherical Fourier transform.

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§1. Direct integral decomposition

1. To motivate our view towards direct integral decomposition of $L^2(X)$, we recall some classical examples.

2. Fourier series. From the representation theoretic point of view the theory of Fourier series may be described as follows. Let $G = \mathbb{R}/2\pi\mathbb{Z}$, $H = \{0\}$, then $X = G/H \simeq \mathbb{R}/2\pi\mathbb{Z}$. Let $dx/2\pi$ denote translation invariant measure on X, normalized by $\int_X \frac{dx}{2\pi} = 1$. For $n \in \mathbb{Z}$ let $L^2(X)_n = \langle e^{in\cdot}\rangle$, the one-dimensional space spanned by the exponential function $x \mapsto e^{inx}$. Then

$$L^2(X) = \widehat{\bigoplus_{n\in\mathbb{Z}}} L^2(X)_n \, ,$$

the sum being orthogonal and G-invariant. The projection operator onto $L^2(X)_n$ is given by $f \mapsto \hat{f}(n)e^{in\cdot}$, wherein $f \mapsto \hat{f}$, the Fourier transform, is given by

$$\hat{f}(n) = \langle f \mid e^{in\cdot}\rangle_{L^2(X)} = \int_0^{2\pi} f(x)e^{-inx}\frac{dx}{2\pi}\,.$$

The Plancherel theorem asserts that the Fourier transform is an isometry from $L^2(X)$ onto $\ell^2(\mathbb{Z})$ (whence the Parseval identities); in other words, the Fourier transform is inverted by its adjoint $\mathcal{I}$. The inverse transform $\mathcal{I}$ is given by

$$(c_n)_{n\in\mathbb{Z}} \mapsto \sum_{n\in\mathbb{Z}} c_n e^{in\cdot}\,.$$

3. The Peter–Weyl theorem generalizes the theory of Fourier series to the case where $X = G$ is a compact group. Let dx be normalized Haar measure, i.e. $\int_G dx = 1$.

Let $\widehat{G}$ be the set of (equivalence classes of) irreducible unitary representations of G. Then we have the G-invariant orthogonal direct sum decomposition

$$L^2(G) = \widehat{\bigoplus_{\delta\in\widehat{G}}} L^2(G)_\delta\,, \tag{3-1}$$

where $L^2(G)_\delta$ can be described as follows. Let V_δ be a finite dimensional Hilbert space in which δ is realized. Then $L^2(G)_\delta$ is the image of the map $\mathrm{M}_\delta : \mathrm{End}(V_\delta) \to C^\infty(G)$ given by

$$\mathrm{M}_\delta(T)(x) = d_\delta \ \mathrm{tr}\,(\delta(x)^{-1}\circ T) \qquad (T \in \mathrm{End}(V_\delta), x \in G)\,.$$

Note that M_δ intertwines the representation $\delta\otimes\delta^*$ on $\mathrm{End}(V_\delta) \simeq V_\delta \otimes V_\delta^*$ with the representation $L\otimes R$ of $G\times G$ on $L^2(G)$. The latter is unitary because dx is bi-G-invariant.

The orthogonal projection $P_\delta : L^2(X) \to L^2(X)_\delta$ is given by

$$P_\delta(f) = \mathrm{M}_\delta(\hat{f}(\delta))\,, \tag{3-2}$$

where $\hat{f}$, the Fourier transform of f, is given by

$$\hat{f}(\delta) = \delta(f) = \int_G f(x)\delta(x)dx \in \mathrm{End}(V_\delta)\,,$$

for $\delta \in \widehat{G}$. We equip $\mathrm{End}(V_\delta)$ with d_δ times the Hilbert-Schmid (or tensor) inner product, i.e. $\langle T_1 \mid T_2 \rangle = d_\delta \,\mathrm{tr}\,(T_1 T_2^*)$.

The Plancherel theorem asserts that $f \mapsto \hat{f}$ is an isometry

$$L^2(X) \xrightarrow{\ \simeq\ } \widehat{\bigoplus_{\delta \in \widehat{G}}} \mathrm{End}(V_\delta)\,.$$

Thus, again, the inverse transform $\mathcal{I}$ is given by the transpose of the Fourier transform;

$$\mathcal{I}((T_\delta \mid \delta \in \widehat{G})) = \sum_{\delta \in \widehat{G}} \mathrm{M}_\delta(T_\delta)\,,$$

in accordance with (3-1).

We note that the Fourier transform intertwines the representation $L \otimes R$ of $G \times G$ in $L^2(G)$ with the direct sum of the representations $\delta \otimes \delta^*$, for $\delta \in \widehat{G}$.

We end this discussion with a different description of the map M_δ.

If V is a complex linear space then by $\overline{V}$ we denote its conjugate. Thus, as a real linear space $\overline{V}$ equals V, but the complex multiplication is given by $(z, v) \mapsto \bar{z}v$, $\mathbb{C} \times \overline{V} \to \overline{V}$.

A sesquilinear inner product $\langle \cdot \mid \cdot \rangle_V$ on V may now be viewed as a complex bilinear map $V \times \overline{V} \to \mathbb{C}$. If V is a Hilbert space for $\langle \cdot \mid \cdot \rangle_V$, and if $c > 0$ is a constant, then the map $\eta \mapsto c\langle \cdot \mid \eta \rangle_V$ induces a linear isomorphism from $\overline{V}$ onto the dual Hilbert space V^*.

We now agree to identify V_δ^* with $\overline{V}_\delta$ via $d_\delta^{-1}\langle \cdot \mid \cdot \rangle_{V_\delta}$. In the corresponding identification of $\mathrm{End}(V_\delta) \simeq V_\delta \otimes V_\delta^*$ with $V_\delta \otimes \overline{V}_\delta$ we then have that $v \otimes \eta \in V_\delta \otimes \overline{V}_\delta$ corresponds with the operator $T(u) = d_\delta^{-1}\langle u \mid \eta \rangle_{V_\delta} v$. Hence M_δ may be described as the matrix coefficient map $V_\delta \otimes \overline{V}_\delta \to C^\infty(G)$ given by

$$\mathrm{M}_\delta(v \otimes \eta)(x) = \langle v \mid \delta(x)\eta \rangle_{V_\delta}\,, \tag{3-3}$$

for $v \in V_\delta$, $\eta \in \overline{V}_\delta$, $x \in G$.

Finally we note that when $\mathrm{End}(V_\delta)$ is equipped with d_δ times the Hilbert-Schmid inner product, the identification of $\mathrm{End}(V_\delta)$ with $V_\delta \otimes \overline{V}_\delta$ is an isometry if $\overline{V}_\delta$ is equipped with the inner product $(v, w) \mapsto d_\delta^{-1}\langle w \mid v \rangle_{V_\delta}$ and $V_\delta \otimes \overline{V}_\delta$ with the tensor product inner product.

4. Compact homogeneous spaces. Let G be a compact group and $H < G$ a closed subgroup. Put $X = G/H$ and let dx be normalized Haar measure of X. Then we may identify $L^2(X)$ with the subspace $L^2(G)^H$ of right-H-invariant functions in $L^2(G)$.

Let $\mathrm{M}_{X,\delta} = \mathrm{M}_\delta \mid V_\delta \otimes \overline{V}_\delta^H$. Then from the right equivariance of M_δ it follows that $\mathrm{M}_{X,\delta}$ maps $V_\delta \otimes \overline{V}_\delta^H$ bijectively onto

$$L^2(X)_\delta := L^2(G)_\delta \cap L^2(G)^H .$$

Moreover, M_δ intertwines the G-representations $\delta \otimes 1$ and L.

Let

$$\widehat{G}_H = \{\delta \in \widehat{G} \mid \overline{V}_\delta^H \neq 0\} .$$

Then it follows from the right equivariance of (3-1) that

$$L^2(X) = \widehat{\bigoplus_{\delta \in \widehat{G}_H}} L^2(X)_\delta ,$$

where the orthogonal projection $P_\delta : L^2(X) \to L^2(X)_\delta$ is now given by $P_\delta(f) = \mathrm{M}_{X,\delta}(\hat{f}_X(\delta))$, and where now $\hat{f}_X$ is defined by

$$\hat{f}_X(\delta) = \int_X f(x)\delta(x)\Big|_{V_\delta^H} dx \in \mathrm{Hom}\left(V_\delta^H, V_\delta\right) \simeq V_\delta \otimes \overline{V}_\delta^H .$$

Here we identify $(V_\delta^H)^* \simeq \overline{V_\delta^H}$ via $\dim(\delta)^{-1}\langle\cdot \mid \cdot\rangle$. From the Peter–Weyl theorem we deduce that the Fourier transform defines an isometry

$$L^2(X) \xrightarrow{\ \simeq\ } \widehat{\bigoplus_{\delta \in \widehat{G}_H}} V_\delta \otimes \overline{V}_\delta^H .$$

Here $\overline{V}_\delta^H$ is equipped with $\dim(\delta)^{-1}$ times the inner product from $\overline{V}_\delta$ and $V_\delta \otimes \overline{V}_\delta^H$ with the tensor inner product. The inverse transform $\mathcal{I}_X$ is the adjoint of $f \mapsto \hat{f}_X$ and given by

$$\mathcal{I}_X(T) = \sum_{\delta \in \widehat{G}_H} \mathrm{M}_{X,\delta}(T_\delta)$$

for $T \in \widehat{\bigoplus}_{\delta \in \widehat{G}_H} V_\delta \otimes \overline{V}_\delta^H$. Put $m_\delta = \dim \overline{V}_\delta^H$, then from the above discussion it follows that

$$(L, L^2(x)) \simeq \bigoplus_{\delta \in \widehat{G}_H} m_\delta \, \delta .$$

5. Remark. If G is a non-compact reductive group and H a non-compact closed subgroup, then one cannot make the identification $L^2(G/H) \simeq L^2(G)^H$

so that the Plancherel theorem for G/H cannot be obtained from that for G. In particular, the Plancherel theorem for a non-Riemannian reductive symmetric space G/H cannot be obtained in the above fashion from Harish–Chandra's formula for the group.

6. Compact symmetric spaces. In 4, assume in addition that G is a connected semisimple Lie group and $H = G^\sigma$, σ an involution. Then $X = G/H$ is a compact symmetric space. In this case it is known that $\dim \overline{V}_\delta^H = 1$ for $\delta \in \widehat{G}_H$. Hence $(L, L^2(X))$ admits the multiplicity free decomposition

$$L \sim \bigoplus_{\delta \in \widehat{G}_H} \delta \,.$$

7. If $G = \mathrm{SO}(n+1)$, $H = \mathrm{SO}(n)$, then $X = S^n$ and one arrives in the theory of spherical harmonics, see [S], p. 97.

8. The Peter–Weyl theorem gives rise to two ways of viewing the Plancherel theorem for a compact group ${}^\backprime G$.

a) One may view ${}^\backprime G$ as the homogeneous space $X = G/H$ with $G = {}^\backprime G$, $H = \{e\}$. Then $\widehat{G}_H = {}^\backprime\widehat{G}$, $\overline{V}_\delta^H = \overline{V}_\delta$ for all $\delta \in {}^\backprime\widehat{G}$ and we see that

$$L \sim \bigoplus_{\delta \in {}^\backprime\widehat{G}} \dim(\delta)\delta \,.$$

Thus, the Plancherel decomposition is not multiplicity free.

b) One may also view ${}^\backprime G$ as $X = G/H$, with $G = {}^\backprime G \times {}^\backprime G$ and $H =$ diagonal $({}^\backprime G \times {}^\backprime G)$; i.e. ${}^\backprime G$ is viewed as a symmetric space for ${}^\backprime G \times {}^\backprime G$, via the left times right action.

Exercise. Show that $L_X = {}^\backprime L \otimes {}^\backprime R$, where ${}^\backprime L$ and ${}^\backprime R$ denote the left and right regular representations of ${}^\backprime G$. Show that

$$\widehat{G}_H = \{\pi \otimes \pi^* \mid \pi \in {}^\backprime\widehat{G}\} \,.$$

Show: if $\delta \in \widehat{G}_H$, then $\dim \overline{V}_\delta^H = 1$; determine $\overline{V}_\delta^H$. Interpret the Peter–Weyl theorem of 3 in the context of 5, 6. In particular, show that L_X decomposes *multiplicity free.*

9. *If X is a symmetric space for a non-compact reductive group G, one cannot expect the Plancherel decomposition to correspond to a direct sum decomposition of $L^2(X)$ into irreducible invariant subspaces.*

This is apparent from the classical example $G = \mathbb{R}^n$, $H = \{0\}$, $X = \mathbb{R}^n$. The irreducible unitary representations of G are all 1-dimensional and given by $\pi_\xi : G \times \mathbb{C} \to \mathbb{C}$, $(x, z) \mapsto e^{\xi(x)}Z$, with $\xi \in i(\mathbb{R}^n)^*$. Fix a choice of Lebesgue measure dx on $\mathbb{R}^n$, then there is a Fourier transform $f \mapsto \hat{f}$ given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\overline{e^{\xi(x)}}dx\,,$$

for $f \in C_c(\mathbb{R}^n)$. The Fourier inversion theorem asserts that there exists a (unique) normalization $d\xi$ of Lebesgue measure on $i(\mathbb{R}^n)^* \simeq \widehat{G}$ such that $f \mapsto \hat{f}$ extends to an isometry

$$\mathcal{F} : L^2(\mathbb{R}^n, dx) \xrightarrow{\ \simeq\ } L^2(i(\mathbb{R}^n)^*, d\xi)\,.$$

In particular, the inverse $\mathcal{I}$ of the Fourier transform is given by its adjoint:

$$\mathcal{I}_\varphi(x) = \int_{i(\mathbb{R}^n)^*} \varphi(\xi)e^{\xi(x)}d\xi\,, \tag{9-1}$$

for $\varphi \in C_c(i(\mathbb{R}^n)^*)$. The Fourier transform has the following intertwining property

$$(L_a f)^\wedge(\xi) = \pi_\xi(a)\hat{f}(\xi) \qquad (\xi \in i\mathbb{R}^{n*}, a \in \mathbb{R}^n)\,.$$

From $\mathcal{I} \circ \mathcal{F} = I$ and (9-1) we obtain, at least for $f \in C_c^\infty(\mathbb{R}^n)$, the inversion formula

$$f(x) = \int_{i(\mathbb{R}^n)^*} \hat{f}(\xi)e^{\xi(x)}d\xi\,, \quad (x \in \mathbb{R}^n)\,,$$

which exhibits f as a superposition of the functions $f_\xi : x \mapsto \hat{f}(\xi)e^{\xi(x)}$, $\xi \in i(\mathbb{R}^n)^*$, which are however not contained in $L^2(\mathbb{R}^n)$, but in the space $C^{-\infty}(\mathbb{R}^n)$ of generalized functions on $\mathbb{R}^n$. (Of course here $f_\xi \in C^\infty(\mathbb{R}^n)$, but in a more general setting $f_\xi \in C^{-\infty}(X)$ will be the natural point of view).

10. Generalized functions. If M is a smooth manifold, we distinguish between $\mathcal{D}'(M) = (C_c^\infty(M))'$, the space of distributions on M, and $C^{-\infty}(M)$, the space of generalized functions on M. The latter is defined as the continuous linear dual of the space of compactly supported smooth densities on M, see [B] for details.

Considering generalized functions has the advantage that

$$C^\infty(M) \hookrightarrow C^{-\infty}(M)$$

naturally. If M is a G-homogeneous space, and dm a G-invariant density on M, then the map $\varphi \mapsto \varphi\, dm$, $\varphi \in C_c^\infty(M)$, induces a continuous linear G-equivariant isomorphism $C^{-\infty}(M) \to \mathcal{D}'(M)$, by transposition.

11. The abstract Plancherel theorem. Let G be a reductive group of class $\mathcal{R}$, let H be as in §0.1, and let dx be invariant measure on $X := G/H$. The goal of Fourier theory on X may be described as follows, using Schwartz' notion of Hilbert subspace.

We note that the left regular representation of G in $C^\infty(X)$ extends naturally to a representation L of G in $C^{-\infty}(X)$.

