

Coherent Versus Incoherent Dynamics During Bose-Einstein Condensation in Atomic Gases

H. T. C. Stoof

*Institute for Theoretical Physics, University of Utrecht,
Princetonplein 5, 3584 CC Utrecht, The Netherlands*

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We review and extend the theory of the dynamics of Bose-Einstein condensation in weakly interacting atomic gases. We present in a unified way both the semiclassical theory as well as the full quantum theory. This is achieved by deriving a Fokker-Planck equation that incorporates both the coherent and incoherent effects of the interactions in a dilute Bose gas. In first instance we focus our attention on the nonequilibrium dynamics of a homogeneous Bose gas with a positive interatomic scattering length. After that we discuss how our results can be generalized to the inhomogeneous situation that exists in the present experiments with magnetically trapped alkali gases, and how we can deal with a negative interatomic scattering length in that case as well. We also show how to arrive at a description of the collective modes of the gas that obeys the Kohn theorem at all temperatures. The theory is based on the many-body T-matrix approximation throughout, since this approximation has the correct physical behavior near the critical temperature and also treats the coherent and incoherent processes taking place in the gas on an equal footing.

I. INTRODUCTION

In the second half of the 1920's the founding fathers of quantum mechanics were mainly concerned with the understanding of the physical world on an atomic scale. In particular the study of the spectra of atomic hydrogen and helium led, besides the formulation of the famous Schrödinger equation¹ and the interpretation of the wavefunction as a probability amplitude,² to the exclusion principle³ or more generally the (anti)symmetrization of the wavefunction in the case of particles with (half)integer spin.⁴ This important postulate of modern quantum theory divides all particles into two distinct families, which are now of course known as fermions and bosons, respectively.

Already before the discovery of the (anti)symmetrization requirement, it was shown on statistical grounds only⁵⁻⁷ that under extreme conditions ideal gases of these indistinguishable particles have also remarkable features on a macroscopic scale. Furthermore, in the case of an interacting gas even more stunning phenomena may occur, however weak the interaction. A well-known example in this context is superconductivity. In a superconducting metal an attractive interaction between two electrons with opposite momenta causes an instability of the Fermi-surface and the formation of Cooper pairs.⁸ The latter are allowed to move freely through the lattice resulting in a superconducting current and a vanishing resistance. According to the successful BCS-theory⁹ describing this, the attractive interaction is the result of a phonon exchange process. It is, however, believed that due to the weakness of this interaction only critical temperatures up to about 30 K are obtainable. Therefore, the underlying mechanism for the high-temperature superconductors cannot be explained in this way and is today still a very active field of investigation.

Although the BCS-theory shows that superconductivity can in a certain sense be regarded as a result of a Bose-Einstein condensation^{6, 10, 11} of the Cooper pairs, the most striking example of this condensation process was until recently always associated with the superfluid phase of liquid ⁴He.¹² Phenomenologically, the characteristics of superfluidity can be explained if the dispersion relation of the elementary excitations differs from the particle-like dispersion $\varepsilon(\mathbf{k}) = \hbar^2 \mathbf{k}^2 / 2m$ and is linear for small momenta.¹³ To see that this feature indeed leads to superfluidity, we consider the situation of liquid ⁴He in a very long cylindrical pipe moving with velocity \mathbf{v} along its symmetry axis.¹⁴ Describing the strongly interacting Bose liquid in the pipe as an assembly of noninteracting quasiparticles in a state of thermal equilibrium, we find from the usual methods of statistical mechanics that in the laboratory frame the number of quasiparticles with momentum $\hbar \mathbf{k}$ and energy $\hbar \omega(\mathbf{k})$ is given by

$$N(\mathbf{k}) = \frac{1}{e^{(\hbar \omega(\mathbf{k}) - \hbar \mathbf{k} \cdot \mathbf{v}) / k_B T} - 1}. \quad (1)$$

The velocity \mathbf{v} is still arbitrary at this point, but its magnitude has an upper bound because the occupation numbers $N(\mathbf{k})$ must be positive. Therefore, we require for all $\mathbf{k} \neq \mathbf{0}$ that $\hbar \omega(\mathbf{k}) > \hbar \mathbf{k} \cdot \mathbf{v}$. In this case the total momentum $\mathbf{P} = \sum_{\mathbf{k}} \hbar \mathbf{k} N(\mathbf{k})$ carried along with the walls of the pipe, is at low temperatures T clearly much smaller than the momentum $Nm\mathbf{v}$ that we obtain when the whole liquid containing N atoms is moving rigidly with the walls. Hence, if the dispersion relation is linear, i.e., $\hbar \omega(\mathbf{k}) = \hbar c k$, we conclude by a Galilean transformation that the fluid can have a stationary

(frictionless) flow if the velocity is small enough and obeys $v < c$.¹⁵ Herewith, both the existence of superfluidity as well as a critical velocity above which the phenomenon cannot take place is explained. It is important to note, however, that the above argument is semiclassical and predicts that a superflow is absolutely stable if the condition $v < c$ is satisfied. This is incorrect in principle, because the superflow can actually decay by means of quantum and thermal fluctuations and is therefore only metastable.¹⁶ As we will see below, the effect of quantum and thermal fluctuations on the semiclassical picture is a recurrent theme throughout the following.

At zero temperature, the required linear behavior of the dispersion relation was first derived on the basis of a microscopic theory by Bogoliubov¹⁷ and was subsequently put on a firm basis by Beliaev,¹⁸ Hugenholtz and Pines,¹⁹ and Gavoret and Nozières.²⁰ Simultaneously, it was also shown to be valid at nonzero temperatures by Lee and Yang²¹ using the pseudopotential method,²² and more rigorously by Hohenberg and Martin²³ and Popov²⁴ using field-theoretical methods. At this point it is important to mention also the work of Nepomnyashchiĭ and Nepomnyashchiĭ, who explicitly showed that the linear dispersion of the elementary excitations also survives if the infrared divergencies in the perturbation theory are taken into account exactly.²⁵

From these famous papers the linear dispersion at long wavelengths in a Bose system with short-range interactions is therefore well established. However, to actually calculate its velocity c from first principles is not feasible in general and in particular not for the strongly interacting ${}^4\text{He}$ liquid. To achieve that we have to consider a dilute Bose gas, for which we can rely on the weakness of the interactions or more precisely on the smallness of the gas parameter $(na^3)^{1/2}$, where n is the density and a the positive s -wave scattering length of the particles in the gas. Unfortunately however, until recently Bose-Einstein condensation in a gas was never realized experimentally and the theory of the weakly interacting Bose gas^{17, 18, 21, 24} could therefore not be applied to any real system. This situation changed dramatically in the summer of 1995 when within a few months three experimental groups reported the observation of Bose-Einstein condensation in a vapor of spin-polarized ${}^{87}\text{Rb}$,²⁶ ${}^7\text{Li}$,²⁷ and ${}^{23}\text{Na}$ (Ref. 28) atoms.

Historically, the first step towards the achievement of Bose-Einstein condensation in a dilute gas was made in 1980 by Silvera and Walraven,²⁹ when they demonstrated the stabilization of spin-polarized atomic hydrogen. After this pioneering experiment, the combined efforts of several experimental and theoretical groups around the world led to a detailed understanding of the atomic and statistical properties of this new quantum system.³⁰ Moreover, just before the reports on the above mentioned experiment with

the rubidium vapor, doubly spin-polarized atomic hydrogen gases still held the record in achieving the required high densities and low temperatures and were generally considered to be the most promising candidates for the observation of Bose-Einstein condensation.³¹ Although, the latter expectation has clearly been proven wrong, it is fair to say that the successful experiments with the alkali vapors would not have been possible without the knowledge acquired by the atomic hydrogen community. In particular the development of magnetic traps^{32, 33} and evaporative cooling techniques^{34, 35} has turned out to be of crucial importance for the actual achievement of Bose-Einstein condensation.³⁶

At present, the most important reason for the interest in degenerate Bose gases is that we can study the nonequilibrium dynamics of these trapped gases in such detail that experimental results can be directly confronted to microscopic many-body theories. In particular, the dynamics of condensate formation and the collective modes of an inhomogeneous Bose condensed gas are two topics that have received considerable attention in the last three years and on which the present paper is also focussed. More generally, however, the aim of this paper is to develop with the techniques of quantum field theory a unified nonequilibrium theory for trapped atomic gases that is capable of treating the dynamics of the gas, not only in the normal and superfluid phases, but also during the phase transition itself. In agreement with the above goal of the paper, we refrain from reviewing other approaches that can and actually have been applied to the topics of interest here. Nevertheless, we have included a rather large amount of references. In this manner we aim to give, to the best of our knowledge, proper reference to important contributions to this field of physics. Furthermore, we also want to provide the reader with additional (and sometimes more advanced) literature that may be useful for a deeper understanding of various technical details, even though this is not necessary for a first reading, since the paper is intended to be self contained and essentially only assumes a working knowledge of the methods of second quantization, the path-integral formulation of quantum mechanics and the Hartree-Fock theory of interacting many-particle systems.

The theoretical challenge that is posed by trapped atomic Bose gases can be stated as follows. On the one hand the interactions between the atoms influence the one-particle wavefunctions and energies of the gas. On the other hand the same interactions also change the occupation numbers of these states. As a result of the strong coupling between these, respectively, coherent and incoherent processes, the nonequilibrium dynamics of the gas is in principle quite complicated and requires a quantum kinetic theory that simultaneously incorporates both effects of the interactions. Such a quantum kinetic theory has recently been developed by Gardiner

and Zoller on the basis of a perturbative treatment of the master equation for the many-body density matrix.³⁷ Below we ultimately also arrive at a quantum kinetic theory, but this is achieved by using field-theoretical tools to derive a single Fokker-Planck equation that nevertheless describes both the coherent and incoherent dynamics of the gas. In principle we could, of course, also first derive a master equation and subsequently the same Fokker-Planck equation by making use of well-known procedures from quantum optics. However, we believe that the approach presented here is more direct and, most importantly, much more convenient if we systematically want to go beyond the pseudopotential method that is not sufficiently accurate for our purposes and that is used by Gardiner and Zoller.

The paper is organized as follows. In Sec. II we start with a discussion of an ideal Bose gas in a trap. In Sec. II.A we consider the isolated case, but in Sec. II.B we couple the gas in the trap also to a thermal reservoir. The reason for treating the ideal Bose gas is that in this way we are able to introduce most easily all the mathematical tools that are required for the discussion of an interacting Bose gas. The necessary extensions for an interacting gas are then developed in Secs. III, IV, and V. In these sections we mainly focus on the problem of understanding the dynamics of condensate formation, because the theory needs to be the most accurate and therefore the most general for this problem. In Secs. III and IV we first treat a homogeneous Bose gas and in Sec. V we then turn to the inhomogeneous case. Discussing the homogeneous Bose gas first is a useful intermediate step, because as a result of the translational symmetry the coherent dynamics of the gas involves only the one-particle energies and not their wavefunctions. Nevertheless, the dynamics is nontrivial. To get a better understanding of the timescales involved in Bose-Einstein condensation we therefore first consider in Sec. III the semiclassical dynamics of the gas, in which fluctuations of the order parameter are neglected. Roughly speaking, the semiclassical theory is the nonequilibrium generalization of the Bogoliubov theory. Next, we discuss in Sec. IV the full quantum theory that incorporates also fluctuations of the order parameter.

Section V is devoted to the quantum kinetic theory for inhomogeneous Bose gases. After our experience with the homogeneous Bose gas we can easily derive the desired Fokker-Planck equation for an interacting Bose gas trapped in an external potential. In contrast to homogeneous gases there are now, however, two cases to consider. If the density of the gas is sufficiently small, the average interaction between the atoms is always smaller than the energy splitting between the one-particle states in the trap and the coherent dynamics of the gas can be neglected. This most simple situation corresponds to the weak-coupling limit and is presented in Sec. V.A. Experimentally, the weak-coupling limit is essentially only

realized for atomic ${}^7\text{Li}$ due to the effectively attractive interatomic interactions of this element. As an application of the weak-coupling theory, we therefore discuss in Sec. V.B the dynamics of Bose-Einstein condensation in a gas with effectively attractive interactions in some detail. It is only after that, that we turn our attention to the most complicated strong-coupling limit and discuss three other applications of our general nonequilibrium approach. Since it follows most naturally from the theory presented in Sec. V.B, we first briefly consider in Sec. V.C an interesting (zero temperature) property of a condensate, namely the so-called “diffusion” of its phase. In Sec. V.D we then return to the dynamics of Bose-Einstein condensation and in Sec. V.E we finally present also a theory for the collective modes of a trapped Bose gas. We end the paper in Sec. VI with some conclusions and an outlook.

II. CALDEIRA-LEGGETT LIKE TOY MODEL

Before we start with the full problem of Bose-Einstein condensation in an interacting gas, we first consider in this section the case of an ideal Bose gas. In particular, we consider an isolated, noninteracting Bose gas in an external trapping potential $V^{ex}(\mathbf{x})$ with one-particle states $\chi_\alpha(\mathbf{x})$ and one-particle energies ε_α . In this manner we try to give an introduction to the Schwinger-Keldysh formalism for nonequilibrium processes^{38, 39} that we are going to use throughout this paper. A review of the Schwinger-Keldysh formalism has already been given by Danielewicz,⁴⁰ but he uses the operator language which turns out to be less convenient for our purposes. Instead, we need a formulation in terms of functional integrals.⁴¹ In addition, we also couple this system to a reservoir in a way that is familiar from the Caldeira-Leggett model for a particle experiencing friction.⁴² This gives us the opportunity to explain how dissipation, and as a result relaxation, is incorporated in the Schwinger-Keldysh formalism. This is especially important in Sec. IV, when we discuss the effect of thermal fluctuations on the semiclassical theory of Bose-Einstein condensation.

A. Isolated Ideal Bose Gas

In textbooks an ideal Bose gas is generally discussed in terms of the average occupation numbers of the one-particle states $\chi_\alpha(\mathbf{x})$.⁴³ Given the density matrix $\hat{\rho}(t_0)$ of the gas at an initial time t_0 , these occupation numbers obey

$$N_\alpha(t) = \text{Tr}[\hat{\rho}(t_0) \hat{\psi}_\alpha^\dagger(t) \hat{\psi}_\alpha(t)], \quad (2)$$

with $\hat{\psi}_\alpha^\dagger(t)$ and $\hat{\psi}_\alpha(t)$ the usual (Heisenberg picture) creation and annihilation operators of second quantization,⁴⁴ respectively. Because the hamiltonian of the gas

$$\hat{H} = \sum_{\alpha} \varepsilon_{\alpha} \hat{\psi}_{\alpha}^{\dagger}(t) \hat{\psi}_{\alpha}(t) \quad (3)$$

commutes with the number operators $\hat{N}_{\alpha}(t) = \hat{\psi}_{\alpha}^{\dagger}(t) \hat{\psi}_{\alpha}(t)$, the nonequilibrium dynamics of the system is trivial and the average occupation numbers are at all times equal to their value at the initial time t_0 . If we are also interested in fluctuations, it is convenient to introduce the eigenstates of the number operators, i.e.,

$$|\{N_{\alpha}\}; t\rangle = \prod_{\alpha} \frac{(\hat{\psi}_{\alpha}^{\dagger}(t))^{N_{\alpha}}}{\sqrt{N_{\alpha}!}} |0\rangle, \quad (4)$$

and to consider the full probability distribution

$$P(\{N_{\alpha}\}; t) = \text{Tr}[\hat{\rho}(t_0) |\{N_{\alpha}\}; t\rangle \langle \{N_{\alpha}\}; t|], \quad (5)$$

which is again independent of time for an ideal Bose gas. The average occupation numbers are then determined by

$$N_{\alpha}(t) = \sum_{\{N_{\alpha}\}} N_{\alpha} P(\{N_{\alpha}\}; t) \quad (6)$$

and the fluctuations can be obtained from similar expressions.

As mentioned in the introduction, we are in the following also interested in the phase dynamics of the gas. To study that it is necessary to introduce so-called coherent states, which are eigenstates of the annihilation operators $\hat{\psi}_{\alpha}(t)$ and are much used both in the field of quantum optics⁴⁵ as well as condensed matter physics.⁴⁶ For our purposes it is even more convenient to consider eigenstates of the field operator

$$\hat{\psi}(\mathbf{x}, t) = \sum_{\alpha} \hat{\psi}_{\alpha}(t) \chi_{\alpha}(\mathbf{x}). \quad (7)$$

An eigenstate of $\hat{\psi}(\mathbf{x}, t)$ with eigenvalue $\phi(\mathbf{x}) = \sum_{\alpha} \phi_{\alpha} \chi_{\alpha}(\mathbf{x})$ is given by

$$|\phi; t\rangle = \exp \left\{ \int d\mathbf{x} \phi(\mathbf{x}) \hat{\psi}^{\dagger}(\mathbf{x}, t) \right\} |0\rangle = \exp \left\{ \sum_{\alpha} \phi_{\alpha} \hat{\psi}_{\alpha}^{\dagger}(t) \right\} |0\rangle \quad (8)$$

and is clearly also an eigenstate of $\hat{\psi}_{\alpha}(t)$ with eigenvalue ϕ_{α} . The latter can also be proved immediately by noting that the equal-time commutation

relation $[\hat{\psi}_\alpha(t), \hat{\psi}_\alpha^\dagger(t)] = \hat{1}$ implies that $\hat{\psi}_\alpha(t)$ acts as $d/d\hat{\psi}_\alpha^\dagger(t)$ on these states. Moreover, from their definition it can easily be shown that these eigenstates obey the inner product

$$\langle \phi; t | \phi'; t \rangle = \exp \left\{ \int d\mathbf{x} \phi^*(\mathbf{x}) \phi'(\mathbf{x}) \right\} = \exp \left\{ \sum_\alpha \phi_\alpha^* \phi'_\alpha \right\} \quad (9)$$

and the completeness relation

$$\int d[\phi^*] d[\phi] \frac{|\phi; t\rangle \langle \phi; t|}{\langle \phi; t | \phi; t \rangle} = \hat{1}, \quad (10)$$

if the integration measure of the functional integral is taken to be

$$\int d[\phi^*] d[\phi] = \int \prod_\alpha \frac{d\phi_\alpha^* d\phi_\alpha}{2\pi i}. \quad (11)$$

In analogy with the occupation number representation in Eq. (5), we can now obtain a completely different description of the Bose gas by making use of the above coherent states and considering the probability distribution

$$P[\phi^*, \phi; t] = \text{Tr} \left[\hat{\rho}(t_0) \frac{|\phi; t\rangle \langle \phi; t|}{\langle \phi; t | \phi; t \rangle} \right]. \quad (12)$$

Although we expect that this probability distribution is again independent of time, let us nevertheless proceed to derive its equation of motion in a way that can be generalized in Secs. III and IV when we consider an interacting Bose gas. First, we need to expand the density matrix $\hat{\rho}(t_0)$ in the above coherent states. For an isolated Bose gas it is appropriate to take an initial density matrix that commutes with the total number operator $\hat{N} = \sum_\alpha \hat{N}_\alpha(t)$ and we then find the expression

$$\hat{\rho}(t_0) = \int d[\phi_0^*] d[\phi_0] \rho[|\phi_0|^2; t_0] \frac{|\phi_0; t_0\rangle \langle \phi_0; t_0|}{\langle \phi_0; t_0 | \phi_0; t_0 \rangle}, \quad (13)$$

in which the expansion coefficients $\rho[|\phi_0|^2; t_0]$ only depend on the amplitude of the field $\phi_0(\mathbf{x})$ but not on its phase. This is equivalent to saying that the initial state of the gas does not have a spontaneously broken $U(1)$ symmetry. Since we are ultimately interested in the dynamics of Bose-Einstein condensation it is of course crucial not to consider an initial state in which this symmetry is already broken and we therefore never use such initial states in the following.

Next, we substitute this expansion into Eq. (12) to obtain

$$P[\phi^*, \phi; t] = \int d[\phi_0^*] d[\phi_0] \rho[|\phi_0|^2; t_0] \frac{|\langle \phi; t | \phi_0; t_0 \rangle|^2}{\langle \phi; t | \phi; t \rangle \langle \phi_0; t_0 | \phi_0; t_0 \rangle}. \quad (14)$$

This is a particularly useful result, because the time dependence is now completely determined by the matrix element $\langle \phi; t | \phi_0; t_0 \rangle$ for which the functional integral representation is well known.⁴⁶ It is given by a “path” integral over all complex field evolutions $\psi(\mathbf{x}, t_+) = \sum_\alpha \psi_\alpha(t_+) \chi_\alpha(\mathbf{x})$ from t_0 to t . More precisely we have

$$\langle \phi; t | \phi_0; t_0 \rangle = \int_{\psi(\mathbf{x}, t_0) = \phi_0(\mathbf{x})}^{\psi^*(\mathbf{x}, t) = \phi^*(\mathbf{x})} d[\psi^*] d[\psi] \exp \left\{ \frac{i}{\hbar} S_+[\psi^*, \psi] \right\}, \quad (15)$$

with the forward action $S_+[\psi^*, \psi]$ given by

$$S_+[\psi^*, \psi] = \sum_\alpha \left\{ -i\hbar \psi_\alpha^*(t) \psi_\alpha(t) + \int_{t_0}^t dt_+ \psi_\alpha^*(t_+) \left(i\hbar \frac{\partial}{\partial t_+} - \varepsilon_\alpha \right) \psi_\alpha(t_+) \right\}. \quad (16)$$

In the same manner the matrix element $\langle \phi; t | \phi_0; t_0 \rangle^* = \langle \phi_0; t_0 | \phi; t \rangle$ can be written as a “path” integral over all field configurations $\psi(\mathbf{x}, t_-) = \sum_\alpha \psi_\alpha(t_-) \chi_\alpha(\mathbf{x})$ evolving backward in time from t to t_0 , i.e.,

$$\langle \phi; t | \phi_0; t_0 \rangle^* = \int_{\psi(\mathbf{x}, t) = \phi(\mathbf{x})}^{\psi^*(\mathbf{x}, t_0) = \phi_0^*(\mathbf{x})} d[\psi^*] d[\psi] \exp \left\{ \frac{i}{\hbar} S_-[\psi^*, \psi] \right\}, \quad (17)$$

with a backward action

$$\begin{aligned} S_-[\psi^*, \psi] &= \sum_\alpha \left\{ -i\hbar \psi_\alpha^*(t_0) \psi_\alpha(t_0) + \int_t^{t_0} dt_- \psi_\alpha^*(t_-) \left(i\hbar \frac{\partial}{\partial t_-} - \varepsilon_\alpha \right) \psi_\alpha(t_-) \right\} \\ &= \sum_\alpha \left\{ -i\hbar \psi_\alpha^*(t) \psi_\alpha(t) + \int_t^{t_0} dt_- \psi_\alpha(t_-) \left(-i\hbar \frac{\partial}{\partial t_-} - \varepsilon_\alpha \right) \psi_\alpha^*(t_-) \right\}. \end{aligned} \quad (18)$$

Putting all these results together, we see that the probability distribution $P[\phi^*, \phi; t]$ can in fact be represented by a functional integral over all fields $\psi(\mathbf{x}, t)$ that evolve backwards from t to t_0 and then forward in time

from t_0 to t . Absorbing the factor $\rho[|\phi_0|^2; t_0]$ into the measure of the functional integral, we thus arrive at the desired result that

$$P[\phi^*, \phi; t] = \int_{\psi(\mathbf{x}, t) = \phi(\mathbf{x})}^{\psi^*(\mathbf{x}, t) = \phi^*(\mathbf{x})} d[\psi^*] d[\psi] \exp \left\{ \frac{i}{\hbar} S[\psi^*, \psi] \right\}, \quad (19)$$

where the total (backward-forward) action in first instance obeys

$$\begin{aligned} S[\psi^*, \psi] &= S_-[\psi^*, \psi] + S_+[\psi^*, \psi] \\ &= -i\hbar \sum_{\alpha} (\psi_{\alpha}^*(t) \psi_{\alpha}(t) - |\phi_{\alpha}|^2) \\ &\quad + \sum_{\alpha} \int_{\mathcal{C}'} dt' \left\{ \frac{1}{2} \left(\psi_{\alpha}^*(t') i\hbar \frac{\partial}{\partial t'} \psi_{\alpha}(t') - \psi_{\alpha}(t') i\hbar \frac{\partial}{\partial t'} \psi_{\alpha}^*(t') \right) \right. \\ &\quad \left. - \varepsilon_{\alpha} \psi_{\alpha}^*(t') \psi_{\alpha}(t') \right\}, \end{aligned} \quad (20)$$

and the integration along the Schwinger-Keldysh contour \mathcal{C}' is defined by $\int_{\mathcal{C}'} dt' = \int_{t_0}^{t_0} dt'_- + \int_{t_0}^t dt'_+$.³⁸⁻⁴⁰ Note also that in Eq. (19) we have explicitly specified the boundary conditions on the functional integral. It is interesting to mention that these boundary conditions are essentially dictated by the topological terms in the action $S[\psi^*, \psi]$, which is a general feature of the path-integral formulation of quantum mechanics due to the fact that the quantum theory should have the correct (semi)classical limit.⁴⁷ Making use of the periodicity of the field $\psi(\mathbf{x}, t)$ on the Schwinger-Keldysh contour, the variational principle $\delta S[\psi^*, \psi]/\delta \psi_{\alpha}^*(t_{\pm}) = 0$ indeed leads not only to the Euler-Lagrange equation

$$i\hbar \frac{\partial}{\partial t_{\pm}} \psi_{\alpha}(t_{\pm}) = \varepsilon_{\alpha} \psi_{\alpha}(t_{\pm}), \quad (21)$$

which agrees with the Heisenberg equation of motion $i\hbar \partial \hat{\psi}_{\alpha}(t_{\pm})/\partial t_{\pm} = [\hat{\psi}_{\alpha}(t_{\pm}), H]$ for the annihilation operators, but also to the appropriate boundary condition $\delta \psi_{\alpha}^*(t) = 0$. In the same way we find the complex conjugate results if we require that $\delta S[\psi^*, \psi]/\delta \psi_{\alpha}(t_{\pm}) = 0$. Substituting the boundary conditions in Eq. (20) and performing a partial integration, we then finally obtain for the action

$$S[\psi^*, \psi] = \sum_{\alpha} \int_{\mathcal{C}'} dt' \psi_{\alpha}^*(t') \left(i\hbar \frac{\partial}{\partial t'} - \varepsilon_{\alpha} \right) \psi_{\alpha}(t'), \quad (22)$$

which we for completeness sake also rewrite as

$$S[\psi^*, \psi] = \int_{\mathcal{C}'} dt' \int d\mathbf{x} \psi^*(\mathbf{x}, t') \left(ih \frac{\partial}{\partial t'} + \frac{\hbar^2 \nabla^2}{2m} - V^{ex}(\mathbf{x}) \right) \psi(\mathbf{x}, t'). \quad (23)$$

We are now in a position to derive the equation of motion, i.e., the Fokker-Planck equation, for the probability distribution $P[\phi^*, \phi; t]$. This is most easily achieved by performing the variable transformation $\psi(\mathbf{x}, t_{\pm}) = \phi(\mathbf{x}, t') \pm \xi(\mathbf{x}, t')/2$ in Eq. (19). In this manner the fields $\psi(\mathbf{x}, t_-)$ and $\psi(\mathbf{x}, t_+)$ that live on the backward and forward branch of the Schwinger-Keldysh contour, respectively, are “projected” onto the real time axis. Moreover, at the same time we effect a separation between the (semi)classical dynamics described by $\phi(\mathbf{x}, t')$ and the quantum fluctuations determined by $\xi(\mathbf{x}, t')$. After the transformation we have

$$P[\phi^*, \phi; t] = \int_{\phi(\mathbf{x}, t) = \phi(\mathbf{x})}^{\phi^*(\mathbf{x}, t) = \phi^*(\mathbf{x})} d[\phi^*] d[\phi] \int d[\xi^*] d[\xi] \times \exp \left\{ \frac{i}{\hbar} S[\phi^*, \phi; \xi^*, \xi] \right\}, \quad (24)$$

with

$$S[\phi^*, \phi; \xi^*, \xi] = \sum_{\alpha} \int_{t_0}^t dt' \left\{ \phi_{\alpha}^*(t') \left(ih \frac{\partial}{\partial t'} - \varepsilon_{\alpha} \right) \xi_{\alpha}(t') + \xi_{\alpha}^*(t') \left(ih \frac{\partial}{\partial t'} - \varepsilon_{\alpha} \right) \phi_{\alpha}(t') \right\}. \quad (25)$$

Because this action is linear in $\xi_{\alpha}(t')$ and $\xi_{\alpha}^*(t')$, the integration over the quantum fluctuations leads just to a constraint and we find that

$$P[\phi^*, \phi; t] = \int_{\phi(\mathbf{x}, t) = \phi(\mathbf{x})}^{\phi^*(\mathbf{x}, t) = \phi^*(\mathbf{x})} d[\phi^*] d[\phi] \times \prod_{\alpha} \delta \left[\left(-i \frac{\partial}{\partial t'} - \frac{\varepsilon_{\alpha}}{\hbar} \right) \phi_{\alpha}^*(t') \cdot \left(i \frac{\partial}{\partial t'} - \frac{\varepsilon_{\alpha}}{\hbar} \right) \phi_{\alpha}(t') \right] \quad (26)$$

or equivalently that

$$P[\phi^*, \phi; t] = \int d[\phi_0^*] d[\phi_0] P[|\phi_0|^2; t_0] \prod_{\alpha} \delta(|\phi_{\alpha} - \phi_{\alpha}^d(t)|^2), \quad (27)$$

where we used that $P[\phi^*, \phi; t_0]$ is only a function of the amplitude $|\phi|^2$ and also introduced the quantity $\phi_\alpha^{cl}(t)$ obeying the semiclassical equation of motion

$$i\hbar \frac{\partial}{\partial t} \phi_\alpha^{cl}(t) = \varepsilon_\alpha \phi_\alpha^{cl}(t) \quad (28)$$

and the initial condition $\phi_\alpha^{cl}(t_0) = \phi_{0,\alpha}$.

The latter equation is thus solved by $\phi_\alpha^{cl}(t) = \phi_{0,\alpha} e^{-i\varepsilon_\alpha(t-t_0)/\hbar}$ and we conclude from a simple change of variables in Eq. (27) that for an ideal Bose gas $P[\phi^*, \phi; t] = P[|\phi|^2; t_0]$, as expected. We also see from Eq. (27) that the desired equation of motion for $P[\phi^*, \phi; t]$ reads

$$i\hbar \frac{\partial}{\partial t} P[\phi^*, \phi; t] = - \left(\sum_\alpha \frac{\partial}{\partial \phi_\alpha} \varepsilon_\alpha \phi_\alpha \right) P[\phi^*, \phi; t] + \left(\sum_\alpha \frac{\partial}{\partial \phi_\alpha^*} \varepsilon_\alpha \phi_\alpha^* \right) P[\phi^*, \phi; t]. \quad (29)$$

This is indeed the correct Fokker-Planck equation for an ideal Bose gas.⁴⁸ To see that we first consider the average $\langle \phi_\alpha \rangle(t) = \int d[\phi^*] d[\phi] \phi_\alpha \times P[\phi^*, \phi; t]$. Multiplying Eq. (29) with ϕ_α and integrating over $\phi(\mathbf{x})$, we easily find after a partial integration that

$$i\hbar \frac{\partial}{\partial t} \langle \phi_\alpha \rangle(t) = \varepsilon_\alpha \langle \phi_\alpha \rangle(t), \quad (30)$$

which precisely corresponds to the equation of motion of $\langle \hat{\psi}_\alpha(t) \rangle = \text{Tr}[\hat{\rho}(t_0) \hat{\psi}_\alpha(t)]$ in the operator formalism. Similarly, we find that

$$i\hbar \frac{\partial}{\partial t} \langle \phi_\alpha^* \rangle(t) = -\varepsilon_\alpha \langle \phi_\alpha^* \rangle(t), \quad (31)$$

in agreement with the result for $\langle \hat{\psi}_\alpha^\dagger(t) \rangle = \text{Tr}[\hat{\rho}(t_0) \hat{\psi}_\alpha^\dagger(t)]$.

Next we consider the average of $|\phi_\alpha|^2$, for which we immediately obtain

$$i\hbar \frac{\partial}{\partial t} \langle |\phi_\alpha|^2 \rangle(t) = 0. \quad (32)$$

We expect this result to be related to the fact that in the operator formalism the occupation numbers $N_\alpha(t)$ are independent of time. Although this turns out to be true, to give the precise relation between $\langle |\phi_\alpha|^2 \rangle(t)$ and $N_\alpha(t)$ is complicated by the fact that at equal times the operators $\hat{\psi}_\alpha(t)$ and $\hat{\psi}_\alpha^\dagger(t)$ do not commute. However, the path-integral formulation of

quantum mechanics is only capable of calculating time-ordered operator products.⁴⁹ In our case this implies that $\langle |\phi_\alpha|^2 \rangle(t)$ is the value at $t' = t$ of

$$\begin{aligned} & \text{Tr}[\hat{\rho}(t_0) T_{\mathcal{G}'}(\hat{\psi}_\alpha(t) \hat{\psi}_\alpha^\dagger(t'))] \\ &= \Theta(t, t') \text{Tr}[\hat{\rho}(t_0) \hat{\psi}_\alpha(t) \hat{\psi}_\alpha^\dagger(t')] + \Theta(t', t) \text{Tr}[\hat{\rho}(t_0) \hat{\psi}_\alpha^\dagger(t') \hat{\psi}_\alpha(t)], \end{aligned}$$

with $T_{\mathcal{G}'}$ the time-ordering operator on the Schwinger-Keldysh contour and $\Theta(t, t')$ the corresponding Heaviside function. Since the Heaviside function is equal to 1/2 at equal times,⁵⁰ we conclude that

$$\langle |\phi_\alpha|^2 \rangle(t) = N_\alpha(t) + \frac{1}{2} \quad (33)$$

and that Eq. (32) is thus fully consistent with the operator formalism. An intuitive understanding of the relation between $\langle |\phi_\alpha|^2 \rangle(t)$ and the occupation numbers can be obtained by noting that, since all our manipulations up to now have been exact, we expect $\langle |\phi_\alpha|^2 \rangle(t)$ to contain both classical and quantum fluctuations. These correspond precisely to the contributions $N_\alpha(t)$ and 1/2, respectively.

