

Quantum Fluctuations of a Vortex in an Optical Lattice

J.-P. Martikainen and H. T. C. Stoof

Institute for Theoretical Physics, Utrecht University, Leuvenlaan 4, 3584 CE Utrecht, The Netherlands

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Using a variational ansatz for the wave function of the Bose-Einstein condensate, we develop a quantum theory of vortices and quadrupole modes in a one-dimensional optical lattice. We study the coupling between the quadrupole modes and Kelvin modes, which turns out to be formally analogous to the theory of parametric processes in quantum optics. This leads to the possibility of squeezing vortices. We solve the quantum multimode problem for the Kelvin modes and quadrupole modes numerically and find properties that cannot be explained with a simple linear-response theory.

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Introduction.—There is a large body of research on vortices in Bose-Einstein condensates, both before and after their first experimental observation [1,2]. A recent review of this vibrant field was given by Fetter and Svidzinsky [3]. One reason for this interest is that vortices play a crucial role in explaining rotational and dissipative properties of superfluids. They are also important for an understanding of classical fluids and wave excitations of a vortex were therefore already studied by Lord Kelvin. Such excitations have been observed in superfluid helium [4] and, most recently, in Bose-Einstein condensates by Bretin *et al.* [5]. The latter experiment also elucidated the role of the quadrupole modes of a Bose-Einstein condensate in exciting the Kelvin modes of the vortex line. A theoretical understanding of the observed phenomena was provided by Mizushima *et al.* [6].

The phenomena of superfluidity is closely connected to the existence of quantized vortices and relies on phase coherence across the condensate. Placing the condensate in an optical lattice makes the establishment of phase coherence more difficult and can even induce a transition into a Mott-insulator state. This quantum phase transition was indeed recently observed experimentally [7]. In a one-dimensional optical lattice, the condensate splits into a stack of weakly coupled disk-shaped condensates, which leads to some intriguing analogs with high- T_c superconductors due to their similar layered structure [8].

In this Letter, we combine the above ideas and study a vortex in a Bose-Einstein condensate in a one-dimensional optical lattice. This problem has not yet been addressed either experimentally or theoretically, but experimental studies are possible with current technical capabilities. Indeed, such an experimental setup appears to be a promising way to achieve the quantum Hall regime in a Bose gas, for which we need to have about one vortex per particle [9]. Also, outside the quantum Hall regime an optical lattice allows for unique possibilities to study vortices and their quantum fluctuations. In a normal three-dimensional condensate, a vortex line is a very complicated dynamical object and it is nearly impossible to tune its properties in a controlled

way. Placing the vortex in an optical lattice provides a way, for example, to reduce the stiffness of the vortex line by increasing the depth of the lattice.

With this goal in mind, we develop a quantum theory for the Kelvin modes in an optical lattice and obtain the Kelvin mode dispersion relation analytically by means of a variational approach. Furthermore, because of their importance for the creation of the Kelvin waves, we study in detail the coupling between the Kelvin modes and quadrupole modes. Our approach leads to a quantum multimode problem with an interaction between Kelvin modes and quadrupole modes that closely resembles squeezing Hamiltonians familiar from quantum optics. We solve this quantum multimode problem and show how the squeezing is reflected in the quantum mechanical uncertainty of the vortex line. In addition, numerical solution of the multimode problem reveals properties of the nonlinear dynamics that cannot be explained with a linear-response theory.

Theory.—The theory we use is based in our previous work [10,11], and we briefly summarize the main points. We consider an elongated Bose-Einstein condensate in a magnetic trap with a radial trapping frequency ω_r that is much larger than the axial trapping frequency ω_z . In addition to this cigar-shaped potential, the condensate experiences a one-dimensional optical lattice $V_0(\mathbf{r}) = V_0 \sin^2(2\pi z/\lambda)$, where V_0 is the lattice depth and λ is the wavelength of the laser light. Furthermore, we assume that the lattice is deep enough so that it dominates over the magnetic trapping potential in the z direction and that the number of sites is very large. Then we can in first instance ignore the magnetic trap in the z direction.