11.1 Definition. By a *Hilbert subspace* of $(C^{-\infty}(X), L)$ we mean a unitary representation $(\mathcal{H}, \pi)$ of G together with a G-intertwining continuous linear embedding $m : \mathcal{H} \hookrightarrow C^{-\infty}(X)$.

The Hilbert subspace is called irreducible if the representation π is irreducible.

11.2 The space $L^2(X)$ is a Hilbert subspace of $C^{-\infty}(X)$. There is a general theory of functional analysis, due to Schwartz, which allows a 'desintegration' of any Hilbert subspace into a superposition of irreducibles, of course under suitable conditions on G, H. We shall not go deeply into this, but will merely use this view point to describe our ultimate goal.

11.3 Definition. Let π be a continuous representation of G in a Hilbert space $\mathcal{H}$. A vector $v \in \mathcal{H}$ is called *smooth* if the map $G \to \mathcal{H}$, $x \mapsto \pi(x)v$ is C^∞. The space of smooth vectors is denoted by $\mathcal{H}^\infty$. It is a natural representation space for G and $\mathfrak{g}$, hence for $\mathcal{U}(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$. The space $\mathcal{H}^\infty$ is equipped with a Fréchet topology by means of the seminorms

$$\|\cdot\|_U : v \mapsto \|U \cdot v\|_\mathcal{H}, \quad (U \in \mathcal{U}(\mathfrak{g}))\,.$$

The continuous linear dual of the conjugate Fréchet space $\overline{\mathcal{H}^\infty}$ is denoted by $\mathcal{H}^{-\infty}$. It is called the space of *generalized vectors* of $\mathcal{H}$. Via the inner product of $\mathcal{H}$ we obtain continuous linear embeddings

$$\mathcal{H}^\infty \subset \mathcal{H} \subset \mathcal{H}^{-\infty}\,.$$

The space $\mathcal{H}^{-\infty}$ naturally carries a G- and a $\mathfrak{g}$-module structure for which the second embedding is equivariant.

11.4 Let $(\pi, \mathcal{H})$ be a unitary representation for G and let $\eta \in (\mathcal{H}^{-\infty})^H$. We define the linear map $m_\eta : \mathcal{H} \to C^{-\infty}(G/H)$ by

$$m_\eta(v)(x) = \langle v \mid \pi(x)\eta\rangle\,.$$

This is to be interpreted in the sense of generalized functions, i.e., if $f \in C_c^\infty(G/H)$, then

$$m_\eta(v)(f\,dx) = \int_{G/H} \langle v \mid \pi(x)\eta\rangle f(x)\,dx = \langle v \mid \pi(\bar{f})\eta\rangle$$

(note that $\pi(f)\eta \in \mathcal{H}^\infty$).

Note that by unitarity, m_η is a G-equivariant continuous linear map. If π is irreducible and $\eta \neq 0$, then m_η is an embedding. Hence $(\mathcal{H}, \pi, m_\eta)$ is a Hilbert subspace of $C^{-\infty}(G/H)$.

11.5 Lemma. *Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of G. Then the map $\eta \mapsto m_\eta$ defines a linear isomorphism from $(\overline{\mathcal{H}^{-\infty}})^H$ onto the space $\mathrm{Hom}_G(\mathcal{H}, C^{-\infty}(G/H))$ of continuous linear G-equivariant maps $\mathcal{H} \to C^{-\infty}(G/H)$.*

11.6 Lemma. *Let $\pi \in \widehat{G}$. Then $\dim_{\mathbb{C}}(\mathcal{H}_\pi^{-\infty})^H < \infty$.*

Proof. See [4].

11.7 In view of the two preceding lemmas, it is reasonable to define, for $\pi \in \widehat{G}$:

$$C^{-\infty}(G/H)_\pi = M_\pi(\mathcal{H}_\pi \otimes (\overline{\mathcal{H}_\pi^{-\infty}})^H)\,,$$

where M_π is the matrix coefficient map determined by

$$M_\pi(v \otimes \eta)(x) = m_\eta(v)(x) = \langle v \mid \pi(x)\eta\rangle\,,$$

which formally looks the same as §1, (3-3) but has to be interpreted in the sense of generalized functions, see 11.4. Notice that the continuous linear isomorphism

$$M_\pi : \mathcal{H}_\pi \otimes (\overline{\mathcal{H}_\pi^{-\infty}})^H \xrightarrow{\ \sim\ } C^{-\infty}(G/H)_\pi,$$

is G-equivariant with respect to the representations $\pi \otimes 1$ and L.

12 We can now describe our goal of obtaining a Plancherel decomposition for G/H. Let

$$\widehat{G}_H := \{\pi \in \widehat{G} \mid (\mathcal{H}_\pi^{-\infty})^H \neq 0\}\,.$$

We wish to obtain a measure $d\mu$ on $\widehat{G}_H$ together with continuous G-equivariant linear operators $L^2(G/H) \to C^{-\infty}(G/H)_\pi$, $f \mapsto f_\pi$, for $\pi \in \widehat{G}_H$, such that

$$f = \int_{\widehat{G}_H} f_\pi \, d\mu(\pi)\,. \tag{12-1}$$

13 The Fourier transform. If (12-1) has been accomplished, we can define the Fourier transform $(\hat{f}(\pi))_{\pi\in\widehat{G}_H}$ of f by

$$\begin{cases} \hat{f}(\pi) \in \mathcal{H}_\pi \otimes (\overline{\mathcal{H}_\pi^{-\infty}})^H\,, \\ M_\pi(\hat{f}(\pi)) = f_\pi\,, \end{cases} \tag{13-1}$$

for $\pi \in \widehat{G}_H$. Since M_π is a G-equivariant embedding, the map $f \mapsto \hat{f}(\pi)$ is linear from $L^2(G/H)$ to $\mathcal{H}_\pi \otimes (\overline{\mathcal{H}_\pi^{-\infty}})^H$ and it intertwines the G-representations L and $\pi \otimes 1$.

In general, it is not the full space $C^{-\infty}(G/H)_\pi = M_\pi(\mathcal{H}_\pi \otimes (\overline{\mathcal{H}_\pi^{-\infty}})^H)$ which is relevant for the decomposition of $L^2(G/H)$. Moreover, it carries no natural Hilbert space structure. Therefore, the requirement that $f \mapsto \hat{f}$ is an isometric isomorphism has to be formulated with care. We require for each $\pi \in \widehat{G}_H$ a linear subspace $\mathcal{V}_\pi \subset (\mathcal{H}_\pi^{-\infty})^H$, equipped with a positive definite inner product, to be given such that $\hat{f}(\pi) \in \mathcal{H}_\pi \otimes \overline{\mathcal{V}_\pi}$ and

$$\|f\|^2_{L^2(X)} = \int_{\widehat{G}_H} \|\hat{f}(\pi)\|^2 d\mu(\pi)\,.$$

Finally, the image of $f \mapsto \hat{f}$ should consist of all families $(T_\pi \in \mathcal{H}_\pi \otimes \overline{\mathcal{V}}_\pi \mid \pi \in \widehat{G}_H)$ that are measurable in a suitable sense, and satisfy $\int_{\widehat{G}_H} \|T_\pi\|^2 d\mu(\pi) < \infty$.

By (12-1) and (13-1), the inverse operator $\mathcal{I}$ should be given as

$$\mathcal{I}T = \int_{\widehat{G}_H} M_\pi(T_\pi) d\mu(\pi)\,.$$

Moreover, it should also be the adjoint of $f \mapsto \hat{f}$. Thus, for $f \in L^2(X)$ we should have

$$\int_{\widehat{G}_H} \langle f \mid M_\pi(T_\pi)\rangle_{L^2(X)} d\mu(\pi) = \int_{\widehat{G}_H} \langle \hat{f}(\pi) \mid T_\pi\rangle d\mu(\pi)\,.$$

This leads to the insight that for reasonably nice functions f the Fourier transform should be given by

$$\langle \hat{f}(\pi) \mid T_\pi\rangle = \langle f \mid M_\pi(T_\pi)\rangle_{L^2(X)}\,,$$

for a given $\pi \in \widehat{G}_H$ and all $T_\pi \in \mathcal{H}_\pi \otimes \overline{\mathcal{V}}_\pi$. If $T_\pi = v \otimes \eta$, then the right-hand side becomes

$$\int_X f(x)\overline{\langle v \mid \pi(x)\eta\rangle} dx = \langle \pi(f)\eta \mid v\rangle\,.$$

Thus, via the identification $\mathcal{H}_\pi \otimes \overline{\mathcal{V}}_\pi \simeq \mathrm{Hom}(\mathcal{V}_\pi, \mathcal{H}_\pi)$ the Fourier transform $\hat{f}(\pi) \in \mathrm{Hom}(\mathcal{V}_\pi, \mathcal{H}_\pi)$ is given by

$$\hat{f}(\pi) = \pi(f)\Big|_{\mathcal{V}_\pi} = \int_{G/H} f(gH)\pi(g)\Big|_{\mathcal{V}_\pi} dgH\,.$$

Together with the choice of the space $\mathcal{V}_\pi$ this constitutes our definition of the abstract Fourier transform. It makes sense because $\mathcal{V}_\pi \subset (\mathcal{H}_\pi^{-\infty})^H$.

14 The discrete series. An irreducible unitary representation π of G is said to belong to the discrete series of G/H if it can be realized on a closed subspace of $L^2(X)$, i.e. if $\mathrm{Hom}_G(\mathcal{H}_\pi, L^2(X)) \neq 0$. The latter space may be viewed as a subspace of $\mathrm{Hom}_G(\mathcal{H}_\pi, C^{-\infty}(X))$ and according to 11.5 determines the subspace

$$(\mathcal{H}_\pi^{-\infty})^H_{ds} \subset (\mathcal{H}_\pi^{-\infty})^H\,.$$

The collection of (equivalence classes of) discrete series representations of G/H is denoted by $\widehat{G}_{H,ds}$. It is at most countable, since $L^2(X)$ is separable. Therefore, it makes sense to define, for $\pi \in \widehat{G}_{H,ds}$,

$$L^2(G/H)_\pi := M_\pi\left(\mathcal{H}_\pi \otimes \overline{(\mathcal{H}_\pi^{-\infty})^H_{ds}}\right)\,,$$

and

$$L^2_d(G/H) := c\ell\left(\bigoplus\nolimits_{\pi\in\widehat{G}_{H,ds}} L^2(G/H)_\pi\right)\,.$$

Exercise. Show that $\pi_1, \pi_2 \in \widehat{G}_{H,ds}$, $\pi_1 \not\sim \pi_2 \Rightarrow L^2(G/H)_{\pi_1} \perp L^2(G/H)_{\pi_2}$.

In the complementary part $L^2_d(G/H)^\perp$, the discrete series will occur with $d\mu$-measure 0 (why?) so we may take

$$\mathcal{V}_\pi = (\mathcal{H}_\pi^{-\infty})^H_{ds}$$

for $\pi \in \widehat{G}_{H,ds}$.

Exercise. Show that $\mathcal{V}_\pi$ has a unique Hilbert space structure s.t. M_π is an isometry from $\mathcal{H}_\pi \otimes \overline{\mathcal{V}}_\pi$ onto $L^2(G/H)_\pi$ (for $\pi \in \widehat{G}_{H,ds}$).

15 Invariant differential operators. In the process of finding the Plancherel formula, the interaction with invariant differential operators on G/H will play an essential role.

15.1 Definition. An *invariant differential operator* on G/H is a linear partial differential operator D with C^∞-coefficients on G/H that commutes with the left action of G on $C^\infty(X)$, i.e.

$$L_g D f = D L_g f\ ,$$

for $f \in C^\infty(X)$, $g \in G$. The algebra of these operators is denoted by $\mathbb{D}(G/H)$ or $\mathbb{D}(X)$.

15.2 Definition. If $D \in \mathbb{D}(X)$, then its *formal adjoint* is the operator $D^* \in \mathbb{D}(X)$ defined by the formula

$$\int_X Df(x)\overline{g(x)}dx = \int_X f(x)\overline{Dg(x)}dx\ ,$$

for $f, g \in C_c^\infty(X)$.

An operator $D \in \mathbb{D}(X)$ with $D = D^*$ is called formally self-adjoint.

15.3 Theorem (see [4]). *Let $D \in \mathbb{D}(X)$ be formally self-adjoint. Then D, viewed as an operator in $L^2(X)$ with domain $C_c^\infty(X)$, is essentially self-adjoint (i.e. it has a symmetric closure).*

16 It follows from the above theorem that every formally self-adjoint operator $D \in \mathbb{D}(X)$ allows a spectral decomposition that commutes with the unitary action of G on $L^2(X)$. Let $\mathcal{U}_D$ be the unitary group with infinitesimal generator iD, then G and $\mathcal{U}_D$ commute. Applying general functional analysis to $G \times \mathcal{U}_D$ one expects that there should be a desintegration of L that diagonalizes G and D at the same time! Let us see what this means in terms of the decomposition theory of subsections 12 and 13.

17. If $u \in \mathcal{U}(\mathfrak{g})^H$, then $R_u : C^\infty(G) \to C^\infty(G)$ leaves the subspace $C^\infty(G)^H$ of right-H-invariant functions invariant. Via the identification $C^\infty(G)^H \simeq C^\infty(X)$, we may view R_u as a smooth differential operator on X which obviously commutes with the G-action. Hence $u \mapsto R_u$ defines an algebra homomorphism $\mathcal{U}(\mathfrak{g})^H \to \mathbb{D}(X)$.

17.1 Lemma. *The map $u \mapsto R_u$, $\mathcal{U}(\mathfrak{g})^H \to \mathbb{D}(X)$ is a surjective homomorphism of algebras. Its kernel equals $\mathcal{U}(\mathfrak{g})^H \cap \mathcal{U}(\mathfrak{g})\mathfrak{h}$.*

Proof. See [S], Prop. 4.1.

17.2 We denote the induced isomorphism by

$$r : \mathcal{U}(\mathfrak{g})^H / \mathcal{U}(\mathfrak{g})^H \cap \mathcal{U}(\mathfrak{g})\mathfrak{h} \xrightarrow{\ \simeq\ } \mathbb{D}(X)\,.$$

In §2 we will use this isomorphism to show that $\mathbb{D}(X)$ is a polynomial algebra, just as in the Riemannian case. In particular, $\mathbb{D}(X)$ is commutative.

18. Let π be a unitary representation of G. Then $\mathcal{U}(\mathfrak{g})$ acts naturally on $\mathcal{H}_\pi^\infty$ and on $\mathcal{H}_\pi^{-\infty}$. Moreover, $\mathcal{U}(\mathfrak{g})^H$ preserves the subspace $(\mathcal{H}_\pi^{-\infty})^H$. This induces the structure of a $\mathcal{U}(\mathfrak{g})^H/\mathcal{U}(\mathfrak{g})^H \cap \mathcal{U}(\mathfrak{g})\mathfrak{h}$-module on $(\mathcal{H}_\pi^{-\infty})^H$. Via the isomorphism r we may thus view $(\mathcal{H}_\pi^{-\infty})^H$ as a $\mathbb{D}(X)$-module. Since every complex linear endomorphism of $(\mathcal{H}_\pi^{-\infty})^H$ is also a complex linear endomorphism of the conjugate space, we see that $\overline{(\mathcal{H}_\pi^{-\infty})^H}$ is a $\mathbb{D}(X)$-module as well.

18.1 Lemma. *Let $\pi \in \widehat{G}_H$. Then $D \circ M_\pi = M_\pi \circ (I \otimes D)$, for all $D \in \mathbb{D}(X)$.*

Proof. *Exercise.*

19. By the discussion in 16 together with Lemma 18.1 we expect the subspaces $\mathcal{V}_\pi \subset (\mathcal{H}_\pi^{-\infty})^H$ to be $\mathbb{D}(X)$-invariant. Moreover, by commutativity of $\mathbb{D}(X)$, we expect the action of $\mathbb{D}(X)$ on $\mathcal{V}_\pi$ to allow a simultaneous diagonalization.

Exercise. Let G be a Lie group, viewed as $X = G \times G/\text{diagonal}$. Show that

$$\mathbb{D}(X) \simeq \mathcal{U}(\mathfrak{g})^G .$$

If G is connected, show that $\mathbb{D}(X) \simeq$ center $\mathcal{U}(\mathfrak{g})$.