Finally we need to discuss the stationary solutions of the Fokker-Planck equation. It is not difficult to show that any functional that only depends on the amplitudes $|\phi_\alpha|^2$ is a solution. As it stands the Fokker-Planck equation, therefore, does not lead to a unique equilibrium distribution. This is not surprising, because for an isolated, ideal Bose gas there is no mechanism for redistributing the particles over the various energy levels and thus for relaxation towards equilibrium. However, the situation changes when we allow the bosons in the trap to tunnel back and forth to a reservoir at a temperature T . The corrections to the Fokker-Planck equation that are required to describe the physics in this case are considered next. However, to determine these corrections in the most convenient way, we have to slightly generalize the above theory because with the probability distribution $P[\phi^*, \phi; t]$ we are only able to study spatial, but not temporal correlations in the Bose gas.

To study also those we follow a well-known procedure in quantum field theory^{51,52} and construct a generating functional $Z[J, J^*]$ for all (time-ordered) correlation functions. It is obtained by performing two steps. First, we introduce the probability distribution $P_J[\phi^*, \phi; t]$ for a Bose gas in the presence of the external currents $J(\mathbf{x}, t)$ and $J^*(\mathbf{x}, t)$ by adding to the hamiltonian the terms

$$\begin{aligned} & -\hbar \int d\mathbf{x} (\hat{\psi}(\mathbf{x}, t) J^*(\mathbf{x}, t) + J(\mathbf{x}, t) \hat{\psi}^\dagger(\mathbf{x}, t)) \\ &= -\hbar \sum_\alpha (\hat{\psi}_\alpha(t) J_\alpha^*(t) + J_\alpha(t) \hat{\psi}_\alpha^\dagger(t)). \end{aligned}$$

As a result we have

$$\begin{aligned}
P_J[\phi^*, \phi; t] &= \int_{\psi(\mathbf{x}, t) = \phi(\mathbf{x})}^{\psi^*(\mathbf{x}, t) = \phi^*(\mathbf{x})} d[\psi^*] d[\psi] \exp \left\{ \frac{i}{\hbar} S[\psi^*, \psi] \right\} \\
&\quad \times \exp \left\{ i \int_{\mathcal{C}'} dt' \int d\mathbf{x} (\psi(\mathbf{x}, t') J^*(\mathbf{x}, t') + J(\mathbf{x}, t') \psi^*(\mathbf{x}, t')) \right\}.
\end{aligned} \tag{34}$$

Second, we integrate this expression over $\phi(\mathbf{x})$ to obtain the desired generating functional. Hence

$$\begin{aligned}
Z[J, J^*] &= \int d[\phi^*] d[\phi] P_J[\phi^*, \phi; t] \\
&= \int d[\psi^*] d[\psi] \exp \left\{ \frac{i}{\hbar} S[\psi^*, \psi] \right\} \\
&\quad \times \exp \left\{ i \int_{\mathcal{C}^\infty} dt \int d\mathbf{x} (\psi(\mathbf{x}, t) J^*(\mathbf{x}, t) + J(\mathbf{x}, t) \psi^*(\mathbf{x}, t)) \right\}.
\end{aligned} \tag{35}$$

It is important to realize that $Z[J, J^*]$ is indeed independent of the time t because of the fact that $P_J[\phi^*, \phi; t]$ is a probability distribution (cf. Eqs. (10) and (12)) and thus properly normalized. We are therefore allowed to deform the contour \mathcal{C}' to any closed contour that runs through t_0 . Since we are in principle interested in all times $t \geq t_0$, the most convenient choice is the contour that runs backward from infinity to t_0 and then forwards from t_0 to infinity. This contour is denoted by \mathcal{C}^∞ and also called the Schwinger-Keldysh contour in the following because there is in practice never confusion with the more restricted contour \mathcal{C}' that is required when we consider a probability distribution. With this choice it is also clear that $Z[J, J^*]$ is a generating functional. Indeed, Eq. (35) shows explicitly that all time-ordered correlation functions can be obtained by functional differentiation with respect to the currents $J(\mathbf{x}, t)$ and $J^*(\mathbf{x}, t)$. We have, for instance, that

$$\text{Tr}[\hat{\rho}(t_0) \hat{\psi}(\mathbf{x}, t)] = \frac{1}{i} \frac{\delta}{\delta J^*(\mathbf{x}, t)} Z[J, J^*] \Big|_{J, J^* = 0} \tag{36}$$

and similarly that

$$\text{Tr}[\hat{\rho}(t_0) T_{\mathcal{C}^\infty}(\hat{\psi}(\mathbf{x}, t) \hat{\psi}^\dagger(\mathbf{x}', t')))] = \frac{1}{i^2} \frac{\delta^2}{\delta J^*(\mathbf{x}, t) \delta J(\mathbf{x}', t')} Z[J, J^*] \Big|_{J, J^* = 0} \tag{37}$$

Note that the times t and t' always have to be larger or equal to t_0 for these identities to be valid.

B. Ideal Bose Gas Coupled to a Reservoir

For the reservoir we take an ideal gas of N bosons in a box with volume V . The states in this box are labeled by the momentum $\hbar\mathbf{k}$ and equal to $\chi_{\mathbf{k}}(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}}/\sqrt{V}$. They have an energy $\varepsilon(\mathbf{k}) = \hbar^2\mathbf{k}^2/2m + \Delta V^{ex}$, where ΔV^{ex} accounts for a possible bias between the potential energies of a particle in the center of the trap and a particle in the reservoir. The reservoir is also taken to be sufficiently large that it can be treated in the thermodynamic limit and is in an equilibrium with temperature T and chemical potential μ for times $t < t_0$. At t_0 it is brought into contact with the trap by means of a tunnel hamiltonian

$$\hat{H}^{int} = \frac{1}{\sqrt{V}} \sum_{\alpha} \sum_{\mathbf{k}} (t_{\alpha}(\mathbf{k}) \hat{\psi}_{\alpha}(t) \hat{\psi}_{\mathbf{k}}^{\dagger}(t) + t_{\alpha}^*(\mathbf{k}) \hat{\psi}_{\mathbf{k}}(t) \hat{\psi}_{\alpha}^{\dagger}(t)), \quad (38)$$

with complex tunneling matrix elements $t_{\alpha}(\mathbf{k})$ that for simplicity are assumed to be almost constant for momenta $\hbar k$ smaller than a cutoff $\hbar k_c$ but to vanish rapidly for momenta larger than this ultraviolet cutoff. Moreover, we consider here only the low-temperature regime in which the thermal de Broglie wavelength $\Lambda = (2\pi\hbar^2/mk_B T)^{1/2}$ of the particles obeys $\Lambda \gg 1/k_c$, since this is the most appropriate limit for realistic atomic gases.

To study the evolution of the combined system for times $t \geq t_0$ we thus have to deal with the action

$$\begin{aligned} S[\psi^*, \psi; \psi_R^*, \psi_R] = & -\frac{1}{\sqrt{V}} \sum_{\alpha} \sum_{\mathbf{k}} \int_{\mathcal{Q}^{\infty}} dt (t_{\alpha}(\mathbf{k}) \psi_{\alpha}(t) \psi_{\mathbf{k}}^*(t) + t_{\alpha}^*(\mathbf{k}) \psi_{\mathbf{k}}(t) \psi_{\alpha}^*(t)) \\ & + \sum_{\alpha} \int_{\mathcal{Q}^{\infty}} dt \psi_{\alpha}^*(t) \left(i\hbar \frac{\partial}{\partial t} - \varepsilon_{\alpha} + \mu \right) \psi_{\alpha}(t) \\ & + \sum_{\mathbf{k}} \int_{\mathcal{Q}^{\infty}} dt \psi_{\mathbf{k}}^*(t) \left(i\hbar \frac{\partial}{\partial t} - \varepsilon(\mathbf{k}) + \mu \right) \psi_{\mathbf{k}}(t), \end{aligned} \quad (39)$$

if we measure all energies relative to the chemical potential and also introduce the complex field $\psi_R(\mathbf{x}, t) = \sum_{\mathbf{k}} \psi_{\mathbf{k}}(t) \chi_{\mathbf{k}}(\mathbf{x})$ for the degrees of freedom of the reservoir. However, we are only interested in the evolution of the Bose gas in the trap and therefore only in the time-ordered correlation

functions of this part of the system. The corresponding generating functional

$$Z[J, J^*] = \int d[\psi^*] d[\psi] \int d[\psi_R^*] d[\psi_R] \exp \left\{ \frac{i}{\hbar} S[\psi^*, \psi; \psi_R^*, \psi_R] \right\} \\ \times \exp \left\{ i \int_{\mathcal{G}^\infty} dt \int d\mathbf{x} (\psi(\mathbf{x}, t) J^*(\mathbf{x}, t) + J(\mathbf{x}, t) \psi^*(\mathbf{x}, t)) \right\}. \quad (40)$$

is of the same form as the functional integral in Eq. (35), but now with an effective action that is defined by

$$\exp \left\{ \frac{i}{\hbar} S^{\text{eff}}[\psi^*, \psi] \right\} \equiv \int d[\psi_R^*] d[\psi_R] \exp \left\{ \frac{i}{\hbar} S[\psi^*, \psi; \psi_R^*, \psi_R] \right\}. \quad (41)$$

Hence, our next task is to integrate out the field $\psi_R(\mathbf{x}, t)$, which can be done exactly because it only requires the integration of a gaussian.

To familiarize ourselves with the Schwinger-Keldysh formalism, we perform the gaussian integration here in some detail. In principle, this can of course be done explicitly by making use of the fact that for the initial density matrix $\rho_R(t_0)$ of the reservoir we have that

$$\rho_R[|\phi_R|^2, t_0] = \prod_{\mathbf{k}} \frac{1}{N(\mathbf{k})} e^{-|\phi_{\mathbf{k}}|^2/N(\mathbf{k})}, \quad (42)$$

with $N(\mathbf{k}) = 1/(e^{\beta(\varepsilon(\mathbf{k}) - \mu)} - 1)$ the appropriate Bose distribution function and $\beta = 1/k_B T$. However, in practice it is much more easy to use a different procedure. It is based on the observation that if we introduce the δ function on the Schwinger-Keldysh contour defined by $\int_{\mathcal{G}^\infty} dt' \delta(t, t') = 1$ and the Green's function $G(\mathbf{k}; t, t')$ obeying

$$\left(ih \frac{\partial}{\partial t} - \varepsilon(\mathbf{k}) + \mu \right) G(\mathbf{k}; t, t') = h\delta(t, t'), \quad (43)$$

the action $S[\psi^*, \psi; \psi_R^*, \psi_R]$ can be written as a complete square, or more precisely as the sum of two squares $S_1[\psi^*, \psi]$ and $S_2[\psi^*, \psi; \psi_R^*, \psi_R]$ that are given by

$$S_1[\psi^*, \psi] = \sum_{\alpha} \int_{\mathcal{G}^\infty} dt \psi_{\alpha}^*(t) \left(ih \frac{\partial}{\partial t} - \varepsilon_{\alpha} + \mu \right) \psi_{\alpha}(t) \\ - \frac{1}{\hbar V} \sum_{\alpha, \alpha'} \sum_{\mathbf{k}} \int_{\mathcal{G}^\infty} dt \int_{\mathcal{G}^\infty} dt' \psi_{\alpha}^*(t) t_{\alpha}^*(\mathbf{k}) G(\mathbf{k}; t, t') t_{\alpha'}(\mathbf{k}) \psi_{\alpha'}(t') \quad (44)$$

and

$$\begin{aligned}
 & S_2[\psi^*, \psi; \psi_R^*, \psi_R] \\
 &= \sum_{\mathbf{k}} \int_{\mathcal{G}^\alpha} dt \left(\psi_{\mathbf{k}}^*(t) - \frac{1}{h\sqrt{V}} \sum_{\alpha} \int_{\mathcal{G}^\alpha} dt' t_{\alpha}^*(\mathbf{k}) \psi_{\alpha}^*(t') G(\mathbf{k}; t', t) \right) \\
 & \quad \times \left(ih \frac{\partial}{\partial t} - \varepsilon(\mathbf{k}) + \mu \right) \left(\psi_{\mathbf{k}}(t) - \frac{1}{h\sqrt{V}} \sum_{\alpha} \int_{\mathcal{G}^\alpha} dt' G(\mathbf{k}; t, t') \psi_{\alpha}(t') t_{\alpha}(\mathbf{k}) \right),
 \end{aligned} \tag{45}$$

respectively. Since the first term is independent of the field $\psi_R(\mathbf{x}, t)$, we only need to evaluate the functional integral $\int d[\psi_R^*] d[\psi_R] \exp(iS_2[\psi^*, \psi; \psi_R^*, \psi_R]/h)$. Performing a shift in the integration variables, we however see that this functional integral is equal to

$$\begin{aligned}
 & \int d[\psi_R^*] d[\psi_R] \exp \left\{ \frac{i}{h} \sum_{\mathbf{k}} \int_{\mathcal{G}^\alpha} dt \psi_{\mathbf{k}}^*(t) \left(ih \frac{\partial}{\partial t} - \varepsilon(\mathbf{k}) + \mu \right) \psi_{\mathbf{k}}(t) \right\} \\
 &= \text{Tr}[\rho_R(t_0)] = 1.
 \end{aligned} \tag{46}$$

As a result the effective action $S^{\text{eff}}[\psi^*, \psi]$ is just equal to $S_1[\psi^*, \psi]$, which can be slightly rewritten to read

$$\begin{aligned}
 S^{\text{eff}}[\psi^*, \psi] &= \sum_{\alpha, \alpha'} \int_{\mathcal{G}^\alpha} dt \int_{\mathcal{G}^\alpha} dt' \psi_{\alpha}^*(t) \\
 & \quad \times \left\{ \left(ih \frac{\partial}{\partial t} - \varepsilon_{\alpha} + \mu \right) \delta_{\alpha, \alpha'} \delta(t, t') - h \Sigma_{\alpha, \alpha'}(t, t') \right\} \psi_{\alpha'}(t'),
 \end{aligned} \tag{47}$$

with the so-called selfenergy $\Sigma_{\alpha, \alpha'}(t, t')$ obeying

$$h \Sigma_{\alpha, \alpha'}(t, t') = \frac{1}{hV} \sum_{\mathbf{k}} t_{\alpha}^*(\mathbf{k}) G(\mathbf{k}; t, t') t_{\alpha'}(\mathbf{k}). \tag{48}$$

This is our first example of an effective action describing the dynamics of a Bose gas. Before we can study its consequences we clearly first need to determine the Green's function $G(\mathbf{k}; t, t')$ in terms of which the selfenergy is expressed. Although we know that this Green's function fulfills Eq. (43), we cannot directly solve this equation because we do not know the appropriate boundary condition at $t = t'$. To calculate $G(\mathbf{k}; t, t')$ we therefore have to proceed differently. It is however clear from Eq. (43) that $G(\mathbf{k}; t, t')$ is a property of the reservoir and we thus expect that it can

somehow be related to a time-ordered correlation function of this reservoir. To see that explicitly we consider again the generating functional of these correlation functions, i.e.,

$$\begin{aligned}
Z_R[J, J^*] &= \int d[\psi_R^*] d[\psi_R] \\
&\times \exp \left\{ \frac{i}{\hbar} \sum_{\mathbf{k}} \int_{\mathcal{Q}^\infty} dt \psi_{\mathbf{k}}^*(t) \left(i\hbar \frac{\partial}{\partial t} - \varepsilon(\mathbf{k}) + \mu \right) \psi_{\mathbf{k}}(t) \right\} \\
&\times \exp \left\{ i \sum_{\mathbf{k}} \int_{\mathcal{Q}^\infty} dt (\psi_{\mathbf{k}}(t) J_{\mathbf{k}}^*(t) + J_{\mathbf{k}}(t) \psi_{\mathbf{k}}^*(t)) \right\}. \quad (49)
\end{aligned}$$

It is again a gaussian integral and can thus be evaluated in the same manner as before. The result is now

$$Z_R[J, J^*] = \exp \left\{ -i \sum_{\mathbf{k}} \int_{\mathcal{Q}^\infty} dt \int_{\mathcal{Q}^\infty} dt' J_{\mathbf{k}}^*(t) G(\mathbf{k}; t, t') J_{\mathbf{k}}(t') \right\}, \quad (50)$$

which shows by means of Eq. (37) that

$$iG(\mathbf{k}; t, t') = \text{Tr}[\hat{\rho}_R(t_0) T_{\mathcal{Q}^\infty}(\hat{\psi}_{\mathbf{k}}(t) \hat{\psi}_{\mathbf{k}}^\dagger(t'))]. \quad (51)$$

Note first of all that this is indeed consistent with Eq. (43), because the right-hand side obeys

$$\begin{aligned}
& i\hbar \frac{\partial}{\partial t} \text{Tr}[\hat{\rho}_R(t_0) T_{\mathcal{Q}^\infty}(\hat{\psi}_{\mathbf{k}}(t) \hat{\psi}_{\mathbf{k}}^\dagger(t'))] \\
&= i\hbar \delta(t, t') \text{Tr}[\hat{\rho}_R(t_0) [\hat{\psi}_{\mathbf{k}}(t), \hat{\psi}_{\mathbf{k}}^\dagger(t)]] \\
&\quad + \text{Tr} \left[\hat{\rho}_R(t_0) T_{\mathcal{Q}^\infty} \left(i\hbar \frac{\partial}{\partial t} \hat{\psi}_{\mathbf{k}}(t) \hat{\psi}_{\mathbf{k}}^\dagger(t') \right) \right] \\
&= i\hbar \delta(t, t') + (\varepsilon(\mathbf{k}) - \mu) \text{Tr}[\hat{\rho}_R(t_0) T_{\mathcal{Q}^\infty}(\hat{\psi}_{\mathbf{k}}(t) \hat{\psi}_{\mathbf{k}}^\dagger(t'))] \quad (52)
\end{aligned}$$

in the operator formalism. Moreover, from this identification we see that the desired solution fulfilling the appropriate boundary conditions is apparently

$$G(\mathbf{k}; t, t') = -ie^{-i(\varepsilon(\mathbf{k}) - \mu)(t - t')/\hbar} \{ \Theta(t, t')(1 + N(\mathbf{k})) + \Theta(t', t) N(\mathbf{k}) \}. \quad (53)$$

The specific dependence on the backward and forward branches of the Schwinger-Keldysh contour is thus solely determined by the Heaviside function $\Theta(t, t')$. As a result it is convenient to decompose the Green's

function into its analytic pieces $G^>(\mathbf{k}; t-t')$ and $G^<(\mathbf{k}; t-t')$ (Ref. 53) by means of

$$G(\mathbf{k}; t, t') = \Theta(t, t') G^>(\mathbf{k}; t-t') + \Theta(t', t) G^<(\mathbf{k}; t-t'). \quad (54)$$

Due to the fact that we are always dealing with time-ordered correlation functions, such a decomposition turns out to be a generic feature of all the functions on the Schwinger-Keldysh contour that we will encounter in the following.⁵⁴ For a general function $F(t, t')$, it is however also possible to have δ -function singularities. If that happens the correct decomposition is

$$F(t, t') = F^\delta(t) \delta(t, t') + \Theta(t, t') F^>(t, t') + \Theta(t', t) F^<(t, t'). \quad (55)$$

This more general decomposition is not required here, but will be needed in Secs. III and IV when we determine the effective interaction between two atoms in a gas.

Having obtained the Green's function of the reservoir, we can now return to our discussion of the effective action $S^{\text{eff}}[\psi^*, \psi]$ for the Bose gas in the trap. Although we have chosen to derive in Eq. (47) the effective action for the generating functional $Z[J, J^*]$, it is straightforward to show that the effective action for the probability distribution $P[\phi^*, \phi; t]$ is obtained by only replacing the contour \mathcal{C}^∞ by \mathcal{C}^t . In the following we therefore no longer always specify all the boundary conditions on the time integration, if the precise details of this integration are not important and the discussion applies equally well to both cases. Keeping this in mind, we now again perform the transformation $\psi_\alpha(t_\pm) = \phi_\alpha(t) \pm \xi_\alpha(t)/2$ to explicitly separate the (semi)classical dynamics from the effect of fluctuations. It leads in first instance to⁵⁵

$$\begin{aligned} & S^{\text{eff}}[\phi^*, \phi; \xi^*, \xi] \\ &= \sum_\alpha \int_{t_0} dt \left\{ \phi_\alpha^*(t) \left(ih \frac{\partial}{\partial t} - \varepsilon_\alpha + \mu \right) \xi_\alpha(t) + \xi_\alpha^*(t) \left(ih \frac{\partial}{\partial t} - \varepsilon_\alpha + \mu \right) \phi_\alpha(t) \right\} \\ & \quad - \sum_{\alpha, \alpha'} \int_{t_0} dt \int_{t_0} dt' \left(\phi_\alpha^*(t) h \Sigma_{\alpha, \alpha'}^{(-)}(t-t') \xi_{\alpha'}(t') + \xi_\alpha^*(t) h \Sigma_{\alpha, \alpha'}^{(+)}(t-t') \phi_{\alpha'}(t') \right) \\ & \quad - \frac{1}{2} \sum_{\alpha, \alpha'} \int_{t_0} dt \int_{t_0} dt' \xi_\alpha^*(t) h \Sigma_{\alpha, \alpha'}^K(t-t') \xi_{\alpha'}(t'), \end{aligned} \quad (56)$$

where we introduced the retarded and advanced components of the self-energy

$$\Sigma_{\alpha, \alpha'}^{(\pm)}(t-t') = \pm \Theta(\pm(t-t')) (\Sigma_{\alpha, \alpha'}^>(t-t') - \Sigma_{\alpha, \alpha'}^<(t-t')) \quad (57)$$

that affect the terms in the action that are linear in $\xi_\alpha^*(t)$ and $\xi_\alpha(t)$, respectively, and also the Keldysh component

$$\Sigma_{\alpha, \alpha'}^K(t-t') = (\Sigma_{\alpha, \alpha'}^>(t-t') + \Sigma_{\alpha, \alpha'}^<(t-t')) \quad (58)$$

that is associated with the part quadratic in the fluctuations.

The physical content of these various components of the selfenergy is understood most clearly if we now apply the beautiful functional-integral procedure that is due to Hubbard and Stratonovich.^{56, 57, 46} The basic idea is to write the factor

$$\exp \left\{ -\frac{i}{2} \sum_{\alpha, \alpha'} \int_{t_0} dt \int_{t_0} dt' \xi_\alpha^*(t) \Sigma_{\alpha, \alpha'}^K(t-t') \xi_{\alpha'}(t') \right\}$$

in the integrant of the functional integral $\int d[\phi^*] d[\phi] \int d[\xi^*] d[\xi] \times \exp(iS^{\text{eff}}[\phi^*, \phi; \xi^*, \xi]/\hbar)$ as a gaussian integral over a complex field $\eta(\mathbf{x}, t)$. It is equivalent, but in practice more convenient, to just multiply the integrant by a factor 1 that is written as a gaussian integral $\int d[\eta^*] \times d[\eta] \exp(iS^{\text{eff}}[\eta^*, \eta]/\hbar)$ with

$$\begin{aligned} S^{\text{eff}}[\eta^*, \eta] = & \frac{1}{2} \sum_{\alpha, \alpha'} \int_{t_0} dt \int_{t_0} dt' \left(2\eta_\alpha^*(t) - \sum_{\alpha''} \int_{t_0} dt'' \xi_{\alpha''}^*(t'') h\Sigma_{\alpha'', \alpha}^K(t''-t) \right) \\ & \times (h\Sigma^K)_{\alpha, \alpha'}^{-1}(t-t') \left(2\eta_{\alpha'}(t') - \sum_{\alpha''} \int_{t_0} dt'' h\Sigma_{\alpha'', \alpha'}^K(t'-t'') \xi_{\alpha''}(t'') \right) \end{aligned} \quad (59)$$

a complete square. Adding this to $S^{\text{eff}}[\phi^*, \phi; \xi^*, \xi]$ the total effective action becomes

$$\begin{aligned} S^{\text{eff}}[\phi^*, \phi; \xi^*, \xi; \eta^*, \eta] &= \sum_{\alpha, \alpha'} \int_{t_0} dt \int_{t_0} dt' \phi_\alpha^*(t) \\ & \times \left\{ \left(ih \frac{\partial}{\partial t} - \varepsilon_\alpha + \mu - \eta_\alpha^*(t) \right) \delta_{\alpha, \alpha'} \delta(t-t') - h\Sigma_{\alpha, \alpha'}^{(-)}(t-t') \right\} \xi_{\alpha'}(t') \\ & + \sum_{\alpha, \alpha'} \int_{t_0} dt \int_{t_0} dt' \xi_\alpha^*(t) \\ & \times \left\{ \left(ih \frac{\partial}{\partial t} - \varepsilon_\alpha + \mu - \eta_\alpha(t) \right) \delta_{\alpha, \alpha'} \delta(t-t') - h\Sigma_{\alpha, \alpha'}^{(+)}(t-t') \right\} \phi_{\alpha'}(t') \\ & + 2 \sum_{\alpha, \alpha'} \int_{t_0} dt \int_{t_0} dt' \eta_\alpha^*(t) (h\Sigma^K)_{\alpha, \alpha'}^{-1}(t-t') \eta_{\alpha'}(t') \end{aligned} \quad (60)$$

and is thus linear in $\xi_\alpha(t)$ and $\xi_\alpha^*(t)$. Integrating over these fluctuations we conclude from this action that the field $\phi(\mathbf{x}, t)$ is constraint to obey the Langevin equations⁵⁸

$$i\hbar \frac{\partial}{\partial t} \phi_\alpha(t) = (\varepsilon_\alpha - \mu) \phi_\alpha(t) + \sum_{\alpha'} \int_{t_0}^{\infty} dt' \hbar \Sigma_{\alpha, \alpha'}^{(+)}(t-t') \phi_{\alpha'}(t') + \eta_\alpha(t) \quad (61)$$

and

$$-i\hbar \frac{\partial}{\partial t} \phi_\alpha^*(t) = (\varepsilon_\alpha - \mu) \phi_\alpha^*(t) + \sum_{\alpha'} \int_{t_0}^{\infty} dt' \phi_{\alpha'}^*(t') \hbar \Sigma_{\alpha', \alpha}^{(-)}(t'-t) + \eta_\alpha^*(t) \quad (62)$$

with gaussian noise terms $\eta_\alpha(t)$ and $\eta_\alpha^*(t)$ that from the last term in the right-hand side of Eq. (60) are seen to have the time correlations

$$\begin{aligned} \langle \eta_\alpha^*(t) \eta_{\alpha'}(t') \rangle &= \frac{i\hbar^2}{2} \Sigma_{\alpha, \alpha'}^K(t-t') \\ &= \frac{1}{2} \int \frac{d\mathbf{k}}{(2\pi)^3} (1 + 2N(\mathbf{k})) t_\alpha^*(\mathbf{k}) e^{-i(\varepsilon(\mathbf{k}) - \mu)(t-t')/\hbar} t_{\alpha'}(\mathbf{k}) \end{aligned} \quad (63)$$

in the thermodynamic limit. Moreover, in that limit the retarded selfenergy is equal to

$$\hbar \Sigma_{\alpha, \alpha'}^{(+)}(t-t') = -\Theta(t-t') \frac{i}{\hbar} \int \frac{d\mathbf{k}}{(2\pi)^3} t_\alpha^*(\mathbf{k}) e^{-i(\varepsilon(\mathbf{k}) - \mu)(t-t')/\hbar} t_{\alpha'}(\mathbf{k}) \quad (64)$$

and the advanced selfenergy can be found from the relation $\Sigma_{\alpha', \alpha}^{(-)}(t'-t) = (\Sigma_{\alpha, \alpha'}^{(+)}(t-t'))^*$. Note that the Heaviside functions in these retarded and advanced selfenergies are precisely such that Eqs. (61) and (62) are causal, i.e., the evolution of $\phi(\mathbf{x}, t)$ and $\phi^*(\mathbf{x}, t)$ depends only on their value at previous times. This is clearly a sensible result and explains why these components of the full selfenergy enter into the terms in the action that are linear in the fluctuations.

To understand why the Keldysh component enters into the terms that are quadratic in the fluctuations is more complicated and is related to the famous fluctuation-dissipation theorem.^{53, 58, 59} Physically, the fluctuation-dissipation theorem ensures in our case that, due to the interactions with the reservoir, the gas in the trap now relaxes towards thermal equilibrium in the limit $t \rightarrow \infty$. Let us therefore study this limit in somewhat more detail. However, before we do so we would like to mention for completeness that if we apply the above Hubbard-Stratonovich transformation to

the functional integral for the probability distribution $P[\phi^*, \phi; t]$, we immediately find that analogous to Eq. (27)

$$P[\phi^*, \phi; t] = \int d[\eta^*] d[\eta] P[\eta^*, \eta] \int d[\phi_0^*] d[\phi_0] P[|\phi_0|^2; t_0] \times \prod_{\alpha} \delta(|\phi_{\alpha} - \phi_{\alpha}^{cl}(t)|^2), \quad (65)$$

where $\phi_{\alpha}^{cl}(t)$ now obeys the Langevin equation in Eq. (61) for a particular realization of the noise $\eta_{\alpha}(t)$ and with the initial condition $\phi_{\alpha}^{cl}(t_0) = \phi_{0, \alpha}$. Furthermore, the probability distribution for each realization of the noise is given by

$$P[\eta^*, \eta] = \exp \left\{ \frac{2i}{h} \sum_{\alpha, \alpha'} \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' \eta_{\alpha}^*(t') (\hbar \Sigma^K)_{\alpha, \alpha'}^{-1} (t' - t'') \eta_{\alpha'}(t'') \right\}. \quad (66)$$

This result shows explicitly that, as expected, the Fokker-Planck equation associated with the Langevin equations in Eqs. (61) and (62) is in fact the desired Fokker-Planck equation for the probability distribution $P[\phi^*, \phi; t]$. Hence, this Fokker-Planck equation can be obtained either from these Langevin equations or directly from the effective action in Eq. (56), without performing a Hubbard-Stratonovich transformation. We now first follow the former route, but lateron also explain the more direct procedure because that is interesting in its own right and also shows the consistency of the theory.

After this slight digression, we now return to the long-time behaviour of the gas. In the long-time limit two important simplifications occur. Looking at the expressions for the selfenergies, we see that their width as a function of the time difference $t - t'$ is at most of order $\mathcal{O}(\hbar/k_B T)$ in the low-temperature regime of interest where the thermal de Broglie wavelength of the particles obeys $\Lambda \gg 1/k_c$. If therefore $t \gg t_0 + \mathcal{O}(\hbar/k_B T)$ we are allowed to take the limit $t_0 \rightarrow -\infty$. Taking this limit means physically that we neglect the initial transients that are due to the precise way in which the contact between the trap and the reservoir is made, and focus on the “universal” dynamics that is independent of these details. In addition, at long times the dynamics of the gas is expected to be sufficiently slow that we can neglect the memory effects altogether. Finally, we consider here first the case of a reservoir that is so weakly coupled to the gas in the trap that we can treat the coupling with second order perturbation theory. As a result we can also neglect the nondiagonal elements of the selfenergies. In total we are then allowed to put (but see below)

$$\Sigma_{\alpha, \alpha'}^{(\pm), K}(t - t') \simeq \Sigma_{\alpha, \alpha}^{(\pm), K}(\varepsilon_{\alpha} - \mu) \delta_{\alpha, \alpha'} \delta(t - t') \equiv \Sigma_{\alpha}^{(\pm), K} \delta_{\alpha, \alpha'} \delta(t - t'), \quad (67)$$

where the Fourier transform of the selfenergies is defined by

$$\Sigma_{\alpha,\alpha'}^{(\pm),K}(t-t') = \int d\varepsilon \Sigma_{\alpha,\alpha'}^{(\pm),K}(\varepsilon) \frac{e^{-i\varepsilon(t-t')/\hbar}}{2\pi\hbar}. \quad (68)$$

Note that this implies in general that $\Sigma_{\alpha,\alpha'}^{(-)}(\varepsilon) = (\Sigma_{\alpha,\alpha'}^{(+)}(\varepsilon))^*$, and in particular therefore that $\Sigma_{\alpha}^{(-)} = (\Sigma_{\alpha}^{(+)})^*$.