We expand the condensate wave function in terms of wave functions $\Phi_n(x, y)\Phi(z)$ that are well localized in the sites labeled by n . We do not yet assume anything about the wave functions $\Phi_n(x, y)$ for the two-dimensional condensates, but for the wave function in the z direction, $\Phi(z)$, we use the ground-state wave function of the harmonic approximation to the lattice potential near the lattice minimum. This harmonic trap has the frequency $\omega_L = (2\pi/\lambda)\sqrt{2V_0/m}$, where m is the atomic mass. These

considerations lead to an energy functional of the weakly coupled two-dimensional condensates, which is characterized by the strength of the on-site particle interactions $4\pi U = 4\pi N(a/l_r)\sqrt{\omega_L/2\pi\omega_r}$ and by the strength of the nearest-neighbor Josephson coupling $J = (1/8\pi^2)(\omega_L/\omega_r)^2(\lambda/l_r)^2[\pi^2/4 - 1]e^{-(\lambda/4 l_L)^2}$ [11]. In these formulas, N is the number of atoms in each site, a is the scattering length, and $l_L = \sqrt{\hbar/m\omega_L}$. Also, we use trap units so that the unit of energy is $\hbar\omega_r$, the unit of time is $1/\omega_r$, and the unit of length is $l_r = \sqrt{\hbar/m\omega_r}$. Our theory is valid when the lattice depth is much bigger than the chemical potential. Also, the strength of the Josephson coupling should be small, but still large enough to support a superfluid state. The superfluid Mott-insulator transition in one-dimensional optical lattice is expected to occur only for very deep lattices [12], and such a transition has indeed not yet been observed experimentally.

Variational ansatz.—A variational ansatz for the two-dimensional condensates should satisfy two obvious physical criteria. First, the ansatz must predict the vortex precession dynamics correctly. Second, the ansatz must predict the quadrupole mode frequencies with sufficient accuracy. In an earlier publication, we used an ansatz that fulfilled the second requirement very well in the full parameter range from the noninteracting limit to the Thomas-Fermi regime [11]. Unfortunately, this ansatz does not fulfill the first requirement because it does not properly account for the size of the vortex core, which is not important for the collective modes but plays a crucial role in the dynamics of the vortex. Therefore, we use the ansatz

$$\Phi_n(x, y) \propto \exp\left[-\frac{B_0}{2}(x^2 + y^2) - \frac{\epsilon_n(t)}{2}(x^2 - y^2) - \epsilon_{xy,n}(t)xy + i \tan^{-1}\left(\frac{y - y_n(t)}{x - x_n(t)}\right)\right], \quad (1)$$

where (x_n, y_n) is the position of the vortex in the n th site and the variational parameters ϵ_n and $\epsilon_{xy,n}$ are complex. In addition, the square of the wave function is normalized to N . We have chosen our variational parameters such that only Kelvin modes and quadrupole modes are included. Our theory can be easily adapted to include the monopole mode as well, but we choose to focus on the quadrupole modes due to their experimental importance for the creation of Kelvin waves [5]. Since our ansatz does not have a core, the average kinetic energy diverges. Therefore, we introduce a small distance cutoff ξ as an additional variational parameter. Together with B_0 , this parameter will be determined by minimizing the equilibrium energy functional. This minimization is a simple numerical task that can be done once in the beginning of the calculation. In that manner, the condensate with a vortex is slightly bigger than a condensate without a vortex, and the vortex core size becomes

smaller with increasing atom numbers. Quantitatively, we find to a very good accuracy that $\xi = \sqrt{B_0}$.

The ansatz in Eq. (1) fulfills both our requirements with physically realistic parameter values and is still simple enough to be analytically tractable. Our ansatz fails for very weakly interacting condensates, but provides an accurate description when the interaction strength U is larger than about 10. In the other regime, our previous ansatz can be used [11]. In this Letter, we do not consider rotation of the trap, but a change to the rotating frame can again be easily included into our approach.