§2. Basic structure theory

1. From now on we always assume that $G \in \mathcal{R}$, see Appendix, that σ is an involution of G and that $(G^\sigma)_e < H < G^\sigma$.

Lemma. *There exists a Cartan involution θ of G that commutes with σ, i.e.*

$$\sigma \circ \theta = \theta \circ \sigma . \tag{1-1}$$

Proof. Fix a Cartan involution θ_0 for G. Then according to [S], Prop. 2.1, there exists a $g \in G$ s.t. the conjugate σ^g commutes with θ_0. Hence σ commutes with $\theta = (\theta_0)^{g^{-1}}$. $\square$

2. From now on we assume θ to be as in (1-1). Then the maximal compact subgroup

$$K = G^\theta$$

of G is σ-stable. We recall the Cartan decomposition

$$G = K \exp \mathfrak{p} \,.$$

Here $\mathfrak{p}$ is the -1 eigenspace of θ in $\mathfrak{g}$ and the map $(k, X) \mapsto k \exp X$ is a diffeomorphism from $K \times \mathfrak{p}$ onto G. By (1-1), both K and $\mathfrak{p}$ are σ-stable, hence from the uniqueness of the Cartan decomposition it follows that $G^\sigma = (K \cap G^\sigma) \exp(\mathfrak{p} \cap \mathfrak{g}^\sigma)$. This in turn implies that $(G^\sigma)_e = (K \cap G^\sigma)_e \exp(\mathfrak{p} \cap \mathfrak{g}^\sigma)$.

Exercise. Show that $(K \cap (G^\sigma)_e) \exp(\mathfrak{p} \cap \mathfrak{g}^\sigma)$ is an open subgroup of G^σ. Conclude that $(K \cap G^\sigma)_e = K \cap (G^\sigma)_e$. Using $(G^\sigma)_e < H < G^\sigma$ conclude that:

$$H = (H \cap K) \exp(\mathfrak{h} \cap \mathfrak{p}) \,.$$

In particular, it follows that H is θ-stable.

Since σ and θ commute, it also follows that g admits the following joint eigenspace decomposition for σ and θ:

$$\mathfrak{g} = \mathfrak{k} \cap \mathfrak{q} \;\oplus\; \mathfrak{k} \cap \mathfrak{h} \;\oplus\; \mathfrak{p} \cap \mathfrak{q} \;\oplus\; \mathfrak{p} \cap \mathfrak{h} \,. \tag{2-1}$$

3. Lemma. *$G = K(\mathfrak{p} \cap \mathfrak{q}) \exp(\mathfrak{p} \cap \mathfrak{h})$. The map $(k, X, Y) \mapsto k \exp X \exp Y$ is a diffeomorphism from $K \times (\mathfrak{p} \cap \mathfrak{q}) \times (\mathfrak{p} \cap \mathfrak{h})$ onto G.*

For a proof, we refer the reader to [S], Prop. 2.2.

It is a good habit to check what a result means for the Riemannian case, which arises for $\sigma = \theta$. In that case the above lemma gives the old Cartan decomposition, since $\mathfrak{p} \cap \mathfrak{q} = \mathfrak{p}$ and $\mathfrak{p} \cap \mathfrak{h} = 0$.

Exercise What is the meaning of the above lemma in the group case?

4. Corollary. *The map $(k, X) \mapsto k \exp X \; H$ is a submersion from $K \times (\mathfrak{p} \cap \mathfrak{q})$ onto G/H which factors to a diffeomorphism*

$$K \underset{K \cap H}{\times} (\mathfrak{p} \cap \mathfrak{q}) \xrightarrow{\;\simeq\;} G/H \,, \tag{4-1}$$

thus exhibiting G/H as a vector bundle with fiber $\mathfrak{p} \cap \mathfrak{q}$ over $K/K \cap H$, which is K-equivariant.

Example. The real hyperbolic space. $G = \mathrm{SO}_e(p,q)$, $H = \mathrm{SO}_e(p-1,q)$. Here the Cartan involution $\theta : A \mapsto (A^t)^{-1}$ commutes with σ. Thus, $K = \mathrm{SO}(p) \times \mathrm{SO}(q)$. Moreover, the decomposition (2-1) is given by [S], Example 2.1.

In the geometric realization of $X_{p,q}$ the map (4-1) means the following

$$K/K \cap H \simeq K \cdot e_1 = S^{p-1} \times \{0\},$$

where S^{p-1} is the unit sphere in $\mathbb{R}^p$. The fiber of the vectorbundle (4-1) over e_1 corresponds to $\exp(\mathfrak{p} \cap \mathfrak{q}) \cdot e_1$ and is given by the equations

$$\begin{cases} x_2 = \cdots = x_p = 0 \\ x_1 = \sqrt{1 + x_{p+1}^2 + \cdots + x_n^2}\,. \end{cases}$$

In an exercise to come you will be asked to show this. Note that the projection $p : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^q$ restricts to a diffeomorphism of this fiber onto $\mathbb{R}^q$. The other fibers of the vector bundle are readily obtained by applying the action of $\mathrm{SO}(p) \times \{I\}$, since $K = \mathrm{SO}(p) \times \mathrm{SO}(q)$ and $\{I\} \times \mathrm{SO}(q)$ stabilizes the fiber over e_1.

$X_{p,q}$ as an $\mathbb{R}^q$-bundle over S^{p-1}

5. We fix a maximal abelian subspace $\mathfrak{a}_q$ of $\mathfrak{p} \cap \mathfrak{q}$. In the Riemannian case $\sigma = \theta$, $\mathfrak{a}_q$ is a maximal abelian subspace of $\mathfrak{p}$ and then the following lemma is well known. For the general case, see [S], Thm. 2.6.

Lemma. *The nonzero weights of* $\mathfrak{a}_q$ *in* $\mathfrak{g}$ *form a possibly non-reduced root system, denoted* $\Sigma(\mathfrak{g}, \mathfrak{a}_q) = \Sigma$.

The natural map $N_K(\mathfrak{a}_q) \to GL(\mathfrak{a}_q)$ *factors to an isomorphism from* $N_K(\mathfrak{a}_q)/Z_K(\mathfrak{a}_q)$ *onto the reflection group* W *of* Σ.

6. Definition. $W_{K\cap H}$ is the natural image of $N_{K\cap H}(\mathfrak{a}_q)$ in W.

From the Cartan decomposition it follows that exp is a diffeomorphism from $\mathfrak{a}_q$ onto a closed abelian subgroup A_q of G. Via this diffeomorphism W acts on A_q. In the Riemannian case ($\sigma = \theta$) we have the $G = KA_qK$ decomposition, where the A_q-part is uniquely determined modulo W. The generalization to the present context is:

Lemma (polar decomposition). $G = KA_qH$. *If $x \in G$, then $x \in KaH$ for an element $a \in A_q$ that is uniquely determined modulo $W_{K\cap H}$. Finally,*

$$X_+ := KA_q^{\mathrm{reg}}H\,,$$

viewed as a subset of X, is open dense.

In the above, $A_q^{\mathrm{reg}} = \exp(\mathfrak{a}_q^{\mathrm{reg}})$, where, of course, $\mathfrak{a}_q^{\mathrm{reg}}$ is the complement of the union of the root hyperplanes $\ker\alpha$, $\alpha \in \Sigma$.

Proof. $K\cap H$ normalizes $\mathfrak{p}\cap\mathfrak{q}$, hence $G_S = (K\cap H)\exp(\mathfrak{p}\cap\mathfrak{q})$ is a (closed) subgroup of G, with the indicated Cartan decomposition. Now $\mathfrak{a}_q$ is maximal abelian in $\mathfrak{p}\cap\mathfrak{q}$, hence

$$G_S = (K\cap H)A_q(K\cap H)\,,$$

where the A_q-part is unique modulo $W_{K\cap H}$. Combine this with $G = K\exp(\mathfrak{p}\cap\mathfrak{q})\exp(\mathfrak{p}\cap\mathfrak{h})$ and $H = (K\cap H)\exp(\mathfrak{p}\cap\mathfrak{h})$. $\square$

7. In the following we will always assume that

$$\mathcal{W} \subset N_K(\mathfrak{a}_q)$$

is a set of representatives for $W/W_{K\cap H}$. By this we mean that the natural map $\mathcal{W} \to W/W_{K\cap H}$ is a bijection.

Corollary. *Let A_q^+ be a chamber in A_q^{reg}. Then*

$$X_+ = \bigcup_{v\in\mathcal{W}} KA_q^+vH \quad \text{(disjoint union)}.$$

Moreover, if $x \in X_+$ then $x \in KavH$ for uniquely determined $v \in \mathcal{W}$ and $a \in A_q^+$.

Example. Let $X = X_{p,q}$ be a real hyperbolic space. Let

$$Y = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 0 \end{pmatrix}.$$

Then $\mathfrak{a}_q = \mathbb{R}Y$ is maximal abelian in $\mathfrak{p} \cap \mathfrak{q}$. One readily checks that

$$a_t = \exp tY = \begin{pmatrix} \cosh t & 0 & \cdots & \sinh t \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ \sinh t & 0 & \cdots & \cosh t \end{pmatrix}$$

from which it follows that

$$\begin{aligned} A_q e_1 &= \left\{ \cosh t \; e_1 + \sinh t \; e_n \middle| \; t \in \mathbb{R} \right\} \\ &= \left\{ x \in \mathbb{R}^n \middle| \; x_i = 0 \, , \; 1 < i < n \, , \; x_1^2 - x_n^2 = 1 \right\} \, . \end{aligned}$$

Exercise. Show that $KA_q e_1 = X_{p,q}$ in this case, and conclude that $G = \mathrm{SO}_e(p,q)$ acts transitively and that $G = KA_qH$ in this case.

Also prove the statements of Example 4.

Example $X_{p,q}$ continued. Here $\Sigma = \{\pm \alpha\}$, where $\alpha(Y) = 1$. Thus, $W = \{\pm I\}$. Moreover, there is a significant difference between the cases $q = 1$ and $q > 1$. If $q = 1$, then $W_{K\cap H} = \{I\}$, but if $q > 1$ then $W_{K \cap H} = W$. See [S], Example 2.2 for details. This is reflected by the fact that $X_+ = X_{p,q} \setminus (\mathbb{R}^p \times \{0\})$ consists of two connected components for $q = 1$ and of one connected component for $q > 1$.

Exercise. Verify these statements.

8. The decomposition $X_+ = \bigcup_{v \in \mathcal{W}} KA_q^+ vH$ gives rise to a formula for the invariant measure on X.

Since σ and θ commute, their composition $\sigma \circ \theta$ is also an involution. Let

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$$

be the associated decomposition into a $+1$ and a -1 eigenspace. (This notation is somewhat unfortunate, $\mathfrak{g}_+$ has nothing to do with X_+.)

If $\alpha \in \Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}_q)$, let $\mathfrak{g}_\alpha$ be the associated root space in $\mathfrak{g}$. Since $\sigma \circ \theta = I$ on $\mathfrak{a}_q$, $\sigma \circ \theta$ leaves every $\mathfrak{g}_\alpha$, $\alpha \in \Sigma$, invariant. It follows that

$$\mathfrak{g}_\alpha = (\mathfrak{g}_\alpha \cap \mathfrak{g}_+) \oplus (\mathfrak{g}_\alpha \cap \mathfrak{g}_-) \, .$$

Accordingly, $m_\alpha := \dim \mathfrak{g}_\alpha = m_\alpha^+ + m_\alpha^-$, where

$$m_\alpha^\pm = \dim \mathfrak{g}_\alpha \cap \mathfrak{g}_\pm \, .$$

We will also need the following notation. Recall that $\exp : \mathfrak{a}_q \to A_q$ is a diffeomorphism. We denote its inverse by log. If $\mu \in \mathfrak{a}^*_{q\mathbb{C}}$, we put

$$a^\mu = e^{\mu(\log a)} \qquad (a \in A_q)\,. \tag{8-1}$$

In other words $(\exp X)^\mu = e^{\mu(X)}\,, \quad X \in \mathfrak{a}_q$.

Theorem. *Let dx be a choice of invariant measure on X and let dk be normalized Haar measure. There exists a unique choice of Haar measure da on A_q s.t., for $f \in L^1(X)$,*

$$\int_X f(x)dx = \sum_{v \in \mathcal{W}} \int_K \int_{A_q^+} f(kavH)J(a)dadk\,,$$

where $J(a) = \prod\limits_{\alpha \in \Sigma^+} (a^\alpha - a^{-\alpha})^{m_\alpha^+}(a^\alpha + a^{-\alpha})^{m_\alpha^-}$.

Here Σ^+ is the positive system determined by $\mathfrak{a}_q^+$.

Remarks. This result is equivalent to Theorem 2.5 in [S], but somewhat differently stated. Our J differs from the one given there by a factor 2^N, $N = \sum_{\alpha \in \Sigma^+} m_\alpha$. Also, in the above formula, A_q^+ is a chamber for Σ, whereas in [S], Theorem 2.5, A_q^+ is a chamber for the smaller root system $\Sigma(\mathfrak{g}_+, \mathfrak{a}_q)$, hence bigger and it can be shown that $X_+ \subset KA_q^+H$ for this chamber. Hence the summation over $\mathcal{W}$ is not needed.

Example. The example $X_{p,q}$ is treated in detail in [S], Ex. 2.3.

§3. Invariant differential operators

Since σ and θ commute, $\sigma\theta$ is an involution as well. Let

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$$

be the associated decomposition into a $+1$ and a -1 eigenspace. Then

$$\begin{aligned} \mathfrak{g}_+ &= \mathfrak{k} \cap \mathfrak{h} \;\oplus\; \mathfrak{p} \cap \mathfrak{q}\,, \\ \mathfrak{g}_- &= \mathfrak{k} \cap \mathfrak{q} \;\oplus\; \mathfrak{p} \cap \mathfrak{h}\,. \end{aligned}$$

One readily checks that $\mathfrak{g}_+$ is a subalgebra and that $[\mathfrak{g}_+, \mathfrak{g}_-] \subset \mathfrak{g}_-$, hence

$$\mathfrak{g}^d := \mathfrak{g}_+ \oplus i\mathfrak{g}_-$$

is a real form of the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$. The nice thing about this form is that the roles of θ and σ become interchanged. More precisely, let $\theta_\mathbb{C}$ and $\sigma_\mathbb{C}$ be the complex linear extensions of θ and σ to $\mathfrak{g}_\mathbb{C}$, and define

$$\theta^d = \sigma_\mathbb{C}\big|\ \mathfrak{g}^d\,, \quad \sigma^d = \theta_\mathbb{C}\big|\ \mathfrak{g}^d\,.$$

Then $\ker(\theta^d - I) = (\mathfrak{k}\cap\mathfrak{h})\oplus i(\mathfrak{p}\cap\mathfrak{h})$. It is well known that $\mathfrak{k}\oplus i\mathfrak{p}$ is a compact real form of $\mathfrak{g}_\mathbb{C}$. Hence θ^d is a Cartan involution ("the Cartan involution for $\mathfrak{g}^d$", whence the notation). Clearly θ^d and σ^d commute!

Fix a complex linear algebraic group $G_\mathbb{C}$ with algebra $\mathfrak{g}_\mathbb{C}$ and let G^d, K^d be the analytic subgroups with Lie algebras $\mathfrak{g}^d$ and

$$\mathfrak{k}^d := \ker(\theta^d - I) = \mathfrak{h}_\mathbb{C}\cap\mathfrak{g}^d\,.$$

Let $H_\mathbb{C}$ be the analytic subgroup with algebra $\mathfrak{h}_\mathbb{C}$. Then $X^d = G^d/K^d$ is a Riemannian real form of the complex symmetric space $G_\mathbb{C}/H_\mathbb{C}$.

Example. Let $X = X_{p\cdot q} = \mathrm{SO}_e(p,q)/\mathrm{SO}_e(p-1,q)$. As a complexification of X we may take $\mathrm{SO}(n)_\mathbb{C}/\mathrm{SO}(n-1)_\mathbb{C}$ and then the dual Riemannian form becomes $X^d = \mathrm{SO}_e(1,n-1)/\mathrm{SO}(n-1)$.