With these simplifications our Langevin equations become

$$i\hbar \frac{\partial}{\partial t} \phi_{\alpha}(t) = (\varepsilon_{\alpha} + \hbar \Sigma_{\alpha}^{(+)} - \mu) \phi_{\alpha}(t) + \eta_{\alpha}(t) \quad (69)$$

and just the complex conjugate equation for $\phi_{\alpha}^*(t)$. The retarded selfenergy in this equation is given by

$$\hbar \Sigma_{\alpha}^{(+)} = \int \frac{d\mathbf{k}}{(2\pi)^3} t_{\alpha}^*(\mathbf{k}) \frac{1}{\varepsilon_{\alpha}^{+} - \varepsilon(\mathbf{k})} t_{\alpha}(\mathbf{k}), \quad (70)$$

with ε_{α}^{+} the usual notation for the limiting procedure $\varepsilon_{\alpha} + i0$. It clearly has real and imaginary parts that we denote by S_{α} and $-R_{\alpha}$, respectively. Denoting the Cauchy principle value part of an integral by \mathcal{P} , they obey

$$S_{\alpha} = \int \frac{d\mathbf{k}}{(2\pi)^3} t_{\alpha}^*(\mathbf{k}) \frac{\mathcal{P}}{\varepsilon_{\alpha} - \varepsilon(\mathbf{k})} t_{\alpha}(\mathbf{k}) \quad (71)$$

and

$$R_{\alpha} = \pi \int \frac{d\mathbf{k}}{(2\pi)^3} \delta(\varepsilon_{\alpha} - \varepsilon(\mathbf{k})) |t_{\alpha}(\mathbf{k})|^2. \quad (72)$$

The interpretation of these results is quite obvious if we consider the average of the Langevin equation, i.e.,

$$i\hbar \frac{\partial}{\partial t} \langle \phi_{\alpha} \rangle(t) = (\varepsilon_{\alpha} + S_{\alpha} - iR_{\alpha} - \mu) \langle \phi_{\alpha} \rangle(t), \quad (73)$$

which is solved by

$$\langle \phi_{\alpha} \rangle(t) = \langle \phi_{\alpha} \rangle(0) e^{-i(\varepsilon_{\alpha} + S_{\alpha} - \mu)t/\hbar} e^{-R_{\alpha}t/\hbar}. \quad (74)$$

Hence the real part of the retarded selfenergy S_{α} represents the shift in the energy of state $\chi_{\alpha}(\mathbf{x})$ due to the coupling with the reservoir. Indeed, Eq. (71) agrees precisely with the value of this shift in second order perturbation theory. Moreover, the fact that $|\langle \phi_{\alpha} \rangle(t)|^2 = |\langle \phi_{\alpha} \rangle(0)|^2 e^{-2R_{\alpha}t/\hbar}$

shows that the average rate of decay Γ_α of the state $\chi_\alpha(\mathbf{x})$ is equal to $2R_\alpha/\hbar$, which, together with Eq. (72), exactly reproduces Fermi's Golden Rule.

From the equation of motion for the average $\langle \phi_\alpha \rangle(t)$ we can also conclude that the right-hand side of the Fokker-Planck equation must contain the “streaming” terms

$$-\left(\sum_\alpha \frac{\partial}{\partial \phi_\alpha} (\varepsilon_\alpha + \hbar \Sigma_\alpha^{(+)} - \mu) \phi_\alpha\right) P[\phi^*, \phi; t] \\ + \left(\sum_\alpha \frac{\partial}{\partial \phi_\alpha^*} (\varepsilon_\alpha + \hbar \Sigma_\alpha^{(-)} - \mu) \phi_\alpha^*\right) P[\phi^*, \phi; t].$$

However, there must now also be a “diffusion” term due to the fact that the average $\langle |\phi_\alpha|^2 \rangle(t)$ is no longer independent of time since the interactions with the reservoir can lead to changes in the occupation numbers $N_\alpha(t)$. To obtain the “diffusion” term we thus need to determine $i\hbar \partial \langle |\phi_\alpha|^2 \rangle(t) / \partial t$. To do so we first formally solve the Langevin equation by

$$\phi_\alpha(t) = e^{-i(\varepsilon_\alpha + \hbar \Sigma_\alpha^{(+)} - \mu)t/\hbar} \left\{ \phi_\alpha(0) - \frac{i}{\hbar} \int_0^t dt' \eta(t') e^{i(\varepsilon_\alpha + \hbar \Sigma_\alpha^{(+)} - \mu)t'/\hbar} \right\}. \quad (75)$$

Multiplying this with the complex conjugate expression and taking the average, we obtain first of all

$$\langle |\phi_\alpha|^2 \rangle(t) = e^{-2R_\alpha t/\hbar} \left\{ \langle |\phi_\alpha|^2 \rangle(0) + \frac{i}{2} \Sigma_\alpha^K \int_0^t dt' e^{2R_\alpha t'/\hbar} \right\}, \quad (76)$$

which shows that

$$i\hbar \frac{\partial}{\partial t} \langle |\phi_\alpha|^2 \rangle(t) = -2iR_\alpha \langle |\phi_\alpha|^2 \rangle(t) - \frac{1}{2} \hbar \Sigma_\alpha^K. \quad (77)$$

Moreover, the Keldysh component of the selfenergy is given by

$$\hbar \Sigma_\alpha^K = -2\pi i \int \frac{d\mathbf{k}}{(2\pi)^3} (1 + 2N(\mathbf{k})) \delta(\varepsilon_\alpha - \varepsilon(\mathbf{k})) |t_\alpha(\mathbf{k})|^2 \quad (78)$$

and therefore obeys

$$\hbar \Sigma_\alpha^K = -2i(1 + 2N_\alpha) R_\alpha \quad (79)$$

with $N_\alpha = (e^{\beta(\varepsilon_\alpha - \mu)} - 1)^{-1}$ the Bose distribution function. This is in fact the fluctuation-dissipation theorem, because it relates the strength of the fluctuations determined by $\hbar \Sigma_\alpha^K$, to the amount of dissipation that is given by R_α .

As mentioned previously, the fluctuation-dissipation theorem ensures that the gas relaxes to thermal equilibrium. This we can already see from Eq. (77), because substitution of the fluctuation-dissipation theorem leads to

$$i\hbar \frac{\partial}{\partial t} \langle |\phi_\alpha|^2 \rangle(t) = -2iR_\alpha \langle |\phi_\alpha|^2 \rangle(t) + iR_\alpha(1 + 2N_\alpha), \quad (80)$$

which tells us that in equilibrium $\langle |\phi_\alpha|^2 \rangle = N_\alpha + 1/2$. In fact, we have argued in Sec. II.A that in general $\langle |\phi_\alpha|^2 \rangle(t) = N_\alpha(t) + 1/2$. Substituting this identity in Eq. (80) and realizing that $\Gamma_\alpha N_\alpha = \Gamma_\alpha N(\mathbf{k})$ due to the energy-conserving δ function in Γ_α , we indeed obtain the correct rate equation for the average occupation numbers

$$\frac{\partial}{\partial t} N_\alpha(t) = -\Gamma_\alpha N_\alpha(t) + \Gamma_\alpha N_\alpha = -\Gamma_\alpha N_\alpha(t)(1 + N(\mathbf{k})) + \Gamma_\alpha(1 + N_\alpha(t)) N(\mathbf{k}), \quad (81)$$

that might justly be called the quantum Boltzmann equation for the gas, because the right-hand side contains precisely the rates for scattering into and out of the reservoir.

Furthermore, Eq. (77) shows that the “diffusion” term in the Fokker-Planck equation is

$$-\left(\frac{1}{2} \sum_\alpha \frac{\partial^2}{\partial \phi_\alpha^* \partial \phi_\alpha} \hbar \Sigma_\alpha^K\right) P[\phi^*, \phi; t],$$

because the first term in the right-hand side of Eq. (77) is also present without the noise and is therefore already accounted for in the “streaming” terms. In total we have thus arrived at

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} P[\phi^*, \phi; t] = & -\left(\sum_\alpha \frac{\partial}{\partial \phi_\alpha} (\varepsilon_\alpha + \hbar \Sigma_\alpha^{(+)} - \mu) \phi_\alpha\right) P[\phi^*, \phi; t] \\ & + \left(\sum_\alpha \frac{\partial}{\partial \phi_\alpha^*} (\varepsilon_\alpha + \hbar \Sigma_\alpha^{(-)} - \mu) \phi_\alpha^*\right) P[\phi^*, \phi; t] \\ & - \left(\frac{1}{2} \sum_\alpha \frac{\partial^2}{\partial \phi_\alpha^* \partial \phi_\alpha} \hbar \Sigma_\alpha^K\right) P[\phi^*, \phi; t]. \end{aligned} \quad (82)$$

Using again the fluctuation-dissipation theorem, it is not difficult to show that the stationary solution of this Fokker-Planck equation is

$$P[\phi^*, \phi; \infty] = \prod_\alpha \frac{1}{N_\alpha + 1/2} \exp \left\{ -\frac{1}{N_\alpha + 1/2} |\phi_\alpha|^2 \right\}. \quad (83)$$

Although this appears to be quite reasonable, it is important to note that this result is not fully consistent because the “streaming” terms in Eq. (82) show that the energies of the states $\chi_\alpha(\mathbf{x})$ are shifted and equal to $\varepsilon_\alpha + S_\alpha$. We therefore expect that the equilibrium occupation numbers are equal to the Bose distribution evaluated at those energies and not at the unshifted values ε_α . The reason for this inconsistency is, of course, that we have taken $\Sigma_\alpha^{(\pm), K}$ equal to $\Sigma_\alpha^{(\pm), K}(\varepsilon_\alpha - \mu)$, which was only justified on the basis of second-order perturbation theory. We now, however, see that to obtain a fully consistent theory we must take instead $\Sigma_\alpha^{(\pm), K} \equiv \Sigma_\alpha^{(\pm), K}(\varepsilon_\alpha + S_\alpha - \mu)$. This implies that to obtain the retarded selfenergy, we first have to solve for the real part

$$S_\alpha = \int \frac{d\mathbf{k}}{(2\pi)^3} t_\alpha^*(\mathbf{k}) \frac{\mathcal{P}}{\varepsilon_\alpha + S_\alpha - \varepsilon(\mathbf{k})} t_\alpha(\mathbf{k}) \quad (84)$$

and then substitute this into

$$R_\alpha = \pi \int \frac{d\mathbf{k}}{(2\pi)^3} \delta(\varepsilon_\alpha + S_\alpha - \varepsilon(\mathbf{k})) |t_\alpha(\mathbf{k})|^2 \quad (85)$$

to obtain the imaginary part. The fluctuation-dissipation theorem then again reads $\hbar \Sigma_\alpha^K = -2i(1 + 2N_\alpha)R_\alpha$ but now with the equilibrium distribution $N_\alpha = (e^{\beta(\varepsilon_\alpha + S_\alpha - \mu)} - 1)^{-1}$. As a consequence the quantum Boltzmann equation and the stationary state of the Fokker-Planck equation will now also contain these correct equilibrium occupation numbers.⁶⁰

This almost completes our discussion of this Caldeira-Leggett like toy model, that nevertheless already shows many of the features that we will encounter in Sec. III and IV when we study a realistic interacting Bose gas. However, we promised to derive the Fokker-Planck equation also directly from the effective action $S^{\text{eff}}[\phi^*, \phi; \xi^*, \xi]$, without making use of a Hubbard-Stratonovich transformation and the Langevin equations that are an immediate result of that transformation. Making the same approximations on the selfenergies as before, the effective action for the probability distribution $P[\phi^*, \phi; t]$ reads

$$\begin{aligned} S^{\text{eff}}[\phi^*, \phi; \xi^*, \xi] = & \sum_\alpha \int_{t_0}^t dt' \phi_\alpha^*(t') \left(i\hbar \frac{\partial}{\partial t'} - \varepsilon_\alpha - \hbar \Sigma_\alpha^{(-)} + \mu \right) \xi_\alpha(t') \\ & + \sum_\alpha \int_{t_0}^t dt' \xi_\alpha^*(t') \left(i\hbar \frac{\partial}{\partial t'} - \varepsilon_\alpha - \hbar \Sigma_\alpha^{(+)} + \mu \right) \phi_\alpha(t') \\ & - \frac{1}{2} \sum_\alpha \int_{t_0}^t dt' \xi_\alpha^*(t') \hbar \Sigma_\alpha^K \xi_\alpha(t'), \end{aligned} \quad (86)$$

and is quadratic in the fluctuation field $\xi(\mathbf{x}, t)$. We can thus again perform the integration over this field exactly. The result is

$$\begin{aligned} S^{\text{eff}}[\phi^*, \phi] &= \sum_{\alpha} \int_{t_0}^t dt' \frac{2}{\hbar \Sigma_{\alpha}^K} \left| \left(i\hbar \frac{\partial}{\partial t'} - \varepsilon_{\alpha} - \hbar \Sigma_{\alpha}^{(+)} + \mu \right) \phi_{\alpha}(t') \right|^2 \\ &\equiv \int_{t_0}^t dt' L(t'). \end{aligned} \quad (87)$$

Since the probability distribution $P[\phi^*, \phi; t]$ is equal to the functional integral

$$P[\phi^*, \phi; t] = \int_{\phi(\mathbf{x}, t) = \phi(\mathbf{x})}^{\phi^*(\mathbf{x}, t) = \phi^*(\mathbf{x})} d[\phi^*] d[\phi] \exp \left\{ \frac{i}{\hbar} S^{\text{eff}}[\phi^*, \phi] \right\}, \quad (88)$$

we know from the general connection between classical mechanics and the functional formulation of quantum mechanics, that $P[\phi^*, \phi; t]$ must obey the Schrödinger equation that results from quantizing the classical theory with the lagrangian $L(t)$.

Fortunately, the quantization of this theory is straightforward. The momentum conjugate to $\phi_{\alpha}(t)$ is

$$\pi_{\alpha}(t) = \frac{\partial L(t)}{\partial(\partial\phi_{\alpha}(t)/\partial t)} = \frac{2i}{\Sigma_{\alpha}^K} \left(-i\hbar \frac{\partial}{\partial t} - \varepsilon_{\alpha} - \hbar \Sigma_{\alpha}^{(-)} + \mu \right) \phi_{\alpha}^*(t), \quad (89)$$

whereas the momentum conjugate to $\phi_{\alpha}^*(t)$, i.e., $\pi_{\alpha}^*(t)$, is given by the complex conjugate expression. The corresponding hamiltonian is therefore

$$\begin{aligned} H &= \sum_{\alpha} \left\{ \pi_{\alpha}(t) \frac{\partial \phi_{\alpha}(t)}{\partial t} + \pi_{\alpha}^*(t) \frac{\partial \phi_{\alpha}^*(t)}{\partial t} \right\} - L(t) \\ &= \sum_{\alpha} \left\{ -\frac{i}{\hbar} \pi_{\alpha}(t) (\varepsilon_{\alpha} + \hbar \Sigma_{\alpha}^{(+)} - \mu) \phi_{\alpha}(t) + \frac{i}{\hbar} \pi_{\alpha}^*(t) (\varepsilon_{\alpha} + \hbar \Sigma_{\alpha}^{(-)} - \mu) \phi_{\alpha}^*(t) \right\} \\ &\quad + \sum_{\alpha} \frac{\Sigma_{\alpha}^K}{2\hbar} |\pi_{\alpha}(t)|^2. \end{aligned} \quad (90)$$

Applying now the usual quantum mechanical recipe of demanding non-vanishing commutation relations between the coordinates and their conjugate momenta, we can put in this case $\pi_{\alpha} = (\hbar/i) \partial/\partial\phi_{\alpha}$ and similarly $\pi_{\alpha}^* = (\hbar/i) \partial/\partial\phi_{\alpha}^*$. The Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} P[\phi^*, \phi; t] = HP[\phi^*, \phi; t] \quad (91)$$

then indeed exactly reproduces the Fokker-Planck equation in Eq. (82). It should be noted at this point that the hamiltonian in Eq. (90) has in principle ordering problems, due to the fact that the coordinates and the conjugate momenta no longer commute after quantization. These ordering problems are, however, resolved by realizing that in the path-integral formulation of quantum mechanics we always deal with so-called normally ordered hamiltonians in which the momenta are positioned to the left of the coordinates. For that reason we have to use the same ordering in Eq. (90).

Having arrived at the same Fokker-Planck equation, we have now demonstrated the use of essentially all the theoretical tools that are required for a discussion of a weakly interacting Bose gas. Before we start with this discussion, however, we want to make a final remark about the effect of the nondiagonal elements of the selfenergies. Physically, including these nondiagonal elements accounts for the change in the wavefunctions $\chi_\alpha(\mathbf{x})$ due to the interaction with the reservoir. It can clearly be neglected if the coupling with the reservoir is sufficiently weak, or more precisely if $|\hbar\Sigma_{\alpha',\alpha}^{(+)}(\varepsilon_\alpha + S_\alpha - \mu)|$ is much smaller than the energy splitting $|\varepsilon_{\alpha'} - \varepsilon_\alpha|$ between the states in the trap. For a quantum dot in solid-state physics⁶¹ this is often the case and for that reason we might call our toy model a bosonic quantum dot. However, in magnetically trapped atomic gases the average interaction energy between the atoms is already slightly below the Bose-Einstein condensation critical temperature larger than the typical energy splitting and the nondiagonal elements of the selfenergies are very important. Such a strong-coupling situation can, of course, also be treated in our Caldeira-Leggett like model. The main difference is that we need to expand our various fields not in the eigenstates of the trapping potential $\chi_\alpha(\mathbf{x})$ but in the eigenstates $\chi'_\alpha(\mathbf{x})$ of the nonlocal Schrödinger equation

$$\varepsilon'_\alpha \chi'_\alpha(\mathbf{x}) = \left(-\frac{\hbar^2 \nabla^2}{2m} + V^{ex}(\mathbf{x}) \right) \chi'_\alpha(\mathbf{x}) + \int d\mathbf{x}' \operatorname{Re}[\hbar\Sigma^{(+)}(\mathbf{x}, \mathbf{x}'; \varepsilon'_\alpha - \mu)] \chi'_\alpha(\mathbf{x}') \quad (92)$$

where ε'_α are the new eigenvalues and $\hbar\Sigma^{(+)}(\mathbf{x}, \mathbf{x}'; e) = \sum_{\alpha, \alpha'} \chi_\alpha(\mathbf{x}) \times \hbar\Sigma_{\alpha, \alpha'}^{(+)}(e) \chi_{\alpha'}^*(\mathbf{x}')$. In this new basis the nondiagonal elements of the selfenergies can now be neglected and we find essentially the same results as before. We only need to replace $\varepsilon_\alpha + S_\alpha$ by ε'_α and $t_\alpha(\mathbf{k})$ by $\sum_{\alpha'} (\int d\mathbf{x} \chi'_\alpha(\mathbf{x}') \chi_{\alpha'}^*(\mathbf{x})) t_{\alpha'}(\mathbf{k})$. Neglecting the nondiagonal elements in this basis only requires that the real part of the retarded selfenergy is much larger than its imaginary part, which is always fulfilled in our case because $k_c \Lambda \gg 1$.

Summarizing, the coherent and incoherent dynamics of the gas in the trap is for our toy model quite generally solved by

$$\langle \phi_\alpha \rangle(t) = \langle \phi_\alpha \rangle(0) e^{-i(\varepsilon'_\alpha - \mu) t/\hbar} e^{-\Gamma_\alpha t/2} \quad (93)$$

and

$$N_\alpha(t) = N_\alpha(0) e^{-\Gamma_\alpha t} + N_\alpha(1 - e^{-\Gamma_\alpha t}), \quad (94)$$

respectively. In the limit $t \rightarrow \infty$, the average of the annihilation operators $\langle \phi_\alpha \rangle(t)$ thus always vanishes but the average occupation numbers $N_\alpha(t)$ relax to the equilibrium distribution $N_\alpha = (e^{\beta(\varepsilon'_\alpha - \mu)} - 1)^{-1}$. Although this appears to be an immediately obvious result, its importance stems from the fact that it is also true if we tune the potential energy bias ΔV^{ex} such that at low temperatures the groundstate $\chi'_g(\mathbf{x})$ acquires a macroscopic occupation, i.e., $N_g \gg 1$. The gas therefore never shows a spontaneous breaking of the $U(1)$ symmetry, in agreement with the notion that we are essentially dealing with an ideal Bose gas in the grand-canonical ensemble.⁶² The reason of the absence of the spontaneous symmetry breaking can also be understood from our stationary solution of the Fokker-Planck equation in Eq. (83), which shows that the probability distribution for $|\phi_g|$ is proportional to the Boltzmann factor $e^{-\beta(\varepsilon'_g - \mu)|\phi_g|^2}$ in the degenerate regime of interest. Hence, the corresponding Landau free energy $F(|\phi_g|) = (\varepsilon'_g - \mu) \times |\phi_g|^2$ never shows an instability towards the formation of a nonzero average of $|\phi_g|$ due to the fact that $\varepsilon'_g - \mu$ can never become less than zero. Once we introduce interactions between the atoms in the gas, this picture fully changes.

III. SEMICLASSICAL THEORY OF BOSE-EINSTEIN CONDENSATION

In the present and the following section we turn our attention to an interacting but homogeneous Bose gas. However, in Sec. V we generalize our results and include also the effects of an external trapping potential. There are several reasons for considering the homogeneous case first. One important reason is that for a homogeneous gas we are allowed to take the thermodynamic limit in which Bose-Einstein condensation becomes a true second-order phase transition. We are then in fact studying the dynamics of a spontaneous symmetry breaking under the most ideal circumstances, which may nevertheless have important implications for cosmology^{63–66} and also lead to a better understanding of the recent experiments performed with liquid ^4He .⁶⁷ Furthermore, the homogeneous case is in a sense also the most difficult one because thermalizing collisions cannot lead to a

macroscopic occupation of the one-particle ground state.^{68,69} Using the language of quantum optics, we also cannot compare Bose-Einstein condensation of a homogeneous gas with a single-mode laser and we are essentially dealing with the (in principle infinitely many) multi-mode situation. Since realistic experiments with atomic rubidium and sodium gases^{26, 28, 70} are usually also in a multi-mode regime, it even turns out that an understanding of the homogeneous gas is important for magnetically trapped gases as well. This is less so for experiments with atomic lithium,²⁷ because then the inhomogeneity plays a much more fundamental role.

In the semiclassical theory of Bose-Einstein condensation in a homogeneous, interacting gas, the main emphasis is on the study of the coherent dynamics of the gas.⁷¹ The reason for this is that if Bose-Einstein condensation is possible at all, it has to be caused by coherent processes since incoherent, i.e., kinetic, processes cannot lead to Bose-Einstein condensation in the thermodynamic limit. The reason why Bose-Einstein condensation cannot be achieved in this way is easily understood. From the appropriate quantum Boltzmann equation (cf. Eq. (81)) we obtain that the production rate for the number of particles in the zero-momentum state is given by

$$\frac{dN(\mathbf{0}; t)}{dt} = \frac{C}{\tau_{el}} (1 + N(\mathbf{0}; t)), \quad (95)$$

where τ_{el} is the average time between two elastic collisions and C is a constant of $\mathcal{O}(1)$. For initial conditions close to equilibrium but with $N(\mathbf{0}; t_0) = 0$, we can in first instance neglect the evolution of the non-condensed part of the gas, i.e., the time dependence of τ_{el} , and we find that

$$N(\mathbf{0}; t) = (e^{C(t-t_0)/\tau_{el}} - 1). \quad (96)$$

Hence, a macroscopic occupation of the one-particle ground state is only achieved at times $t > t_0 + \mathcal{O}(\tau_{el} \ln(N))$, which indeed diverge if $N \rightarrow \infty$. For the initial conditions of interest this condition can also be written as $t > t_0 + \mathcal{O}(\tau_{el} \ln(L/\Lambda_c))$, with L the (linear) system size and $\Lambda_c = (2\pi\hbar^2/mk_B T_c)^{1/2}$ the thermal de Broglie wavelength of the atoms at the critical temperature T_c .

Physically, we thus envisage the dynamics of the gas in response to evaporative cooling in the following way. In the nondegenerate regime the gas just thermalizes by elastic collisions in such a manner that the occupation numbers $N(\mathbf{k}; t)$ are in a good approximation at all times equal to a truncated equilibrium distribution.^{72,73} Since the temperature of the gas is decreasing, this distribution becomes more sharply peaked at low momenta and we enter at some point the degenerate regime where $n\Lambda^3 = \mathcal{O}(1)$ but no

condensate has been formed yet because we know that this is impossible by means of elastic collisions. At this stage the most important question to be answered is: Is it nevertheless possible that for a noncondensed but degenerate distribution function $N(\mathbf{k}; t)$, the energy $\varepsilon'(\mathbf{0}; t)$ of the zero-momentum state becomes less than the instantaneous chemical potential $\mu(t)$ and the gas becomes unstable towards Bose-Einstein condensation? If the answer to this question is no, Bose-Einstein condensation is impossible and the study of the dynamics of the gas is restricted to the normal phase and relatively well understood.⁷⁴⁻⁷⁶ However, if the answer is yes, the dynamics of the gas after the point of instability is clearly more complicated and deserves further investigation.

In view of the central importance of the above question for the topic of this paper, we now first of all show in Sec. III.A that the gas indeed develops the required instability for Bose-Einstein condensation. We then turn to the dynamics of the unstable gas. In Sec. III.B the dynamics of the phase transition is first studied in the semiclassical approximation, in which fluctuations in the order parameter are neglected. Due to this neglect of fluctuations, the time at which the instability occurs in the gas is equivalent to the time at which Bose-Einstein condensation takes place. In Sec. IV, however, we include the fluctuations into our theory and discuss in detail how this affects the semiclassical picture.

A. Instability of the Bose Gas

From our Caldeira-Leggett toy model we remember that the (renormalized) energies ε'_α and the corresponding eigenstates $\chi'_\alpha(\mathbf{x})$ can be determined from a nonlocal Schrödinger equation once we know the retarded selfenergy $\hbar\Sigma^{(+)}(\mathbf{x}, \mathbf{x}'; t - t')$. The same is true for a homogeneous atomic Bose gas, only with this exception that the nonzero selfenergy is now due to the interatomic interactions and not due to the presence of a reservoir. In a sense an interacting gas plays also the role of its own reservoir. Moreover, the homogeneity of the gas leads to an important simplification because translational invariance requires that the retarded selfenergy is only a function of the relative distance $\mathbf{x} - \mathbf{x}'$. Therefore, the Schrödinger equation in Eq. (92) is solved by $\chi'_\mathbf{k}(\mathbf{x}) = \chi_\mathbf{k}(\mathbf{x})$ and

$$\varepsilon'(\mathbf{k}) = \varepsilon(\mathbf{k}) + \text{Re}[\hbar\Sigma^{(+)}(\mathbf{k}; \varepsilon'(\mathbf{k}) - \mu)], \quad (97)$$

where from now on $\varepsilon(\mathbf{k}) = \hbar^2 \mathbf{k}^2 / 2m$ again and

$$\Sigma^{(+)}(\mathbf{x} - \mathbf{x}'; t - t') = \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\varepsilon}{2\pi\hbar} \Sigma^{(+)}(\mathbf{k}; \varepsilon) e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}') - \varepsilon(t - t')/\hbar}. \quad (98)$$

Having arrived at this result, it is important to briefly remind ourselves what it implies physically. Mathematically, we have assumed that the gas has at the initial time t_0 the occupation numbers $N(\mathbf{k})$ that correspond to a truncated equilibrium distribution with chemical potential μ . Subsequently, we have taken the limit of $t_0 \rightarrow -\infty$. The energy $\varepsilon'(\mathbf{k})$ is therefore the energy of the one-particle state $\chi_{\mathbf{k}}(\mathbf{x})$ after all the transients have died out. However, as is shown explicitly in Ref. 71, the memory time in the selfenergy is for the same reason as in our Caldeira-Leggett toy model only of $\mathcal{O}(\hbar/k_B T)$ and hence much smaller than all the other relevant time scales in the problem. As a consequence we are for all practical purposes allowed to consider $\varepsilon'(\mathbf{k})$ as the energy of the state $\chi_{\mathbf{k}}(\mathbf{x})$, given all the occupation numbers $N(\mathbf{k})$. The time dependence of $\varepsilon'(\mathbf{k}; t)$ can then be seen as solely due to the time dependence of the occupation numbers $N(\mathbf{k}; t)$.

Keeping the last remark in mind, we now proceed to solve Eq. (97). To do that we of course need an expression for the retarded selfenergy of a weakly interacting Bose gas, which follows once we know the full selfenergy $\hbar\Sigma(\mathbf{k}; t, t')$ defined on the Schwinger-Keldysh contour \mathcal{C}^∞ . Unfortunately, even for a dilute system this quantity cannot be calculated exactly and some approximation is called for. The approximation that we will make here is the so-called many-body T-matrix approximation that has recently been reviewed in the literature.^{77, 78} The main motivation for this approximation is that due to the smallness of the gasparameter $(na^3)^{1/2}$ it is very unlikely for three or more particles to be within the range of the interaction and we only need to take account of all possible two-body processes taking place in the gas. It goes beyond the much used Popov approximation^{24, 79} by including also quite accurately the effect of the surrounding medium on the effective interaction between the atoms in the gas,^{80, 81} which turns out to be of crucial importance for our purposes.

Given this effective interaction $V(\mathbf{k}, \mathbf{k}', \mathbf{K}; t, t')$ for the scattering of two atoms that at time t' have the momenta $\hbar(\mathbf{K}/2 \pm \mathbf{k}')$ and at time t the momenta $\hbar(\mathbf{K}/2 \pm \mathbf{k})$, respectively, the exact selfenergy obeys the Hartree-Fock-like relation

$$\hbar\Sigma(\mathbf{k}; t, t') = 2i \int \frac{d\mathbf{k}'}{(2\pi)^3} V(\mathbf{k} - \mathbf{k}', \mathbf{k} - \mathbf{k}', \mathbf{k} + \mathbf{k}'; t, t') G(\mathbf{k}'; t', t), \quad (99)$$

where the Green's function equals again Eq. (53) but with $\varepsilon(\mathbf{k})$ replaced by $\varepsilon'(\mathbf{k})$ to make the theory selfconsistent. The retarded selfenergy therefore obeys

$$\begin{aligned} \hbar\Sigma^{(+)}(\mathbf{k}; t - t') &= 2i \int \frac{d\mathbf{k}'}{(2\pi)^3} (V^{(+)}(\mathbf{k} - \mathbf{k}', \mathbf{k} - \mathbf{k}', \mathbf{k} + \mathbf{k}'; t - t') G^<(\mathbf{k}'; t' - t) \\ &\quad + V^<(\mathbf{k} - \mathbf{k}', \mathbf{k} - \mathbf{k}', \mathbf{k} + \mathbf{k}'; t - t') G^{(-)}(\mathbf{k}'; t' - t)), \quad (100) \end{aligned}$$

leading to

$$\begin{aligned}
& \hbar \Sigma^{(+)}(\mathbf{k}; \varepsilon'(\mathbf{k}) - \mu) \\
&= 2 \int \frac{d\mathbf{k}'}{(2\pi)^3} V^{(+)}(\mathbf{k} - \mathbf{k}', \mathbf{k} - \mathbf{k}', \mathbf{k} + \mathbf{k}'; \varepsilon'(\mathbf{k}) + \varepsilon'(\mathbf{k}') - 2\mu) N(\mathbf{k}') \\
&+ 2i \int \frac{d\mathbf{k}'}{(2\pi)^3} \int \frac{d\varepsilon}{2\pi\hbar} V^{<}(\mathbf{k} - \mathbf{k}', \mathbf{k} - \mathbf{k}', \mathbf{k} + \mathbf{k}'; \varepsilon) \frac{\hbar}{\varepsilon^- - (\varepsilon'(\mathbf{k}) + \varepsilon'(\mathbf{k}') - 2\mu)},
\end{aligned} \tag{101}$$

if we also introduce the notation ε^- for the limiting procedure $\varepsilon - i0$. Note that the two terms in the right-hand side correspond roughly speaking to the contributions to the selfenergy from scattering processes out and into the state $\chi_{\mathbf{k}}(\mathbf{x})$, respectively. This will be more clear in Sec. IV when we discuss the imaginary part of this expression that describes the (incoherent) dynamics of the occupation numbers. To see the instability of the gas, however, we only need to consider the real part.