Quantization of the vortex and the quadrupole modes.—If we expand the Lagrangian for our system to second order in the deviations from equilibrium, we are in a position to study the eigenmodes of the system. The eigenmodes we find are the Kelvin modes and two quadrupole modes with quantum numbers $m = \pm 2$. To second order, there is no coupling between the vortex positions and the quadrupole modes, and we can treat the two independently. We transform to momentum space using the convention $f_k = (1/\sqrt{N_s})\sum_n e^{-ikz_n} f_n$, where N_s is the number of sites. We then proceed by defining the vortex positions in terms of bosonic creation and annihilation operators as $\hat{x}_k = (\hat{v}_{-k}^\dagger + \hat{v}_k)/2\sqrt{NB_0}$ and $\hat{y}_k = i(\hat{v}_{-k}^\dagger - \hat{v}_k)/2\sqrt{NB_0}$ [13]. With this definition, the Lagrangian for the vortex position takes the desired form:

$$\hat{L}_K = \sum_k (i\hat{v}_k^\dagger \hat{v}_k - \omega_K(k)\hat{v}_k^\dagger \hat{v}_k), \quad (2)$$

where, to leading order in B_0 , the Kelvin mode dispersion is given by $\omega_K(k) = B_0/2 + \Gamma[0, B_0^2][2J(k) - B_0/2]$. In this expression $J(k) = J[1 - \cos(kd)]$, $d = \lambda/2$, and $\Gamma[a, z]$ is the incomplete gamma function. Our result for the Kelvin mode dispersion relation has some interesting features as seen in Fig. 1(b). First, the initially negative precession frequency, which indicates that a straight vortex line is unstable without rotation, can change sign if the momentum k of the Kelvin mode is large enough. Moreover, the dispersion is quadratic for small momenta. Both features are due to the attractive interactions between neighboring pancake vortices, which is harmonic for small separations and due to the energy cost for having phase differences between sites. The quadratic Kelvin dispersion in a lattice contrasts with the behavior of the vortex line in a three-dimensional bulk superfluid, where the Kelvin mode dispersion relation behaves similar to $k^2 \ln(1/k\xi)$ and has a logarithmic dependence for small k . Third, the vortex position is “smeared” by quantum fluctuations. In particular, we find for the Kelvin mode vacuum state that $\langle \hat{x}_n^2 \rangle = \langle \hat{y}_n^2 \rangle = 1/(4NB_0)$. Therefore, the quantum properties of the vortex become more important in a lattice. This is due to the reduced on-site particle number, as opposed to the total number of particles, and

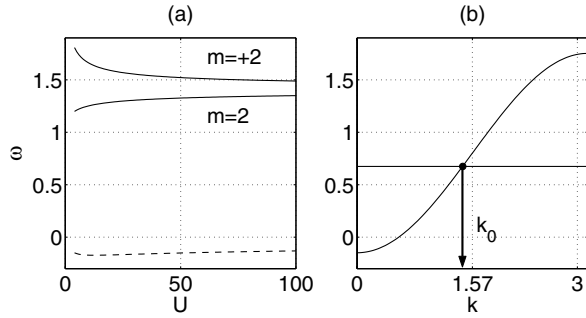


FIG. 1. (a) Frequencies of the $m = \pm 2$ quadrupole modes and the Kelvin mode at $k = 0$ as a function of interaction strength. (b) The constant line is $\omega_{-2}(0)/2$ and the curve is the Kelvin mode dispersion. In linear response, the energy transfer between the $m = -2$ quadrupole modes and the Kelvin modes becomes nonvanishing if the Kelvin mode frequency matches $\omega_{-2}(0)/2$, i.e., when the lines cross. We used $U = 100$ and $J = 0.1$.

the spreading out of the condensate wave function as the lattice depth is increased.