2. Lemma. *There is a natural isomorphism $\mathbb{D}(X)\simeq\mathbb{D}(X^d)$.*

Proof. If H is connected, the proof is straightforward:

$$\begin{aligned}\mathbb{D}(X) &\simeq \mathcal{U}(\mathfrak{g})^H/\mathcal{U}(\mathfrak{g})^H\cap\mathcal{U}(\mathfrak{g})\mathfrak{h}\\ &= \mathcal{U}(\mathfrak{g}^\mathfrak{h})/\mathcal{U}(\mathfrak{g})^\mathfrak{h}\cap\mathcal{U}(\mathfrak{g})\mathfrak{h}\\ &= \mathcal{U}(\mathfrak{g}^d)^{\mathfrak{k}^d}/\mathcal{U}(\mathfrak{g}^d)^{\mathfrak{k}^d}\cap\mathcal{U}(\mathfrak{g}^d)\mathfrak{k}^d \simeq \mathbb{D}(X^d)\,.\end{aligned}$$

If H is non-connected, the second identity is not completely obvious, but can be proved, using the structure of H/H_e. □

Exercise. Show that $\mathbb{D}(X_{p,q})$ consists of all polynomials in the Laplace–Beltrami operator of $X_{p,q}$ by using the knowledge of this fact for $\mathbb{D}(X^d) = \mathbb{D}(\mathrm{SO}_e(1,n-1)/\mathrm{SO}(n-1))$.

From the theory of Riemannian symmetric spaces, we recall the existence of a Harish–Chandra isomorphism $\gamma^d : \mathbb{D}(X^d) \xrightarrow{\simeq} I(\mathfrak{a}^d_\mathfrak{p})$, where $\mathfrak{a}^d_\mathfrak{p}$ is maximal abelian in $\mathfrak{p}$, and where $I(\mathfrak{a}^d_\mathfrak{p})$ is the collection of $W(\mathfrak{g},\mathfrak{a}^d_\mathfrak{p})$-invariants in $S(\mathfrak{a}^d_\mathfrak{p})$. It follows from this that $\mathbb{D}(X^d)$ is a polynomial algebra of rank $\dim\mathfrak{a}^d_\mathfrak{p}$.

3. Corollary. *$\mathbb{D}(X)$ is a polynomial algebra. In particular it is commutative.*

4. Definition. By a *Cartan subspace* of $\mathfrak{q}$ we mean a subspace $\mathfrak{b} \subset \mathfrak{q}$ that is maximal subject to the conditions that (i) it is abelian, (ii) it consists of semisimple elements.

By using the above method of complexification it can be shown that $\dim \mathfrak{b}$ is independent of $\mathfrak{b}$, though in general there are several, but finitely many, H-conjugacy classes of Cartan subspaces. We call the number $\dim \mathfrak{b}$ the *rank of* X.

Exercise. Interpret the definition of rank for the case of the group.

Let now $\mathfrak{b} \subset \mathfrak{q}$ be a θ-stable Cartan subspace. Such a Cartan subspace may for instance be obtained as follows. Let $\mathfrak{a}_q \subset \mathfrak{p} \cap \mathfrak{q}$ be maximal abelian. Then $\mathfrak{a}_q$ consists of semisimple elements. Let $\mathfrak{m}_1 =$ centralizer$(\mathfrak{a}_q)$. Then $\mathfrak{m}_1 \cap \mathfrak{q} \subset (\mathfrak{m}_1 \cap \mathfrak{k} \cap \mathfrak{q}) \oplus \mathfrak{a}_q$. Let $\mathfrak{t} \subset \mathfrak{m}_1 \cap \mathfrak{k} \cap \mathfrak{q}$ be maximal abelian. Then $\mathfrak{t}$ consists of semisimple elements, hence so does $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{a}_q$. We see that $\mathfrak{b}$ is a θ-stable Cartan subspace. We call it maximally split, since $\mathfrak{g}$ splits maximally for the action of $\mathfrak{b}$ by ad. Note that the dimension of $\mathfrak{a}_q$ (the rank of the Riemannian pair $(\mathfrak{g}_+, \mathfrak{k} \cap \mathfrak{h})$) is independent of $\mathfrak{a}_q$. It is called the *split rank of* X.

To the θ-stable Cartan subspace $\mathfrak{b}$ we associate $\mathfrak{a}_{\mathfrak{p}}^d := \mathfrak{b} \cap \mathfrak{p} \oplus i(\mathfrak{b} \cap \mathfrak{k})$ which is maximally abelian in $\mathfrak{g}^d$. Let $\Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b})$ be the root system of $\mathfrak{b}$ in $\mathfrak{g}_{\mathbb{C}}$, $W(\mathfrak{b})$ the associated Weyl group and $I(\mathfrak{b})$ the associated collection of $W(\mathfrak{b})$-invariants in $\mathcal{S}(\mathfrak{b})$. Then obviously $I(\mathfrak{b}) = I(\mathfrak{a}_{\mathfrak{p}}^d)$. Let $\gamma : \mathbb{D}(X) \to I(\mathfrak{b})$ be the map which makes the following diagram commutative

$$\begin{tikzcd} \mathbb{D}(X) \arrow[r, dashed, "\gamma"] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\circlearrowleft"] & I(\mathfrak{b}) \arrow[d, "="] \\ \mathbb{D}(X^d) \arrow[r, "\gamma^d"] & I(\mathfrak{a}_{\mathfrak{p}}^d) \end{tikzcd}$$

It clearly is an isomorphism, called the Harish–Chandra isomorphism for $\mathbb{D}(X), \mathfrak{b}$. The description of γ^d in terms of the universal enveloping algebra leads to the following description of γ.

Let $\Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b})$ be a choice of positive roots, and let $\mathfrak{g}_{\mathbb{C}}^+$ be the associated sum of positive root spaces. Then

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{b}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}^+$$

complexifies the Iwasawa decomposition $\mathfrak{g}^d = \mathfrak{k}^d \oplus \mathfrak{a}_{\mathfrak{p}^d} \oplus (\mathfrak{g}_{\mathbb{C}}^+ \cap \mathfrak{g}^d)$ for $\mathfrak{g}^d$. Via Poincaré–Birkhoff–Witt,

$$\mathcal{U}(\mathfrak{g}) = (\mathfrak{g}_{\mathbb{C}}^+ \mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\mathfrak{h}_{\mathbb{C}}) \oplus \mathcal{U}(\mathfrak{b}).$$

Let $D \in \mathbb{D}(X)$. Then $D = R_u$ for a $u \in U(\mathfrak{g})^H$. There is a unique $u_0 \in \mathcal{U}(\mathfrak{b})$, only depending on D, such that

$$u - u_0 \in \mathfrak{g}_{\mathbb{C}}^{+}\mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\mathfrak{h}_{\mathbb{C}}\,.$$

The element $\gamma(D) \in I(\mathfrak{b})$ is now given by $\gamma(D) = T_{\rho_{\mathfrak{b}}} u_0$, where $\rho_{\mathfrak{b}} = \frac{1}{2}\mathrm{tr}_{\mathbb{C}}\,\mathrm{ad}(\cdot)\big|\mathfrak{g}_{\mathbb{C}}^{+}$ and where $T_{\rho_{\mathfrak{b}}}$ denotes the the automorphism of $\mathcal{S}(\mathfrak{b})$ induced by the translation $x \mapsto x + \rho_{\mathfrak{b}}$. For more details we refer to [S], Lemma 4.6.

5. Some remarks on the discrete series. The idea of passing to the dual Riemannian form X^d plays an important role in the theory of the discrete series of X. Flensted-Jensen applied this idea in [30] to translate the problem of determining $\widehat{G}_{H,ds}$ into an equivalent problem of classifying eigenfunctions of $\mathbb{D}(X^d)$ on X^d satisfying certain growth conditions at infinity. In this way he was able to show that if $\mathrm{rk}(G/H) = \mathrm{rk}(K/K \cap H)$, then $\widehat{G}_{H,ds}$ has infinitely many elements. Moreover, he constructed infinitely many discrete series representations by using the Poisson transform of X^d.

In [45], Oshima & Matsuki established the necessity of the above rank condition for the existence of discrete series. Therefore:

5.1. Theorem. $\widehat{G}_{H,ds} \neq \varnothing \Leftrightarrow \mathrm{rk}(G/H) = \mathrm{rk}(K/K \cap H)$.

Exercise. Show: if $\mathrm{rk}(G/H) = 1$ then $\widehat{G}_{H,ds} \neq \varnothing$. Show that $\mathrm{rk}(X_{p,q}) = 1$.

Moreover, by using the theory of the Poisson transform and the associated boundary value maps on X^d, Oshima & Matsuki were able to construct all the discrete series for G/H. In the proof of the Plancherel theorem we do not need the full description of $\widehat{G}_{H,ds}$, but we do need the following information on the infinitesimal characters.

5.2. Theorem ([45]). *For every $\widehat{G}_{H,ds}$ the eigenvalues of the $\mathbb{D}(X)$-module $(\mathcal{H}_\pi^{-\infty})^H_{ds}$ (see §1 n° 14) are all of the form*

$$D \mapsto \gamma(D, \Lambda)\,,$$

with $\Lambda \in \mathfrak{b}_{\mathbb{C}}^*$ real and regular, i.e.

$$\langle \Lambda, \alpha \rangle \in \mathbb{R} \setminus \{0\} \quad (\forall_{\alpha \in \Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b})})\,.$$

§4. Parabolic subgroups

1. In the proof of the Plancherel formula, the asymptotic behavior of K-finite eigenfunctions of $\mathbb{D}(X)$ plays a crucial role. For a proper description of this behavior it is necessary to use a description of asymptotic directions to ∞ in $A_q \simeq \mathfrak{a}_q$. (Think of the $G = KA_qH$-decomposition.) The description of these directions relies on the following notion of a parabolic subset for the root system Σ.

2. Definition. A *parabolic subset* of $\mathfrak{a}_q$ is defined to be an equivalence class for the eigenvalue relation $\sim$ on $\mathfrak{a}_q$ defined by

$$X \sim Y \Leftrightarrow \{\alpha \in \Sigma \mid \alpha(X) > 0\} = \{\alpha \in \Sigma \mid \alpha(Y) > 0\}\,.$$

Example. Consider the root system A_2 in $\mathbb{R}^2$.

The parabolic subsets have been labeled
(2.1) – (2.6) (dimension 2)
(1.1) – (1.6) (dimension 1)
and $\{0\}$ (dimension 0)

3. The collection of parabolic subsets in $\mathfrak{a}_q$ is denoted by $\mathcal{P}(\Sigma)$. Obviously, W acts on $\mathcal{P}(\Sigma)$. If $X \in \mathfrak{a}_q$, let $C_X \in \mathcal{P}(\Sigma)$ denote its class. If $C \in \mathcal{P}(\Sigma)$, put

$$\begin{aligned}\Sigma(C) &= \{\alpha \in \Sigma \mid \alpha > 0 \text{ on } C\}\,,\\ \Sigma_C &= \{\alpha \in \Sigma \mid \alpha = 0 \text{ on } C\}\,.\end{aligned}$$

We note that

$$\Sigma = -\Sigma(C) \cup \Sigma_C \cup \Sigma(C) \quad \text{(disjoint)} \tag{3-1}$$

Let S be the intersection of the root hyperplanes $\ker\alpha$, $\alpha \in \Sigma_C$. Then, clearly, the set

$$D = \{X \in S \mid \forall_{\alpha\in\Sigma(C)}\alpha(X) > 0\}$$

contains C, hence is a non-empty open subset of S, hence spans S. On the other hand, from (3-1) it follows that $D \in \mathcal{P}(\Sigma)$. Hence $C = D$ and we conclude that

$$\bigcap_{\alpha \in \Sigma_C} \ker \alpha = \operatorname{span}(C)\,.$$

Exercise. Let $\overline{\mathfrak{a}}_q^+$ be a closed Weyl chanber for Σ, and let $C \in \mathcal{P}(\Sigma)$. Show that there exists a unique $D \in \mathcal{P}(\Sigma)$ such that

(i) C is W-conjugate to D

(ii) $D \subset \overline{\mathfrak{a}}_q^+$.

4. Let Σ^+ be a positive system for Σ, let Δ be the associated collection of simple roots, and $\overline{\mathfrak{a}}_q^+$ the associated closed Weyl chamber.

If $F \subset \Delta$, we put

$$\begin{aligned} \mathfrak{a}_{Fq} &= \bigcap_{\alpha \in F} \ker \alpha\,, \text{ and} \\ \mathfrak{a}_{Fq}^+ &= \{X \in \mathfrak{a}_{Fq} \mid \forall \alpha \in \Sigma^+ : \alpha(X) > 0 \text{ or } \alpha = 0 \text{ on } \mathfrak{a}_{Fq}\}\,. \end{aligned}$$

Then $\mathfrak{a}_{Fq}^+ \in \mathcal{P}(\Sigma)$. Moreover

Lemma. *$F \to \mathfrak{a}_{Fq}^+$ is a bijection from the collection of all subsets of Δ onto the collection of all $C \in \mathcal{P}(\Sigma)$ contained in $\overline{\mathfrak{a}}_q^+$. Finally, $\overline{\mathfrak{a}}_q^+$ is the disjoint union of all $\mathfrak{a}_{Fq}^+$, $F \subset \Delta$.*

Proof. Exercise.

Definition. A *standard parabolic subset* (relative to Σ^+) is a parabolic subset C satisfying one of the following equivalent conditions

(a) $C \subset \overline{\mathfrak{a}}_q^+$

(b) $C = \mathfrak{a}_{Fq}^+$ for some $F \subset \Delta$.

5. To a parabolic subset $C \in \mathcal{P}(\Sigma)$ we associate the following subalgebra of $\mathfrak{g}$:

$$\mathfrak{p}_C := \mathfrak{g}_0 \oplus \bigoplus_{\alpha : \alpha|_C \geq 0} \mathfrak{g}_\alpha\,.$$

(Here $\mathfrak{g}_0$ denotes the centralizer of $\mathfrak{a}_q$). This algebra is its own normalizer in $\mathfrak{g}$ and for this reason called a parabolic subalgebra of $\mathfrak{g}$. Note that $\sigma\theta = I$ on $\mathfrak{a}_q$ implies that $\mathfrak{p}_C$ is $\sigma\theta$-stable.

Lemma. *$C \mapsto \mathfrak{p}_C$ is a bijective correspondence between $\mathcal{P}(\Sigma)$ and the set of $\sigma\theta$-stable parabolic subalgebras containing $\mathfrak{a}_q$.*

If $\mathfrak{p}$ is a parabolic subalgebra, then its normalizer $P = N_G(\mathfrak{p})$ in G is a closed subgroup with Lie algebra $\mathfrak{p}$. Thus, P is the normalizer of its Lie algebra. Any group with this property is called a *parabolic subgroup* of G. Clearly $\mathfrak{p} \mapsto N_G(\mathfrak{p})$ defines a bijective correspondence between the collection of all parabolic subalgebras of $\mathfrak{g}$ and the collection of all parabolic subgroups of G. We put $P_C = N_G(\mathfrak{p}_C)$.

Corollary. *$C \mapsto P_C$ is a bijective correspondence between $\mathcal{P}(\Sigma)$ and the collection $\mathcal{P}_\sigma$ of all $\sigma\theta$-stable parabolic subgroups of G containing A_q.*

6. If $P \in \mathcal{P}_\sigma$, let $C \in \mathcal{P}(\Sigma)$ be the unique parabolic set with $P_C = P$. We agree to write

$$\mathfrak{a}_{Pq}^+ := C\,, \quad \Sigma_P = \Sigma_C\,, \quad \Sigma(P) = \Sigma(C)\,.$$

We also agree to write $\mathfrak{a}_{Pq} = \operatorname{span}(C)$; then

$$\begin{aligned} \mathfrak{a}_{Pq} &= \bigcap_{\alpha\in\Sigma_P} \ker\alpha\,, \\ \mathfrak{a}_{Pq}^+ &= \{X \in \mathfrak{a}_{Pq} \mid \forall_{\beta\in\Sigma(P)} \beta(X) > 0\}\,. \end{aligned}$$

Put $\mathfrak{m}_{1P} = \text{centralizer}(\mathfrak{a}_{Pq})$. Then

$$\mathfrak{m}_{1P} = \mathfrak{g}_0 \oplus \sum_{\alpha\in\Sigma_P} \mathfrak{g}_\alpha\,. \tag{6-1}$$

Put $\mathfrak{n}_P = \sum_{\alpha\in\Sigma(P)} \mathfrak{g}_\alpha$, then

$$\operatorname{Lie}(P) = \mathfrak{m}_{1P} \oplus \mathfrak{n}_P\,.$$

Exercise. Show that $\mathfrak{n}_P$ is the nilpotent radical of $\operatorname{Lie}(P)$. Show that the above decomposition is a Levi decomposition.