Hence, our next task is to determine the effective interaction $V(\mathbf{k}, \mathbf{k}', \mathbf{K}; t, t')$ by considering all possible two-body scattering processes. Because we are dealing with bosons, the effective interaction is a sum of a direct and an exchange term and can be written as $V(\mathbf{k}, \mathbf{k}', \mathbf{K}; t, t') = (T(\mathbf{k}, \mathbf{k}', \mathbf{K}; t, t') + T(-\mathbf{k}, \mathbf{k}', \mathbf{K}; t, t'))/2$ in terms of the many-body T-matrix that obeys the Lippmann-Schwinger equation⁸²

$$\begin{aligned}
T(\mathbf{k}, \mathbf{k}', \mathbf{K}; t, t') &= V(\mathbf{k} - \mathbf{k}') \delta(t, t') \\
&+ \frac{i}{\hbar} \int_{-\infty}^{\infty} dt'' \int \frac{d\mathbf{k}''}{(2\pi)^3} V(\mathbf{k} - \mathbf{k}'') G(\mathbf{K}/2 + \mathbf{k}''; t, t'') \\
&\times G(\mathbf{K}/2 - \mathbf{k}''; t, t'') T(\mathbf{k}'', \mathbf{k}', \mathbf{K}; t'', t'),
\end{aligned} \tag{102}$$

with $V(\mathbf{k} - \mathbf{k}')$ the Fourier transform of the interatomic interaction potential. By iterating this equation, we immediately see that the many-body T-matrix indeed sums all possible collisions between two particles. Moreover, the Green's functions $G(\mathbf{K}/2 \pm \mathbf{k}''; t, t'')$ describe the propagation of an atom with momentum $\hbar(\mathbf{K}/2 \pm \mathbf{k}'')$ from time t'' to time t in the gas. Therefore, we also see that the many-body T-matrix incorporates the effect of the surrounding gaseous medium on the propagation of the atoms between two collisions. This in contrast to the two-body T-matrix, that is well-known from scattering theory⁸³ and obeys the same Lippman-Schwinger equation but with the Green's functions $G(\mathbf{K}/2 \pm \mathbf{k}''; t, t'')$ replaced by

$$G_0(\mathbf{K}/2 \pm \mathbf{k}''; t, t'') = -ie^{-i\varepsilon(\mathbf{K}/2 \pm \mathbf{k}'')(t-t'')/\hbar} \Theta(t, t'')$$

describing the propagation of an atom in the absence of a surrounding gas, i.e., in a vacuum. Note also that the many-body T-matrix has a δ -function singularity on the Schwinger-Keldysh contour. As a result the retarded and advanced components are now given by the more general expression

$$T^{(\pm)}(\mathbf{k}, \mathbf{k}', \mathbf{K}; t - t') = T^\delta(\mathbf{k}, \mathbf{k}', \mathbf{K}; t) \delta(t - t') \pm \Theta(\pm(t - t')) \\ \times (T^>(\mathbf{k}, \mathbf{k}', \mathbf{K}; t - t') - T^<(\mathbf{k}, \mathbf{k}', \mathbf{K}; t - t')) \quad (103)$$

but the Keldysh component still obeys

$$T^K(\mathbf{k}, \mathbf{k}', \mathbf{K}; t - t') = (T^>(\mathbf{k}, \mathbf{k}', \mathbf{K}; t - t') + T^<(\mathbf{k}, \mathbf{k}', \mathbf{K}; t - t')). \quad (104)$$

Before we turn to the solution of the Lippmann-Schwinger equation in Eq. (102), it is instructive to briefly mention what we obtain if we now apply the pseudopotential method.²² In this method it is assumed that the effective interaction equals

$$V(\mathbf{k}, \mathbf{k}', \mathbf{K}; t, t') = \frac{4\pi a \hbar^2}{m} \delta(t, t'), \quad (105)$$

which implies that only the retarded and advanced components of the effective interaction are nonzero and both equal to $4\pi a \hbar^2 \delta(t - t')/m$. Substituting this in Eq. (101), we simply find that

$$\hbar \Sigma^{(+)}(\mathbf{k}; \varepsilon'(\mathbf{k}) - \mu) = \frac{8\pi n a \hbar^2}{m} \quad (106)$$

and thus that $\varepsilon'(\mathbf{k}; t) = \varepsilon(\mathbf{k}) + 8\pi n(t) a \hbar^2/m$. On the basis of the pseudopotential method we therefore conclude that the Bose gas never shows an instability towards Bose-Einstein condensation because a momentum independent selfenergy only leads to a shift in the chemical potential, which plays no role in the dynamics of the gas, even if we are evaporatively cooling the gas and the density $n(t)$ is a time-dependent quantity. For our purposes it is thus important to determine a more accurate expression for the effective interaction.

As a first step towards this goal, we consider the retarded and advanced parts of Eq. (102). They are given by

$$T^{(\pm)}(\mathbf{k}, \mathbf{k}', \mathbf{K}; t - t') = V(\mathbf{k} - \mathbf{k}') \delta(t - t') + \frac{i}{\hbar} \int dt'' \int \frac{d\mathbf{k}''}{(2\pi)^3} V(\mathbf{k} - \mathbf{k}'') \\ \times G^{(\pm)}(\mathbf{k}'', \mathbf{K}; t - t'') T^{(\pm)}(\mathbf{k}'', \mathbf{k}', \mathbf{K}; t'' - t'), \quad (107)$$

where $G^{(\pm)}(\mathbf{k}'', \mathbf{K}; t - t'')$ are the retarded and advanced components of the two-particle Green's function $G(\mathbf{k}'', \mathbf{K}; t, t'') \equiv G(\mathbf{K}/2 + \mathbf{k}''; t, t'') G(\mathbf{K}/2 - \mathbf{k}''; t, t'')$ describing the propagation of two (independent) atoms with momenta $\hbar(\mathbf{K}/2 + \mathbf{k}'')$ and $\hbar(\mathbf{K}/2 - \mathbf{k}'')$ from time t'' to time t in the gas. Explicitly, we have that

$$\begin{aligned} G^{(\pm)}(\mathbf{k}'', \mathbf{K}; t - t'') \\ = \pm \Theta(\pm(t - t''))(1 + N(\mathbf{K}/2 + \mathbf{k}'') + N(\mathbf{K}/2 - \mathbf{k}'')) e^{-i(\varepsilon'(\mathbf{k}'', \mathbf{K}) - 2\mu)(t - t'')/\hbar}, \end{aligned} \quad (108)$$

with $\varepsilon'(\mathbf{k}'', \mathbf{K}) = \varepsilon'(\mathbf{K}/2 + \mathbf{k}'') + \varepsilon'(\mathbf{K}/2 - \mathbf{k}'')$ the appropriate two-particle energy. Performing a Fourier transformation we then obtain

$$\begin{aligned} T^{(\pm)}(\mathbf{k}, \mathbf{k}', \mathbf{K}; \varepsilon) = V(\mathbf{k} - \mathbf{k}') + \int \frac{d\mathbf{k}''}{(2\pi)^3} V(\mathbf{k} - \mathbf{k}'') \\ \times \frac{1 + N(\mathbf{K}/2 + \mathbf{k}'') + N(\mathbf{K}/2 - \mathbf{k}'')}{\varepsilon^\pm - (\varepsilon'(\mathbf{k}'', \mathbf{K}) - 2\mu)} T^{(\pm)}(\mathbf{k}'', \mathbf{k}', \mathbf{K}; \varepsilon), \end{aligned} \quad (109)$$

which is well-known in nuclear physics as the Bethe-Salpeter equation and was first used in the context of Bose-Einstein condensation by Brueckner and Sawada.⁸⁴ Compared with the usual Lippmann-Schwinger equation

$$T^{(\pm)}(\mathbf{k}, \mathbf{k}'; \varepsilon) = V(\mathbf{k} - \mathbf{k}') + \int \frac{d\mathbf{k}''}{(2\pi)^3} V(\mathbf{k} - \mathbf{k}'') \frac{1}{\varepsilon^\pm - 2\varepsilon(\mathbf{k}'')} T^{(\pm)}(\mathbf{k}'', \mathbf{k}'; \varepsilon) \quad (110)$$

for the two-body T-matrix, we see that the Bethe-Salpeter equation incorporates the effect that if the intermediate states are already occupied it is more likely for bosons to scatter into these states. Indeed the factor $1 + N(\mathbf{K}/2 + \mathbf{k}'') + N(\mathbf{K}/2 - \mathbf{k}'')$ arises as the net difference between two atoms scattering into and scattering out of the intermediate states $\chi_{\mathbf{K}/2 + \mathbf{k}''}(\mathbf{x})$ and $\chi_{\mathbf{K}/2 - \mathbf{k}''}(\mathbf{x})$, which are proportional to $(1 + N(\mathbf{K}/2 + \mathbf{k}'')) \times (1 + N(\mathbf{K}/2 - \mathbf{k}''))$ and $N(\mathbf{K}/2 + \mathbf{k}'') N(\mathbf{K}/2 - \mathbf{k}'')$, respectively. In a real space picture the effect of the Bose enhancement factors can be seen as a result of the fact that the scattering wavefunction of the two colliding atoms has to be matched to the many-body wavefunction of the gas at large interatomic distances.⁸⁵ Seen in this manner, it will be clear that this effect is not small in the gas parameter $(na^3)^{1/2}$ and cannot be neglected beforehand if we aim to include all two-body processes in the gas.

In general the solution of a Bethe-Salpeter equation is quite complicated. However, for the quantum gases of interest we can basically solve it analytically by making use of the fact in the quantum regime the temperatures are always such that the thermal de Broglie wavelength obeys $a/\Lambda \ll 1$. As a result the many-body T-matrix at the relevant momenta and energies that are much smaller than \hbar/a and $\hbar^2/2ma^2$, respectively, can in a very good approximation be expressed as⁷⁷

$$T^{(\pm)}(\mathbf{k}, \mathbf{k}', \mathbf{K}; \varepsilon) = \frac{T^{(\pm)}(\mathbf{0}, \mathbf{0}; 0)}{1 - T^{(\pm)}(\mathbf{0}, \mathbf{0}; 0) \Xi(\mathbf{K}; \varepsilon)}, \quad (111)$$

where $T^{(\pm)}(\mathbf{0}, \mathbf{0}; 0) = 4\pi a \hbar^2/m$ and we have also introduced the function

$$\Xi(\mathbf{K}; \varepsilon) = \int \frac{d\mathbf{k}''}{(2\pi)^3} (N(\mathbf{K}/2 + \mathbf{k}'') + N(\mathbf{K}/2 - \mathbf{k}'')) \frac{\mathcal{P}}{\varepsilon - (\varepsilon'(\mathbf{k}'', \mathbf{K}) - 2\mu)}. \quad (112)$$

Because $\Xi(\mathbf{K}; \varepsilon)$ contains only the principle value part of the integral, the above approximation gives only the real part of the many-body T-matrix. This is all we need for the semiclassical theory, but for the full quantum theory we need also the imaginary part. The latter is found from the optical theorem,⁸³ which in the many-body case reads

$$\begin{aligned} & T^{(+)}(\mathbf{k}, \mathbf{k}', \mathbf{K}; \varepsilon) - T^{(-)}(\mathbf{k}, \mathbf{k}', \mathbf{K}; \varepsilon) \\ &= -2\pi i \int \frac{d\mathbf{k}''}{(2\pi)^3} \delta(\varepsilon - (\varepsilon'(\mathbf{k}'', \mathbf{K}) - 2\mu)) T^{(+)}(\mathbf{k}, \mathbf{k}'', \mathbf{K}; \varepsilon) \\ & \quad \times (1 + N(\mathbf{K}/2 + \mathbf{k}'') + N(\mathbf{K}/2 - \mathbf{k}'')) T^{(-)}(\mathbf{k}'', \mathbf{k}', \mathbf{K}; \varepsilon). \end{aligned} \quad (113)$$

Note that no approximation is made to obtain this result.

We are now in the position to evaluate the contribution of the $V^{(+)}G^<$ -term to the real part of the retarded selfenergy. Combining Eqs. (101) and (111) we obtain in first instance

$$2 \int \frac{d\mathbf{k}'}{(2\pi)^3} \frac{T^{(+)}(\mathbf{0}, \mathbf{0}; 0)}{1 - T^{(+)}(\mathbf{0}, \mathbf{0}; 0) \Xi(\mathbf{k} + \mathbf{k}'; \varepsilon'(\mathbf{k}) + \varepsilon'(\mathbf{k}') - 2\mu)} N(\mathbf{k}').$$

However, the dominant contribution to the integral comes from thermal momenta due to the presence of the distribution function $N(\mathbf{k}')$. It is not difficult to show that for these momenta $T^{(+)}(\mathbf{0}, \mathbf{0}; 0) \Xi(\mathbf{k} + \mathbf{k}'; \varepsilon'(\mathbf{k}) + \varepsilon'(\mathbf{k}') - 2\mu) = \mathcal{O}(na\Lambda^2)$ and therefore, for temperatures near the critical temperature where $n\Lambda^3 = \mathcal{O}(1)$, always much smaller than one. Under these

conditions this contribution to the retarded selfenergy is essentially equal to $8\pi n a \hbar^2/m$, i.e., the same as the full result obtained by the pseudopotential method. At this point we thus conclude that if the gas becomes unstable, it is due to the second term in the right-hand side of Eq. (101). This contribution we consider next.

We first have to determine the component $T^{< \cdot >}(\mathbf{k}, \mathbf{k}', \mathbf{K}; \varepsilon)$ of the many-body T-matrix. In general Eq. (102) shows that

$$\begin{aligned} T^{< \cdot >}(\mathbf{k}, \mathbf{k}', \mathbf{K}; t-t') &= \frac{i}{\hbar} \int dt'' \int \frac{d\mathbf{k}''}{(2\pi)^3} V(\mathbf{k}-\mathbf{k}'') G^{< \cdot >}(\mathbf{k}'', \mathbf{K}; t-t'') T^{(-)}(\mathbf{k}'', \mathbf{k}', \mathbf{K}; t''-t') \\ &+ \frac{i}{\hbar} \int dt'' \int \frac{d\mathbf{k}''}{(2\pi)^3} V(\mathbf{k}-\mathbf{k}'') G^{(+)}(\mathbf{k}'', \mathbf{K}; t-t'') T^{< \cdot >}(\mathbf{k}'', \mathbf{k}', \mathbf{K}; t''-t'), \end{aligned} \quad (114)$$

which is immediately solved by

$$\begin{aligned} T^{< \cdot >}(\mathbf{k}, \mathbf{k}', \mathbf{K}; t-t') &= \frac{i}{\hbar} \int dt'' \int dt''' \int \frac{d\mathbf{k}''}{(2\pi)^3} T^{(+)}(\mathbf{k}, \mathbf{k}'', \mathbf{K}; t-t'') \\ &\times G^{< \cdot >}(\mathbf{k}'', \mathbf{K}; t''-t''') T^{(-)}(\mathbf{k}'', \mathbf{k}', \mathbf{K}; t'''-t'), \end{aligned} \quad (115)$$

if we use the Bethe-Salpeter equation for the retarded component of the many-body T-matrix. A Fourier transformation leads, therefore, in particular to

$$\begin{aligned} T^{< \cdot >}(\mathbf{k}, \mathbf{k}', \mathbf{K}; \varepsilon) &= -2\pi i \int \frac{d\mathbf{k}''}{(2\pi)^3} \delta(\varepsilon - (\varepsilon'(\mathbf{k}'', \mathbf{K}) - 2\mu)) T^{(+)}(\mathbf{k}, \mathbf{k}'', \mathbf{K}; \varepsilon) \\ &\times N(\mathbf{K}/2 + \mathbf{k}'') N(\mathbf{K}/2 - \mathbf{k}'') T^{(-)}(\mathbf{k}'', \mathbf{k}', \mathbf{K}; \varepsilon) \end{aligned} \quad (116)$$

but also to

$$\begin{aligned} T^K(\mathbf{k}, \mathbf{k}', \mathbf{K}; \varepsilon) &= -2\pi i \int \frac{d\mathbf{k}''}{(2\pi)^3} \delta(\varepsilon - (\varepsilon'(\mathbf{k}'', \mathbf{K}) - 2\mu)) T^{(+)}(\mathbf{k}, \mathbf{k}'', \mathbf{K}; \varepsilon) \\ &\times T^{(-)}(\mathbf{k}'', \mathbf{k}', \mathbf{K}; \varepsilon) (1 + N(\mathbf{K}/2 + \mathbf{k}'') + N(\mathbf{K}/2 - \mathbf{k}'') \\ &+ 2N(\mathbf{K}/2 + \mathbf{k}'') N(\mathbf{K}/2 - \mathbf{k}'')), \end{aligned} \quad (117)$$

which we need later on in Sec. IV.

The real part of the $V^{<G^{(-)}}$ -term in the expression for the retarded selfenergy thus becomes

$$-2 \int \frac{d\mathbf{k}'}{(2\pi)^3} \int \frac{d\mathbf{k}''}{(2\pi)^3} |V^{(+)}(\mathbf{k}, \mathbf{k}', \mathbf{k}'')|^2 N(\mathbf{k}') N(\mathbf{k}'') \\ \times \frac{\mathcal{P}}{\varepsilon'(\mathbf{k}) + \varepsilon'(\mathbf{k}' + \mathbf{k}'' - \mathbf{k}) - \varepsilon'(\mathbf{k}') - \varepsilon'(\mathbf{k}'')},$$

with $V^{(\pm)}(\mathbf{k}, \mathbf{k}', \mathbf{k}'')$ a convenient shorthand notation for the following matrix element of the effective interaction $V^{(\pm)}(\mathbf{k} - (\mathbf{k}' + \mathbf{k}'')/2, (\mathbf{k}' - \mathbf{k}'')/2, \mathbf{k}' + \mathbf{k}''; \varepsilon'(\mathbf{k}') + \varepsilon'(\mathbf{k}'') - 2\mu)$. For momenta $\hbar\mathbf{k}$ that are of the order of the thermal momenta \hbar/Λ , this contribution to the retarded selfenergy is easily estimated to be only of $\mathcal{O}(2nT^{(+)}(\mathbf{0}, \mathbf{0}; 0)(na\Lambda^2))$ and therefore negligible compared to the contribution from the $V^{(+)}G^{<}$ -term that we considered previously and was shown to be equal to $2nT^{(+)}(\mathbf{0}, \mathbf{0}; 0)$. For thermal momenta we thus conclude that

$$\varepsilon'(\mathbf{k}; t) = \varepsilon(\mathbf{k}) + \frac{8\pi n(t) a\hbar^2}{m}. \quad (118)$$

However, this conclusion is not valid for momenta $\hbar\mathbf{k}$ that are much smaller than the thermal momenta, because in that case the energy denominator in the integrand favors the small momenta where the occupation numbers are especially large in the degenerate regime. For such momenta, and in particular for the momenta obeying $\hbar k < \hbar\sqrt{8\pi na} \ll \hbar/\Lambda$ that are shown in Sec. III.B to be the most important states for the dynamics of the gas after the occurrence of the instability, we find in a good approximation that

$$\varepsilon'(\mathbf{k}; t) = \varepsilon(\mathbf{k}) + \frac{8\pi n(t) a\hbar^2}{m} - 2 \int \frac{d\mathbf{k}'}{(2\pi)^3} \\ \times \int \frac{d\mathbf{k}''}{(2\pi)^3} |V^{(+)}(\mathbf{0}, \mathbf{k}', \mathbf{k}'')|^2 N(\mathbf{k}'; t) N(\mathbf{k}''; t) \frac{\mathcal{P}}{\hbar^2 \mathbf{k}' \cdot \mathbf{k}''/m}. \quad (119)$$

Note that in the right-hand side $V^{(+)}(\mathbf{0}, \mathbf{k}', \mathbf{k}'')$ also implicitly depends on t , because it depends on the distribution function $N(\mathbf{k}; t)$ as shown for example in detail in Eq. (111).

In principle, Eqs. (118) and (119) already clearly show the tendency of the gas to become unstable towards Bose-Einstein condensation, because the energy of the one-particle ground state is shifted less upwards as compared

to the one-particle states with thermal energies. To actually show when the gas is unstable we, however, need to compare the energy of the zero-momentum state with the instantaneous chemical potential. Using Eq. (118) and our picture that during evaporative cooling the occupation numbers of the gas are almost equal to a truncated equilibrium distribution, the chemical potential becomes

$$\mu(t) \simeq \frac{8\pi n(t) a\hbar^2}{m} + \mu_0(t), \quad (120)$$

where the time dependence of the ideal gas chemical potential $\mu_0(t) \equiv \mu_0(n(t), T(t))$ depends on the precise path in the density-temperature plane that is followed during the cooling process. An instability therefore occurs once the quantity

$$\begin{aligned} \varepsilon'(\mathbf{0}; t) - \mu(t) \simeq & -\mu_0(t) - 2 \int \frac{d\mathbf{k}'}{(2\pi)^3} \int \frac{d\mathbf{k}''}{(2\pi)^3} |V^{(+)}(\mathbf{0}, \mathbf{k}', \mathbf{k}'')|^2 \\ & \times N(\mathbf{k}'; t) N(\mathbf{k}''; t) \frac{\mathcal{P}}{\hbar^2 \mathbf{k}' \cdot \mathbf{k}''/m} \end{aligned}$$

becomes less than zero. It was the single most important conclusion of Ref. 71 that the gas indeed develops the required instability for Bose-Einstein condensation if $a > 0$ and the temperature is less than a critical temperature $T_c = T_0(1 + \mathcal{O}(a/\Lambda_0))$ that is slightly higher than the critical temperature T_0 of the ideal Bose gas. Although we can argue that this might be an artefact of the many-body T-matrix approximation, which certainly does not take proper account of all the critical fluctuations near the critical temperature, it should be noted that all the predictions of the many-body T-matrix approximation are fully confirmed by a renormalization group calculation that does take critical fluctuations into account and only neglects the effects of a mass renormalization.⁸¹ Moreover, the upward shift of the interacting critical temperature by an amount of $\mathcal{O}((a/\Lambda_0) T_0) = \mathcal{O}((na^3)^{1/3} T_0)$ has recently also been found by path-integral Monte-Carlo simulations.⁸⁶ We therefore believe that the results obtained here give an, even more than qualitatively, correct picture of the behavior of the gas near the critical temperature.

B. Semiclassical Dynamics

Having positively answered the question if an interacting Bose gas can be experimentally quenched into the regime in which it is unstable for a

spontaneous breaking of the $U(1)$ symmetry, our next task is to consider the dynamics of the gas after it has developed the instability. At the semiclassical level our results obtained in Sec. III.A show that the effective action for the long-wavelength dynamics of the gas, i.e., for the states with momenta $\hbar \mathbf{k} < \hbar \sqrt{8\pi n a}$, is given by

$$\begin{aligned}
S^{cl}[\phi^*, \phi] &= \int dt \left\{ \sum_{\mathbf{k}} \phi_{\mathbf{k}}^*(t) \left(i\hbar \frac{\partial}{\partial t} - \varepsilon'(\mathbf{k}; t) + \mu(t) \right) \phi_{\mathbf{k}}(t) \right. \\
&\quad \left. - \frac{1}{2V} \sum_{\mathbf{k}, \mathbf{k}', \mathbf{K}} T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) \phi_{\mathbf{K}/2+\mathbf{k}}^*(t) \phi_{\mathbf{K}/2-\mathbf{k}}^*(t) \phi_{\mathbf{K}/2-\mathbf{k}'}(t) \phi_{\mathbf{K}/2+\mathbf{k}'}(t) \right\}.
\end{aligned} \tag{121}$$

We can understand that this is the correct semiclassical action in two different ways. First, it is found from our Schwinger-Keldysh formalism if we realize that at the semiclassical level the “path” followed by the gas on the backward branch of the Schwinger-Keldysh contour is identical to the “path” followed on the forward branch. Applying this observation to the full effective action $S^{\text{eff}}[\psi^*, \psi]$ immediately leads to the action $S^{cl}[\phi^*, \phi]$ describing the semiclassical “path” on either branch. Second, we can make use of the fact that $\phi(\mathbf{x}, t)$ can also be seen as the order parameter of the Bose gas. The corresponding time-dependent Landau-Ginzburg theory can easily be derived in the many-body T-matrix approximation, if we use the imaginary-time approach to the equilibrium theory of interacting Bose gases.^{46, 52} Performing then the usual Wick rotation to real time, we again obtain $S^{cl}[\phi^*, \phi]$ for the effective action of the order parameter.

In particular, we thus find that the dynamics of the zero-momentum part of the order parameter, i.e., the condensate, is in first instance determined by

$$\begin{aligned}
S_0^{cl}[\phi_0^*, \phi_0] &= \int dt \left\{ \phi_0^*(t) \left(i\hbar \frac{\partial}{\partial t} - \varepsilon'(\mathbf{0}; t) + \mu(t) \right) \phi_0(t) \right. \\
&\quad \left. - \frac{T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0)}{2V} |\phi_0(t)|^4 \right\}.
\end{aligned} \tag{122}$$

Introducing the density $n_0(t)$ of the condensate and its phase θ_0 by means of the relation $\phi_0(t) \equiv \sqrt{n_0(t)} V e^{i\theta_0}$, this simply leads to

$$S_0^{cl}[n_0, \mu] = V \int dt \left(-\varepsilon'(\mathbf{0}; t) + \mu(t) - \frac{T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0)}{2} n_0(t) \right) n_0(t). \tag{123}$$

Notice that in the process of deriving the last equation, we have omitted the topological term $\int dt(i\hbar \partial n_0(t)/\partial t)$ that does not affect the equations of motion and is therefore irrelevant at the semiclassical level. It is in principle only important when we also want to consider the quantum fluctuations of the condensate. Note also that we have made use of the definition of the chemical potential, or equivalently of the Josephson relation,⁸⁷ to take the condensate phase independent of time. If we do not measure our energies relative to the chemical potential, we must use $\phi_0(t) \equiv \sqrt{n_0(t)V} e^{i\theta_0(t)}$ and $\hbar \partial \theta_0/\partial t = -\mu(t)$ instead, which then leads to exactly the same action. Clearly, the above action is minimized by $n_0(t) = 0$ if $\varepsilon'(\mathbf{0}; t) - \mu(t) > 0$. However, in agreement with our previous stability analysis, we have a non-trivial minimum at

$$n_0(t) = -\frac{\varepsilon'(\mathbf{0}; t) - \mu(t)}{T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0)}, \quad (124)$$

if $\varepsilon'(\mathbf{0}; t) - \mu(t) < 0$. This result gives the desired evolution of the condensate density after the gas has become unstable in terms of the time-dependent chemical potential $\mu(t)$. Our next task is therefore to determine an equation of motion for the chemical potential.

To achieve this we now also have to consider the interactions between the condensed and the noncondensed parts of the gas, which have been neglected thusfar. Substituting $\phi_{\mathbf{k}}(t) = \phi_0(t) \delta_{\mathbf{k}, \mathbf{0}} + \phi'_{\mathbf{k}}(t)(1 - \delta_{\mathbf{k}, \mathbf{0}})$ into the semiclassical action and integrating over the fluctuations $\phi'_{\mathbf{k}}(t)$ describing the noncondensed part of the gas, we find that the exact semiclassical action for the condensate obeys

$$S^{cl}[n_0, \mu] = S_0^{cl}[n_0, \mu] - i\hbar \ln(Z^{cl}[n_0, \mu]), \quad (125)$$

where $Z^{cl}[n_0, \mu]$ represents the functional integral over the fluctuations for given evolutions of the condensate density and the chemical potential. Writing $S^{cl}[\phi_0^* + \phi'^*, \phi_0 + \phi']$ as $S_0^{cl}[\phi_0^*, \phi_0] + S_1^{cl}[\phi'^*, \phi'; n_0, \mu]$ we thus have explicitly that

$$Z^{cl}[n_0, \mu] = \int d[\phi'^*] d[\phi'] \exp \left\{ \frac{i}{\hbar} S_1^{cl}[\phi'^*, \phi'; n_0, \mu] \right\}. \quad (126)$$

With this action the total density of the gas is calculated in the thermodynamic limit as

$$n(t) = \frac{1}{V} \frac{\delta S^{cl}[n_0, \mu]}{\delta \mu(t)} = n_0(t) + \int \frac{d\mathbf{k}}{(2\pi)^3} N(\mathbf{k}; t), \quad (127)$$

where the occupation numbers are found from

$$N(\mathbf{k}; t) = \frac{\int d[\phi'^*] d[\phi'] \phi'_{\mathbf{k}}{}^*(t) \phi'_{\mathbf{k}}(t) \exp\{iS_1^{cl}[\phi'^*, \phi'; n_0, \mu]/\hbar\}}{Z^{cl}[n_0, \mu]} \equiv \langle \phi'_{\mathbf{k}}{}^*(t) \phi'_{\mathbf{k}}(t) \rangle^{cl}. \quad (128)$$

The latter two equations, together with Eq. (124), in principle give both the condensate density $n_0(t)$ and the chemical potential $\mu(t)$ as a function of the total density $n(t)$ and formally thus completely solve the semiclassical dynamics of the gas. However, the above derivation of the semiclassical theory appears to be not fully consistent. On the one hand we have in the calculation of the total density varied the full semiclassical action $S^{cl}[n_0, \mu]$ with respect to the chemical potential, but on the other hand we have varied only $S_0^{cl}[n_0, \mu]$ with respect to $n_0(t)$ to obtain the equation for the condensate. The following question therefore arises: Why does the term $-i\hbar \ln(Z^{cl}[n_0, \mu])$ not contribute to the dynamics of the condensate? Fortunately, the reason for this is well-known.⁸⁸ Expanding $-i\hbar \ln(Z^{cl}[n_0, \mu])$ in powers of $n_0(t)$ we can easily convince ourselves that the fluctuations only lead to corrections to the linear and quadratic terms in $S_0^{cl}[n_0, \mu]$ that in fact have already been accounted for by using the renormalized energy $\varepsilon'(\mathbf{0}; t)$ and the renormalized interaction $T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0)$. To avoid a double counting of the effects of the interaction, we should therefore indeed neglect the fluctuations when calculating the condensate density. Formally, this is equivalent to the requirement that the condensate density fulfills the Hugenholtz-Pines theorem,¹⁹ because Eq. (124) can be rewritten as $\mu(t) = \hbar\Sigma_{11}^{(+)}(\mathbf{0}; t) - \hbar\Sigma_{12}^{(+)}(\mathbf{0}; t)$ in terms of the usual normal and anomalous self-energies in $S_1^{cl}[\phi'^*, \phi'; n_0, \mu]$.

Determining the occupation numbers $\langle \phi'_{\mathbf{k}}{}^*(t) \phi'_{\mathbf{k}}(t) \rangle^{cl}$ requires solving an interacting quantum field theory, which cannot be done exactly. An approximation is thus called for. Taking only the quadratic terms in $S_1^{cl}[\phi'^*, \phi'; n_0, \mu]$ into account amounts to the Bogoliubov approximation. Indeed, the action for the fluctuations then becomes equal to

$$S_B[\phi'^*, \phi'] = \int dt \left\{ \sum_{\mathbf{k} \neq \mathbf{0}} \phi'_{\mathbf{k}}{}^*(t) \left(i\hbar \frac{\partial}{\partial t} - \varepsilon(\mathbf{k}) - n_0(t) T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) \right) \phi'_{\mathbf{k}}(t) - \frac{1}{2} T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) n_0(t) \sum_{\mathbf{k} \neq \mathbf{0}} (\phi'_{\mathbf{k}}{}^*(t) \phi'_{-\mathbf{k}}{}^*(t) + \phi'_{-\mathbf{k}}(t) \phi'_{\mathbf{k}}(t)) \right\}, \quad (129)$$

if we use Eq. (119) to evaluate the energy difference $\varepsilon'(\mathbf{k}; t) - \varepsilon'(\mathbf{0}; t) = \varepsilon(\mathbf{k})$ at the long wavelengths of interest here. The energies of the Bogoliubov quasiparticles thus obey

$$\hbar\omega(\mathbf{k}; t) = \sqrt{\varepsilon^2(\mathbf{k}) + 2n_0(t) T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) \varepsilon(\mathbf{k})}, \quad (130)$$

and are purely real at this level of approximation. However, in the Bogoliubov approximation the quasiparticles are noninteracting. This is reasonable sufficiently far below the critical temperature when the condensate density is large, but not very close to the critical temperature.²¹ In that case the interactions between the quasiparticles are very important and cannot be neglected.

A qualitative understanding of the effect of these interactions can be obtained by making use of a simple mean-field theory. Since the condensate density is relatively small we can approximate the average interaction between the quasiparticles by

$$\begin{aligned} & \left\langle \frac{1}{2V} \sum_{\mathbf{k}, \mathbf{k}', \mathbf{K} \neq \mathbf{0}} T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) \phi'_{\mathbf{K}/2+\mathbf{k}}(t) \phi'_{\mathbf{K}/2-\mathbf{k}}(t) \phi'_{\mathbf{K}/2-\mathbf{k}'}(t) \phi'_{\mathbf{K}/2+\mathbf{k}'}(t) \right\rangle^{cl} \\ & \simeq VT^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) \left(n^2(t) - n(t) n_0(t) - n_0(t) \frac{1}{V} \sum_{\mathbf{k} \neq \mathbf{0}} \langle \phi_{\mathbf{k}}'(t) \phi_{\mathbf{k}}'(t) \rangle^{cl} \right). \end{aligned} \quad (131)$$

Within the context of a mean-field approximation, we therefore see that the interactions effectively renormalize the Bogoliubov theory to

$$\begin{aligned} S_{RB}[\phi'^*, \phi'] = & \int dt \left\{ \sum_{\mathbf{k} \neq \mathbf{0}} \phi_{\mathbf{k}}'^*(t) \left(i\hbar \frac{\partial}{\partial t} - \varepsilon(\mathbf{k}) \right) \phi_{\mathbf{k}}'(t) \right. \\ & \left. - \frac{1}{2} T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) n_0(t) \sum_{\mathbf{k} \neq \mathbf{0}} (\phi_{\mathbf{k}}'^*(t) \phi_{-\mathbf{k}}'^*(t) + \phi_{-\mathbf{k}}'(t) \phi_{\mathbf{k}}'(t)) \right\} \end{aligned} \quad (132)$$

and lead to quasiparticle energies

$$\hbar\omega(\mathbf{k}; t) = \sqrt{\varepsilon^2(\mathbf{k}) - (n_0(t) T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0))^2} \quad (133)$$

that are purely imaginary at long wavelengths. The occupation numbers for these momentum states thus decay in favor of the condensate density. The typical time scale for the decay is of $\mathcal{O}(\hbar/n_0 T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0))$ and always much slower than $\hbar/k_B T_c$ because $a/\Lambda_c \ll 1$ for the quantum gases of interest.

This *a posteriori* justifies our neglect of the memory effects in the above. In summary, the semiclassical picture of Bose-Einstein condensation can then be visualized as in Fig. 1, where we have also tried to indicate the dynamics of the phase $\theta_0(t)$ due to the Josephson relation. After the condensate density has been formed in this fully coherent manner, the distribution function $N(\mathbf{k}; t)$ is clearly not in equilibrium since all the particles in the low-energy states have been transferred to the condensate. In the third and last stage of the condensation process this distribution function relaxes to thermal equilibrium due to elastic collisions between the Bogoliubov quasiparticles. This can again be accurately described by a quantum Boltzmann equation.^{89,90} We do not discuss this third stage in any detail here (see, for example, Refs. 76 and 90) and just end with a brief overview of the semiclassical picture of Bose-Einstein condensation.