The quadrupole modes are technically somewhat more complicated, but can be quantized in a similar way as the Kelvin modes. The quadrupole mode frequencies are

$$\omega_{\pm 2}(k) = \frac{B_0}{A(k)} + A(k) \left(\frac{1}{B_0} + \frac{B_0}{4} - UB_0 - \frac{B_0 \Gamma[0, B_0^2]}{2} \right) + 2J(k) \left(A(k) + \frac{1}{A(k)} \right) \pm B_0, \quad (3)$$

where

$$A(k) = \sqrt{\frac{4B_0^2 + 8B_0J(k)}{4 + 8B_0J(k) + B_0^2(1 - 4U - 2\Gamma[0, B_0^2])}}. \quad (4)$$

The Lagrangian for the quadrupole modes is then

$$\hat{L}_Q = \sum_{m=\pm 2} \sum_k [i\hat{q}_{m,k}^\dagger \dot{\hat{q}}_{m,k} - \omega_m(k) \hat{q}_{m,k}^\dagger \hat{q}_{m,k}]. \quad (5)$$

In Fig. 1(a), we plot the behavior of the mode frequencies at $k = 0$ as a function of the interaction strength. The quadrupole mode frequencies are in good agreement with the ones calculated using the Bogoliubov-de Gennes equation [11], and with realistic interaction strengths the vortex precession frequency is close to the result using a Thomas-Fermi wave function for the disk-shaped condensate [3].

Coupled Kelvin modes and quadrupole modes.—We now proceed to apply our previous results to study the coupling between the Kelvin modes and the quadrupole modes. To obtain coupling between the quadrupole modes and the Kelvin modes, we must expand the Lagrangian up to third order. At third order, there are contributions from the kinetic energy, the time-derivative term in the Lagrangian, and the Josephson coupling term. Since we

assume small J , we can safely ignore the last one. The remaining terms describe the coupling between the Kelvin modes and both quadrupole modes. The contribution from the time-derivative term in the Lagrangian includes the time derivative of the Kelvin mode operators. In the rotating-wave approximation, we can replace these time derivatives with the Kelvin mode dispersion. In addition, this eliminates the nonresonant terms from the Hamiltonian, which, in particular, removes the coupling between the Kelvin modes and the $m = 2$ quadrupole mode. In this way, we obtain the following interaction Hamiltonian for the Kelvin modes and the quadrupole modes:

$$\hat{H}_I = \sum_{k,k'} E_c(k, k') (\hat{q}_{-2,k'}^\dagger \hat{v}_{k+k'}^\dagger \hat{v}_{-k}^\dagger + \hat{q}_{-2,k'}^\dagger \hat{v}_{-k-k'}^\dagger \hat{v}_k), \quad (6)$$

where $E_c(k, k') = \sqrt{A(k')/8N_s(B_0\Gamma[0, B_0^2] + \omega_K(k))}$. This expression is accurate as long as the coupling between the Kelvin modes and the quadrupole modes is sharply peaked around some resonant value k_0 . Note that the angular momentum is conserved, since the quadrupole mode has an angular momentum of -2 whereas each Kelvin mode has an angular momentum of -1 .

Multimode problem.—Our interaction Hamiltonian shows how the coupling singles out a wave number for the Kelvin modes and thus selects the wavelength for the ensuing wiggles in the vortex line [6]. We demonstrate this in Fig. 1(b). The width of the resonance is still unknown, but we can give a simple estimate. Assume that initially we have a straight vortex line and a coherent quadrupole field $\langle \hat{q}_{-2,k}(0) \rangle = q_{-2,0}(0) \delta_{k,0}$. At short times, we can assume that the quadrupole field stays constant and then solve the operator equation for the Kelvin modes $\hat{v}_{\pm k}$. It turns out that the solutions are dynamically unstable only when $|\Delta\omega(k)| < |E_c(k, 0)q_{-2,0}(0)|$, where $\Delta\omega(k) = \omega_{-2}(0) - 2\omega_K(k)$. Our numerical solutions of the multimode problem confirm this simple estimate with a reasonable accuracy. The number of modes within the interval depends on the number of sites in the lattice and on the quadrupole amplitude, but with realistic parameter values the multimode behavior is more likely than a single-mode behavior.