7. Let $C \in \mathcal{P}(\Sigma)$, then $-C \in \mathcal{P}(\Sigma)$ as well. If $P \in \mathcal{P}_\sigma$, then the *opposite* parabolic subgroup P is determined by

$$\mathfrak{a}_{\bar{P}q}^+ = -\mathfrak{a}_{Pq}^+\,.$$

Exercise. Show that $\overline{P} = \theta(P) = \sigma(P)$, for $P \in \mathcal{P}_\sigma$. Show that $\mathfrak{m}_{1\overline{P}} = \mathfrak{m}_{1P}$, $\theta\mathfrak{m}_{1P} = \mathfrak{m}_{1P} = \sigma\mathfrak{m}_{1P}$, show that $\mathfrak{g} = \mathfrak{n}_{\overline{P}} \oplus \mathfrak{m}_{1P} \oplus \mathfrak{n}_P$.

We define

$$M_{1P} = Z_G(\mathfrak{a}_{Pq}) ,$$
$$N_P = \exp \mathfrak{n}_P .$$

8. Proposition. *Let $P \in \mathcal{P}_\sigma$.*

(a) N_P is a closed subgroup of G.

(b) M_{1P} is a group of class $\mathcal{R}$.

(c) $P = M_{1P}N_P$, the multiplication map is a diffeomorphism from $M_{1P} \times N_P$ onto P.

We do not give the proof here, but refer to standard text books (e.g. [K]). Assertion (b) is important, since it allows induction with respect to dimension of groups of class $\mathcal{R}$.

9. Lemma. *$G = PK$ (for $P \in \mathcal{P}_\sigma$).*

This is a rather straightforward consequence of the Iwasawa decomposition $G = NAK$. See [S], Lecture 3, for details.

10. We can now describe the so-called *Langlands* decomposition of a parabolic subgroup $P \in \mathcal{P}_\sigma$. We first do this on the infinitesimal level.

Let $P \in \mathcal{P}_\sigma$. We observe that $\mathfrak{m}_{1P}$ is θ-invariant (see Exercise 7), hence $\mathfrak{m}_{1P} = (\mathfrak{m}_{1P} \cap \mathfrak{k}) \oplus (\mathfrak{m}_{1P} \cap \mathfrak{p})$. We define

$$\mathfrak{a}_P = \text{ center}(\mathfrak{m}_{1P}) \cap \mathfrak{p} .$$

From (6-1) it follows that

$$\mathfrak{a}_{Pq} = \mathfrak{a}_P \cap \mathfrak{q} ;$$

this justifies the notation in hindsight. We call

$$A_P = \exp \mathfrak{a}_P$$

the split component of P, and $A_{Pq} = \exp \mathfrak{a}_{Pq}$ the σ-split component.

Define

$$\mathfrak{m}_P = (\mathfrak{m}_{1P} \cap \mathfrak{k}) \oplus ([\mathfrak{m}_{1P}, \mathfrak{m}_{1P}] \cap \mathfrak{p}) .$$

Then $\mathfrak{m}_P$ is a reductive Lie algebra with $\mathfrak{p} \cap \operatorname{center}(\mathfrak{m}_P) = 0$. Moreover,

$$\mathfrak{m}_{1P} = \mathfrak{m}_P \oplus \mathfrak{a}_P .$$

It follows that $\operatorname{Lie}(P) = \mathfrak{m}_P \oplus \mathfrak{a}_P \oplus \mathfrak{n}_P$. This is called the infinitesimal Langlands decomposition. Define

$$M_P = (M_{1P} \cap K) \exp(\mathfrak{m}_P \cap \mathfrak{p}) .$$

Proposition. (Langlands decomposition). *Let $P \in \mathcal{P}_\sigma$. Then $M_P \in \mathcal{R}$. Moreover,*

$$M_{1P} = M_P A_P , \quad P = M_P A_P N_P.$$

The multiplication maps give diffeomorphisms $M_P \times A_P \to M_{1P}$ and $M_P \times A_P \times N_P \to P$.

11. **Remark**. P. Delorme uses the above notation exclusively for the so-called *σ-Langlands decomposition*. Let $A_{Ph} = A_P \cap H$, then $A_P = A_{Ph} A_{Pq}$. Put $M_{P\sigma} = M_P A_{Ph}$. Then

$$P = M_{P\sigma} A_{Pq} N_P$$

is called the σ-Langlands decomposition. Delorme writes M_P for $M_{P\sigma}$ and A_P for A_{Pq}.

§5. Parabolically induced representations

1. Let $P \in \mathcal{P}_\sigma$. (Note that everything that follows also holds for $\sigma = \theta$. In that case $\mathcal{P}_\sigma$ consists of *all* parabolic subgroups containing $A_\mathfrak{p}$).

Let $\xi \in \widehat{M_P}$ and $\lambda \in \mathfrak{a}^*_{p\mathbb{C}}$. We define the representation

$$\xi \otimes \lambda \otimes 1$$

of $P = M_P A_P N_P$ in $\mathcal{H}_\xi$ (Hilbert representation space of ξ) by

$$(\xi \otimes \lambda \otimes 1)(man) = a^\lambda \xi(m) ,$$

for $m \in M_P$, $a \in A_p$, $n \in N_P$. This defines indeed a representation of P, since M_P centralizes A_P, and $M_{1P} = M_P A_P$ normalizes N_P.

We define the parabolically induced representation

$$\pi_{P,\xi,\lambda} := \operatorname{ind}_P^G(\xi \otimes (\lambda + \rho_P) \otimes 1) .$$

Here $\rho_P \in \mathfrak{a}_P^*$ is defined by

$$\begin{aligned}\rho_P(x) &= \frac{1}{2}\mathrm{tr}\ \mathrm{ad}(x)\big|_{\mathfrak{n}_P} \\ &= \frac{1}{2}\sum_{\alpha\in\Sigma(P)} \dim(\mathfrak{g}_\alpha)\alpha\,.\end{aligned}$$

The translation over ρ_P is needed to ensure that the representation $\pi_{P,\xi,\lambda}$ is unitary for $\lambda \in i\mathfrak{a}_{Pq}^*$. To describe the representation space for $\pi_{P,\xi,\lambda}$ we define

$$C(P:\xi:\lambda)$$

to be the space of continuous functions $f : G \to \mathcal{H}_\xi$ transforming according to the rule

$$f(man\ x) = a^{\lambda+\rho_P}\xi(m)f(x)\,, \tag{1-1}$$

for $x \in G$, $m \in M_P$, $a \in A_P$, $n \in N_P$.

On $C(P:\xi:\lambda)$ the representation $\pi_{P,\xi,\lambda}$ is defined by restricting the right regular representation, i.e. if $f \in C(P:\xi:\lambda)$, $x \in G$ then

$$\pi_{P,\xi,\lambda}(x)f(y) = f(yx)\,, \qquad (y \in G)\,.$$

2. Our next goal is to extend $\pi_{P,\xi,\lambda}$ to a suitable Hilbert completion of $C(P:\xi:\lambda)$. We recall that $G = PK$. Hence a function f in $C(P:\xi:\lambda)$ is completely determined by its restriction $f\big|_K$ to K. Let $C(K:\xi)$ denote the space of continuous functions $\varphi : K \to \mathcal{H}_\xi$ transforming according to the rule

$$\varphi(mk) = \xi(m)\varphi(k)\,,$$

for $k \in K$, $m \in K \cap P = K \cap M_P$.

Lemma. *The map $f \mapsto f\big|_K$ defines a topological linear isomorphism $C(P:\xi:\lambda) \xrightarrow{\simeq} C(K:\xi)$.*

Via the above isomorphism, $\pi_{P,\xi,\lambda}$ may be viewed as a (λ-dependent) representation of G on the (λ-independent) space $C(K:\xi)$.

Accordingly, we may equip $C(P:\xi:\lambda)$ with the pre-Hilbert structure defined by

$$\begin{aligned}\langle f \mid g\rangle &= \langle f\big|_K \mid g\big|_K\rangle_{L^2(K,\mathcal{H}_\xi)} \\ &= \int_K \langle f(k) \mid g(k)\rangle_{\mathcal{H}_\xi} dk\,.\end{aligned}$$

The Hilbert completion of $C(P:\xi:\lambda)$ for this structure is denoted by $\mathcal{H}_{P,\xi,\lambda}$. It can be shown that $\pi_{P,\xi,\lambda}$ extends uniquely to a continuous representation of G in $\mathcal{H}_{P,\xi,\lambda}$.

Alternatively, the Hilbert space $\mathcal{H}_{P,\xi,\lambda}$ may also be chacterized as the space of measurable functions $f : G \to \mathcal{H}_\xi$ that transform according to the rule (1-1) and satisfy $f\big|_K \in L^2(K, \mathcal{H}_\xi)$, equipped with the inner product given by $\langle \cdot \mid \cdot \rangle$.

3. Proposition. *Let $\lambda \in \mathfrak{a}^*_{P\mathbb{C}}$. Then the sesquilinear pairing $\mathcal{H}_{P,\xi,\lambda} \times \mathcal{H}_{P,\xi,-\overline{\lambda}} \to \mathbb{C}$ defined by*

$$\langle f \mid g \rangle = \int_K \langle f(k) \mid g(k) \rangle_{\mathcal{H}_\xi} dk$$

is G-equivariant.

In particular, it follows that the representation $\pi_{P,\xi,\lambda}$ is unitary for $\lambda \in i\mathfrak{a}^*_P$.

4. The space of smooth vectors for $\pi_{P,\xi,\lambda}$ equals the space

$$C^\infty(P:\xi:\lambda)$$

of smooth functions $G \to \mathcal{H}_\xi^\infty$ transforming according to the rule (1-1). The sesquilinear pairing of the above proposition induces a G-invariant linear embedding

$$\mathcal{H}_{P,\xi,-\overline{\lambda}} \hookrightarrow (\overline{C^\infty(P:\xi:\lambda)})' = \mathcal{H}^{-\infty}_{P,\xi,\lambda}\,.$$

This motivates us to use the notation

$$C^{-\infty}(P:\xi:-\overline{\lambda}) := (\overline{C^\infty(P:\xi:\lambda)})'\,.$$

The sesquilinear pairing of Proposition (3) then naturally extends to a pairing

$$C^\infty(P:\xi:\lambda) \times C^{-\infty}(P:\xi:-\overline{\lambda}) \to \mathbb{C}\,,$$

also denoted by $\langle \cdot \mid \cdot \rangle$.

Similar definitions give us the spaces $C^\infty(K:\xi)$ and $C^{-\infty}(K:\xi)$, and we have that the map $f \mapsto f\big|_K$ induces topological linear isomorphisms $C^{\pm\infty}(P:\xi:\lambda) \simeq C^{\pm\infty}(K:\xi)$. The representations $\pi^{\pm\infty}_{P,\xi,\lambda}$ may then be realized in the λ-independent spaces $C^{\pm\infty}(K:\xi)$.

The Plancherel formula will essentially be built from the representations from $\widehat{G}_{H,ds}$ and from the induced representations $\pi_{P,\xi,\lambda}$, where

$$\begin{aligned} &P \in \mathcal{P}_\sigma\,, \quad P \neq G \\ &\xi \in (M_P)^\wedge_{M_P \cap H, ds}\,, \quad \lambda \in i\mathfrak{a}^*_{Pq}\,. \end{aligned}$$

§6. H–fixed generalized vectors

1. We assume that $P \in \mathcal{P}_\sigma$, $\xi \in \widehat{M_P}$ and $\lambda \in \mathfrak{a}^*_{P\mathbb{C}}$ and will try to describe sufficiently many H-fixed elements in $C^{-\infty}(P : \xi : \lambda)$. With this in mind it is important to have knowledge of the H-orbits on $P \setminus G$. The following result is a direct consequence of results of Matsuki.

We agree to write W_P for the centralizer of $\mathfrak{a}_{Pq}$ in W. We fix a collection ${}^P\mathcal{W}$ of representatives for $W_P \setminus W/W_{K\cap H}$, contained in $N_K(\mathfrak{a}_q)$.

Proposition. *The map ${}^P\mathcal{W} \to P \setminus G/H$, $v \mapsto PvH$ is a bijection from ${}^P\mathcal{W}$ onto the collection $(P\setminus G/H)_{\text{open}}$ of open H-orbits on $P\setminus G$. Moreover, $\#(P\setminus G/H) < \infty$. Hence $\bigcup\limits_{v\in {}^P\mathcal{W}} PvH$ is open dense.*

2. On the open H-orbits one expects the elements of $C^{-\infty}(P : \xi : \lambda)^H$ to be just functions, which may be evaluated in points. Let $\varphi \in C^{-\infty}(P : \xi : \lambda)^H$. Then, for $v \in^P \mathcal{W}$, one expects that $\varphi(v) \in \mathcal{H}_\xi^{-\infty}$ is a vector that is fixed for $\xi \otimes (\lambda + \rho_P) \otimes 1\big|_{P\cap vHv^{-1}}$, because of the formal identity:

$$\begin{aligned}(\xi \otimes (\lambda + \rho_p) \otimes 1)(p)\varphi(v) &= \varphi(pv)\\ &= \varphi(vv^{-1}pv) = (\pi_{\xi,\lambda}(v^{-1}pv)\varphi)(v)\\ &= \varphi(v)\,, \quad (p \in P\cap vHv^{-1})\end{aligned}$$

since $v^{-1}pv \in H$. Now this implies that $\varphi(v) \in (\mathcal{H}_\xi^{-\infty})^{M_P\cap vHv^{-1}}$ and, if $\varphi(v) \neq 0$, $\lambda + \rho_P \mid \mathfrak{a}_{Ph} = 0$.

Lemma. *ρ_P vanishes on $\mathfrak{a}_{Ph} = \mathfrak{a}_P \cap \mathfrak{h}$.*

Proof. Since $\theta\sigma(\mathfrak{m}_P) = \mathfrak{m}_P$, it follows that $\rho_P(\theta\sigma X) = \rho_P(X)$ for all $X \in \mathfrak{a}_P$. Hence $\rho_P = -\rho_P$ on $\mathfrak{a}_P \cap \mathfrak{h}$. $\square$

Thus, the above leads to $\lambda\big|_{\mathfrak{a}_{Ph}} = 0$. Identifying $\mathfrak{a}^*_{Pq\mathbb{C}}$ with a subspace of $\mathfrak{a}^*_{P\mathbb{C}}$ via the direct sum decomposition

$$\mathfrak{a}_P = \mathfrak{a}_{Ph} \oplus \mathfrak{a}_{Pq}\,,$$

we see that we should require $\lambda \in \mathfrak{a}^*_{Pq\mathbb{C}}$.

3. We note that, for $v \in^P \mathcal{W}$, the space

$$M_P/M_P \cap vHv^{-1}$$

is reductive symmetric. Indeed, $\sigma^v : x \mapsto v\sigma(v^{-1}xv)v^{-1}$ is an involution for M_P, having $vG^\sigma v^{-1} \cap M_P$ as its set of fixed points. The involution σ^v commutes with θ.

We now agree to define the finite dimensional Hilbert space $V(P,\xi,v)$, for $v \in^P \mathcal{W}$ by

$$\begin{aligned} V(P,\xi,v) &= (\mathcal{H}_\xi^{-\infty})^H_{ds} \quad \text{if } \xi \in (M_P)^\wedge_{M_P\cap vHv^{-1},ds} \\ &= 0 \qquad\qquad\quad \text{otherwise}\,. \end{aligned}$$

(See §1, 14 for notation used.)