C. Overview

In the semiclassical theory the divergence of the time required for Bose-Einstein condensation is avoided, because the incoherent dynamics of the occupation numbers $N(\mathbf{k}; t)$ for the one-particle states is coupled to the coherent dynamics of the energies $\epsilon'(\mathbf{k}; t)$ of these states, and the fluctuations of the condensate are neglected. As a result the process of Bose-Einstein condensation now proceeds in three stages. In the first stage the

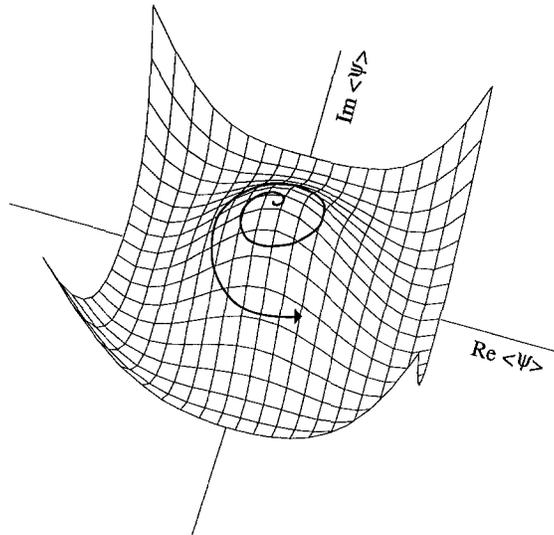


Fig. 1. Semiclassical picture of the coherent stage of Bose-Einstein condensation.

gas moves, for example as a response to quickly removing the hottest atoms, towards the critical region of the phase transition. This part of the evolution can be accurately described by a quantum Boltzmann equation, in which in a selfconsistent manner collisions between the atoms lead to changes in the occupation numbers $N(\mathbf{k}; t)$ and these changes in their turn affect the mean-field corrections to the energies $\varepsilon'(\mathbf{k}; t)$. As long as the gas is outside the critical region, however, these mean-field corrections are constant and we simply have $\varepsilon'(\mathbf{k}; t) = \varepsilon(\mathbf{k}) + 8\pi n a \hbar^2/m$. Hence, if initially not enough energy is removed, the gas always remains in this regime and then typically evolves towards equilibrium on a time scale of $\mathcal{O}(\tau_{el})$, which near the critical temperature is of $\mathcal{O}((\Lambda_c/a)^2 \hbar/k_B T_c)$.⁷²⁻⁷⁶

If on the other hand sufficient energy is removed initially, the gas enters the critical region and the second stage towards Bose-Einstein condensation begins. In this second stage, the gas first develops the required instability for the phase transition. Physically, the instability is a result of the fact that in the critical region, fluctuations tend to decrease the mean-field corrections to the energies of the low-momentum states with $\hbar k < \mathcal{O}(\hbar \sqrt{8\pi n a})$,^{80, 81} which makes it possible for the zero-momentum state to acquire an energy which is less than the instantaneous chemical potential. It is then clearly energetically favorable to put a macroscopic number of particles in that state. It is important to note that for this instability mechanism, it is crucial that the typical time scale on which the mean-field corrections are established is only of $\mathcal{O}(\hbar/k_B T_c)$ and therefore much faster than the time scale on which elastic collisions can change the occupations numbers. In that sense it can be said that Bose-Einstein condensation is “nucleated” on a time scale of $\mathcal{O}(\hbar/k_B T_c)$.⁶²

As we have seen above, incoherent collisions alone cannot lead to a macroscopic occupation of the zero-momentum state. After the gas has become unstable, the actual growth of the condensate density is therefore in the semiclassical theory caused by a coherent depletion of the low-lying excited states in favor of the one-particle ground state. At the end of the second stage, which also has a duration of $\mathcal{O}((\Lambda_c/a)^2 \hbar/k_B T_c)$, the gas has thus acquired a highly nonequilibrium energy distribution and has to come to equilibrium in a third and final stage. This last stage of the condensation process is again of a kinetic nature and can be accurately described by a quantum Boltzmann equation for the Bogoliubov quasiparticles.^{89, 90} Due to the linear dispersion of these quasiparticles at long wavelengths the typical time scale for relaxation towards equilibrium is exceptionally long and of $\mathcal{O}((\Lambda_c/a)^3 \hbar/k_B T_c)$, as was shown by Eckern in the context of spin-polarized atomic hydrogen.⁹⁰

For completeness sake, we should mention that Kagan, Svistunov, and Shlyapnikov have put forward a different semiclassical picture and

assert that during the second, coherent stage of the evolution, the gas does not form a condensate but, just as in a two dimensional gas, only a quasi-condensate.⁹¹ Mathematically, this implies that the gas does not acquire a nonzero average of the annihilation operator of the zero-momentum state $\hat{\psi}_0(t)$ alone, but only of the sum $\sum_{k < k_0} \hat{\psi}_k(t) e^{i\mathbf{k}\cdot\mathbf{x}}$, where $\hbar k_0$ is a momentum cut-off of $\mathcal{O}(\hbar \sqrt{8\pi n a})$.²⁴ This semiclassical picture is however not confirmed by our microscopic calculation, because from the above discussion of the instability mechanism it is clear that a possible instability for a quasicondensate is always preceded by an instability for a real condensate since a removal of the long-wavelength fluctuations with $k < k_0$ always increases the mean-field corrections to the energies $\varepsilon'(\mathbf{k}; t)$.

The semiclassical theory neglects in particular the effect of thermal fluctuations on the condensate dynamics, which are nevertheless anticipated to be very important for conditions near the critical temperature when a large fraction of the gas is still uncondensed. For a quantitative comparison with experiments that are presently being performed,⁹² the inclusion of these thermal fluctuations is therefore clearly of interest. There are also several theoretical reasons why we need to consider the effect of thermal fluctuations. Although the semiclassical theory of Bose-Einstein condensation gives important insight into the dynamics of this phase transition, it has two important drawbacks. First, it makes a sharp distinction between the kinetic and coherent stages of the evolution, which in principle of course overlap with each other. Second, the discussion of the growth of the condensate actually makes use of the presence of fluctuations, without taking these explicitly into account. This is due to the fact that at the semiclassical level we always have a solution to the equations of motion in which the gas remains at all times in the metastable, uncondensed phase. By considering the effect of thermal fluctuations, both these problems can indeed be resolved and we finally arrive at a single Fokker-Planck equation that describes all the three stages of the condensation process simultaneously. How this is achieved is reviewed next.

IV. QUANTUM THEORY OF BOSE-EINSTEIN CONDENSATION

In the semiclassical theory of the previous section the main emphasis is only on the average $\langle \phi_0 \rangle(t)$, whereas the aim of the quantum theory is to consider also the fluctuations and determine the full probability distribution $P[\phi^*, \phi; t]$. Combining the results of Secs. II and III, we find that this probability distribution can be expressed as a functional integral with an effective action that in the many-body T-matrix approximation is given by

$$\begin{aligned}
 S^{\text{eff}}[\psi^*, \psi] &= \int_{\varphi'} dt' \int_{\varphi''} dt'' \left\{ \sum_{\mathbf{k}} \psi_{\mathbf{k}}^*(t') \left[\left(i\hbar \frac{\partial}{\partial t'} - \varepsilon(\mathbf{k}) + \mu(t') \right) \right. \right. \\
 &\quad \left. \left. \times \delta(t', t'') - \hbar \Sigma(\mathbf{k}; t', t'') \right] \psi_{\mathbf{k}}(t'') \right. \\
 &\quad \left. - \frac{1}{2V} \sum_{\mathbf{k}, \mathbf{k}', \mathbf{K}} V(\mathbf{k}, \mathbf{k}', \mathbf{K}; t', t'') \psi_{\mathbf{K}/2+\mathbf{k}}^*(t') \right. \\
 &\quad \left. \times \psi_{\mathbf{K}/2-\mathbf{k}}^*(t') \psi_{\mathbf{K}/2-\mathbf{k}'}(t'') \psi_{\mathbf{K}/2+\mathbf{k}'}(t'') \right\}. \quad (134)
 \end{aligned}$$

We next have to project the field $\psi(\mathbf{x}, t)$ onto the real axis by means of $\psi(\mathbf{x}, t_{\pm}) = \phi(\mathbf{x}, t) \pm \xi(\mathbf{x}, t)/2$. Up to quadratic terms in the fluctuations $\xi(\mathbf{x}, t)$ and neglecting again the memory effects, which is also known as the Markovian approximation^{45, 58} and is justified here because the memory time is at most of $\mathcal{O}(\hbar/k_B T)$, we then easily obtain

$$\begin{aligned}
 S^{\text{eff}}[\phi^*, \phi; \xi^*, \xi] &= \sum_{\mathbf{k}, \mathbf{k}'} \int_{-\infty}^t dt' \phi_{\mathbf{k}}^*(t') \left\{ \left(i\hbar \frac{\partial}{\partial t'} - \varepsilon(\mathbf{k}) + \mu(t') - \hbar \Sigma^{(-)}(\mathbf{k}) \right) \delta_{\mathbf{k}, \mathbf{k}'} \right. \\
 &\quad \left. + \frac{1}{V} \sum_{\mathbf{k}''} V^{(-)}(\mathbf{k}, \mathbf{k}', \mathbf{k}'') \phi_{\mathbf{k}'+\mathbf{k}''-\mathbf{k}}^*(t') \phi_{\mathbf{k}''}(t') \right\} \xi_{\mathbf{k}}(t') \\
 &\quad + \sum_{\mathbf{k}, \mathbf{k}'} \int_{-\infty}^t dt' \xi_{\mathbf{k}}^*(t') \left\{ \left(i\hbar \frac{\partial}{\partial t'} - \varepsilon(\mathbf{k}) + \mu(t') - \hbar \Sigma^{(+)}(\mathbf{k}) \right) \delta_{\mathbf{k}, \mathbf{k}'} \right. \\
 &\quad \left. + \frac{1}{V} \sum_{\mathbf{k}''} V^{(+)}(\mathbf{k}, \mathbf{k}', \mathbf{k}'') \phi_{\mathbf{k}'+\mathbf{k}''-\mathbf{k}}^*(t') \phi_{\mathbf{k}''}(t') \right\} \phi_{\mathbf{k}}(t') \\
 &\quad - \frac{1}{2} \sum_{\mathbf{k}, \mathbf{k}'} \int_{-\infty}^t dt' \xi_{\mathbf{k}}^*(t') \left\{ \hbar \Sigma^K(\mathbf{k}) \delta_{\mathbf{k}, \mathbf{k}'} \right. \\
 &\quad \left. + \frac{2}{V} \sum_{\mathbf{k}''} V^K(\mathbf{k}, \mathbf{k}', \mathbf{k}'') \phi_{\mathbf{k}'+\mathbf{k}''-\mathbf{k}}^*(t') \phi_{\mathbf{k}''}(t') \right\} \xi_{\mathbf{k}}(t'), \quad (135)
 \end{aligned}$$

introducing the shorthand notation $\Sigma^{(\pm), K}(\mathbf{k})$ for the quantities $\Sigma^{(\pm), K}(\mathbf{k}; \varepsilon'(\mathbf{k}) - \mu)$ that, just like the matrix elements $V^{(\pm), K}(\mathbf{k}, \mathbf{k}', \mathbf{k}'')$, implicitly also depend on time due to the time dependence of the occupation numbers. At this point it is possibly not immediately clear why we do not need

to consider cubic and quartic terms in the fluctuations. The reason for that has been explicitly mentioned in Ref. 71 and boils down to the general problem of working with an effective theory, i.e., we must be careful not to double count the effects of the interaction. This almost happened in Sec. III.B when we derived the semiclassical equation of motion for the condensate density and it would also happen here if we do not restrict ourselves to the linear and quadratic terms in the fluctuations.

As long as all the one-particle states are stable, we can in first instance neglect the nonlinear terms in $S^{\text{eff}}[\phi^*, \phi; \xi^*, \xi]$ that are proportional to $V^{(\pm), K}(\mathbf{k}, \mathbf{k}', \mathbf{k}'')$. Physically, this implies that we can treat the interacting atomic Bose gas as an (almost) noninteracting gas of dressed quasiparticles, i.e., as a ‘‘Bose liquid.’’ As a result the effective action becomes essentially identical to that of our Caldeira-Leggett toy model in Sec. II.B and we can immediately write down the corresponding Fokker-Planck equation. Denoting the imaginary part of $\hbar\Sigma^{(\pm)}(\mathbf{k})$ again by $\mp R(\mathbf{k}; t)$, we have

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} P[\phi^*, \phi; t] &= - \sum_{\mathbf{k}} \frac{\partial}{\partial \phi_{\mathbf{k}}} (\varepsilon'(\mathbf{k}; t) - iR(\mathbf{k}; t) - \mu(t)) \phi_{\mathbf{k}} P[\phi^*, \phi; t] \\ &\quad + \sum_{\mathbf{k}} \frac{\partial}{\partial \phi_{\mathbf{k}}^*} (\varepsilon'(\mathbf{k}; t) + iR(\mathbf{k}; t) - \mu(t)) \phi_{\mathbf{k}}^* P[\phi^*, \phi; t] \\ &\quad - \frac{1}{2} \sum_{\mathbf{k}} \frac{\partial^2}{\partial \phi_{\mathbf{k}}^* \partial \phi_{\mathbf{k}}} \hbar\Sigma^K(\mathbf{k}) P[\phi^*, \phi; t]. \end{aligned} \quad (136)$$

However, to extract the physics from this Fokker-Planck equation, we now need to determine the quantities $R(\mathbf{k}; t)$ and $\hbar\Sigma^K(\mathbf{k})$. Using our results from the semiclassical theory, in particular Eqs. (101), (113) and (116), it is not difficult to show that the imaginary part of the retarded and advanced selfenergies obeys

$$\begin{aligned} R(\mathbf{k}; t) &= 2\pi \int \frac{d\mathbf{k}'}{(2\pi)^3} \int \frac{d\mathbf{k}''}{(2\pi)^3} \delta(\varepsilon'(\mathbf{k}'; t) + \varepsilon'(\mathbf{k}''; t) \\ &\quad - \varepsilon'(\mathbf{k}' + \mathbf{k}'' - \mathbf{k}; t) - \varepsilon'(\mathbf{k}; t)) |V^{(+)}(\mathbf{k}, \mathbf{k}', \mathbf{k}'')|^2 \\ &\quad \times [(1 + N(\mathbf{k}'; t) + N(\mathbf{k}''; t)) N(\mathbf{k}' + \mathbf{k}'' - \mathbf{k}; t) - N(\mathbf{k}'; t) N(\mathbf{k}''; t)], \end{aligned} \quad (137)$$

which has recently also been obtained by an equation-of-motion method instead of the Schwinger-Keldysh formalism that is employed here.⁹³ To obtain the Keldysh component of the selfenergy we first have to realize that

from the general Hartree-Fock-like relation for the selfenergy $\hbar\Sigma(\mathbf{k}; t, t')$ on the Schwinger-Keldysh contour it follows that

$$\begin{aligned} & \hbar\Sigma^K(\mathbf{k}; \varepsilon'(\mathbf{k}) - \mu) \\ &= 2 \int \frac{d\mathbf{k}'}{(2\pi)^3} V^K(\mathbf{k} - \mathbf{k}', \mathbf{k} - \mathbf{k}', \mathbf{k} + \mathbf{k}'; \varepsilon'(\mathbf{k}) + \varepsilon'(\mathbf{k}') - 2\mu) N(\mathbf{k}') \\ & \quad + 2 \int \frac{d\mathbf{k}'}{(2\pi)^3} V^<(\mathbf{k} - \mathbf{k}', \mathbf{k} - \mathbf{k}', \mathbf{k} + \mathbf{k}'; \varepsilon'(\mathbf{k}) + \varepsilon'(\mathbf{k}') - 2\mu)(1 + 2N(\mathbf{k}')), \end{aligned} \quad (138)$$

after a Fourier transformation of the time difference $t - t'$. Substituting now Eqs. (116) and (117) we then finally obtain

$$\begin{aligned} & \hbar\Sigma^K(\mathbf{k}) \\ &= -4\pi i \int \frac{d\mathbf{k}'}{(2\pi)^3} \int \frac{d\mathbf{k}''}{(2\pi)^3} \delta(\varepsilon'(\mathbf{k}'; t) + \varepsilon'(\mathbf{k}''; t) - \varepsilon'(\mathbf{k}' + \mathbf{k}'' - \mathbf{k}; t) - \varepsilon'(\mathbf{k}; t)) \\ & \quad \times |V^{(+)}(\mathbf{k}, \mathbf{k}', \mathbf{k}'')|^2 [(1 + N(\mathbf{k}'; t) + N(\mathbf{k}''; t)) N(\mathbf{k}' + \mathbf{k}'' - \mathbf{k}; t) \\ & \quad + N(\mathbf{k}'; t) N(\mathbf{k}''; t)(1 + 2N(\mathbf{k}' + \mathbf{k}'' - \mathbf{k}; t))]. \end{aligned} \quad (139)$$

What does the above imply for the dynamics of the gas? Considering first the equation of motion of $\langle \phi_{\mathbf{k}} \rangle(t)$, the Fokker-Planck equation tells us that

$$i\hbar \frac{\partial}{\partial t} \langle \phi_{\mathbf{k}} \rangle(t) = (\varepsilon'(\mathbf{k}; t) - iR(\mathbf{k}; t) - \mu(t)) \langle \phi_{\mathbf{k}} \rangle(t). \quad (140)$$

Hence the rate of decay of a particle with momentum $\hbar\mathbf{k}$ due to collisions with other particles in the gas equals $\Gamma(\mathbf{k}; t) = 2R(\mathbf{k}; t)/\hbar$. This indeed precisely agrees with a Fermi's Golden Rule calculation, because the rate for a particle to scatter out of the momentum state $\chi_{\mathbf{k}}(\mathbf{x})$ is

$$\begin{aligned} & \Gamma^{\text{out}}(\mathbf{k}; t) \\ &= \frac{4\pi}{\hbar} \int \frac{d\mathbf{k}'}{(2\pi)^3} \int \frac{d\mathbf{k}''}{(2\pi)^3} \delta(\varepsilon'(\mathbf{k}'; t) + \varepsilon'(\mathbf{k}''; t) - \varepsilon'(\mathbf{k}' + \mathbf{k}'' - \mathbf{k}; t) - \varepsilon'(\mathbf{k}; t)) \\ & \quad \times |V^{(+)}(\mathbf{k}, \mathbf{k}', \mathbf{k}'')|^2 (1 + N(\mathbf{k}'; t))(1 + N(\mathbf{k}''; t)) N(\mathbf{k}' + \mathbf{k}'' - \mathbf{k}; t), \end{aligned} \quad (141)$$

whereas the rate to scatter into that state is

$$\begin{aligned} & \Gamma^{\text{in}}(\mathbf{k}; t) \\ &= \frac{4\pi}{\hbar} \int \frac{d\mathbf{k}'}{(2\pi)^3} \int \frac{d\mathbf{k}''}{(2\pi)^3} \delta(\varepsilon'(\mathbf{k}'; t) + \varepsilon'(\mathbf{k}''; t) - \varepsilon'(\mathbf{k}' + \mathbf{k}'' - \mathbf{k}; t) - \varepsilon'(\mathbf{k}; t)) \\ & \quad \times |V^{(+)}(\mathbf{k}, \mathbf{k}', \mathbf{k}'')|^2 N(\mathbf{k}'; t) N(\mathbf{k}''; t) (1 + N(\mathbf{k}' + \mathbf{k}'' - \mathbf{k}; t)). \end{aligned} \quad (142)$$

Note that both the in and the out rate are a factor of two larger than might have been expected in first instance due to the fact that the direct and exchange contributions interfere constructively in the collision process.

Although the value of the decay rate $\Gamma(\mathbf{k}; t) = \Gamma^{\text{out}}(\mathbf{k}; t) - \Gamma^{\text{in}}(\mathbf{k}; t)$ is clearly a correct consequence of our Fokker-Planck equation, what is missing at this point is the dynamics of the occupation numbers $N(\mathbf{k}; t)$ themselves. This omission can be repaired by considering also the average $\langle |\phi_{\mathbf{k}}|^2 \rangle(t) = N(\mathbf{k}; t) + 1/2$. For that quantity the Fokker-Planck equation gives

$$i\hbar \frac{\partial}{\partial t} \langle |\phi_{\mathbf{k}}|^2 \rangle(t) = -2iR(\mathbf{k}; t) \langle |\phi_{\mathbf{k}}|^2 \rangle(t) - \frac{1}{2} \hbar \Sigma^{\mathcal{K}}(\mathbf{k}). \quad (143)$$

As expected, this leads to the appropriate quantum Boltzmann equation

$$\begin{aligned} \frac{\partial}{\partial t} N(\mathbf{k}; t) &= -(\Gamma^{\text{out}}(\mathbf{k}; t) - \Gamma^{\text{in}}(\mathbf{k}; t)) \left(N(\mathbf{k}; t) + \frac{1}{2} \right) + \frac{1}{2} i \Sigma^{\mathcal{K}}(\mathbf{k}) \\ &= -\Gamma^{\text{out}}(\mathbf{k}; t) N(\mathbf{k}; t) + \Gamma^{\text{in}}(\mathbf{k}; t) (1 + N(\mathbf{k}; t)), \end{aligned} \quad (144)$$

if we make use of the observation that $i\Sigma^{\mathcal{K}}(\mathbf{k}) = \Gamma^{\text{out}}(\mathbf{k}; t) + \Gamma^{\text{in}}(\mathbf{k}; t)$. The gas, therefore, relaxes for $t \rightarrow \infty$ to an equilibrium in which the occupation numbers $N(\mathbf{k})$ are equal to the Bose distribution evaluated at $\varepsilon'(\mathbf{k}) - \mu$, where both $\varepsilon'(\mathbf{k})$ and μ are determined selfconsistently by the retarded selfenergy in the same way as in the semiclassical theory. An identical conclusion is reached by noting that in equilibrium the fluctuation-dissipation theorem is satisfied, i.e., $\hbar \Sigma^{\mathcal{K}}(\mathbf{k}) = -2i(1 + 2N(\mathbf{k})) R(\mathbf{k})$. Therefore, the stationary solution of the Fokker-Planck equation equals

$$P[\phi^*, \phi; \infty] = \prod_{\mathbf{k}} \frac{1}{N(\mathbf{k}) + 1/2} \exp \left\{ -\frac{1}{N(\mathbf{k}) + 1/2} |\phi_{\mathbf{k}}|^2 \right\}. \quad (145)$$

In the degenerate regime and at long wavelengths the probability distribution for $|\phi_{\mathbf{k}}|$ is thus proportional to the Boltzmann factor $e^{-\beta(\varepsilon'(\mathbf{k}) - \mu)|\phi_{\mathbf{k}}|^2}$. Once again, this shows that the Landau free energy for the zero-momentum

state $F(|\phi_0\rangle) = (\varepsilon'(\mathbf{0}) - \mu)|\phi_0|^2$ is unstable when $\varepsilon'(\mathbf{0}) < \mu$. If that happens we are no longer allowed to neglect the nonlinear terms in $S^{\text{eff}}[\phi^*, \phi; \xi^*, \xi]$, because they are essential to ultimately stabilize the large fluctuations in the amplitude of $\phi(\mathbf{x})$.

The linearized version of the full quantum theory given by $S^{\text{eff}}[\phi^*, \phi; \xi^*, \xi]$ thus accurately describes the first kinetic stage of the condensation process in which the gas is quenched into the unstable region of the phase diagram. The next question is: How does the quantum theory deal with the subsequent coherent stage? To answer that question we can neglect all the imaginary parts in the effective action, which are associated with the kinetic processes as we have seen, but now retain the nonlinear terms. The Fokker-Planck equation for the long-wavelength dynamics of interest then reads

$$\begin{aligned}
 i\hbar \frac{\partial}{\partial t} P[\phi^*, \phi; t] = & - \sum_{\mathbf{k}} \frac{\partial}{\partial \phi_{\mathbf{k}}} \left((\varepsilon'(\mathbf{k}; t) - \mu(t)) \phi_{\mathbf{k}} \right. \\
 & \left. + \frac{1}{V} \sum_{\mathbf{k}', \mathbf{k}''} T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) \phi_{\mathbf{k}'+\mathbf{k}''-\mathbf{k}}^* \phi_{\mathbf{k}'} \phi_{\mathbf{k}''} \right) P[\phi^*, \phi; t] \\
 & + \sum_{\mathbf{k}} \frac{\partial}{\partial \phi_{\mathbf{k}}^*} \left((\varepsilon'(\mathbf{k}; t) - \mu(t)) \phi_{\mathbf{k}}^* \right. \\
 & \left. + \frac{1}{V} \sum_{\mathbf{k}', \mathbf{k}''} T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) \phi_{\mathbf{k}'}^* \phi_{\mathbf{k}''}^* \phi_{\mathbf{k}'+\mathbf{k}''-\mathbf{k}} \right) P[\phi^*, \phi; t].
 \end{aligned} \tag{146}$$

Since it contains only streaming terms the solution is immediately seen to be

$$P[\phi^*, \phi; t] = \int d[\phi_1^*] d[\phi_1] P[|\phi_1|^2; t_1] \prod_{\mathbf{k}} \delta(|\phi_{\mathbf{k}} - \phi_{\mathbf{k}}^{cl}(t)|^2), \tag{147}$$

where $P[|\phi_1|^2; t_1]$ is the probability distribution after the kinetic stage, $\phi^{cl}(\mathbf{x}, t)$ is the solution of the nonlinear Schrödinger equation

$$i\hbar \frac{\partial \phi^{cl}(\mathbf{x}, t)}{\partial t} = \left(-\frac{\hbar^2 \nabla^2}{2m} + S(\mathbf{0}; t) - \mu(t) + T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) |\phi^{cl}(\mathbf{x}, t)|^2 \right) \phi^{cl}(\mathbf{x}, t) \tag{148}$$

with the initial condition $\phi^{cl}(\mathbf{x}, t_1) = \phi_1(\mathbf{x})$, and $S(\mathbf{k}; t)$ denotes the real part of the retarded selfenergy that is well approximated by the constant $S(\mathbf{0}; t)$ for small momenta $\hbar k \ll \hbar/\Lambda$ and given explicitly in Eq. (119).

The behavior of the solution in Eq. (147) has recently been investigated numerically by Damle, Majumdar, and Sachdev.⁹⁴ In agreement with the semiclassical picture, they find that the nonlinear Schrödinger equation has the property of depleting the states with momenta $\hbar k < \hbar \sqrt{8\pi n_0 a}$ in favor of the growth of the condensate. In contrast with the semiclassical theory, however, the time it actually takes to form the condensate in a truly infinite system is infinite also. More precisely, the time to form a condensate in a system with linear system size L diverges as

$$\frac{\hbar}{n_0 T^{(+)}(\mathbf{0}, \mathbf{0}; 0)} \left(\frac{L}{\xi}\right)^z,$$

introducing the correlation length of the condensate $\xi = 1/\sqrt{8\pi n_0 a}$ and the usual dynamical exponent z (Ref. 95) that was shown to be equal to 1 in this case.⁹⁶ From this outcome of the quantum theory we conclude that the fluctuations with wavelengths larger than the correlation length ξ play a crucial role. In the semiclassical theory the effect of these fluctuations is only accounted for in a mean-field sense, which allows the condensation to take place in a finite time. Put differently, not taking fully account of the fluctuations on the longest length scales has the same effect as considering a finite-size system. For a truly infinite system the thermodynamic limit in fact prevents the formation of the condensate even though the gas becomes unstable in a finite amount of time.

To understand the last remarks in a somewhat different manner, let us now consider the gas in a box with volume $V = L^3$. Because of the energy gap of $(2\pi\hbar)^2/2mL^2$ between the one-particle ground state and the first excited state, there is always a regime (the details are given below) in which it is allowed to single out the zero-momentum state. In that regime the Fokker-Planck equation following from the zero-momentum part of $S^{\text{eff}}[\phi^*, \phi; \xi^*, \xi]$ is

$$\begin{aligned} & i\hbar \frac{\partial}{\partial t} P[\phi_{\mathbf{0}}^*, \phi_{\mathbf{0}}; t] \\ &= -\frac{\partial}{\partial \phi_{\mathbf{0}}} \left(S(\mathbf{0}; t) - iR(\mathbf{0}; t) - \mu(t) + \frac{1}{V} T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) |\phi_{\mathbf{0}}|^2 \right) \phi_{\mathbf{0}} P[\phi_{\mathbf{0}}^*, \phi_{\mathbf{0}}; t] \\ &+ \frac{\partial}{\partial \phi_{\mathbf{0}}^*} \left(S(\mathbf{0}; t) + iR(\mathbf{0}; t) - \mu(t) + \frac{1}{V} T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) |\phi_{\mathbf{0}}|^2 \right) \phi_{\mathbf{0}}^* P[\phi_{\mathbf{0}}^*, \phi_{\mathbf{0}}; t] \\ &- \frac{1}{2} \frac{\partial^2}{\partial \phi_{\mathbf{0}}^* \partial \phi_{\mathbf{0}}} \hbar \Sigma^K(\mathbf{0}) P[\phi_{\mathbf{0}}^*, \phi_{\mathbf{0}}; t]. \end{aligned} \quad (149)$$

Anticipating now that the condensate dynamics is much slower than the dynamics of the noncondensed part of the gas, we can take for the self-energies $\hbar\Sigma^{(\pm),\mathbf{K}}(\mathbf{0})$ and the chemical potential their equilibrium values. There is, however, again a slight complication because the streaming terms in the right-hand side of Eq. (149) show that the energy of the condensate is $\varepsilon'(\mathbf{0}) + T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) |\phi_{\mathbf{0}}|^2/V$. To be fully consistent we must therefore evaluate $\hbar\Sigma^{(\pm),\mathbf{K}}(\mathbf{0}; \varepsilon)$ at these energies to obtain the correct equilibrium values. Making then also use of the fluctuation-dissipation theorem that now becomes

$$iR(\mathbf{0}) = -\frac{\beta}{4} \hbar\Sigma^{\mathbf{K}}(\mathbf{0}) \left(\varepsilon'(\mathbf{0}) - \mu + \frac{1}{V} T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) |\phi_{\mathbf{0}}|^2 \right) \quad (150)$$

in the degenerate regime of interest, we find that the Fokker-Planck equation for a $U(1)$ invariant probability distribution $P[|\phi_{\mathbf{0}}|; t]$ is

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} P[|\phi_{\mathbf{0}}|; t] &= -\frac{\beta}{4} \hbar\Sigma^{\mathbf{K}}(\mathbf{0}) \frac{\partial}{\partial \phi_{\mathbf{0}}} \left(\varepsilon'(\mathbf{0}) - \mu + \frac{1}{V} T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) |\phi_{\mathbf{0}}|^2 \right) \phi_{\mathbf{0}} P[|\phi_{\mathbf{0}}|; t] \\ &\quad -\frac{\beta}{4} \hbar\Sigma^{\mathbf{K}}(\mathbf{0}) \frac{\partial}{\partial \phi_{\mathbf{0}}^*} \left(\varepsilon'(\mathbf{0}) - \mu + \frac{1}{V} T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) |\phi_{\mathbf{0}}|^2 \right) \phi_{\mathbf{0}}^* P[|\phi_{\mathbf{0}}|; t] \\ &\quad -\frac{1}{2} \hbar\Sigma^{\mathbf{K}}(\mathbf{0}) \frac{\partial^2}{\partial \phi_{\mathbf{0}}^* \partial \phi_{\mathbf{0}}} P[|\phi_{\mathbf{0}}|; t]. \end{aligned} \quad (151)$$

The stationary solution therefore obeys

$$P[|\phi_{\mathbf{0}}|; \infty] \propto \exp \left\{ -\beta \left(\varepsilon'(\mathbf{0}) - \mu + \frac{T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0)}{2V} |\phi_{\mathbf{0}}|^2 \right) |\phi_{\mathbf{0}}|^2 \right\},$$

and shows explicitly that the nonlinear terms in the effective action $S^{\text{eff}}[\phi^*, \phi; \xi^*, \xi]$ have indeed stabilized the Landau free energy for the condensate $F(|\phi_{\mathbf{0}}|) = (\varepsilon'(\mathbf{0}) - \mu) |\phi_{\mathbf{0}}|^2 + T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) |\phi_{\mathbf{0}}|^4/2V$ in the case that $\varepsilon'(\mathbf{0}) < \mu$. Note that this result also shows why the nonlinearities can be neglected in the stable regime when $\varepsilon'(\mathbf{0}) > \mu$, because the most important feature of this stationary solution is that if we consider the probability distribution for the condensate density $n_0 = V |\phi_{\mathbf{0}}|^2$, i.e.,

$$P[n_0; \infty] \propto \exp \left\{ -\beta V \left(\varepsilon'(\mathbf{0}) - \mu + \frac{T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0)}{2} n_0 \right) n_0 \right\},$$

it is for a large volume V very sharply peaked around zero if $\varepsilon'(\mathbf{0}) > \mu$. On the other hand, if $\varepsilon'(\mathbf{0}) < \mu$ it is very sharply peaked around the semiclassical solution $n_0 = (\mu - \varepsilon'(\mathbf{0}))/T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0)$. This represents the phenomenon of spontaneous symmetry breaking in the context of Bose-Einstein condensation.