In linear response, we can use Fermi's golden rule to calculate the transition rate from the initially coherent state $|\psi_i\rangle = |q_{-2,0}, \{v_k\}\rangle = |q_{-2,0}, \{0\}\rangle$. After a short calculation, we get the transition rate as

$$\Gamma_{Q \rightarrow K} = \frac{\pi E_c(k_0, 0)^2 |q_{-2,0}(0)|^2}{2J\Gamma[0, B_0^2] \sin(k_0 d)}. \quad (7)$$

In Fig. 2, we show an example of the time evolution of the system, when the quadrupole field is treated as a classical field, but Kelvin modes are treated fully quantum mechanically. In the same figure, we also compare the exponential $q_{-2,0}(0)e^{-\Gamma_{Q \rightarrow K} t}$ with the numerical solution of

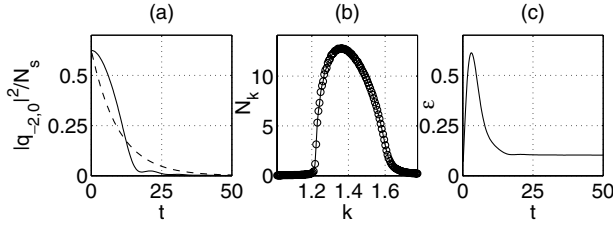


FIG. 2. (a) The time evolution of the quadrupole mode (solid line) compared with the result based on Fermi's golden rule (dashed line). (b) The number distribution of the Kelvin modes at the end of the simulation, when the quadrupole amplitude has essentially vanished. The circles indicate the individual Kelvin modes included in the simulation and the solid line is an interpolation between the data points. (c) The time evolution of the asymmetry factor $\epsilon(t)$. We used $U = 100$, $J = 0.1$, $N_s = 1001$, $N = 100$, and $\langle \hat{q}_{-2,0}(0) \rangle = 25$.

the nonlinear multimode problem and show the number distribution of Kelvin modes at long times. It should be noted that at these time scales the system appears irreversible due to the large number of modes present. If the number of sites in the lattice is smaller, the oscillatory behavior becomes stronger. It should also be realized that, depending on parameters, the result from Fermi's golden rule can be unreliable since it ignores the finite lifetime of the final states. It is interesting to observe that the number distribution of the Kelvin modes is asymmetric. This is due to the asymmetric nature of the coupling constant $E_c(k, 0)$ and the detuning $\Delta\omega(k)$.

The interaction Hamiltonian in Eq. (6) closely resembles the squeezing Hamiltonian encountered for parametric processes in quantum optics [14]. Therefore, it is not surprising that we can observe a dramatic squeezing under certain conditions. For example, if we have only a single resonant mode then the initial minimum uncertainty state, i.e., the vacuum, will remain a minimum uncertainty state, but can become strongly squeezed. Such a squeezing is reflected in the vortex position distribution. This is shown in Fig. 2, where we also monitor the time evolution of the deformation $\epsilon(t) = \max\{[\langle \hat{x}_n^2(t) \rangle - \langle \hat{y}_n^2(t) \rangle] / [\langle \hat{x}_n^2(t) \rangle + \langle \hat{y}_n^2(t) \rangle]\}$ of the vortex position distribution. Because the uncertainty ellipse that characterizes the vortex position distribution precesses in time, the deformation is maximized with respect to the rotation angle of the coordinate system.

Discussion.—We have developed a quantum theory of a vortex in a one-dimensional optical lattice and applied the theory to study the coupled quadrupole-Kelvin mode dynamics. Our solution of the Kelvin mode dispersion relation essentially amounts to a solution of the vortex line dynamics in an optical lattice. Among other things, our study revealed the possibility of squeezed vortex

states and asymmetric number distributions for the Kelvin modes. As yet, there have been no experiments on vortices in optical lattices reported. Detailed studies of the Kelvin mode dynamics would require imaging of the three-dimensional vortex line which is difficult, but not impossible [5,15]. An alternative, but less direct way to probe the system, would be to track the evolution of the quadrupole oscillations. If the resonance is narrow enough, our theory predicts revivals of these oscillations. In addition, the multimode nature of the Kelvin mode distribution can cause a decay rate of the quadrupole mode that is quite different from Fermi's golden rule. Our theory can be applied to a variety of phenomenon. For example, the interplay between the superfluid Mott-insulator transition, the quantum melting of a vortex lattice, and the quantum Hall regime in an optical lattice are very interesting topics for further research.

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