Definition. We define $V(P,\xi)$ to be the formal direct Hilbert sum

$$V(P,\xi) = \bigoplus_{v\in^P\mathcal{W}} V(P,\xi,v)\,.$$

If $\eta \in V(P,\xi)$, then η_v denotes its component in $V(P,\xi,v)$.

4. The idea now is to invert the map $\varphi \mapsto (\varphi(v))_{v\in^P\mathcal{W}}$ described above.

4.1. Definition. An element $\mu \in \mathfrak{a}^*_{Pq}$ is called strictly P-dominant if

$$\langle \mu, \alpha\rangle > 0\,\forall_{\alpha\in\Sigma(P)}\,.$$

4.2. Definition. Let $\eta \in V(P,\xi)$. For $\lambda \in \mathfrak{a}^*_{Pq\mathbb{C}}$ with $-(\mathrm{Re}\,\lambda+\rho_P)$ strictly P-dominant we define the function $j(P:\xi:\lambda:\eta): G \to \mathcal{H}_\xi^{-\infty}$ by

$$j(P:\xi:\lambda:\eta)(man\;vh) = a^{\lambda+\rho_P}\xi(m)\eta_v$$

for $v \in^P \mathcal{W}$, $man \in P$, $h \in H$ and by 0 outside $\bigcup_{v\in^P\mathcal{W}} PvH$ (the union of the open H-orbits).

5. Theorem. *Let $\xi \in \widehat{M_P}$ and let $\eta \in V(P,\xi)$. For every $\lambda \in \mathfrak{a}^*_{Pq\mathbb{C}}$ with $-(\mathrm{Re}\,\lambda+\rho_P)$ strictly P-dominant, the function $j(P:\xi:\lambda:\eta)$ defines an element of $C^{-\infty}(P:\xi:\lambda)^H$. Moreover, $\lambda \mapsto j(P:\xi:\lambda:\eta)$ extends meromorphically to $\mathfrak{a}^*_{Pq\mathbb{C}}$ as a function with values in $C^{-\infty}(K:\xi)$. The singular locus of this extended function is the union of a locally finite collection of hyperplanes of the form $\langle\lambda,\alpha\rangle = c$, with $\alpha \in \Sigma(P)$ and $c \in \mathbb{C}$.*

Finally, if λ is a regular value for the meromorphic extension, then

$$j(P:\xi:\lambda:\eta) \in C^{-\infty}(P:\xi:\lambda)^H\,.$$

6. Remarks. The meromorphic continuation is absolutely crucial, since the set $i\mathfrak{a}_{Pq}^*$ (where the $\pi_{P,\xi,\lambda}$ are unitary) is not contained in the region $\langle \mathrm{Re}\,\lambda + \rho_P, \alpha\rangle < 0$ ($\alpha \in \Sigma(P)$).

By meromorphic continuation one still has that $j(P:\xi:\lambda:\eta)(v) = \eta_v$, showing that $j(P:\xi:\lambda) = j(P:\xi:\lambda:\cdot)$ defines an injective homomorphism

$$V(P,\xi) \hookrightarrow C^{-\infty}(P:\xi:\lambda)^H\,,$$

for regular λ. Thus $V(P,\xi)$ becomes a model for the set $\mathcal{V}_{\pi_{P,\xi,\lambda}}$ mentioned in the introduction.

The inner product of $V(P,\xi)$ may be transferred to an inner product on $\mathcal{V}_{\pi_{P,\xi,\lambda}}$. However, it is more convenient to keep working with $V(P,\xi)$, since this space is independent of λ.

7. Remarks. From the description of the singularities it follows that

$$j(P:\xi): i\mathfrak{a}_{Pq}^* \to \mathrm{Hom}(V(P,\xi), C^{-\infty}(K:\xi))$$

is continuous outside a set of measure 0.

8. Definition. We define $\widehat{M}_{P,ps}$ to be the set of $\xi \in \widehat{M}_P$ for which $V(P,\xi) \neq 0$. Equivalently,

$$\exists v \in^P \mathcal{W} : \xi \in (M_P)^\wedge_{M_P \cap vHv^{-1},ds}\,.$$

9. Remark. Let P be a minimal element of $\mathcal{P}_\sigma$ (for inclusion). Then $\mathfrak{a}_{Pq}^+$ is maximal among the parabolic subsets of $\mathfrak{a}_q$, hence an open Weyl chamber. Thus, $\mathfrak{a}_{Pq} = \mathfrak{a}_q$ is maximal abelian in $\mathfrak{p} \cap \mathfrak{q}$. From this one can derive that $M_P/M_P \cap vHv^{-1}$ is compact, for $v \in^P \mathcal{W} = \mathcal{W}$. It follows that $\widehat{M}_{P,ps}$ consists of finite dimensional unitary representations of M. This makes the functional analysis in Theorem 5 considerably simpler. Under the assumption $\#W/W_{K\cap H} = 1$ this case is discussed in [S].

Remark. For the case of minimal $P \in \mathcal{P}_\sigma$, Theorem 5 is due to [6]. For general P it is due to [23].

10. Example. **Riemannian case**, $\sigma = \theta$. We assume $P \in \mathcal{P}_\sigma$ to be minimal. Then $M_P \subset K$, hence $\widehat{M}_{P,ps}$ consists of the trivial representation 1. Moreover, one may take ${}^P\mathcal{W} = \{e\}$ and $V(P,1) = \mathbb{C}$, equipped with the standard inner product.

Now:

$$j(P:1:\lambda:1)(nak) = a^{\lambda+\rho_P}$$

hence $j(P:1:\lambda:1) = \mathbf{1}_\lambda$, the unique K-fixed vector in $\pi_{P,1,\lambda}$ determined by $\mathbf{1}_\lambda(e) = 1$.

11. Definition. Let $P \in \mathcal{P}_\sigma$. The series of unitary representations

$$\pi_{P,\xi,\lambda}\,, \quad \xi \in \widehat{M}_{P,ps}\,, \quad \lambda \in i\mathfrak{a}_{Pq}^*$$

is called the generalized σ-principal series attached to P.

The Plancherel measure will be supported by the generalized σ-principal series.

12. Remark. Assume that center$(\mathfrak{g}) = \{0\}$ and that $P = G$ $(= P_{\{0\}})$. Then one may identify $\widehat{M}_{P,ps}$ with $\widehat{G}_{H,ds}$. Moreover, for $\xi \in \widehat{M}_{P,ps}$, $\mathcal{H}_{P,\xi,0} \simeq \mathcal{H}_\xi^{H,ds}$ (note: $\mathfrak{a}_{Pq} = \{0\}$) and $\pi_{P,\xi,0} \sim \xi$.

Thus, the discrete series might be thought of as the generalized σ-principal series attached to G.

13. Definition. The Fourier transform $\hat{f} = \hat{f}_u$ of a function $f \in C_c^\infty(G/H)$ is defined by

$$\begin{aligned}\hat{f}_u(P:\xi:\lambda) &= \int_{G/H} f(x)\pi_{P,\xi,\lambda}(x) j(P:\xi:\lambda)dx \\ &\in \mathrm{Hom}(V(P,\xi), \mathcal{H}_{P,\xi,\lambda}) \\ &\simeq \mathcal{H}_{P,\xi,\lambda} \otimes \overline{V(P,\xi)}\,,\end{aligned}$$

for $P \in \mathcal{P}_\sigma$, $\xi \in \widehat{M}_{P,ps}$, generic $\lambda \in i\mathfrak{a}_{Pq}^*$. It follows from this definition that $f \mapsto \hat{f}(P:\xi:\lambda)$ intertwines L with $\pi_{P,\xi,\lambda} \otimes I_{\overline{V(P,\xi)}}$.

14. To define the Plancherel measure, we need to introduce the so-called standard intertwining operators.

Let $P, Q \in \mathcal{P}_\sigma$ and assume that $A_{Pq} = A_{Qq}$. Then also $M_P = M_Q$ and $A_P = A_Q$. Let $\xi \in \widehat{M}_{P,ps} = \widehat{M}_{Q,ps}$. Let

$$\Sigma(Q:P) = \{\alpha \in \Sigma \mid \mathfrak{g}_\alpha \subset \overline{\mathfrak{n}}_Q \cap \mathfrak{n}_P\}\,.$$

Then for $\lambda \in \mathfrak{a}_{Pq\mathbb{C}}$ with $\mathrm{Re}\,\langle\lambda, \alpha\rangle$ sufficiently large for all $\alpha \in \Sigma(Q:P)$, the following integral converges absolutely, for $f \in C^\infty(P:\xi:\lambda)$, $x \in G$:

$$A(Q:P:\xi:\lambda)f(x) = \int_{N_Q \cap \overline{N}_P} f(nx)dn\,.$$

Here dn is suitably normalized Haar measure of $N_Q \cap \overline{N}_P$. Moreover, it can be shown that $A(Q : P : \xi : \lambda)$ defined above is a continuous linear operator $C^\infty(P : \xi : \lambda) \to C^\infty(Q : \xi : \lambda)$. It obviously intertwines the representations $\pi_{P,\xi,\lambda}$ and $\pi_{Q,\xi,\lambda}$. Moreover, $A(Q : P : \xi : \lambda)$ can be meromorphically extended in the parameter $\lambda \in \mathfrak{a}_{Pq\mathbb{C}}$.

15. Theorem. *Let $\xi \in \widehat{M}_{P,ps}$. Then for every $\lambda \in i\mathfrak{a}_{Pq}$ with $\langle \lambda, \alpha \rangle \neq 0$ for every $\alpha \in \Sigma(P)$, the representation $\pi_{P,\xi,\lambda}$ is irreducible (unitary).*

Proof. This follows from a result of Bruhat-Harish-Chandra, using the information of type §3, 5.2 on ξ.

16. The adjoint operator

$$A(Q : P : \xi : -\overline{\lambda})^* : C^{-\infty}(Q : \xi : \lambda) \to C^{-\infty}(P : \xi : \lambda)$$

depends meromorphically on λ and is G-equivariant. Therefore

$$A(\overline{P} : P : \xi : -\overline{\lambda})^* \circ A(\overline{P} : P : \xi : \lambda) \tag{16-1}$$

is a G-intertwining operator from $C^\infty(P : \xi : \lambda)$ to $C^{-\infty}(P : \xi : \lambda)$ for generic $\lambda \in i\mathfrak{a}_{Pq}^*$, hence a scalar by the above theorem. By meromorphy it follows that (16-1) equals

$$\eta(P : \xi : \lambda)I$$

with $\eta(P : \xi : \cdot) : \mathfrak{a}_{Pq\mathbb{C}}^* \to \mathbb{C}$ a meromorphic function. Clearly, from the expression (16-1) it follows that $\eta \geq 0$ on $i\mathfrak{a}_{Pq}^*$. Hence $\eta(P : \xi : \cdot)^{-1}$ defines a measurable function on $i\mathfrak{a}_{Pq}^*$ with values in $[0, \infty[$ (almost everywhere). We define the measure $d\mu_{P,\xi}$ on $i\mathfrak{a}_{Pq}^*$ by

$$d\mu_{P,\xi}(\lambda) = \frac{1}{\eta(P : \xi : \lambda)} d\lambda_P\,, \tag{16-2}$$

where $d\lambda_P$ is Lebesgue measure in $i\mathfrak{a}_{Pq}^*$, suitably normalized.

17. Finally, let

$$W^*(\mathfrak{a}_{Pq}) = N_W(\mathfrak{a}_{Pq})$$

and recall that $W_P = Z_W(\mathfrak{a}_{Pq})$. We define

$$W(\mathfrak{a}_{Pq}) = W^*(\mathfrak{a}_{Pq})/W_P\,.$$

18. Definition. Two parabolic subgroups $P, Q \in \mathcal{P}_\sigma$ are said to be associated if $\mathfrak{a}_{Pq}$ and $\mathfrak{a}_{Qq}$ are W-conjugate.

We denote the equivalence relation of being associated by $\sim$.

Let $\mathbb{P}_\sigma$ be a set of representatives in $\mathcal{P}_\sigma$ for the classes of $\sim$. Thus, $\mathbb{P}_\sigma$ is in (1-1) correspondence with $\mathcal{P}_\sigma / \sim$.

19. Theorem (pre-Plancherel theorem). *Let $C_c^\infty(G/H)$. Then*

$$\|f\|^2_{L^2(x)} = \sum_{P \in \mathbb{P}_\sigma} [W : W^*(\mathfrak{a}_{Pq})] \sum_{\xi \in \widehat{M}_{P,ps}} \int_{i\mathfrak{a}^*_{Pq}} \|\hat{f}_u(P : \xi : \lambda)\|^2 d\mu_{P,\xi}(\lambda)\,.$$

20. Corollary. *$f \mapsto \hat{f}_u$ extends to an isometry $\mathfrak{F}$ from $L^2(G/H)$ into the Hilbert space*

$$\mathfrak{H}_{\mathrm{pre}} := \bigoplus_{P \in \mathbb{P}_\sigma} \widehat{\bigoplus}_{\xi \in \widehat{M}_{P,ps}} L^2(i\mathfrak{a}^*_{Pq}, L^2(K : \xi) \otimes \overline{V(P : \xi)}\,, [W : W^*(\mathfrak{a}_{Pq})] d\mu_{P,\xi})\,,$$

intertwining L with the representation π_{pre} in $\mathfrak{H}_{\mathrm{pre}}$ given by

$$[\pi_{\mathrm{pre}}(x)\varphi]_{P,\xi}(\lambda) = (\pi_{P,\xi,\lambda}(x) \otimes I)\varphi_{P,\xi}(\lambda)\,.$$

21. Remarks. The reason for passing to $\mathbb{P}_\sigma$ is that the principal series for associated P, Q are related by intertwining operators.

First, assume that $\mathfrak{a}_{Pq} = \mathfrak{a}_{Qq}$. Then the standard intertwining operator $A(Q : P : \xi : \lambda)$ intertwines the representations $\pi_{P,\xi,\lambda}$ and $\pi_{Q,\xi,\lambda}$ for $\xi \in \widehat{M}_{P,ps} = \widehat{M}_{Q,ps}$ and generic $\lambda \in i\mathfrak{a}^*_{Pq}$.

If $Q = wPw^{-1}$ for some Weyl group element $w \in W$, then there also is an intertwining operator between principal series representations for P and Q. It is defined as follows. We observe that $wM_Pw^{-1} = M_Q$. Hence if $\xi \in \widehat{M}_P$, then the representation $w \cdot \xi$ defined by $w\xi(m) = \xi(w^{-1}mw)$ belongs to $\widehat{M}_Q$ (here we have abused notation, w should be replaced by a representative in $N_K(\mathfrak{a}_q)$). Now the map $L(w)$ given by

$$L(w)\varphi(x) = \varphi(w^{-1}x)$$

defines an intertwining operator from $\mathcal{H}_{P,\xi,\lambda}$ to $\mathcal{H}_{Q,w\xi,w\lambda}$ which is obviously unitary.

Finally, if ${}^P\mathcal{W}$ is a set of representatives for $W_P \setminus W/W_{K\cap H}$ in $N_K(\mathfrak{a}_q)$, then ${}^Q\mathcal{W} = w^P\mathcal{W}$ is a set of representatives for $W_Q \setminus W/W_{K\cap H}$. This implies that $\xi \mapsto w\xi$ is a bijection from $\widehat{M}_{P,ps}$ onto $\widehat{M}_{Q,ps}$.

In general, if $Q \sim P$, then there exists $w \in W$ such that $\mathfrak{a}_{Qq} = w(\mathfrak{a}_{Pq}) = \mathfrak{a}_{wPw^{-1},q}$. Combining the above two cases we see that for each such w, the operator

$$A(Q : wPw^{-1} : w\xi : w\lambda) \circ L(w) : \mathcal{H}_{P,\xi,\lambda} \to \mathcal{H}_{Q,w\xi,w\lambda} \tag{21-1}$$

intertwines the principal series $\pi_{P,\xi,\lambda}$ and $\pi_{Q,w\xi,w\lambda}$. Therefore, we need not use both of these series.