To extract the time scale for the probability distribution to relax to equilibrium, it is convenient to realize that Eq. (151) is equivalent to the well-known Fokker-Planck equation for a single-mode laser.⁴⁵ Indeed, introducing the dimensionless time variable $\tau = t(i\Sigma^K(\mathbf{0})/8)(2\beta T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0)/V)^{1/2}$ and in addition the dimensionless quantities $I = |\phi_0|^2 (2\beta T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0)/V)^{1/2}$ and $a = 2\beta(\mu - \varepsilon'(\mathbf{0}))(V/2\beta T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0))^{1/2}$, that in the laser theory correspond respectively to the “laser intensity” and the so-called “pump parameter”, it acquires the standard form

$$\frac{\partial}{\partial \tau} P[I; \tau] = \frac{\partial}{\partial I} \left\{ 2(I - a)I + 4I \frac{\partial}{\partial I} \right\} P[I; \tau] \quad (152)$$

that has been studied numerically by Risken.⁹⁷ From the linear part of this equation we observe that initially the average “intensity” behaves as

$$\langle I \rangle(\tau) \simeq \frac{2}{a} (e^{2a\tau} - 1). \quad (153)$$

Hence, $\langle I \rangle(\tau)$ is of the order of its equilibrium value, i.e., of $\mathcal{O}(a)$, when τ is of $\mathcal{O}(\ln(a)/a)$. The time it takes to form a condensate is therefore at least of $\mathcal{O}(\tau_{el} \ln(L/\Lambda_c))$ and again diverges for $L \rightarrow \infty$. At this point it is important to remember that the same time scale for the formation of a condensate was obtained in Sec. III from the quantum Boltzmann equation. This is not surprising because by considering only the dynamics of the zero-momentum state, a condensate can only be formed by incoherent collisions. Physically this is, however, only a good approximation as long as the mean-field interaction $n_0 T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0)$ due to the condensate is small compared to the energy splitting of the one-particle states in the box or if

$$\mu - \varepsilon'(\mathbf{0}) \ll \left(\frac{2\pi}{L} \right)^2 \frac{\hbar^2}{2m}.$$

If this condition is not fulfilled, which is generally the case for a large system size, we are no longer allowed to single out the zero-momentum state since all the states with momenta $\hbar k < \hbar \sqrt{8\pi n_0 a}$ are now coupled by

the mean-field interaction. As we have seen in Eq. (147), and in an approximate sense also in the semiclassical theory, the condensate can then also be formed in a fully coherent manner. In the language of laser theory this means that Bose-Einstein condensation is then analogous to a multi-mode laser instead of a single-mode laser described by Eq. (152).

The competition between the coherent and incoherent processes trying to form a condensate is of course contained in the Fokker-Planck equation associated with the full effective action $S^{\text{eff}}[\phi^*, \phi; \xi^*, \xi]$ given in Eq. (135). By performing a Hubbard-Stratonovich transformation or by integrating over the fluctuations and quantizing the resulting theory, we ultimately find that⁹⁸

$$\begin{aligned}
 i\hbar \frac{\partial}{\partial t} P[\phi^*, \phi; t] = & - \sum_{\mathbf{k}} \frac{\partial}{\partial \phi_{\mathbf{k}}} \left((\varepsilon(\mathbf{k}) + \hbar \Sigma^{(+)}(\mathbf{k}) - \mu(t)) \phi_{\mathbf{k}} \right. \\
 & \left. + \frac{1}{V} \sum_{\mathbf{k}', \mathbf{k}''} V^{(+)}(\mathbf{k}, \mathbf{k}', \mathbf{k}'') \phi_{\mathbf{k}'+\mathbf{k}''-\mathbf{k}}^* \phi_{\mathbf{k}'} \phi_{\mathbf{k}''} \right) P[\phi^*, \phi; t] \\
 & + \sum_{\mathbf{k}} \frac{\partial}{\partial \phi_{\mathbf{k}}^*} \left((\varepsilon(\mathbf{k}) + \hbar \Sigma^{(-)}(\mathbf{k}) - \mu(t)) \phi_{\mathbf{k}}^* \right. \\
 & \left. + \frac{1}{V} \sum_{\mathbf{k}', \mathbf{k}''} V^{(-)}(\mathbf{k}, \mathbf{k}', \mathbf{k}'') \phi_{\mathbf{k}'}^* \phi_{\mathbf{k}''}^* \phi_{\mathbf{k}'+\mathbf{k}''-\mathbf{k}} \right) P[\phi^*, \phi; t] \\
 & - \frac{1}{2} \sum_{\mathbf{k}, \mathbf{k}'} \frac{\partial^2}{\partial \phi_{\mathbf{k}} \partial \phi_{\mathbf{k}'}^*} \left(\hbar \Sigma^K(\mathbf{k}) \delta_{\mathbf{k}, \mathbf{k}'} \right. \\
 & \left. + \frac{2}{V} \sum_{\mathbf{k}''} V^K(\mathbf{k}, \mathbf{k}', \mathbf{k}'') \phi_{\mathbf{k}'+\mathbf{k}''-\mathbf{k}}^* \phi_{\mathbf{k}''} \right) P[\phi^*, \phi; t]. \quad (154)
 \end{aligned}$$

We have already seen in detail how this equation describes the first and second stages of Bose-Einstein condensation. What remains to be shown is that it in fact also describes the third kinetic stage in which the condensed and noncondensed parts of the gas come to equilibrium. To do so, we make the ansatz that the probability distribution $P[\phi^*, \phi; t]$ separates as $P_0[\phi_0^*, \phi_0; t] P_1[\phi_1^*, \phi_1; t]$. As will be clear shortly, this essentially amounts to a Hartree-Fock approximation. Substituting this ansatz in Eq. (154) and integrating over the field $\phi'(\mathbf{x})$ that is again associated with the noncondensed part of the gas, we recover Eq. (149) for the probability distribution of the condensate

$$\begin{aligned}
& i\hbar \frac{\partial}{\partial t} P_0[\phi_{\mathbf{0}}^*, \phi_{\mathbf{0}}; t] \\
&= -\frac{\partial}{\partial \phi_{\mathbf{0}}} \left(\hbar \Sigma^{(+)}(\mathbf{0}) - \mu(t) + \frac{1}{V} T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) |\phi_{\mathbf{0}}|^2 \right) \phi_{\mathbf{0}} P_0[\phi_{\mathbf{0}}^*, \phi_{\mathbf{0}}; t] \\
&+ \frac{\partial}{\partial \phi_{\mathbf{0}}^*} \left(\hbar \Sigma^{(-)}(\mathbf{0}) - \mu(t) + \frac{1}{V} T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) |\phi_{\mathbf{0}}|^2 \right) \phi_{\mathbf{0}}^* P_0[\phi_{\mathbf{0}}^*, \phi_{\mathbf{0}}; t] \\
&- \frac{1}{2} \frac{\partial^2}{\partial \phi_{\mathbf{0}}^* \partial \phi_{\mathbf{0}}} \hbar \Sigma^K(\mathbf{0}) P_0[\phi_{\mathbf{0}}^*, \phi_{\mathbf{0}}; t], \tag{155}
\end{aligned}$$

if we are careful not to double count the effects of the noncondensed part of the gas that are already included in the selfenergies $\hbar \Sigma^{(\pm), K}(\mathbf{0})$. Integrating over $\phi_{\mathbf{0}}$ instead, we also arrive at a Fokker-Planck equation for the probability distribution $P_1[\phi'^*, \phi'; t]$ that reads

$$\begin{aligned}
& i\hbar \frac{\partial}{\partial t} P_1[\phi'^*, \phi'; t] \\
&= - \sum_{\mathbf{k} \neq \mathbf{0}} \frac{\partial}{\partial \phi'_{\mathbf{k}}} ((\varepsilon(\mathbf{k}) + \hbar \Sigma^{(+)}(\mathbf{k}) + 2V^{(+)}(\mathbf{k}, \mathbf{k}, \mathbf{0}) n_0(t) \\
&- \mu(t)) \phi'_{\mathbf{k}}) P_1[\phi'^*, \phi'; t] + \sum_{\mathbf{k} \neq \mathbf{0}} \frac{\partial}{\partial \phi'^*_{\mathbf{k}}} ((\varepsilon(\mathbf{k}) + \hbar \Sigma^{(-)}(\mathbf{k}) \\
&+ 2V^{(-)}(\mathbf{k}, \mathbf{k}, \mathbf{0}) n_0(t) - \mu(t)) \phi'^*_{\mathbf{k}}) P_1[\phi'^*, \phi'; t] \\
&- \frac{1}{2} \sum_{\mathbf{k} \neq \mathbf{0}} \frac{\partial^2}{\partial \phi'^*_{\mathbf{k}} \partial \phi'_{\mathbf{k}}} (\hbar \Sigma^K(\mathbf{k}) + V^K(\mathbf{k}, \mathbf{k}, \mathbf{0}) n_0(t)) P_1[\phi'^*, \phi'; t]. \tag{156}
\end{aligned}$$

From the real part of the streaming terms we observe that in the presence of a condensate the renormalized energies of the states $\chi_{\mathbf{k}}(\mathbf{x})$ with $\mathbf{k} \neq \mathbf{0}$ are well approximated by

$$\varepsilon''(\mathbf{k}; t) = \varepsilon(\mathbf{k}) + S(\mathbf{k}; t) + 2n_0(t) T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0), \tag{157}$$

as expected from a Hartree-Fock approximation. Furthermore, the quantum Boltzmann equation that follows from the imaginary terms in the right-hand side of Eq. (156) is again of the form

$$\frac{\partial}{\partial t} N(\mathbf{k}; t) = -\Gamma^{\text{out}}(\mathbf{k}; t) N(\mathbf{k}; t) + \Gamma^{\text{in}}(\mathbf{k}; t)(1 + N(\mathbf{k}; t)), \tag{158}$$

but Γ^{out} and Γ^{in} are now given by Eqs. (141) and (142) with $N(\mathbf{k}' + \mathbf{k}'' - \mathbf{k}; t)$ replaced by $N(\mathbf{k}' + \mathbf{k}'' - \mathbf{k}; t) + n_0(t)(2\pi)^3 \delta(\mathbf{k}' + \mathbf{k}'' - \mathbf{k})$ and $\varepsilon'(\mathbf{k}; t)$ by $\varepsilon''(\mathbf{k}; t)$ to account for the macroscopic occupation of the zero-momentum state. Apart from the use of the renormalized one-particle energies in the expressions for the rates for particles to scattering out of and into the momentum state $\chi_{\mathbf{k}}(\mathbf{x})$, this kinetic equation is identical to the one studied by Semikoz and Tkachev.⁷⁶

In equilibrium the occupation numbers are thus equal to a Bose distribution evaluated at $\varepsilon(\mathbf{k}) + S(\mathbf{k}) + 2n_0 T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) - \mu$. However, for a large system size we have seen that the chemical potential is given by the Hugenholtz-Pines relation $\mu = S(\mathbf{k}) + n_0 T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0)$. Substituting this we obtain that the Bose distribution is in effect evaluated at $\hbar\omega(\mathbf{k}) = \varepsilon(\mathbf{k}) + T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) n_0$, which precisely corresponds to the high-momentum (or Hartree-Fock) limit of the expected Bogoliubov dispersion $\hbar\omega(\mathbf{k}) = (\varepsilon^2(\mathbf{k}) + 2n_0 T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) \varepsilon(\mathbf{k}))^{1/2}$. For the quantum gases of interest the thermal momenta at the critical temperature are always much larger than $\hbar \sqrt{8\pi n_0 a}$ and the Hartree-Fock approximation presented above is very accurate. It can thus safely be used to describe the third stage of the condensation process. For applications of Eq. (154) at temperatures very far below the critical temperature, it is in principle also of interest to know how we can recover the Bogoliubov approximation from our Fokker-Planck equation.⁹⁹ Such extreme low temperatures have experimentally, however, only been obtained because of the inhomogeneity of the magnetically trapped atomic gases. Therefore, we do not discuss the Bogoliubov approximation here and return to this point in the next section when we have generalized the quantum theory to the inhomogeneous case. For now we only mention that the derivation of the Bogoliubov theory is not difficult in principle but unnecessary for a homogeneous atomic Bose gas, because our previous ways of looking at the Fokker-Planck equation in Eq. (154) have already shown that it can accurately describe all the coherent and incoherent processes during Bose-Einstein condensation in that case.

V. BOSE-EINSTEIN CONDENSATION IN AN EXTERNAL POTENTIAL

Up to now we have only considered atomic Bose gases with effectively repulsive interactions, i.e., a positive scattering length a . The reason for this restriction is that a homogeneous gas with effectively attractive interactions never develops an instability towards Bose-Einstein condensation as we have seen.¹⁰⁰ Instead it develops an instability towards a BCS-like transition that for bosons is known as an Evans-Rashid transition.¹⁰¹ More

precisely, this state of affairs depends on the strength of the three-body interaction, which in principle can stabilize a condensate if it is sufficiently repulsive. Using the renormalization group approach of Ref. 81, this can easily be studied quantitatively and the result is shown here in Fig. 2. It is important to mention that the renormalization group calculation predicts the Evans-Rashid transition to take place in the mechanically stable region of the phase diagram, in contrast to the mean-field treatment.¹⁰² Note also that in general the three-body interaction must be anomalously large to be able to lead to Bose-Einstein condensation and this is therefore not expected to occur in realistic atomic gases with a negative scattering length.

Although this essentially rules out the achievement of Bose-Einstein condensation in a homogeneous gas with $a < 0$, this is no longer true if the gas is confined in an external trapping potential, as was first argued by Sackett and Hulet¹⁰³ and subsequently shown convincingly by Ruprecht, Holland, Burnett, and Edwards.¹⁰⁴ In fact, the quantum theory for Bose-Einstein condensation in a trapped gas with attractive interactions is even more simple than for a gas with repulsive interactions, because the average mean-field interaction $n |T^{(+)}(\mathbf{0}, \mathbf{0}; 0)|$ can never become much larger than the energy splitting of the trap due to the intrinsic instability of the gas to collapse to a dense phase. We therefore treat this case first and then turn to the discussion of Bose-Einstein condensation in a harmonically trapped gas with repulsive interactions, where the average mean-field interaction is

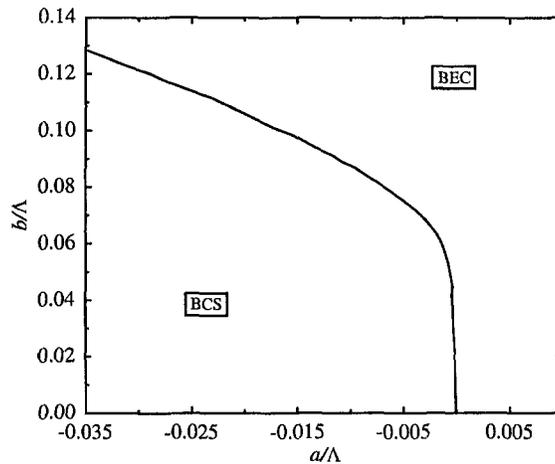


Fig. 2. Renormalization group calculation for the (meta) stable phases of a homogeneous Bose gas with two and three-body interactions. The two-body T-matrix is equal to $4\pi\hbar^2 a/m$, whereas the three-body T-matrix equals $4\pi\hbar^2 b^4/m$.

usually much larger than the energy splitting in the trap and we cannot neglect the effect of the interactions on the one-particle eigenstates in the external potential. These conditions are in particular realized in the recent experiment by H.-J. Miesner *et al.* that has resulted in the first quantitative data on the dynamics of condensate formation.⁹²

A. Weak-Coupling Limit

For a trapped atomic gas the appropriate one-particle states are no longer the momentum states $\chi_{\mathbf{k}}(\mathbf{x})$ but instead the eigenstates $\chi_{\alpha}(\mathbf{x})$ of the external potential $V^{ex}(\mathbf{x})$. As a result the Fokker-Planck equation for the interacting gas will now quite generally be given by

$$\begin{aligned}
 i\hbar \frac{\partial}{\partial t} P[\phi^*, \phi; t] = & - \sum_{\alpha, \alpha'} \frac{\partial}{\partial \phi_{\alpha}} \left\{ ((\varepsilon_{\alpha} - \mu(t)) \delta_{\alpha, \alpha'} + \hbar \Sigma_{\alpha, \alpha'}^{(+)}) \phi_{\alpha} \right. \\
 & + \left. \sum_{\beta, \beta'} V_{\alpha, \beta; \alpha', \beta'}^{(+)} \phi_{\beta}^* \phi_{\beta'} \phi_{\alpha'} \right\} P[\phi^*, \phi; t] \\
 & + \sum_{\alpha, \alpha'} \frac{\partial}{\partial \phi_{\alpha}^*} \left\{ ((\varepsilon_{\alpha} - \mu(t)) \delta_{\alpha, \alpha'} + \hbar \Sigma_{\alpha', \alpha}^{(-)}) \phi_{\alpha}^* \right. \\
 & + \left. \sum_{\beta, \beta'} V_{\alpha', \beta'; \alpha, \beta}^{(-)} \phi_{\alpha'}^* \phi_{\beta'}^* \phi_{\beta} \right\} P[\phi^*, \phi; t] \\
 & - \frac{1}{2} \sum_{\alpha, \alpha'} \frac{\partial^2}{\partial \phi_{\alpha} \partial \phi_{\alpha'}^*} \left\{ \hbar \Sigma_{\alpha, \alpha'}^K + \sum_{\beta, \beta'} V_{\alpha, \beta, \alpha', \beta'}^K \phi_{\beta}^* \phi_{\beta'} \right\} P[\phi^*, \phi; t].
 \end{aligned} \tag{159}$$

For this equation to be useful, however, we now need to determine the various selfenergies and interactions. To do so, let us for simplicity first consider the normal state of the gas which is usually in the weak-coupling limit $|\hbar \Sigma_{\alpha, \alpha'}^{(+)}| \ll |\varepsilon_{\alpha} - \varepsilon_{\alpha'}|$ for the experiments with atomic alkali gases that are of interest here. In this regime we can neglect the nondiagonal elements of the selfenergies and also the nonlinear terms in the right-hand side of Eq. (159). The Fokker-Planck equation then reduces to

$$\begin{aligned}
 i\hbar \frac{\partial}{\partial t} P[\phi^*, \phi; t] = & - \sum_{\alpha} \frac{\partial}{\partial \phi_{\alpha}} (\varepsilon_{\alpha} + \hbar \Sigma_{\alpha}^{(+)} - \mu(t)) \phi_{\alpha} P[\phi^*, \phi; t] \\
 & + \sum_{\alpha} \frac{\partial}{\partial \phi_{\alpha}^*} (\varepsilon_{\alpha} + \hbar \Sigma_{\alpha}^{(-)} - \mu(t)) \phi_{\alpha}^* P[\phi^*, \phi; t] \\
 & - \frac{1}{2} \sum_{\alpha} \frac{\partial^2}{\partial \phi_{\alpha}^* \partial \phi_{\alpha}} \hbar \Sigma_{\alpha}^K P[\phi^*, \phi; t],
 \end{aligned} \tag{160}$$

with $\hbar\Sigma_{\alpha}^{(\pm), K} = \hbar\Sigma_{\alpha, \alpha}^{(\pm), K}(\varepsilon'_{\alpha} - \mu)$ and $\varepsilon'_{\alpha} = \varepsilon_{\alpha} + S_{\alpha}$. This equation looks identical to the one we derived for our Caldeira-Leggett toy model but describes in fact quite different physics because now the selfenergies are not due to an interaction with a reservoir but due to the interactions between the atoms themselves. This is, however, completely hidden in the selfenergies as we show now explicitly. Performing the same manipulations as in the homogeneous case, we find in the many-body T-matrix approximation first of all that

$$\begin{aligned} \hbar\Sigma_{\alpha}^{(+)} &= (\hbar\Sigma_{\alpha}^{(-)})^* = 2 \sum_{\beta} V_{\alpha, \beta; \alpha, \beta}^{(+)}(\varepsilon'_{\alpha} + \varepsilon'_{\beta} - 2\mu) N_{\beta} \\ &\quad + 2i \sum_{\beta} \int \frac{d\varepsilon}{2\pi\hbar} V_{\alpha, \beta; \alpha, \beta}^{<}(\varepsilon) \frac{\hbar}{\varepsilon - (\varepsilon'_{\alpha} + \varepsilon'_{\beta} - 2\mu)} \end{aligned} \quad (161)$$

and similarly that

$$\begin{aligned} \hbar\Sigma_{\alpha}^K &= 2 \sum_{\beta} V_{\alpha, \beta; \alpha, \beta}^K(\varepsilon'_{\alpha} + \varepsilon'_{\beta} - 2\mu) N_{\beta} \\ &\quad + 2 \sum_{\beta} V_{\alpha, \beta; \alpha, \beta}^{<}(\varepsilon'_{\alpha} + \varepsilon'_{\beta} - 2\mu)(1 + 2N_{\beta}). \end{aligned} \quad (162)$$

Our next task is therefore the evaluation of the various many-body T-matrix elements $V_{\alpha, \beta; \alpha', \beta'}^{(+), <, K}(\varepsilon) = (T_{\alpha, \beta; \alpha', \beta'}^{(+), <, K}(\varepsilon) + T_{\beta, \alpha; \alpha', \beta'}^{(+), <, K}(\varepsilon))/2$ that occur in these expressions, which is easily carried out with our experience of the homogeneous gas and by noting that the Lippmann-Schwinger equation in Eq. (102) is in fact an operator equation that only needs to be expressed in a new basis. In terms of the appropriate matrix elements of the interatomic interaction, i.e.,

$$V_{\alpha, \beta; \alpha', \beta'} = \int d\mathbf{x} \int d\mathbf{x}' \chi_{\alpha}^*(\mathbf{x}) \chi_{\beta}^*(\mathbf{x}') V(\mathbf{x} - \mathbf{x}') \chi_{\alpha'}(\mathbf{x}) \chi_{\beta'}(\mathbf{x}'), \quad (163)$$

the retarded and advanced components of the many-body T-matrix thus obey the Bethe-Salpeter equation⁹³

$$T_{\alpha, \beta; \alpha', \beta'}^{(\pm)}(\varepsilon) = V_{\alpha, \beta; \alpha', \beta'} + \sum_{\alpha'', \beta''} V_{\alpha, \beta; \alpha'', \beta''} \frac{1 + N_{\alpha''} + N_{\beta''}}{\varepsilon^{\pm} - (\varepsilon'_{\alpha''} + \varepsilon'_{\beta''} - 2\mu)} T_{\alpha'', \beta''; \alpha', \beta'}^{(\pm)}(\varepsilon) \quad (164)$$

and the optical theorem

$$\begin{aligned}
 T_{\alpha, \beta; \alpha', \beta'}^{(+)}(\varepsilon) - T_{\alpha, \beta; \alpha', \beta'}^{(-)}(\varepsilon) &= -2\pi i \sum_{\alpha'', \beta''} \delta(\varepsilon - (\varepsilon'_{\alpha''} + \varepsilon'_{\beta''} - 2\mu)) \\
 &\quad \times T_{\alpha, \beta; \alpha'', \beta''}^{(+)}(\varepsilon) (1 + N_{\alpha''} + N_{\beta''}) T_{\alpha'', \beta''; \alpha', \beta'}^{(-)}(\varepsilon).
 \end{aligned} \tag{165}$$

Finally, we also need

$$\begin{aligned}
 T_{\alpha, \beta; \alpha', \beta'}^{<}(\varepsilon) &= -2\pi i \sum_{\alpha'', \beta''} \delta(\varepsilon - (\varepsilon'_{\alpha''} + \varepsilon'_{\beta''} - 2\mu)) \\
 &\quad \times T_{\alpha, \beta; \alpha'', \beta''}^{(+)}(\varepsilon) N_{\alpha''} N_{\beta''} T_{\alpha'', \beta''; \alpha', \beta'}^{(-)}(\varepsilon)
 \end{aligned} \tag{166}$$

and the Keldysh component

$$\begin{aligned}
 T_{\alpha, \beta; \alpha', \beta'}^K(\varepsilon) &= -2\pi i \sum_{\alpha'', \beta''} \delta(\varepsilon - (\varepsilon'_{\alpha''} + \varepsilon'_{\beta''} - 2\mu)) T_{\alpha, \beta; \alpha'', \beta''}^{(+)}(\varepsilon) \\
 &\quad \times (1 + N_{\alpha''} + N_{\beta''} + 2N_{\alpha''} N_{\beta''}) T_{\alpha'', \beta''; \alpha', \beta'}^{(-)}(\varepsilon).
 \end{aligned} \tag{167}$$

Notice that the appearance of a δ function in the last three equations in practice never leads to any problems, because in the experiments with trapped atomic gases the number of particles is always so large that the critical temperature of the gas is much larger than the energy splitting of the trap and the discrete sum over states is well approximated by an integral over a continuous spectrum.

Making use of the above relations we again find that for the states with thermal energies the renormalized energy is given by the pseudopotential result

$$\varepsilon'_{\alpha}(t) = \varepsilon_{\alpha} + \frac{8\pi a \hbar^2}{m} \int d\mathbf{x} \chi_{\alpha}^*(\mathbf{x}) n(\mathbf{x}, t) \chi_{\alpha}(\mathbf{x}), \tag{168}$$

with $n(\mathbf{x}, t)$ the density profile of the gas cloud. For the states near the one-particle ground state there is, however, a correction given by

$$-2 \sum_{\beta} \sum_{\alpha'', \beta''} |V_{\alpha, \beta; \alpha'', \beta''}^{(+)}(t)|^2 N_{\alpha''}(t) N_{\beta''}(t) \frac{\mathcal{P}}{\varepsilon'_{\alpha}(t) + \varepsilon'_{\beta}(t) - \varepsilon'_{\alpha''}(t) - \varepsilon'_{\beta''}(t)}$$

and $V_{\alpha, \beta; \alpha'', \beta''}^{(+)}(t) = V_{\alpha, \beta; \alpha'', \beta''}^{(+)}(\varepsilon'_{\alpha''} + \varepsilon'_{\beta''} - 2\mu(t))$. In the weak-coupling limit, the renormalization of the one-particle energies plays by definition not such

an important role and we are much more interested in the imaginary part of the retarded and advanced selfenergies. They are given by

$$R_\alpha(t) = 2\pi \sum_\beta \sum_{\alpha'', \beta''} \delta(\varepsilon'_{\alpha''}(t) + \varepsilon'_{\beta''}(t) - \varepsilon'_\alpha(t) - \varepsilon'_\beta(t)) |V_{\alpha, \beta; \alpha'', \beta''}^{(+)}(t)|^2 \\ \times [(1 + N_{\alpha''}(t) + N_{\beta''}(t)) N_\beta(t) - N_{\alpha''}(t) N_{\beta''}(t)]. \quad (169)$$

Furthermore, the Keldysh component of the selfenergies equals

$$\hbar \Sigma_\alpha^K = -4\pi \sum_\beta \sum_{\alpha'', \beta''} \delta(\varepsilon'_{\alpha''}(t) + \varepsilon'_{\beta''}(t) - \varepsilon'_\alpha(t) - \varepsilon'_\beta(t)) |V_{\alpha, \beta; \alpha'', \beta''}^{(+)}(t)|^2 \\ \times [(1 + N_{\alpha''}(t) + N_{\beta''}(t)) N_\beta(t) + N_{\alpha''}(t) N_{\beta''}(t)(1 + 2N_\beta(t))]. \quad (170)$$

As a result the quantum Boltzmann equation for the gas becomes

$$\frac{\partial}{\partial t} N_\alpha(t) = -\Gamma_\alpha^{\text{out}}(t) N_\alpha(t) + \Gamma_\alpha^{\text{in}}(t)(1 + N_\alpha(t)), \quad (171)$$

with the rate to scatter out of the state $\chi_\alpha(\mathbf{x})$ given by

$$\Gamma_\alpha^{\text{out}}(t) = \frac{4\pi}{\hbar} \sum_\beta \sum_{\alpha'', \beta''} \delta(\varepsilon'_{\alpha''}(t) + \varepsilon'_{\beta''}(t) - \varepsilon'_\alpha(t) - \varepsilon'_\beta(t)) \\ \times |V_{\alpha, \beta; \alpha'', \beta''}^{(+)}(t)|^2 (1 + N_{\alpha''}(t))(1 + N_{\beta''}(t)) N_\beta(t) \quad (172)$$

and the rate to scatter into this state by

$$\Gamma_\alpha^{\text{in}}(t) = \frac{4\pi}{\hbar} \sum_\beta \sum_{\alpha'', \beta''} \delta(\varepsilon'_{\alpha''}(t) + \varepsilon'_{\beta''}(t) - \varepsilon'_\alpha(t) - \varepsilon'_\beta(t)) \\ \times |V_{\alpha, \beta; \alpha'', \beta''}^{(+)}(t)|^2 N_{\alpha''}(t) N_{\beta''}(t)(1 + N_\beta(t)), \quad (173)$$

as expected from Fermi's Golden Rule.

The quantum Boltzmann equation found above, in combination with the expressions for the shift in the energy levels, fully describes the dynamics of the gas in the normal state. However, close to the critical temperature when the occupation numbers of the one-particle groundstate $\chi_g(\mathbf{x})$ start to become large, we are no longer allowed to neglect the non-linear terms in the Fokker-Planck equation in Eq. (159). Applying again the Hartree-Fock approximation, which is essentially exact in the weak-coupling limit as we have seen in the homogeneous case, and substituting the ansatz $P[\phi^*, \phi; t] = P_0[\phi_g^*, \phi_g; t] P_1[\phi'^*, \phi'; t]$ into Eq. (159), we obtain for the probability distribution of the condensate

$$\begin{aligned}
 & i\hbar \frac{\partial}{\partial t} P_0[\phi_g^*, \phi_g; t] \\
 &= -\frac{\partial}{\partial \phi_g} (\varepsilon_g + \hbar \Sigma_g^{(+)} - \mu(t) + T_{g, g; g, g}^{(+)}(t) |\phi_g|^2) \phi_g P_0[\phi_g^*, \phi_g; t] \\
 &+ \frac{\partial}{\partial \phi_g^*} (\varepsilon_g + \hbar \Sigma_g^{(-)} - \mu(t) + T_{g, g; g, g}^{(+)}(t) |\phi_g|^2) \phi_g^* P_0[\phi_g^*, \phi_g; t] \\
 &- \frac{1}{2} \frac{\partial^2}{\partial \phi_g^* \partial \phi_g} \hbar \Sigma_g^K P_0[\phi_g^*, \phi_g; t], \tag{174}
 \end{aligned}$$

where it should be remembered that for consistency reasons the selfenergies need to be evaluated at the energies $\varepsilon'_g(t) + T_{g, g; g, g}^{(+)}(t) |\phi_g|^2$. In equilibrium the fluctuation-dissipation theorem then reads

$$iR_\alpha = -\frac{\beta}{4} \hbar \Sigma_g^K (\varepsilon'_g - \mu + T_{g, g; g, g}^{(+)} |\phi_g|^2), \tag{175}$$

which guarantees that the probability distribution for the condensate relaxes to the correct equilibrium distribution

$$P_0[\phi_g^*, \phi_g; \infty] \propto \exp \left\{ -\beta \left(\varepsilon'_g - \mu + \frac{T_{g, g; g, g}^{(+)} |\phi_g|^2}{2} \right) |\phi_g|^2 \right\}.$$

The Fokker-Planck equation, and hence the Boltzmann equation, for the noncondensed part of the gas is the same as in the normal state because for a discrete system the contributions from the condensate to the selfenergies is already included in our expressions for $\hbar \Sigma_\alpha^{(\pm), K}$ and needs not be separated out explicitly. This implies of course, that if we evaluate the rates $\Gamma_\alpha^{\text{out}}(t)$ and $\Gamma_\alpha^{\text{in}}(t)$ by replacing the sum over states by an integral over a continuum, we first need to separate the contribution from the condensate to obtain the correct result.

In summary we have thus found that in the weak-coupling limit the dynamics of Bose-Einstein condensation in an external trapping potential is described by the coupled equations in Eq. (171) and Eq. (174), i.e., by a Boltzmann equation for the noncondensed part of the gas and a Fokker-Planck equation for the condensate, respectively. If we apply these equations to a Bose gas with a positive scattering length, we essentially arrive at the theory recently put forward by Gardiner and coworkers.¹⁰⁵ Indeed, neglecting the energy shifts and assuming that the noncondensed cloud is in

equilibrium with a chemical potential $\mu > \varepsilon_g$, our Fokker-Planck equation for a $U(1)$ invariant probability distribution of the condensate becomes

$$\begin{aligned}
i\hbar \frac{\partial}{\partial t} P_0[|\phi_g|; t] &= -\frac{\beta}{4} \hbar \Sigma_g^K \frac{\partial}{\partial \phi_g} (\varepsilon_g - \mu + T_{g, g; g, g}^{(+)} |\phi_g|^2) \phi_g P_0[|\phi_g|; t] \\
&\quad - \frac{\beta}{4} \hbar \Sigma_g^K \frac{\partial}{\partial \phi_g^*} (\varepsilon_g - \mu + T_{g, g; g, g}^{(+)} |\phi_g|^2) \phi_g^* P_0[|\phi_g|; t] \\
&\quad - \frac{1}{2} \hbar \Sigma_g^K \frac{\partial^2}{\partial \phi_g^* \partial \phi_g} P_0[|\phi_g|; t].
\end{aligned} \tag{176}$$

It is again equivalent to the Fokker-Planck equation for a single-mode laser and thus incorporates in a slightly different language exactly the same physics as the theory of Gardiner *et al.* A comparison with the experiments of H.-J. Miesner *et al.* shows that the theory of Gardiner and coworkers is unfortunately not fully satisfactory.⁹² The reason for this is presumably that these experiments are really in the strong-coupling limit, where the above theory does not apply. As mentioned previously, the theory is however applicable to gases with a negative scattering length. Since we know from our experience with homogeneous gases that the chemical potential will never be larger than ε'_g in this case, we clearly see from the equilibrium probability distribution $P_0[\phi_g^*, \phi_g; \infty]$ that a condensate in a gas with effectively attractive interactions is only metastable and will ultimately collapse to a dense state after it has overcome an energy barrier, either by thermal fluctuations or by quantum mechanical tunneling. In the next section we study the dynamics of this collapse by an extension of the method that has been devised in Ref. 106 and that is more accurate than the weak-coupling theory discussed so far. A brief review of this approach also turns out to be a useful intermediate step towards the discussion of the strong-coupling limit presented in Sec. V.D.