22. We have called Theorem 19 the pre-Plancherel theorem, since we have not described the image of $\mathfrak{F}$. In fact, $\mathfrak{F}$ is not onto $\mathfrak{H}_{\mathrm{pre}}$, due to the presence of intertwining operators, even when we have passed to $\mathbb{P}_\sigma$ as explained in the previous subsection.

In fact, let $w \in W^*(\mathfrak{a}_{Pq})$. Then $\mathfrak{a}_{Pq} = w(\mathfrak{a}_{Pq}) = \mathfrak{a}_{wPw^{-1},q}$ and it follows from (21-1) with $Q = P$ that $A(P : wPw^{-1} : w\xi : w\lambda) \circ L(w)$ intertwines $\pi_{P,\xi,\lambda}$ with $\pi_{P,w\xi,w\lambda}$.

23. Proposition. *Let $P \in \mathcal{P}_\sigma$, $w \in W^*(\mathfrak{a}_{Pq})$ and $\xi \in \widehat{M}_{P,ps}$. Then $w\xi \in \widehat{M}_{P,ps}$, and $d\mu_{P,w\xi}(w\lambda) = d\mu_{P,\xi}(\lambda)$. Moreover, there exists a unique unitary isomorphism*

$$\mathfrak{C}_{P,w}(\xi,\lambda) : L^2(K : \xi) \otimes \overline{V(\xi)} \to L^2(K : w\xi) \otimes \overline{V(w\xi)}$$

depending on $\lambda \in i\mathfrak{a}_{Pq}$ in a measurable way, such that

$$\hat{f}(P : w\xi : w\lambda) = \mathfrak{C}_{P,w}(\xi,\lambda)\hat{f}(P : \xi : \lambda) .$$

The operator $\mathfrak{C}_{P,w}(\xi,\lambda)$ intertwines $\pi_{P,\xi,\lambda} \otimes 1$ with $\pi_{P,w\xi,w\lambda} \otimes 1$.

24. We now define $\mathfrak{H}$ to be the closed subspace of $\mathfrak{H}_{\mathrm{pre}}$ consising of all $\varphi \in \mathfrak{H}_{\mathrm{pre}}$ satisfying

$$\varphi_P(w\xi, w\lambda) = \mathfrak{C}_{P,w}(\xi,\lambda)\varphi_P(\xi,\lambda) ,$$

for all $P \in \mathcal{P}_\sigma$, $w \in W^*(\mathfrak{a}_{Pq})$, $\xi \in \widehat{M}_{P,ps}$ and almost all $\lambda \in i\mathfrak{a}^*_{Pq}$.

From the equivariance of the $\mathfrak{C}_{P,w}(\xi,\lambda)$ it follows that $\mathfrak{H}$ is a G-invariant subspace of $\mathfrak{H}_{\mathrm{pre}}$. Let $\pi = \pi_{\mathrm{pre}}\big|_{\mathfrak{H}}$.

25. Theorem (Plancherel theorem). *The map $\mathfrak{F}$ is an isometry from $L^2(X)$ onto $\mathfrak{H}$, intertwining L with π, establishing the Plancherel decomposition.*

For every $P \in \mathcal{P}_\sigma$ we put

$$\mathfrak{a}_{Pq}^{*\mathrm{reg}} = \{\lambda \in \mathfrak{a}_{Pq}^* \mid \langle \lambda, \alpha \rangle \neq 0 \, \forall \alpha \in \Sigma(P)\} .$$

The group $W(\mathfrak{a}_{Pq}) = W^*(\mathfrak{a}_{Pq})/W_P$ acts freely, but not transitively (in general) on the components of $\mathfrak{a}_{Pq}^{*\mathrm{reg}}$ (which are the parabolic subsets in $\mathfrak{a}_q^*$ whose span equals $\mathfrak{a}_{Pq}^*$).

Let $\mathfrak{a}_{Pq}^{*0}$ be a fundamental domain for the action of $W(\mathfrak{a}_{Pq})$ on $\mathfrak{a}_{Pq}^{*\mathrm{reg}}$, consisting of connected components of $\mathfrak{a}_{Pq}^{*\mathrm{reg}}$. Then the above theorem leads to the following direct integral decomposition of the left regular representation L of G in $L^2(X)$:

$$L \sim \bigoplus_{P \in \mathbb{P}_\sigma} \bigoplus_{\xi \in \widehat{M}_{P,ps}} \int_{i\mathfrak{a}_{Pq}^{*0}} \pi_{P,\xi,\lambda} \otimes 1_{\overline{V(P,\xi)}} \frac{|W|}{|W_P|} d\mu_{P,\xi}(\lambda) .$$

In particular, it follows that the representation $\pi_{P,\xi,\lambda}$ occurs with multiplicity $\dim V(P,\xi)$.

26. The fact that $\hat{f}_u(P : \xi : \lambda)$ may have singularities as a function of $\lambda \in i\mathfrak{a}_{Pq}^*$, even for $f \in C_c^\infty(G/H)$ might be considered as a less appealing aspect from the esthetic point of view. It is a remarkable fact that this can be remedied by normalizing $\hat{f}$ in a different way. As a matter of fact, this is a crucial aspect of the proofs.

The idea is to replace $j(P : \xi : \lambda)$ by a differently normalized element of $\mathrm{Hom}(V(P,\xi), C^{-\infty}(P : \xi : \lambda)^H)$.

27. Definition. Let $P \in \mathcal{P}_\sigma$, $\xi \in \widehat{M}_{P,ps}$. For generic $\lambda \in \mathfrak{a}_{Pq\mathbb{C}}^*$ we put

$$j^0(P : \xi : \lambda) = A(P : \overline{P} : \xi : -\overline{\lambda})^{*-1} j(P : \xi : \lambda) .$$

Then $j^0(P : \xi : \lambda)$ is meromorphic as a function of $\lambda \in i\mathfrak{a}_{Pq}^*$ with values in $C^{-\infty}(K : \xi) \otimes \overline{V(P : \xi)}$. The remarkable fact is:

28. Theorem (regularity theorem). *The meromorphic function $\lambda \mapsto j^0(P : \xi : \lambda)$ is regular on $i\mathfrak{a}_{Pq}^*$.*

We now define the normalized Fourier transform $\hat{f} = \hat{f}_n$ of a function $f \in C_c^\infty(G/H)$ as $\hat{f}$, but with everywhere $j(P : \xi : \lambda)$ replaced by $j^0(P : \xi : \lambda)$ (see 13).

29. Corollary. *Let $f \in C_c^\infty(G/H)$, $P \in \mathcal{P}_\sigma$, $\xi \in \widehat{M}_{P,ps}$. Then $\lambda \mapsto \hat{f}_n(P : \xi : \lambda)$ is analytic as a function of $\lambda \in i\mathfrak{a}_{Pq}^*$ with values in $C^\infty(K : \xi) \otimes \overline{V(\xi)}$.*

30. A simple computation leads to the following relation between $\hat{f}_u$ and $\hat{f}_n$, for $f \in C_c^\infty(G/H)$.

$$\begin{aligned}
&\|\hat{f}_n(P:\xi:\lambda)\|^2 = \\
&\qquad = \langle \hat{f}_u(\overline{P}:\xi:\lambda) \mid A(P:\overline{P}:\xi:-\overline{\lambda})^{-1} A(P:\overline{P}:\xi:\overline{\lambda})^{*-1} \hat{f}_u(\overline{P}:\xi:\lambda)\rangle \\
&\qquad = \eta(\overline{P}:\xi:\lambda)^{-1} \|\hat{f}_u(\overline{P}:\xi:\lambda)\|^2 .
\end{aligned}$$

This has the effect that the Plancherel measure becomes ordinary Lebesgue measure for the normalized Fourier transform.

From now on we will always write $\hat{f}$ instead of $\hat{f}_n$ and we replace (16-2) by

$$d\mu_{P,\xi,\lambda} = d\lambda_P \qquad (P \in \mathcal{P}_\sigma, \xi \in \widehat{M}_{P,ps}) .$$

31. Theorem ('normalized' Plancherel theorem). *Theorem 19, Corollary 20, Proposition 23 (with different $\mathfrak{C}_{P,w}$) and Theorem 25 are valid with the normalized version of $f \mapsto \hat{f}$ and with the above normalization of measure. Hence*

$$L \sim \bigoplus_{P \in \mathbb{P}_\sigma} \bigoplus_{\xi \in \widehat{M}_{P,ps}} \int_{i\mathfrak{a}_{Pq}^{*0}} \pi_{P,\xi,\lambda} \otimes 1_{\overline{V(P,\xi)}} \frac{|W|}{|W_P|} d\lambda_P .$$

32. Remark. In Delorme's paper the above formula occurs without the constants $|W_P|^{-1}|W|$. This is due to a different normalization of measures. Here we agree to normalize measures as follows. If G/H is any reductive symmetric space in our class, we will always normalize dx on G/H and da on A_q so that the formula of Theorem §2, 8 (polar decomposition) is valid. Let $d\lambda_{ft}$ be the choice of Lebesgue measure on $i\mathfrak{a}_q^*$ that turns the Euclidean Fourier transform into an isometry $L^2(A_q) \to L^2(i\mathfrak{a}_q^*)$. We choose $d\lambda = |W|^{-1} d\lambda_{ft}$, whereas Delorme choses $d\lambda = d\lambda_{ft}$.

Let $P \in \mathcal{P}_\sigma$, $v \in {}^P\mathcal{W}$. Then the convention mentioned above, applied to $M_P/M_P \cap vHv^{-1}$ links the measures $dx_{P,v}$ on $M_P/M_P \cap vHv^{-1}$ and $da_{P,v}$ on ${}^*A_{Pq}$. Here we have written ${}^*A_{Pq} = \exp({}^*\mathfrak{a}_{Pq})$, ${}^*\mathfrak{a}_{Pq} = \mathfrak{a}_{Pq}^\perp \cap \mathfrak{a}_q$. The latter is the $\mathfrak{a}_q$ of $\mathfrak{m}'_P$. A choice of measure $da_{P,v}$ induces a choice of measure $d\lambda_{P,v}$ on $i^*\mathfrak{a}_{Pq}^*$ as indicated above. Choices can be made so that $d\lambda_{P,v}$ is independent of $v \in {}^P\mathcal{W}$. We now choose $d\mu_{P,\xi}$ to be the quotient of $d\lambda$ and $d\lambda_{P,v}$ (note that $\mathfrak{a}_q^* = {}^*\mathfrak{a}_{Pq}^* \oplus \mathfrak{a}_{Pq}^*$). Thus, $d\mu_{P,\xi} = d\mu_P$ is independent of ξ.

§7. The spherical Plancherel theorem

1. Let $\delta \in \widehat{K}$. Then the Plancherel theorem is proved by showing that $\mathfrak{F}$ is an isometry from $L^2(X)_\delta$, the space of K-finite functions of type δ in $L^2(X)$, onto $\mathfrak{H}_\delta$, the similar subspace of $\mathfrak{H}$. The restriction of $\mathfrak{F}$ to $L^2(X)_\delta$ naturally leads to the concept of the Eisenstein integral, as we will now indicate.

We start by observing that

$$\begin{aligned} L^2(X)_\delta &\simeq \mathrm{Hom}_K(V_\delta, L^2(X)) \otimes V_\delta \\ &\simeq (L^2(X) \otimes V_\delta^*)^K \otimes V_\delta \,. \end{aligned}$$

Put $V_\tau = V_{\tau_\delta} = V_\delta^* \otimes V_\delta$ and $\tau = \tau_\delta := \delta^* \otimes 1$, then it follows that

$$\begin{aligned} L^2(X)_\delta &\simeq (L^2(X) \otimes V_\tau)^K \\ &= L^2(X : \tau)\,, \end{aligned}$$

where the latter space is the space of functions φ in $L^2(X, V_\tau)$ that are τ-spherical, i.e.,

$$\varphi(kx) = \tau(k)\varphi(x)\,, \qquad (x \in X\,,\, k \in K)\,.$$

We note that similar considerations hold for $C_c^\infty(X)$, $C^\infty(X)$, $C(X)$. In the L^2-context all the natural isomorphisms are isometric if we agree to equip $V_\delta^* \otimes V_\delta \simeq \mathrm{End}(V_\delta)$ with d_δ^{-1} times the Hilbert–Schmid inner product. (Note that we have used nothing special about X; the whole construction works for a manifold X equipped with a smooth K-action and with a K-invariant density).

We shall call the natural isomorphism $L^2(X)_\delta \to L^2(X : \tau_\delta)$ sphericalization, and denote it $f \mapsto f^{\mathrm{sph}}$.

2. We recall that Fourier transform $f \mapsto \hat{f}(P : \xi : \lambda)$ may also be given by testing f with a matrix coefficient (see §1, n^o 13). In the present context, define

$$\mathrm{M}_{P,\xi,\lambda} : C^\infty(K : \xi) \otimes \overline{V(P,\xi)} \to C^\infty(G/H)$$

by

$$\begin{aligned} &\mathrm{M}_{P,\xi,\lambda}(\varphi \otimes \eta)(x) = \\ &\qquad = \langle \varphi \mid \pi_{P,\xi,-\overline{\lambda}}(x) j^0(P : \xi : -\overline{\lambda})\eta \rangle\,, \end{aligned}$$

then, for $f \in C_c^\infty(G/H)$, $T \in C^\infty(K : \xi) \otimes \overline{V(P,\xi)}$,

$$\langle \hat{f}(P : \xi : \lambda) \mid T \rangle = \langle f \mid \mathrm{M}_{P,\xi,-\overline{\lambda}}(T) \rangle\,.$$

Note that $\mathrm{M}_{P,\xi,\lambda}$ intertwines $\pi_{P,\xi,\lambda} \otimes I$ with L. Hence, $\mathrm{M}_{P,\xi,\lambda}$ maps $C^\infty(K : \xi)_\delta \otimes \overline{V(P, \xi)}$ into $C^\infty(G/H)_\delta \simeq C^\infty(G/H : \tau_\delta)$. An Eisenstein integral is essentially an element in the image of $\mathrm{M}_{P,\xi,\lambda}$, viewed as an element of $C^\infty(G/H : \tau_\delta)$. The Eisenstein integral becomes a very practical tool if we parametrize $C^\infty(K : \xi)_\delta \otimes \overline{V(P, \xi)}$ differently.

3. If $\xi \in \widehat{M}_{P,ps}$, $v \in {}^P\mathcal{W}$, we put

$$L^2(M_P/M_P \cap vHv^{-1})_\xi$$

for the image of $\mathcal{H}_\xi \otimes \overline{V(P, \xi, v)}$ under the matrix coefficient map (see §1, n^o 14).

Recall that if $\xi \notin (M_P)^\wedge_{M_P \cap vHv^{-1}, ds}$, then $V(P, \xi, v) = 0$, hence the above space is trivial. Recall that $L^2(M_P/M_P \cap vHv^{-1} : \tau_\delta) = L^2(M_P/M_P \cap vHv^{-1}) \otimes V_{\tau_\delta})^{K_P}$. Accordingly we put

$$L^2(M_P/M_P \cap vHv^{-1} : \tau_\delta)_\xi := (L^2(M_P/M_P \cap vHv^{-1})_\xi \otimes V_{\tau_\delta})^{K_P} .$$

4. Lemma. *The space $C^\infty(K : \xi)_\delta \otimes \overline{V(P, \xi)}$ is finite dimensional and equals the finite dimensional Hilbert space $L^2(K : \xi)_\delta \otimes \overline{V(P, \xi)}$. Moreover, there is a natural isometrical isomorphism*

$$C^\infty(K : \xi)_\delta \otimes \overline{V(P : \xi)} \to \bigoplus_{v \in {}^P\mathcal{W}} L^2(M_P/M_{P \cap vHv^{-1}} : \tau_\delta)_\xi \,,$$

where $\oplus$ denotes the formal Hilbert sum.

Proof. First, we note that $L^2(K : \xi)$ is the representation space for $\mathrm{ind}_{K_P}^K(\xi \mid K_P)$, where $K_P = K \cap M_P$. By Frobenius reciprocity we have

$$\mathrm{Hom}_K(V_\delta, L^2(K : \xi)) \simeq \mathrm{Hom}_{K_P}(V_\delta, \mathcal{H}_\xi) \,. \tag{4-1}$$

hence

$$\begin{aligned} L^2(K : \xi)_\delta &\simeq \mathrm{Hom}_{K_P}(V_\delta, \mathcal{H}_\xi) \otimes V_\delta \\ &\simeq (\mathcal{H}_\xi \otimes V_{\tau_\delta})^{K_P} \,. \end{aligned} \tag{4-2}$$

It is a standard fact from representation theory that each K_P-type occurs with finite multiplicity in $\xi \in \widehat{M}_P$. Hence the space in (4-1) is finite dimensional. It follows that $L^2(K : \xi)_\delta$ is finite dimensional hence equals its dense subspace $C^\infty(K : \xi)_\delta$.

This establishes the first assertions. From (4-2) it follows that, for $v \in {}^P\mathcal{W}$,

$$\begin{aligned} L^2(K:\xi)_\delta \otimes \overline{V(P,\xi,v)} &\simeq \\ &\simeq (\mathcal{H}_\xi \otimes \underset{(1)}{\overline{V(P,\xi,v)}} \otimes V_{\tau_\delta})^{K_P} \\ &\simeq (L^2(M_P/M_P \cap vHv^{-1})_\xi \otimes V_{\tau_\delta})^{K_P} \end{aligned}$$

by the matrix coefficient map of ξ. Here (1) under a tensor component means that the action of the group K_P is trivial on that component. The argument is completed by taking the direct sum over $v \in {}^P\mathcal{W}$. $\square$

We denote the isomorphism of Lemma 4 by $T \mapsto \psi_T$, a notation that is compatible with Harish-Chandra's notation in the group case.

5. Definition. Let $\psi \in \bigoplus_{v \in {}^P\mathcal{W}} L^2(M_P/M_P \cap vHv^{-1} : \tau_\delta)_\xi$. Then the normalized Eisenstein integral $E^0(P:\psi:\lambda)$ is defined by ($\lambda \in \mathfrak{a}^*_{P\mathbb{C}}$):

$$E^0(P:\psi:\lambda) = (\mathrm{M}_{P,\xi,-\lambda}(T))^{\mathrm{sph}} \in C^\infty(G/H:\tau_\delta)\,,$$

where $T \in C^\infty(K:\xi) \otimes \overline{V(P,\xi)}$ is such that $\psi = \psi_T$.

6. The above definition of the Eisenstein integral can be extended to a bigger ψ-space, by collecting all $\xi \in \widehat{M}_{P,ps}$ together. We need some preparation for this.

Let (τ, V_τ) be any finite dimensional unitary representation of K. Then by

$$\mathcal{A}_2(G/H:\tau)$$

we denote the space of smooth functions $f \in C^\infty(G/H:\tau)$ that satisfy

(1) $f \in L^2(G/H:\tau)$,

(2) $\mathbb{D}(X)f$ finite dimensional.

Theorem. *The space $\mathcal{A}_2(G/H:\tau)$ is finite dimensional. Moreover, it decomposes as the orthogonal direct sum*

$$\mathcal{A}_2(G/H:\tau) = \bigoplus_{\xi \in \widehat{G}_{H,ds}} L^2(G/H:\tau)_\xi\,.$$

In particular, only finitely many summands in the direct sum are non-zero.

Proof. This is a deep result, which is equivalent to the assertion that for a given $\delta \in \widehat{K}$ only finitely many representations from $\widehat{G}_{H,ds}$ contain the K-type δ. It follows from the classification of the discrete series by Oshima and Matsuki, and also from our proof of the Plancherel formula, if one only uses the information on the discrete series mentioned in §3, 5.2.

7. We define the finite dimensional Hilbert space

$$\mathcal{A}_{2,P} := \bigoplus_{v \in {}^P\mathcal{W}} \mathcal{A}_2(M_P/M_P \cap vHv^{-1} : \tau_\delta\big|_{K_P}) .$$

Then by the above theorem, applied to $M_P/M_P \cap vHv^{-1}$, for $v \in {}^P\mathcal{W}$,

$$\mathcal{A}_{2,P} = \bigoplus_{\xi \in \widehat{M}_{P,ps}} \mathcal{A}_{2,P,\xi} ,$$

where

$$\mathcal{A}_{2,P,\xi} = \bigoplus_{v \in {}^P\mathcal{W}} L^2(M_P/M_P \cap vHv^{-1} : \tau_\delta)_\xi .$$

8. Definition. For $\psi \in \mathcal{A}_{2,P}$ we define the normalized Eisenstein integral $E^0(P : \psi : \lambda) \in C^\infty(G/H : \tau)$ by

$$E^0(P : \psi : \lambda) = \sum_{\xi \in \widehat{M}_{P,ps}} E^0(P : \psi_\xi : \lambda) .$$

9. Proposition. *The Eisenstein integral $E^0(P : \psi : \lambda)$ is meromorphic as a function of $\lambda \in \mathfrak{a}^*_{Pq\mathbb{C}}$ with values in $C^\infty(G/H : \tau)$. Moreover, it behaves finitely under the action of $\mathbb{D}(G/H)$, for generic $\lambda \in \mathfrak{a}^*_{Pq\mathbb{C}}$.*

10. Theorem (Regularity theorem). *$\lambda \mapsto E^0(P : \psi : \lambda)$ is regular on $i\mathfrak{a}^*_{Pq}$, for every $\psi \in \mathcal{A}_{2,P}$.*

The regularity theorem for j^0 follows from this regularity theorem, which in turn is proved by a careful analysis of the asymptotic behavior of the Eisenstein integral, see 18.

11. The above enables us to encode the Fourier transform in terms of the Eisenstein integral.

Lemma. *Let $F \in C_c^\infty(G/H : \tau_\delta)$, and let f be the corresponding function in $C_c^\infty(G/H)_\delta$. So $F = f^{\mathrm{sph}}$. Then*

$$\langle \hat{f}(P : \xi : \lambda) \mid T\rangle = \int_{G/H} \langle F(x) \mid E^0(P : \psi_T : \bar{\lambda})(x)\rangle_{V_\tau} dx\,.$$

Proof.

$$\begin{aligned}\langle \hat{f}(P : \xi : \lambda) \mid T\rangle &= \langle f \mid \mathrm{M}_{P,\xi - \overline{\lambda}}(T)\rangle \\ &= \langle F \mid \mathrm{M}_{P,\xi, -\overline{\lambda}}(T)^{\mathrm{sph}}\rangle \\ &= \int_{G/H} \langle F(x) \mid E^0(P : \psi_T : \overline{\lambda})(x)\rangle_{V_\tau} dx.\end{aligned}$$

□

12. The above motivates the following definition of the spherical Fourier transform. We write $E^0(P : \lambda)$ for the $\mathrm{Hom}(\mathcal{A}_{2,P}, V_\tau)$-valued function on G/H given by

$$E^0(P : \lambda)(x)\psi = E^0(P : \psi : \lambda)(x)\,,$$

$\psi \in \mathcal{A}_{2,P}$, $x \in G/H$. Moreover, we define

$$E^*(P : \lambda : x) := E^0(P : -\overline{\lambda} : x)^* \in \mathrm{Hom}(V_\tau, \mathcal{A}_{,P})\,.$$

Definition. If $F \in C_c^\infty(G/H : \tau)$, we define the spherical Fourier transform $\mathcal{F}_P F : i\mathfrak{a}_{Pq}^* \to \mathcal{A}_{2,P}$ by

$$\mathcal{F}_P F(\lambda) = \int_{G/H} E^*(P : \lambda : x)F(x)dx\,.$$

It follows from this definition, that

$$\langle \hat{f}(P : \xi : \lambda) \mid T\rangle = \langle \mathcal{F}_P F(-\lambda) \mid \psi_T\rangle$$

in the notation of 6. The change of λ to $-\lambda$ is somewhat ackward, but incorporated to guarantee that the asymptotic expressions for the Eisenstein integrals have a traditional form.

13. Theorem. *$\mathcal{F}_P$ maps $C_c^\infty(G/H : \tau)$ continuous linearly into $\mathcal{S}(i\mathfrak{a}_{Pq}^*) \otimes \mathcal{A}_{2,P}$. Here $\mathcal{S}(i\mathfrak{a}_{Pq}^*)$ denotes the classical Euclidean Schwartz space.*

The operator $\mathcal{F}_P$ has as its adjoint the so-called *wave parket* operator $\mathcal{I}_P : \mathcal{S}(i\mathfrak{a}_{Pq}^*) \to C^\infty(G/H : \tau)$ defined by

$$\mathcal{I}_P\varphi(x) = \int_{i\mathfrak{a}_{Pq}^*} E^0(P : \lambda : x)\varphi(\lambda)d\lambda\,.$$

14. Theorem (pre-Plancherel theorem). *Let $f \in C_c^\infty(G/H : \tau)$. Then*

$$f = \sum_{P\in\mathbb{P}_\sigma} [W : W^*(\mathfrak{a}_{Pq})]\,\mathcal{I}_P \cdot \mathcal{F}_P f\,.$$

The pre-Plancherel theorem of the privious section follows from the above theorem by reduction to K-types.

15. Remark. There is a notion of Schwartz space on G/H. Let $\tau : G/H \to [0,\infty[$ be defined by $\tau(kah) = \|\log a\|$. Then the L^2-Schwartz space of G/H is defined by

$$\mathcal{C}(G/H) = \{f \in C^\infty(G/H) \mid \forall_{\substack{u\in\mathcal{U}(\mathfrak{g})\\ n\in\mathbb{N}}} : (1+\tau)^n L_u f \in L^2(G/H)\}\,.$$

It can be shown that the operators $\mathcal{F}_P$ and $\mathcal{I}_P$ extend continuously to operators $\mathcal{C}(G/H : \tau) \to \mathcal{S}(i\mathfrak{a}_{Pq}^*) \otimes \mathcal{A}_{2,P}$ and "$\leftarrow$", respectively.

16. For the full Plancherel theorem, we need to give a description of the image of $(\mathcal{F}_P)_{P\in\mathbb{P}_\sigma}$. This involves the asymptotic behavior of the Eisenstein integral.

17. If $P, Q \in \mathcal{P}_\sigma$ are associated, we define

$$W(\mathfrak{a}_{Qq} \mid \mathfrak{a}_{Pq}) = \{s\big|_{\mathfrak{a}_{Pq}} \mid s \in W, s(\mathfrak{a}_{Pq}) \subset \mathfrak{a}_{Q_q}\} \subset \mathrm{Hom}(\mathfrak{a}_{Pq}, \mathfrak{a}_{Qq})\,.$$

Theorem. *Let $P \in \mathcal{P}_\sigma$. Let $Q \in \mathcal{P}_\sigma$ be associated with P. Then there exist unique meromorphic functions*

$$\mathfrak{a}_{Pq\mathbb{C}}^* \ni \lambda \mapsto C_{Q|P}^0(s : \lambda) \in Hom(\mathcal{A}_{2,P}, \mathcal{A}_{2,Q})\,,$$

for $s \in W(\mathfrak{a}_{Qq} \mid \mathfrak{a}_{Pq})$, such that, for $\lambda \in i\mathfrak{a}_{Pq}^*$, every $v \in {}^Q\mathcal{W}$, $m \in M_Q/M_Q \cap vHv^{-1}$, $\psi \in \mathcal{A}_{2,P}$,

$$\begin{aligned}&E^0(P : \lambda : mav)\psi \\ &\sim \sum_{s\in W(\mathfrak{a}_{Qq}|\mathfrak{a}_{Pq})} a^{s\lambda-\rho_Q}[C_{Q|P}^0(s : \lambda)\psi]_v(m)\,.\end{aligned}$$

(Here $s\lambda := \lambda \circ s^{-1}$).

Proposition. $C^0_{P|P}(1 : \lambda) = I$.

This follows from the chosen normalization of j^0.

18. The proof of the regularity theorem is based on asymptotic analysis together with the following important fact.

Theorem (The Maass-Selberg relations). *Let $P, Q \in \mathcal{P}_\sigma$ be associated, $s \in W(\mathfrak{a}_{Qq} \mid \mathfrak{a}_{Pq})$. Then*

$$C^0_{Q|P}(s : -\overline{\lambda})^* C^0_{Q|P}(s : \lambda) = I\,.$$

In particular, if $\lambda \in i\mathfrak{a}^*_{Pq}$, then $C^0_{Q|P}(s : \lambda)$ is unitary.

Note. Theorem 18 is due to vdB for P minimal and to Delorme for general P (see the historical notes of §1, n^o 6).

19. We can now define the subspace $(\mathcal{S}(i\mathfrak{a}^*_{Pq}) \otimes \mathcal{A}_{2,P})^{W(\mathfrak{a}_{Pq})}$ of $\mathcal{S}(i\mathfrak{a}^*_{Pq}) \otimes \mathcal{A}_{2,P}$ consisting of the functions φ satisfying

$$\varphi(s\lambda) = C^0_{P|P}(s : \lambda)\varphi(\lambda)\,,$$

for all $\lambda \in i\mathfrak{a}^*_{Pq}$, $s \in W(\mathfrak{a}_{Pq})$.

20. Theorem (spherical Plancherel theorem). *The map $\mathcal{F} = \oplus_{P \in \mathbb{P}_\sigma} \mathcal{F}_P$ is a topological linear isomorphism:*

$$\mathcal{C}(G/H : \tau) \xrightarrow{\simeq} \bigoplus_{P \in \mathbb{P}_\sigma} (\mathcal{S}(i\mathfrak{a}^*_{Pq}) \otimes \mathcal{A}_{2,P})^{W(\mathfrak{a}_{Pq})}\,.$$

Its inverse is given by

$$\mathcal{I} = \oplus_{P \in \mathbb{P}_\sigma} [W : W^*(\mathfrak{a}_{Pq})]\, \mathcal{I}_P\,.$$

Moreover, for every $f \in \mathcal{C}(G/H : \tau)$,

$$\|f\|^2_{L^2(X:\tau)} = \sum_{P \in \mathbb{P}_\sigma} [W : W^*(\mathfrak{a}_{Pq})]\, \|\mathcal{F}_P f\|^2_{L^2}\,.$$

$\mathbb{P}_\sigma$ contains exactly one minimal parabolic, since all such are associated. Let us denote it by P_0. We denote by $\mathcal{C}_{mc}(G/H : \tau)$ (most continuous part) the image of $\mathcal{I}_{P_0}$. The following results were proved in our earlier work on the most continuous part of the Plancherel theorem.

21. Theorem (B & S [15]): *There exists a $D \in \mathbb{D}(X)$, depending on τ, such that*

(1) D *is injective on* $C_c^\infty(G/H:\tau)$*,*

(2) $D \circ \mathcal{I}_{P_0} \circ \mathcal{F}_{P_0} = D$ *on* $\mathcal{C}(G/H:\tau)$*.*

The above result implies that

$$D \circ \mathcal{I}_P = 0 \qquad (\forall P \in \mathbb{P}_\sigma \setminus \{P_0\})\,.$$

We derive the above Plancherel formula from Theorem 21 by means of a residue calculus for root systems. Sketches of ideas of the proofs may be found in [S] and [BS].

Appendix. Reductive symmetric spaces of Harish-Chandra class

Basically our intention is to study *semisimple symmetric spaces*, that is, reductive symmetric spaces G/H with G connected semisimple. However, several arguments in the heart of the proof of the Plancherel theorem proceed by induction on the rank of the symmetric space. The symmetric spaces of lower rank that are involved are constructed from certain parabolic subgroups of G (cf. §4), and they are in general not connected semisimple, even when G is. Therefore, we are forced to consider a larger class of reductive symmetric spaces. On the other hand, if we pose no restriction on G, except that its Lie algebra is reductive, the spaces are too wild to handle (for example, any discrete group would qualify).

A suitable class of groups is the class $\mathcal{R}$ of *reductive groups of Harish-Chandra class*. For its definition, see [K], p. 384-385. By definition then, a reductive symmetric space $X = G/H$ belongs to *Harish-Chandra's class*, if $G \in \mathcal{R}$.

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Numbers [1],[2]... refer to the list in [BS]