B. Negative Scattering Length

For definiteness we now consider an external potential in the form of an isotropic harmonic oscillator. Thus $V^{ex}(\mathbf{x}) = m\omega^2 \mathbf{x}^2/2$ and the weak-coupling limit then requires that $8\pi\hbar^2 |a| n(\mathbf{0}, t)/m \ll \hbar\omega$. During the growth of the condensate this condition is essentially always fulfilled but during the collapse, that occurs after the number of condensate particles has become too large, the density in the center of the trap quickly increases and we leave the weak-coupling regime. To study also the dynamics of the collapse requires therefore a more accurate theory. However, an important simplification occurs because the typical time scale for the collapse turns

out to be of $\mathcal{O}(1/\omega)$ which is much shorter than the time between collisions τ_{el} . Indeed, their ratio is of order

$$\frac{n(\mathbf{0}, t) T^{(+)}(\mathbf{0}, \mathbf{0}; 0)}{\hbar\omega} \left(\frac{a}{\Lambda}\right) \ll 1$$

and the dynamics of the condensate is thus collisionless. From our general Fokker-Planck equation in Eq. (159) we then see that the condensate wavefunction obeys

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \langle \phi_\alpha \rangle(t) = & \sum_{\alpha'} \left\{ ((\varepsilon_\alpha - \mu) \delta_{\alpha, \alpha'} + \text{Re}[\hbar \Sigma_{\alpha, \alpha'}^{(+)}]) \langle \phi_{\alpha'} \rangle(t) \right. \\ & \left. + \sum_{\beta, \beta'} V_{\alpha, \beta; \alpha', \beta'}^{(+)} \langle \phi_\beta^* \rangle(t) \langle \phi_{\beta'} \rangle(t) \langle \phi_{\alpha'} \rangle(t) \right\}. \end{aligned} \quad (177)$$

In coordinate space this corresponds, in a reasonable first approximation, to the expected nonlinear Schrödinger equation at nonzero temperatures

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \langle \phi(\mathbf{x}) \rangle(t) = & \left\{ -\frac{\hbar^2 \nabla^2}{2m} + V^{ex}(\mathbf{x}) - \mu + \frac{8\pi a \hbar^2}{m} n'(\mathbf{x}, t) \right. \\ & \left. + \frac{4\pi a \hbar^2}{m} |\langle \phi(\mathbf{x}) \rangle(t)|^2 \right\} \langle \phi(\mathbf{x}) \rangle(t), \end{aligned} \quad (178)$$

that was first studied by Goldman, Silvera, and Leggett¹⁰⁷ and Huse and Siggia¹⁰⁸ for a gas with positive scattering length, because these authors were at that time interested in spin-polarized atomic hydrogen. It was recently applied to atomic ⁷Li by Houbiers and Stoof¹⁰⁹ and Bergeman.¹¹⁰ Notice also that we have evaluated the sum over states by an integral over a continuum to obtain the noncondensate density $n'(\mathbf{x}, t)$ in the right-hand side of this equation.

Neglecting the variation of the noncondensate density on the size of the condensate,¹⁰⁹ we conclude that the dynamics of the collapse is, apart from an unimportant shift in the chemical potential, determined by the Gross-Pitaevskii equation¹¹¹

$$i\hbar \frac{\partial}{\partial t} \langle \phi(\mathbf{x}) \rangle(t) = \left\{ -\frac{\hbar^2 \nabla^2}{2m} + V^{ex}(\mathbf{x}) - \mu + \frac{4\pi a \hbar^2}{m} |\langle \phi(\mathbf{x}) \rangle(t)|^2 \right\} \langle \phi(\mathbf{x}) \rangle(t). \quad (179)$$

More precisely, this determines only the semiclassical dynamics. If we also want to study the quantum fluctuations, which is necessary if we are also interested in how the condensate tunnels through the macroscopic energy barrier, it is most convenient to calculate the grand canonical partition

function of the condensate.¹¹² Quantizing the Gross-Pitaevskii equation we obtain for this partition function the functional integral

$$Z_{gr}(\mu) = \int d[\psi^*] d[\psi] \exp \left\{ -\frac{1}{\hbar} S_E[\psi^*, \psi] \right\}, \quad (180)$$

over the complex field $\psi(\mathbf{x}, \tau)$ and with the Euclidian action

$$\begin{aligned} S_E[\psi^*, \psi] = & \int_0^{\hbar\beta} d\tau \int d\mathbf{x} \psi^*(\mathbf{x}, \tau) \\ & \times \left(\hbar \frac{\partial}{\partial \tau} + \frac{\hbar^2 \nabla^2}{2m} + V^{ex}(\mathbf{x}) - \mu + \frac{2\pi a \hbar^2}{m} |\psi(\mathbf{x}, \tau)|^2 \right) \psi(\mathbf{x}, \tau). \end{aligned} \quad (181)$$

As always for Bose systems, the integration is only over fields that are periodic on the imaginary time axis.

Although it has recently been shown by Freire and Arovas that the tunneling process can also be studied in terms of the complex field $\psi(\mathbf{x}, \tau)$,¹¹³ we believe that it leads to somewhat more physical insight if we use instead the fields $\rho(\mathbf{x}, \tau)$ and $\theta(\mathbf{x}, \tau)$ that correspond to the density and phase fluctuations of the condensate, respectively. They are introduced by performing the canonical variable transformation²⁴

$$\psi(\mathbf{x}, \tau) = \sqrt{\rho(\mathbf{x}, \tau)} e^{i\theta(\mathbf{x}, \tau)}$$

in the functional integral for the partition function. As a result we find

$$Z_{gr}(\mu) = \int d[\rho] d[\theta] \exp \left\{ -\frac{1}{\hbar} S_E[\rho, \theta; \mu] \right\}, \quad (182)$$

with

$$\begin{aligned} S_E[\rho, \theta; \mu] = & \int_0^{\hbar\beta} d\tau \int d\mathbf{x} \left(i\hbar \rho(\mathbf{x}, \tau) \frac{\partial \theta(\mathbf{x}, \tau)}{\partial \tau} + \frac{\hbar^2 \rho(\mathbf{x}, \tau)}{2m} (\nabla \theta(\mathbf{x}, \tau))^2 \right. \\ & + \frac{\hbar^2}{8m\rho(\mathbf{x}, \tau)} (\nabla \rho(\mathbf{x}, \tau))^2 + V^{ex}(\mathbf{x}) \rho(\mathbf{x}, \tau) \\ & \left. - \mu \rho(\mathbf{x}, \tau) + \frac{2\pi a \hbar^2}{m} \rho^2(\mathbf{x}, \tau) \right). \end{aligned} \quad (183)$$

Next, we notice that this action is only quadratic in the phase fluctuations. The field $\theta(\mathbf{x}, \tau)$ can therefore be integrated over exactly, because it only involves the evaluation of a gaussian integral.

Compared to ordinary gaussian integrals there is, however, one slight complication which is associated with the fact that $\theta(\mathbf{x}, \tau)$ are phase variables. This implies that the periodicity of the original field $\psi(\mathbf{x}, \tau)$ only constraints the phase field $\theta(\mathbf{x}, \tau)$ to be periodic up to a multiple of 2π . To evaluate the grand canonical partition function in Eq. (182) we must therefore first integrate over all fields $\theta(\mathbf{x}, \tau)$ that obey the boundary condition $\theta(\mathbf{x}, \hbar\beta) = \theta(\mathbf{x}, 0) + 2\pi j$ and subsequently sum over all possible integers j . Because these different boundary conditions only affect the zero-momentum part of $\theta(\mathbf{x}, \tau)$ we first have to evaluate the sum

$$\sum_j \int^{\theta_0(\hbar\beta) = \theta_0(0) + 2\pi j} d[\theta_0] \exp \left\{ -i \int_0^{\hbar\beta} d\tau N_0(\tau) \frac{\partial \theta_0(\tau)}{\partial \tau} \right\},$$

with $N_0(\tau) = \int d\mathbf{x} \rho(\mathbf{x}, \tau)$ the number of condensate particles. After performing a partial integration on the integral in the exponent, we can carry out the path integration over $\theta_0(\tau)$ to obtain

$$\sum_j e^{2\pi i N_0 j} \delta \left[\frac{\partial N_0(\tau)}{\partial \tau} \right].$$

As expected, the integration over the global phase of the condensate leads to the constraint of a constant number of condensate particles, i.e., $N_0(\tau) = N_0$. Moreover, we have $\sum_j e^{2\pi i N_0 j} = \sum_j \delta(N_0 - j)$, which restricts the number of condensate particles to an integer. Putting all these results together, we see that the integration over the zero-momentum part of $\rho(\mathbf{x}, \tau)$ is only a sum over the number of condensate particles and we have that

$$Z_{gr}(\mu) = \sum_{N_0} e^{\beta \mu N_0} Z_{N_0}. \quad (184)$$

Here we introduced the canonical partition function of the condensate, which is apparently equal to the functional integral

$$Z_{N_0} = \int d[\rho] d[\theta] \exp \left\{ -\frac{1}{\hbar} S_E[\rho, \theta; 0] \right\} \quad (185)$$

over all the nonzero momentum components of the density and phase fields.

The integration over the nonzero momentum components of the phase field $\theta(\mathbf{x}, \tau)$ is easily performed, because it now involves an ordinary

gaussian integral. Introducing the Green's function for the phase fluctuations $G(\mathbf{x}, \mathbf{x}'; \rho)$ by

$$\frac{\hbar}{m} ((\nabla\rho) \cdot \nabla + \rho \nabla^2) G(\mathbf{x}, \mathbf{x}'; \rho) = \delta(\mathbf{x} - \mathbf{x}'), \quad (186)$$

we immediately obtain the desired effective action for the density field

$$\begin{aligned} S_E[\rho] = & \int_0^{\hbar\beta} d\tau \int d\mathbf{x} \int d\mathbf{x}' \left(-\frac{\hbar\partial\rho(\mathbf{x}, \tau)}{2} G(\mathbf{x}, \mathbf{x}'; \rho) \frac{\partial\rho(\mathbf{x}', \tau)}{\partial\tau} \right) \\ & + \int_0^{\hbar\beta} d\tau \int d\mathbf{x} \left(\frac{\hbar^2}{8m\rho(\mathbf{x}, \tau)} (\nabla\rho(\mathbf{x}, \tau))^2 \right. \\ & \left. + V^{ex}(\mathbf{x}) \rho(\mathbf{x}, \tau) + \frac{2\pi a\hbar^2}{m} \rho^2(\mathbf{x}, \tau) \right). \end{aligned} \quad (187)$$

Being an action for the density fluctuations of the condensate, $S_E[\rho]$ also describes all the collisionless modes of the condensate. This is important for our purposes, because the mode which becomes unstable first, determines precisely how the condensate collapses. Moreover, it determines the probability with which the collapse is going to take place, both for quantum and thermal fluctuations, since the energy barrier is smallest in that direction of the configuration space. It should be noted that as long as we can neglect the interaction between the condensate and the thermal cloud, the action $S_E[\rho]$ describes also the collective modes of a gas with positive scattering length. For various other theoretical approaches that have been applied under these conditions see, for example, Refs. 114–122. The actual measurements of the collective mode frequencies have been performed by Jin *et al.*¹²³ and Mewes *et al.*¹²⁴ and are at sufficiently low temperatures indeed in good agreement with the theoretical predictions.^{125, 126} We expect the same to be true for a gas with effectively attractive interactions and, therefore, the action $S_E[\rho]$ to be a good starting point for the following discussion.

To obtain the collisionless modes explicitly we consider first the case of an ideal Bose gas by putting $a = 0$. For the ideal Bose gas we expect the gaussian profile

$$\rho(\mathbf{x}; q(\tau)) = N_0 \left(\frac{1}{\pi q^2(\tau)} \right)^{3/2} \exp \left(-\frac{\mathbf{x}^2}{q^2(\tau)} \right) \quad (188)$$

to describe an exact mode of the condensate. The reason is that in the non-interacting case we can make a density fluctuation by taking one particle

from the condensate and putting that in one of the excited states of the external potential. The corresponding density fluctuation obeys

$$\delta\rho(\mathbf{x}, t) \propto e^{-i(\varepsilon_\alpha - \varepsilon_g) t/\hbar} \chi_\alpha^*(\mathbf{x}) \chi_g(\mathbf{x}).$$

For our isotropic harmonic oscillator α combines the two angular momentum quantum numbers ℓ and m with the quantum number n that counts the number of nodes in the radial wavefunction $\chi_{n\ell}(x)$. The density fluctuation then becomes

$$\delta\rho(\mathbf{x}, t) \propto e^{-i(2n+\ell)\omega t} \chi_{n\ell}(x) Y_{\ell m}^*(\hat{\mathbf{x}}) \frac{e^{-x^2/2l^2}}{(\pi l^2)^{3/4}},$$

with $\varepsilon_{n\ell m} - \varepsilon_{000} = (2n + \ell) \hbar\omega$ the excitation energy and $l = (\hbar/m\omega)^{1/2}$ the size of the condensate wavefunction. Comparing this now with the expansion of the gaussian profile in Eq. (188) around the groundstate density profile, which is obtained by substituting $q(\tau) = l + \delta q(\tau)$, we find that

$$\delta\rho(\mathbf{x}, \tau) = -\sqrt{6} N_0 \frac{\delta q(\tau)}{l} \chi_{10}(x) Y_{00}^*(\hat{\mathbf{x}}) \frac{e^{-x^2/2l^2}}{(\pi l^2)^{3/4}} \quad (189)$$

has precisely the same form as a density fluctuation in which one particle is taken from the condensate and put into the harmonic oscillator state with quantum numbers $(n\ell m) = (100)$. The frequency of the so-called ‘‘breathing’’ mode described by the gaussian density profile must therefore be equal to 2ω .

To proof that this is indeed correct, we need to evaluate the effective action $S_E[\rho]$, and hence the Green’s function $G(\mathbf{x}, \mathbf{x}'; \rho)$, for a gaussian density profile. Substituting such a profile in Eq. (186) immediately leads to $G(\mathbf{x}, \mathbf{x}'; \rho) = G(\mathbf{x}, \mathbf{x}'; q)/\rho(\mathbf{x}'; q)$, with

$$\frac{\hbar}{m} \left(-\frac{2}{q^2} \mathbf{x} \cdot \nabla + \nabla^2 \right) G(\mathbf{x}, \mathbf{x}'; q) = \delta(\mathbf{x} - \mathbf{x}'). \quad (190)$$

The latter equation can be solved if we can solve the eigenvalue problem

$$\left(\nabla^2 - \frac{2x}{q^2} \frac{\partial}{\partial x} \right) \xi(\mathbf{x}) = \lambda \xi(\mathbf{x}). \quad (191)$$

This turns out to be an easy task, because substituting

$$\xi_{n\ell m}(\mathbf{x}) = \xi_{n\ell}(x) \frac{e^{x^2/2q^2}}{x} Y_{\ell m}(\hat{\mathbf{x}}) \quad (192)$$

gives essentially the radial Schrödinger equation for an isotropic harmonic oscillator with frequency $\omega_q = \hbar/mq^2$, i.e.,

$$-\frac{2m}{\hbar^2} \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m \omega_q^2 x^2 + \frac{\hbar^2 \ell(\ell+1)}{2m x^2} - \frac{3}{2} \hbar \omega_q \right) \xi_{n\ell}(x) = \lambda_{n\ell} \xi_{n\ell}(x). \quad (193)$$

The desired eigenfunctions are therefore $\xi_{n\ell m}(\mathbf{x}; q) = \varphi_{n\ell m}(\mathbf{x}) e^{x^2/2q^2}$, with $\varphi_{n\ell m}(\mathbf{x})$ the usual (properly normalized) harmonic oscillator states with the energies $(2n + \ell + 3/2) \hbar \omega_q$, and the corresponding eigenvalues are $\lambda_{n\ell}(q) = -2(2n + \ell)/q^2$. Introducing finally the “dual” eigenfunctions $\bar{\xi}_{n\ell m}(\mathbf{x}; q) \equiv \varphi_{n\ell m}^*(\mathbf{x}) e^{-x^2/2q^2}$, the Green’s function $G(\mathbf{x}, \mathbf{x}'; q)$ is given by

$$G(\mathbf{x}, \mathbf{x}'; q) = \sum'_{n\ell m} \xi_{n\ell m}(\mathbf{x}; q) \frac{m}{\hbar \lambda_{n\ell}(q)} \bar{\xi}_{n\ell m}(\mathbf{x}'; q). \quad (194)$$

Note that prime on the summation sign indicates that the sum is over all quantum numbers except $(n\ell m) = (000)$. The latter is excluded because the associated eigenfunction $\xi_{000}(\mathbf{x}; q)$ is just a constant and thus does not contribute to $G(\mathbf{x}, \mathbf{x}'; q)$, which is defined as the Green’s function for all phase fluctuations with nonvanishing momenta.

Putting all these results together, we see that the dynamics of the collective variable $q(\tau)$ is determined by the action

$$\begin{aligned} S_E[q] &= \int_0^{\hbar\beta} d\tau \left\{ \frac{3mN_0}{4} \left(\frac{dq}{d\tau} \right)^2 + N_0 \left(\frac{3\hbar^2}{4mq^2} + \frac{3}{4} m\omega^2 q^2 \right) \right\} \\ &\equiv \int_0^{\hbar\beta} d\tau \left\{ \frac{1}{2} m^* \left(\frac{dq}{d\tau} \right)^2 + V(q) \right\}, \end{aligned} \quad (195)$$

that is equivalent to the action of a particle with effective mass $m^* = 3mN_0/2$ in a potential $V(q) = N_0(3\hbar^2/4mq^2 + 3m\omega^2 q^2/4)$. As expected from our previous remarks, this potential has a minimum for $q = l$ and can be expanded near its minimum as

$$V(q) \simeq \frac{3}{2} N_0 \hbar \omega + \frac{1}{2} m^* (2\omega)^2 (\delta q)^2. \quad (196)$$

It thus confirms that the gaussian profile describes a breathing mode with frequency 2ω around an equilibrium density profile that is given by $\rho(\mathbf{x}; l) = N_0 |\chi_{000}(\mathbf{x})|^2$.

Up to now we have only considered the breathing mode of the condensate. However, the explicit expression for the Green’s function $G(\mathbf{x}, \mathbf{x}'; q)$ in Eq. (194), together with our argument for the form of the

density fluctuations in the various modes, presents us with a clue on how to obtain all the collisionless modes. It suggests that we should expand the density profile of the condensate as

$$\rho(\mathbf{x}, \tau) = \rho(\mathbf{x}; q(\tau)) + \sum'_{n\ell m} C_{n\ell m}(\tau) \frac{\bar{\xi}_{n\ell m}(\mathbf{x}; q(\tau))}{(\pi q^2(\tau))^{3/4}}, \quad (197)$$

where the sum is now over all quantum numbers except $(n\ell m) = (000)$ and $(n\ell m) = (100)$. Substituting this in the effective action $S_E[\rho]$ and expanding up to second order in the complex coefficients $C_{n\ell m}$, we find after somewhat tedious but straightforward algebra¹²⁷ that

$$\begin{aligned} S_E[\rho] \simeq S_E[q] + \sum'_{n\ell m} \int_0^{\hbar\beta} d\tau \frac{1}{4N_0} \\ \times \left(\frac{mq^2}{2n+\ell} \left| \frac{dC_{n\ell m}}{d\tau} \right|^2 + \frac{\hbar^2}{mq^2} (2n+\ell) |C_{n\ell m}|^2 \right). \end{aligned} \quad (198)$$

Neglecting then the interaction between the breathing mode and all the other modes of the condensate, we conclude that the coefficients $C_{n\ell m}$ are harmonic oscillators with the frequencies

$$\omega_{n\ell m} = (2n+\ell) \frac{\hbar}{ml^2} = (2n+\ell) \omega. \quad (199)$$

Hence, we have indeed succeeded in correctly describing also all the other collective modes of the condensate. At this point, we might wonder why we have chosen in Eq. (197) to still treat the breathing mode with the collective variable $q(\tau)$ and not with an additional coefficient $C_{100}(\tau)$ and the expansion

$$\rho(\mathbf{x}, \tau) = \rho(\mathbf{x}; l) + \sum'_{n\ell m} C_{n\ell m}(\tau) \frac{\bar{\xi}_{n\ell m}(\mathbf{x}; l)}{(\pi l^2)^{3/4}}. \quad (200)$$

Doing that is clearly also a valid procedure and results at the quadratic level in

$$\begin{aligned} S_E[\rho] \simeq \hbar\beta \left(\frac{3}{2} N_0 \hbar\omega \right) + \sum'_{n\ell m} \int_0^{\hbar\beta} d\tau \frac{1}{4N_0} \\ \times \left(\frac{ml^2}{2n+\ell} \left| \frac{dC_{n\ell m}}{d\tau} \right|^2 + \frac{\hbar^2}{ml^2} (2n+\ell) |C_{n\ell m}|^2 \right). \end{aligned} \quad (201)$$

The reason for using the former, more general, choice is related to the fact that when we include the attractive interaction, the breathing mode turns out to be the unstable mode of the condensate. It therefore plays a special role in the dynamics of the collapse, as we will see now.

Considering first again only gaussian density profiles, the action $S_E[q]$ is again that of a particle with effective mass $m^* = 3mN_0/2$ but now in the potential¹²⁸

$$V(q) = N_0 \left(\frac{3\hbar^2}{4mq^2} + \frac{3}{4} m\omega^2 q^2 - \frac{N_0}{\sqrt{2\pi}} \frac{\hbar^2 |a|}{mq^3} \right). \quad (202)$$

The physically most important feature of this potential is that it is unbounded from below, since $V(q) \rightarrow -\infty$ if $q \downarrow 0$. Hence, the condensate always has the tendency to collapse to the high-density state $\lim_{q \downarrow 0} \rho(\mathbf{x}; q) = N_0 \delta(\mathbf{x})$. However, if the number of condensate particles is sufficiently small, or more precisely if¹²⁹

$$N_0 < \frac{2\sqrt{2\pi}}{5^{5/4}} \frac{l}{|a|} \simeq 0.68 \frac{l}{|a|}, \quad (203)$$

the condensate has to overcome a macroscopic energy barrier before it can collapse. Under these conditions the condensate is therefore really metastable and can in principle be observed experimentally. The most important question in this respect is of course: How metastable is the condensate? Within the gaussian approximation this question is easily answered, because then the dynamics of the condensate is equivalent to the dynamics of a particle in an unstable potential, as we have just seen. We therefore only need to evaluate the WKB-expression for the tunneling rate¹³⁰ and compare this to the rate of decay due to thermal fluctuations by calculating also the height of the energy barrier. The outcome of this comparison for the conditions of the experiment with atomic ${}^7\text{Li}$ is presented in Ref. 131 and shows that for the relatively high temperatures $T \gg \hbar\omega/k_B$ that have been obtained thusfar²⁷ the decay by means of thermal fluctuations over the energy barrier is the dominant decay mechanism of the condensate.

More important for our purposes, however, is that sufficiently close to the maximum number of condensate particles N_m the collective decay of the condensate discussed above is always much more probable than the decay due to two and three-body collisions that lead to a spin-flip or the formation of ${}^7\text{Li}$ molecules, respectively. As a result the collapse of the condensate should be observable within the finite lifetime of the gas. In fact, on the basis of this separation of time scales we expect the condensate

to go through a number of growth and collapse cycles.^{131, 132} Theoretically this physical picture arises as follows. Starting from a gas with a number of atoms $N \gg N_m$, the initial growth of the condensate as a response to evaporative cooling is described by the Boltzmann equation in Eq. (171) and the Fokker-Planck equation for the condensate in Eq. (174). However, if the number of condensate atoms starts to come close to N_m , the condensate fluctuates over the energy barrier and collapses in a very short time of $\mathcal{O}(1/\omega)$.¹³³ During the collapse the condensate density increases rapidly and two and three-body inelastic processes quickly remove almost all the atoms from the condensate. After this has occurred the condensate grows again from the noncondensed part of the gas and a new growth and collapse cycle begins. It is only after many of these cycles that enough atoms are removed for the gas to relax to an equilibrium with a number of condensate particles that is less than N_m . This is shown quantitatively in Fig. 3 for the experimental conditions of interest.

A final issue which needs to be addressed at this point is the actual dynamics of the collapse and in particular how we must include the effect of the inelastic decay processes on this dynamics. From our Caldeira-Leggett toy model we know that the inclusion of inelastic processes in first instance means that we must replace the collisionless Gross-Pitaevskii equation for the condensate by a dissipative nonlinear Schrödinger equation with noise. Considering for simplicity only two-body relaxation since

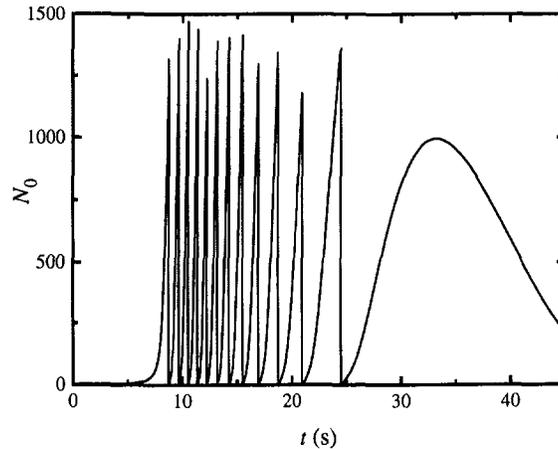


Fig. 3. Typical evolution of condensate number N_0 in response to evaporative cooling. The long-time decay of the condensate is due to two and three-body inelastic collisions.

three-body recombination processes can be treated analogously, we now have to deal with the stochastic equation

$$\left\{ i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2 \nabla^2}{2m} - V^{ex}(\mathbf{x}) + \mu - \left(\frac{4\pi a \hbar^2}{m} - i\hbar G_0 \right) |\phi(\mathbf{x}, t)|^2 \right\} \phi(\mathbf{x}, t) = \eta(\mathbf{x}, t), \quad (204)$$

where the strength of the noise $\eta(\mathbf{x}, t)$ is such that

$$\begin{aligned} \frac{dN_0(t)}{dt} &\equiv \frac{d}{dt} \int d\mathbf{x} n_0(\mathbf{x}, t) \\ &= \frac{d}{dt} \int d\mathbf{x} \langle |\phi(\mathbf{x})|^2 \rangle(t) \\ &= -2G_0 \int d\mathbf{x} n_0^2(\mathbf{x}, t). \end{aligned} \quad (205)$$

The latter equation shows that the condensate density $n_0(\mathbf{x}, t)$ obeys the expected rate equation

$$\frac{\partial}{\partial t} n_0(\mathbf{x}, t) = -2G_0 n_0^2(\mathbf{x}, t) \quad (206)$$

irrespective of the fact that for atomic ${}^7\text{Li}$ we have that $G_0 \ll 4\pi |a| \hbar/m$ and hence that the equation for $\langle \phi(\mathbf{x}) \rangle(t)$ is in a good approximation still equal to the Gross-Pitaevskii equation in Eq. (179). On the basis of these arguments, which are in fact quite similar to the ones used for the strong-coupling limit of our Caldeira-Leggett toy model, we then conclude that the dynamics of the collapse can still be studied by the effective action $S_E[\rho]$. We only need to add the constraint that

$$\frac{dN_0(t)}{dt} = -2G_0 \int d\mathbf{x} \rho^2(\mathbf{x}, t). \quad (207)$$

In the gaussian approximation this simply becomes

$$\frac{dN_0(t)}{dt} = -2G_0 \frac{1}{(2\pi q^2(t))^{3/2}} N_0^2(t), \quad (208)$$

which needs to be solved in combination with the dynamics of the condensate “width” $q(t)$ obeying Newton’s law

$$m^* \frac{d^2 q(t)}{dt^2} = -\frac{dV}{dq}(q(t)). \quad (209)$$

A numerical solution of these two coupled equations for $N_0(t)$ and $q(t)$ shows that, in contrast to the case when there is no decay,¹³³ the collapse is not complete and is finally arrested when the number of condensate particles becomes too small.¹³¹ In the gaussian approximation we find that the final number of condensate atoms is of $\mathcal{O}(1)$. Preliminary experimental results indicate that this number is presumably too small and about 1% of the condensate atoms remain after a single collapse.¹³⁴ This difference between theory and experiment is presumably due to the fact that the gaussian approximation to the density profile of the condensate overestimates the densities in the final stage of the collapse when the decay of the condensate is most severe.¹³⁵

Fortunately, our discussion of the ideal Bose gas shows how one can systematically improve on the gaussian approximation by including also the effect of the other collective variables $C_{n\ell m}(t)$ in the density expansion in Eq. (197). Of course, if one is only interested in the dynamics of the collapse it is also possible to (numerically) solve the Gross-Pitaevskii equation together with

$$\frac{dN_0(t)}{dt} = -2G_0 \int d\mathbf{x} |\langle \phi(\mathbf{x}) \rangle(t)|^4, \quad (210)$$

However, to gain more physical insight into the solution of the Gross-Pitaevskii equation it is helpful to apply the collective coordinate method mentioned above. This is in particular true because for a gas with effectively attractive interactions the breathing mode is the only unstable mode of the condensate.^{115, 119} Therefore, it appears to be an excellent approximation to the effective action $S_E[\rho]$ to expand up to quadratic terms in the coefficients $C_{n\ell m}(t)$, but to take all orders of $q(t)$ into account. Clearly such an approximation is not possible if we describe the breathing mode with an additional coefficient $C_{100}(t)$ as in Eq. (200).

To complete our discussion of the collisionless dynamics of a condensate with negative scattering length, we thus need to add to the noninteracting action in Eq. (198) the various contributions from the interaction term

$$\frac{2\pi a\hbar^2}{m} \int_0^{\hbar\beta} d\tau \int d\mathbf{x} \rho^2(\mathbf{x}, \tau)$$

in the action $S_E[\rho]$. In lowest order in the coefficients $C_{n\ell m}(\tau)$ this only adds the mean-field or Hartree term $-N_0^2 \hbar^2 |a|/\sqrt{2\pi} m q^3$ to $V(q)$ as we have already discussed in detail above. The contributions linear in $C_{n\ell m}(\tau)$ are

$$\begin{aligned} & \frac{4\pi a\hbar^2}{m} \sum_n'' \int_0^{\hbar\beta} d\tau C_{n00} \int d\mathbf{x} \rho(\mathbf{x}; q) \frac{\bar{\xi}_{n00}(\mathbf{x}; q)}{(\pi q^2)^{3/4}} \\ &= N_0 \frac{4\pi a\hbar^2}{m} \sum_n'' \int_0^{\hbar\beta} d\tau C_{n00} \frac{1}{(\pi q^2)^{3/2}} \frac{(-1)^n \sqrt{(2n+1)!}}{2^{2n+3/2} n!}. \end{aligned}$$

Note that we have only linear terms for the coefficients $C_{n00}(\tau)$. This is as expected because in the metastable state the average density profile of the condensate is given by

$$\langle \rho(\mathbf{x}) \rangle = \rho(\mathbf{x}; \langle q \rangle) + \sum_{n\ell m} \langle C_{n\ell m} \rangle \frac{\bar{\xi}_{n\ell m}(\mathbf{x}; \langle q \rangle)}{(\pi \langle q \rangle^2)^{3/4}}, \quad (211)$$

where due to the rotational symmetry of the isotropic harmonic oscillator potential the average value of the coefficients $C_{n\ell m}(\tau)$ must be zero if the angular momentum variables $(\ell m) \neq (00)$. The average value of $C_{n00}(\tau)$ can be calculated if we know the quadratic terms in $C_{n00}(\tau)$, which become

$$\frac{2\pi a\hbar^2}{m} \sum_{n, n'}'' \int_0^{\hbar\beta} d\tau C_{n00} C_{n'00} \frac{1}{(\pi q^2)^{3/2}} \frac{\Gamma(n+n'+3/2)}{\sqrt{2\pi(2n+1)! (2n'+1)!}}.$$

For the calculation of all the static properties of the condensate and in particular a more accurate value of the maximum number of atoms N_m , the latter two contributions are all that we need. However, for a study of its dynamics we require the full quadratic expression

$$\begin{aligned} & \frac{2\pi a\hbar^2}{m} \sum_{n, n'}'' \sum_{\ell m} \int_0^{\hbar\beta} d\tau C_{n\ell m}^* C_{n'\ell m} \frac{1}{(\pi q^2)^{3/2}} \int d\mathbf{x} \bar{\xi}_{n\ell m}^*(\mathbf{x}; q) \bar{\xi}_{n'\ell m}(\mathbf{x}; q) \\ &= \frac{2\pi a\hbar^2}{m} \sum_{n, n'}'' \sum_{\ell m} \int_0^{\hbar\beta} d\tau C_{n\ell m}^* C_{n'\ell m} \frac{1}{(\pi q^2)^{3/2}} \\ & \quad \times \frac{\Gamma(n+n'+\ell+3/2)}{2^{n+n'+\ell+3/2} \sqrt{n! \Gamma(n+\ell+3/2) n'! \Gamma(n'+\ell+3/2)}}. \end{aligned}$$

As an application of the above approach, we consider the decay rate of the condensate due to thermal fluctuations. This rate equals¹⁰⁶

$$\Gamma(N_0, T) = \frac{\omega_{000}}{2\pi} \exp \left\{ -\frac{\Delta V}{k_B T} \right\}, \quad (212)$$

where ΔV represents the macroscopic energy barrier for the condensate. Calculating the average $\langle C_{n\ell m} \rangle$ as a function of q and substituting the

result back into the action $S_E[\rho]$, we obtain a more accurate potential $V(q)$ that now incorporates the fact that the average density profile of the condensate is not gaussian near the point of instability. With this more accurate potential we can again calculate the frequency of the “breathing” mode ω_{000} and the macroscopic energy barrier ΔV as a function of the number of condensate particles. For the experimental conditions of interest the results are given in Fig.4. We see that the maximum number of condensate particles is $N_m \simeq 1316$ in this case, which corresponds to $N_m \simeq 0.60l/|a|$ and should be compared to the exact result $N_m \simeq 0.57l/|a|$ obtained by Ruprecht *et al.*¹⁰⁴ The difference is due to the fact that we have included in the action only terms up to quadratic order in $C_{nlm}(\tau)$. By including higher order terms we should in principle be able to reproduce the exact value of N_m , although this will be rather tedious in practice. The advantage of the present approach is that we can systematically incorporate the effect of fluctuations of the condensate, which cannot be achieved by a solution of the Gross-Pitaevskii equation. Hereby, we conclude our treatment of the dynamics of Bose-Einstein condensation in an interacting Bose gas with negative scattering length. From now on we will again focus on a gas with effectively repulsive interactions, in which case the experiments are in general in the more complicated strong-coupling limit.

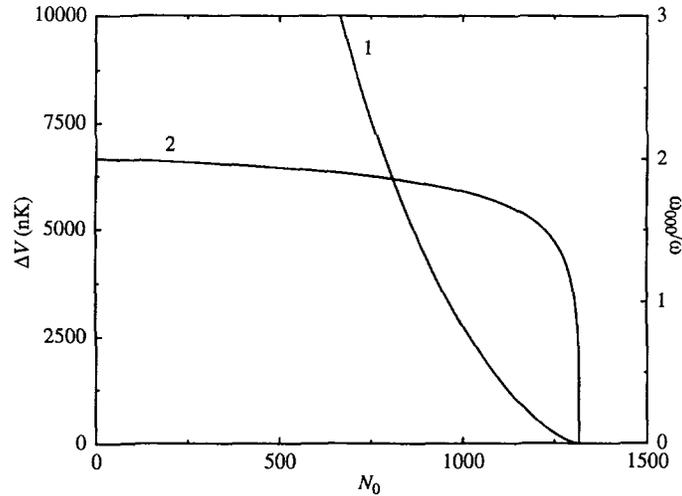


Fig. 4. Dynamical properties of a ^7Li condensate. Curve 1 gives the energy barrier ΔV for thermal fluctuations and curve 2 the frequency for the unstable breathing mode. Together these quantities determine the decay rate of the metastable condensate at the relatively high temperatures achieved in current experiments.

C. Phase “Diffusion”

As we have just seen explicitly, a particularly interesting consequence of the finite size of the gas is that quantum fluctuations play a much more important role. Although this is especially true for the case of attractive interactions, it is also true for a Bose gas with repulsive interactions. A striking example in this respect is the phenomenon of phase “diffusion,” which was recently discussed by Lewenstein and You.¹³⁶ We rederive their results for a trapped Bose gas in a moment, but first consider also the same phenomenon for a neutral and homogeneous superconductor. Not only is it possible in this manner to bring out the physics involved more clearly, but we are then also able to point out an important distinction between a fermionic and a bosonic superfluid that is also crucial for a good understanding of the concept of a quasicondensate in weakly-interacting Bose gases.²⁴

At zero temperature the dynamics of the superconducting order parameter, i.e., the BCS gap parameter $\Delta(\mathbf{x}, t)$ that is proportional to the wavefunction of the condensate of Cooper pairs, is in a good approximation determined by a time-dependent Ginzburg-Landau theory^{137–139} with the action

$$S[\Delta^*, \Delta] = \frac{N(0)}{4} \int dt \int d\mathbf{x} \times \left\{ \frac{\hbar^2}{|\Delta_0|^2} \left| \frac{\partial \Delta}{\partial t} \right|^2 - \frac{\hbar^2 v_F^2}{3 |\Delta_0|^2} |\nabla \Delta|^2 + 2 |\Delta|^2 \left(1 - \frac{|\Delta|^2}{2 |\Delta_0|^2} \right) \right\}, \quad (213)$$

where $N(0)$ is the density of states for one spin projection at the Fermi energy $\varepsilon_F = mv_F^2/2$ and Δ_0 is the equilibrium value of the order parameter.¹⁴⁰ Writing the complex order parameter in terms of an amplitude and a phase, we immediately observe that the amplitude fluctuations are gapped¹⁴¹ and can, therefore, be safely neglected at large length scales. The long-wavelength dynamics of the superconductor is thus dominated by the phase fluctuations, according to the action

$$S[\theta] = \frac{N(0)\hbar^2}{4} \int dt \int d\mathbf{x} \left\{ \left(\frac{\partial \theta}{\partial t} \right)^2 - \frac{v_F^2}{3} (\nabla \theta)^2 \right\}. \quad (214)$$

This also implies that the global phase $\theta_0(t) = \int d\mathbf{x} \theta(\mathbf{x}, t)/V$ of the superconductor has a dynamics that is governed by

$$S[\theta_0] = \frac{N(0) N \hbar^2}{4n} \int dt \left(\frac{d\theta_0}{dt} \right)^2, \quad (215)$$

using the fact that the total volume V of the system is given by N/n .

Up to now our discussion has again been semiclassical. To consider also the quantum fluctuations, we have to quantize this theory by applying the usual rules of quantum mechanics. Doing so, we find that the wavefunction of the overall phase obeys a Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi(\theta_0; t) = -\frac{n}{N(0)N} \frac{\partial^2}{\partial \theta_0^2} \Psi(\theta_0; t), \quad (216)$$

with a “diffusion” constant that can easily be shown to be equal to $(2/\hbar) \partial \varepsilon_F / \partial N$ (Ref. 142) and is, most importantly for our purposes, proportional to $1/N$. In the thermodynamic limit $N \rightarrow \infty$ a state with a well defined stationary phase is clearly a solution and we are then dealing with a system having a spontaneously broken $U(1)$ symmetry. However, for a finite (and fixed) number of particles the global phase cannot be well defined at all times and always has to “diffuse” in accordance with the above Schrödinger equation. Note also that in the groundstate the phase is fully undetermined and $|\Psi(\theta_0; t)|^2 = 1/2\pi$. Maybe surprisingly, the same calculation is somewhat more complicated for a Bose gas because the amplitude fluctuations of the order parameter cannot be neglected even at the largest length scales. However, taking these amplitude fluctuations into account properly, we nevertheless arrive at an action that is equivalent to Eq. (215) and hence again leads to the phenomenon of phase “diffusion”.

We start again from the action $S_E[\rho, \theta; \mu]$ for the condensate. The difference with the previous subsection is, however, that now we are not so much interested in the dynamics of the density but in the phase dynamics instead. Therefore we now want to integrate over the density field $\rho(\mathbf{x}, \tau)$. This cannot be done exactly and we therefore here consider only the strong-coupling limit, which was also treated by Lewenstein and You.¹³⁶ In that limit we are allowed to neglect the gradient of the average density profile¹⁰⁷ and the action $S_E[\rho, \theta; \mu]$ is for the longest wavelengths well approximated by

$$S_E[\rho, \theta; \mu] = \int_0^{\hbar\beta} d\tau \int d\mathbf{x} \left(i\hbar \rho(\mathbf{x}, \tau) \frac{\partial \theta(\mathbf{x}, \tau)}{\partial \tau} + V^{ex}(\mathbf{x}) \rho(\mathbf{x}, \tau) - \mu \rho(\mathbf{x}, \tau) + \frac{2\pi a \hbar^2}{m} \rho^2(\mathbf{x}, \tau) \right). \quad (217)$$

In equilibrium the average density profile of the condensate thus obeys

$$\langle \rho(\mathbf{x}) \rangle = \frac{m}{4\pi a \hbar^2} (\mu - V^{ex}(\mathbf{x})) \Theta(\mu - V^{ex}(\mathbf{x})). \quad (218)$$

Performing now the shift $\rho(\mathbf{x}, \tau) = \langle \rho(\mathbf{x}) \rangle + \delta\rho(\mathbf{x}, \tau)$, we find for the zero-momentum part of the action¹⁴³

$$S_E[\delta N_0, \theta_0; \mu] = \hbar\beta E_0(\mu) + \int_0^{\hbar\beta} d\tau \left(i\hbar\delta N_0 \frac{d\theta_0}{d\tau} + \frac{2\pi a\hbar^2}{mV_0(\mu)} (\delta N_0)^2 \right), \quad (219)$$

where $E_0(\mu)$ and $V_0(\mu)$ correspond, respectively, to the energy and the volume of the condensate in the so-called Thomas-Fermi approximation.¹⁴⁴ Moreover, $\delta N_0(\tau) = \int d\mathbf{x} \delta\rho(\mathbf{x}, \tau)$ represents the fluctuations in the total number of condensate particles in that same approximation, implying that the density fluctuations $\delta\rho(\mathbf{x}, \tau)$ are only nonzero in that region of space where the condensate density does not vanish.

Performing now the integration over the number fluctuations $\delta N_0(\tau)$ and the usual Wick rotation to real times, we immediately see that the effective action for the global phase of the condensate has precisely the same form as in Eq. (215), i.e.,

$$S[\theta_0; \mu] = \frac{mV_0(\mu)}{8\pi a} \int dt \left(\frac{d\theta_0}{dt} \right)^2. \quad (220)$$

The appropriate “diffusion” constant is therefore equal to $2\pi a\hbar/mV_0(\mu)$, which can easily be shown to be equal to $(1/2\hbar) \partial\mu/\partial N_0$ if we make use of the fact that in the Thomas-Fermi approximation the chemical potential obeys $\mu = m\omega^2 R_{TF}^2/2$ and the radius of the condensate is given by $R_{TF} = (15a\hbar^2 N_0/m^2\omega^2)^{1/5}$.¹²⁸ Hence, the “diffusion” constant is proportional to $1/N_0^{3/5}$. Note that if the condensate were contained in a box the “diffusion” constant would be proportional to $1/N_0$ instead. It is important to note also that, in contrast to the case of a fermionic superfluid, we have to integrate over the amplitude fluctuations of the order parameter to arrive at a quadratic action for the phase fluctuations. This leads to the important conclusion that for a bosonic superfluid it is impossible to be in a state with only phase fluctuations and no density fluctuations, even at the largest length scales. A more implicit way to arrive at the same conclusion was in fact already presented in Sec. IV, since it is precisely the coupling between the amplitude and phase fluctuations that is required to understand that the dynamical exponent z is equal to 1.

D. Strong-Coupling Limit

The difficulty with the strong-coupling limit is that the appropriate one-particle states of the gas are no longer the states $\chi_\alpha(\mathbf{x})$ of the trapping potential because these states are now strongly coupled by the mean-field

interaction. For our Caldeira-Leggett model this problem could be easily resolved by introducing the states $\chi'_\alpha(\mathbf{x})$ that incorporate the average interaction with the reservoir. However, for a trapped Bose gas the same procedure would lead to an expansion into states $\chi'_\alpha(\mathbf{x}; t)$ that parametrically depend on time since the mean-field interaction depends on time also. In principle, we thus have to deal with the situation that not only the occupation numbers vary with time but also the states to which these occupation numbers belong. This results in highly nontrivial dynamics since it is *a priori* clear that an adiabatic approximation, in which we neglect the non-diagonal matrix elements $\int d\mathbf{x} \chi'^*_\alpha(\mathbf{x}; t)(\partial/\partial t) \chi'_\alpha(\mathbf{x}; t)$, is not valid, because of the lack of a separation of time scales. Notice that we were able to circumvent the above problems for a Bose gas with effectively attractive interactions due to the fact that on the one hand the growth of the condensate essentially takes place in the weak-coupling regime and that on the other hand the collapse, which certainly occurs in the strong-coupling regime, requires in first instance only knowledge of the collisionless dynamics of the condensate.

Notwithstanding the above remarks, we can make progress in the strong-coupling limit if we realize that in this limit the typical length scale on which the (noncondensate) density of the gas varies is always much larger than the trap size. We are therefore justified in performing a gradient expansion. To do so in a systematic fashion, it is convenient to express the various selfenergies and interactions in the coordinate representation, i.e., by introducing

$$\hbar\Sigma^{(\pm), K}(\mathbf{x}, \mathbf{x}'; \varepsilon) = \sum_{\alpha, \alpha'} \chi_\alpha(\mathbf{x}) \hbar\Sigma_{\alpha, \alpha'}^{(\pm), K}(\varepsilon) \chi'^*_\alpha(\mathbf{x}') \quad (221)$$

and similarly

$$V^{(\pm), K}(\mathbf{x}, \mathbf{x}'; \mathbf{y}, \mathbf{y}'; \varepsilon) = \sum_{\alpha, \alpha'} \sum_{\beta, \beta'} \chi_\alpha(\mathbf{x}) \chi_\beta(\mathbf{y}) V_{\alpha, \beta; \alpha', \beta'}^{(\pm), K}(\varepsilon) \chi'^*_\alpha(\mathbf{x}') \chi'^*_\beta(\mathbf{y}'). \quad (222)$$

In this coordinate representation, our Fokker-Planck equation acquires in all generality the form

$$\begin{aligned} & i\hbar \frac{\partial}{\partial t} P[\phi^*, \phi; t] \\ & = - \int d\mathbf{x} d\mathbf{x}' \frac{\delta}{\delta\phi(\mathbf{x})} \\ & \quad \times \left\{ \left(\left(-\frac{\hbar^2 \nabla^2}{2m} + V^{ex}(\mathbf{x}) - \mu(t) \right) \delta(\mathbf{x} - \mathbf{x}') + \hbar\Sigma^{(+)}(\mathbf{x}, \mathbf{x}') \right) \phi(\mathbf{x}') \right\} \end{aligned}$$

$$\begin{aligned}
& + \int d\mathbf{y} d\mathbf{y}' V^{(+)}(\mathbf{x}, \mathbf{x}'; \mathbf{y}, \mathbf{y}') \phi^*(\mathbf{y}) \phi(\mathbf{y}') \phi(\mathbf{x}') \Big\} P[\phi^*, \phi; t] \\
& + \int d\mathbf{x} d\mathbf{x}' \frac{\delta}{\delta\phi^*(\mathbf{x})} \\
& \times \left\{ \left(\left(-\frac{\hbar^2 \nabla^2}{2m} + V^{ex}(\mathbf{x}) - \mu(t) \right) \delta(\mathbf{x} - \mathbf{x}') + \hbar \Sigma^{(-)}(\mathbf{x}', \mathbf{x}) \right) \phi^*(\mathbf{x}') \right. \\
& + \left. \int d\mathbf{y} d\mathbf{y}' V^{(-)}(\mathbf{x}', \mathbf{x}; \mathbf{y}', \mathbf{y}) \phi^*(\mathbf{x}') \phi^*(\mathbf{y}') \phi(\mathbf{y}) \right\} P[\phi^*, \phi; t] \\
& - \frac{1}{2} \int d\mathbf{x} d\mathbf{x}' \frac{\delta^2}{\delta\phi(\mathbf{x}) \delta\phi^*(\mathbf{x}')} \\
& \times \left\{ \hbar \Sigma^K(\mathbf{x}, \mathbf{x}') + \int d\mathbf{y} d\mathbf{y}' V^K(\mathbf{x}, \mathbf{x}'; \mathbf{y}, \mathbf{y}') \phi^*(\mathbf{y}) \phi(\mathbf{y}') \right\} P[\phi^*, \phi; t].
\end{aligned} \tag{223}$$

To study its physical content and to familiarize ourselves with how to perform the desired gradient expansion, we first analyze again the normal state of the gas in which we can neglect the nonlinear terms representing the effect of the condensate.

In the normal state the condensate is absent and we are only interested in the behavior of the quantity $\langle \phi(\mathbf{x}) \phi^*(\mathbf{x}') \rangle(t)$. From the above Fokker-Planck equation we see that it obeys the equation of motion

$$\begin{aligned}
& i\hbar \frac{\partial}{\partial t} \langle \phi(\mathbf{x}) \phi^*(\mathbf{x}') \rangle(t) \\
& = \int d\mathbf{x}'' \left(\left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \mathbf{x}^2} + V^{ex}(\mathbf{x}) - \mu(t) \right) \delta(\mathbf{x} - \mathbf{x}'') + \hbar \Sigma^{(+)}(\mathbf{x}, \mathbf{x}'') \right) \\
& \quad \times \langle \phi(\mathbf{x}'') \phi^*(\mathbf{x}') \rangle(t) \\
& - \int d\mathbf{x}'' \left(\left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \mathbf{x}'^2} + V^{ex}(\mathbf{x}') - \mu(t) \right) \delta(\mathbf{x}'' - \mathbf{x}') + \hbar \Sigma^{(-)}(\mathbf{x}'', \mathbf{x}') \right) \\
& \quad \times \langle \phi(\mathbf{x}) \phi^*(\mathbf{x}'') \rangle(t) - \frac{1}{2} \hbar \Sigma^K(\mathbf{x}, \mathbf{x}').
\end{aligned} \tag{224}$$

Introducing now the Wigner distribution¹⁴⁵ by means of

$$N(\mathbf{x}, \mathbf{k}; t) + \frac{1}{2} = \int d\mathbf{x}' e^{-i\mathbf{k} \cdot \mathbf{x}'} \langle \phi(\mathbf{x} + \mathbf{x}'/2) \phi^*(\mathbf{x} - \mathbf{x}'/2) \rangle(t) \tag{225}$$

and similarly the selfenergies $\hbar\Sigma^{(\pm),K}(\mathbf{x}, \mathbf{k})$ by

$$\hbar\Sigma^{(\pm),K}(\mathbf{x}, \mathbf{k}) = \int d\mathbf{x}' e^{-i\mathbf{k}\cdot\mathbf{x}'} \hbar\Sigma^{(\pm),K}(\mathbf{x} + \mathbf{x}'/2, \mathbf{x} - \mathbf{x}'/2), \quad (226)$$

we find after substituting these relations in lowest nonvanishing order in the gradient $\partial/\partial\mathbf{x}$ the expected quantum Boltzmann equation^{40,53}

$$\begin{aligned} \frac{\partial}{\partial t} N(\mathbf{x}, \mathbf{k}; t) + \frac{1}{\hbar} \frac{\partial \varepsilon'(\mathbf{x}, \mathbf{k}; t)}{\partial \mathbf{k}} \cdot \frac{\partial}{\partial \mathbf{x}} N(\mathbf{x}, \mathbf{k}; t) - \frac{1}{\hbar} \frac{\partial \varepsilon'(\mathbf{x}, \mathbf{k}; t)}{\partial \mathbf{x}} \cdot \frac{\partial}{\partial \mathbf{k}} N(\mathbf{x}, \mathbf{k}; t) \\ = -\Gamma^{\text{out}}(\mathbf{x}, \mathbf{k}; t) N(\mathbf{x}, \mathbf{k}; t) + \Gamma^{\text{in}}(\mathbf{x}, \mathbf{k}; t)(1 + N(\mathbf{x}, \mathbf{k}; t)), \end{aligned} \quad (227)$$

where the energies of the atoms have to be determined selfconsistently from

$$\varepsilon'(\mathbf{x}, \mathbf{k}; t) = \varepsilon(\mathbf{k}) + V^{\text{ex}}(\mathbf{x}) + \text{Re}[\hbar\Sigma^{(+)}(\mathbf{x}, \mathbf{k}; \varepsilon'(\mathbf{x}, \mathbf{k}; t) - \mu(t))]. \quad (228)$$

Moreover, the expressions for the retarded energy $\hbar\Sigma^{(+)}(\mathbf{x}, \mathbf{k}; \varepsilon'(\mathbf{x}, \mathbf{k}; t) - \mu(t))$, and the collision rates $\Gamma^{\text{out}}(\mathbf{x}, \mathbf{k}; t)$ and $\Gamma^{\text{in}}(\mathbf{x}, \mathbf{k}; t)$ turn out to be identical to the ones derived for a homogeneous gas in Secs. III and IV. We only need to replace $\varepsilon'(\mathbf{k}; t)$ by $\varepsilon'(\mathbf{x}, \mathbf{k}; t)$ and $N(\mathbf{k}; t)$ by $N(\mathbf{x}, \mathbf{k}; t)$. Notice also that the same substitution has to be made in the Bethe-Salpeter equation for the many-body T-matrix, which implies that the effective interaction $V^{(+)}(\mathbf{k}, \mathbf{k}', \mathbf{k}'')$ now implicitly depends both on the position \mathbf{x} and the time t .

As an example, let us consider the gas not too close to the critical temperature. Then the effective interaction is well approximated by the two-body T-matrix and we simply have

$$\varepsilon'(\mathbf{x}, \mathbf{k}; t) = \varepsilon(\mathbf{k}) + V^{\text{ex}}(\mathbf{x}) + \frac{8\pi n(\mathbf{x}, t) \hbar^2}{m}. \quad (229)$$

As a result the equilibrium solution $N(\mathbf{x}, \mathbf{k}; \infty)$ of the quantum Boltzmann equation equals a Bose distribution evaluated at $\varepsilon(\mathbf{k}) + V^{\text{ex}}(\mathbf{x}) + 8\pi n(\mathbf{x}) \times \hbar^2/m - \mu$. The equilibrium density profile $n(\mathbf{x}) = \int d\mathbf{k} N(\mathbf{x}, \mathbf{k}; \infty)/(2\pi)^3$ thus corresponds to a local-density approximation but is indeed very accurate for the atomic gases of interest. Apart from describing the relaxation of the gas towards equilibrium, the quantum Boltzmann equation can also be used to study the collective modes of the trapped gas above the critical temperature. Using the same methods that have recently been applied to inhomogeneous Bose gases by Zaremba, Griffin, and Nikuni,¹⁴⁶

we can for example derive the hydrodynamic equations of motion the gas. Moreover, the collisionless modes are obtained from

$$\begin{aligned} \frac{\partial}{\partial t} N(\mathbf{x}, \mathbf{k}; t) + \frac{\hbar \mathbf{k}}{m} \cdot \frac{\partial}{\partial \mathbf{x}} N(\mathbf{x}, \mathbf{k}; t) \\ - \frac{1}{\hbar} \frac{\partial}{\partial \mathbf{x}} \left(V^{ex}(\mathbf{x}) + \frac{8\pi n(\mathbf{x}, t) a \hbar^2}{m} \right) \cdot \frac{\partial}{\partial \mathbf{k}} N(\mathbf{x}, \mathbf{k}; t) = 0. \end{aligned} \quad (230)$$

Solving this equation is equivalent to the random-phase approximation.⁵³ More important is that the above collisionless Boltzmann equation can be shown to satisfy the Kohn theorem¹⁴⁷ by substituting the shifted equilibrium profile $N(\mathbf{x} - \mathbf{x}_{cm}(t), \mathbf{k} - (m/\hbar) d\mathbf{x}_{cm}(t)/dt; \infty)$. For an harmonic external potential the center-of-mass of the gas cloud then precisely follows Newton's law $d^2\mathbf{x}_{cm}(t)/dt^2 = -\nabla V^{ex}(\mathbf{x}_{cm}(t))$, which shows that the gas has three collisionless modes with frequencies that are equal to the three frequencies of the external trapping potential.

The full quantum Boltzmann equation in Eq. (227) also describes the relaxation of the gas towards equilibrium. It can thus be used to study forced evaporative cooling by adding an appropriate loss term to account for the escaping high-energetic atoms. In agreement with our previous picture we then expect that, if the evaporative cooling is performed slowly, the distribution function $N(\mathbf{x}, \mathbf{k}; t)$ can be well approximated by a truncated equilibrium distribution with a time-dependent temperature and chemical potential. As a result, we can use the results from Sec. III to show that in the critical regime the gas again becomes unstable towards Bose-Einstein condensation. This is the case when $\varepsilon'(\mathbf{x}, \mathbf{0}; t) < \mu(t)$ for positions near the center of the trap where the density of the gas is highest. If that happens we are no longer allowed to neglect the nonlinear terms in the Fokker-Planck equation and some corrections are necessary. Applying again the Hartee-Fock approximation, which is correct sufficiently close to the critical temperature where $k_B T \gg n_0(\mathbf{x}, t) T^{(+)}(\mathbf{0}, \mathbf{0}; 0)$, we find first of all that the quantum Boltzmann equation for the noncondensed part of the gas has the same form as in Eq. (227). We only have to add the mean-field interaction $2n_0(\mathbf{x}, t) T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0)$ due to the condensate to the renormalized energy $\varepsilon'(\mathbf{x}, \mathbf{k}; t)$ and in the same way as in the homogeneous case also account for the presence of the condensate in the scattering rates $\Gamma^{\text{out}}(\mathbf{x}, \mathbf{k}; t)$ and $\Gamma^{\text{in}}(\mathbf{x}, \mathbf{k}; t)$.

In addition, the resulting quantum Boltzmann equation for the noncondensed atoms is coupled to a Fokker-Planck equation for the condensate. After a gradient expansion the latter equals

$$\begin{aligned}
 i\hbar \frac{\partial}{\partial t} P_0[\phi_0^*, \phi_0; t] = & - \int d\mathbf{x} \frac{\delta}{\delta\phi_0(\mathbf{x})} \left(-\frac{\hbar^2 \nabla^2}{2m} + V^{ex}(\mathbf{x}) - \mu(t) + \hbar\Sigma^{(+)}(\mathbf{x}, \mathbf{0}) \right. \\
 & \left. + T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) |\phi_0(\mathbf{x})|^2 \right) \phi_0(\mathbf{x}) P_0[\phi_0^*, \phi_0; t] \\
 & + \int d\mathbf{x} \frac{\delta}{\delta\phi_0^*(\mathbf{x})} \left(-\frac{\hbar^2 \nabla^2}{2m} + V^{ex}(\mathbf{x}) - \mu(t) + \hbar\Sigma^{(-)}(\mathbf{x}, \mathbf{0}) \right. \\
 & \left. + T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) |\phi_0(\mathbf{x})|^2 \right) \phi_0^*(\mathbf{x}) P_0[\phi_0^*, \phi_0; t] \\
 & - \frac{1}{2} \int d\mathbf{x} \frac{\delta^2}{\delta\phi_0(\mathbf{x}) \delta\phi_0^*(\mathbf{x})} \hbar\Sigma^K(\mathbf{x}, \mathbf{0}) P_0[\phi_0^*, \phi_0; t], \tag{231}
 \end{aligned}$$

where $\hbar\Sigma^{(\pm), K}(\mathbf{x}, \mathbf{0})$ denotes $\hbar\Sigma^{(\pm), K}(\mathbf{x}, \mathbf{k} = \mathbf{0})$. In more detail we have for the real part of the retarded and advanced selfenergies that $S(\mathbf{x}, \mathbf{0}) = \text{Re}[\hbar\Sigma^{(\pm)}(\mathbf{x}, \mathbf{0}; \varepsilon''(\mathbf{x}, \mathbf{0}; t) - \mu(t))]$. Moreover, using our picture of the evaporative cooling of the gas, the imaginary parts follow most conveniently from the fluctuation-dissipation theorem

$$\begin{aligned}
 iR(\mathbf{x}, \mathbf{0}) = & -\frac{\beta(t)}{4} \hbar\Sigma^K(\mathbf{x}, \mathbf{0}) \\
 & \times \left(-\frac{\hbar^2 \nabla^2}{2m} + \varepsilon'(\mathbf{x}, \mathbf{0}; t) - \mu(t) + T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) |\phi_0(\mathbf{x})|^2 \right) \tag{232}
 \end{aligned}$$

where $\hbar\Sigma^K(\mathbf{x}, \mathbf{0}) = \hbar\Sigma^{(\pm), K}(\mathbf{x}, \mathbf{0}; \varepsilon''(\mathbf{x}, \mathbf{0}; t) - \mu(t))$ can in a good approximation be considered as independent of $|\phi_0(\mathbf{x})|$. Near equilibrium the Fokker-Planck equation thus simplifies to

$$\begin{aligned}
 i\hbar \frac{\partial}{\partial t} P_0[|\phi_0|; t] = & -\frac{\beta}{4} \int d\mathbf{x} \hbar\Sigma^K(\mathbf{x}, \mathbf{0}) \frac{\delta}{\delta\phi_0(\mathbf{x})} \left(-\frac{\hbar^2 \nabla^2}{2m} + \varepsilon'(\mathbf{x}, \mathbf{0}) - \mu \right. \\
 & \left. + T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) |\phi_0(\mathbf{x})|^2 \right) \phi_0(\mathbf{x}) P_0[|\phi_0|; t] \\
 & - \frac{\beta}{4} \int d\mathbf{x} \hbar\Sigma^K(\mathbf{x}, \mathbf{0}) \frac{\delta}{\delta\phi_0^*(\mathbf{x})} \left(-\frac{\hbar^2 \nabla^2}{2m} + \varepsilon'(\mathbf{x}, \mathbf{0}) - \mu \right. \\
 & \left. + T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0) |\phi_0(\mathbf{x})|^2 \right) \phi_0^*(\mathbf{x}) P_0[|\phi_0|; t] \\
 & - \frac{1}{2} \int d\mathbf{x} \hbar\Sigma^K(\mathbf{x}, \mathbf{0}) \frac{\delta^2}{\delta\phi_0(\mathbf{x}) \delta\phi_0^*(\mathbf{x})} P_0[|\phi_0|; t] \tag{233}
 \end{aligned}$$

and the probability distribution of the condensate relaxes to the expected stationary state

$$P_0[|\phi_0|; \infty] \propto \exp \left\{ -\beta \int d\mathbf{x} \phi_0^*(\mathbf{x}) \left(-\frac{\hbar^2 \nabla^2}{2m} + \varepsilon'(\mathbf{x}, \mathbf{0}) - \mu + \frac{T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0)}{2} |\phi_0(\mathbf{x})|^2 \right) \phi_0(\mathbf{x}) \right\}.$$

From this result we conclude that Eqs. (227) and (231) give an accurate description of the dynamics of Bose-Einstein condensation in the strong-coupling regime. Unfortunately, the solution of these coupled equations is not possible analytically and we therefore have to refer to a future publication for a numerical study and a comparison of this theory with experiment. We here only remark that the semiclassical approximation in this case amounts to the Thomas-Fermi result for the condensate density

$$n_0(\mathbf{x}; t) = \frac{1}{T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0)} (\mu(t) - \varepsilon'(\mathbf{x}, \mathbf{0}; t)) \Theta(\mu(t) - \varepsilon'(\mathbf{x}, \mathbf{0}; t)). \quad (234)$$

Such an approximation implicitly assumes, however, that the dominant mechanism for the formation of the condensate is by means of a coherent evolution and not by incoherent collisions. This might be expected not to be very accurate on the basis of the homogeneous result that the second mechanism only diverges logarithmically with the system size, whereas the first diverges linearly. Nevertheless it seems that only a full solution of Eqs. (227) and (231) can settle this important issue.

E. Collective Modes

Although we have up to now primarily been focussed on the dynamics of Bose-Einstein condensation itself, it is interesting that we have in fact also arrived at an accurate description of the collective modes of a condensed gas at nonzero temperatures. Since these collective modes are receiving much attention at present, we now briefly want to discuss how such a description emerges from our Fokker-Planck equation.¹⁴⁸ Since most experiments at present are performed in the collisionless limit, we consider only this limit here. It is in principle possible, however, to derive the hydrodynamic equations of motions as well.¹⁴⁶ Let us for simplicity first assume that the temperature of the gas is sufficiently far below the critical temperature that the many-body T-matrix is well approximated by the two-body T-matrix, but not so low that the Hartree-Fock approximation is no longer valid. In this

regime we have that $\hbar\Sigma^{(\pm)}(\mathbf{x}, \mathbf{k}) = 8\pi n'(\mathbf{x}, t) a\hbar^2/m$ and the dynamics of the noncondensed part of the gas is again determined by the collisionless quantum Boltzmann equation given in Eq. (230), where the total density now consists of two contributions

$$n(\mathbf{x}, t) = n_0(\mathbf{x}, t) + n'(\mathbf{x}, t) \equiv |\langle \phi(\mathbf{x}) \rangle(t)|^2 + \int \frac{d\mathbf{k}}{(2\pi)^3} N(\mathbf{x}, \mathbf{k}; t). \quad (235)$$

The dynamics of the condensate follows again from the nonlinear Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \langle \phi(\mathbf{x}) \rangle(t) = \left\{ -\frac{\hbar^2 \nabla^2}{2m} + V^{ex}(\mathbf{x}) - \mu + \frac{4\pi a\hbar^2}{m} (2n'(\mathbf{x}, t) + n_0(\mathbf{x}, t)) \right\} \langle \phi(\mathbf{x}) \rangle(t), \quad (236)$$

which can be rewritten as the continuity equation

$$\frac{\partial}{\partial t} n_0(\mathbf{x}, t) = -\nabla \cdot (\mathbf{v}_s(\mathbf{x}, t) n_0(\mathbf{x}, t)) \quad (237)$$

together with the Josephson relation for the superfluid velocity

$$\frac{\partial}{\partial t} \mathbf{v}_s(\mathbf{x}, t) = -\nabla \left(V^{ex}(\mathbf{x}) - \mu + \frac{4\pi a\hbar^2}{m} (2n'(\mathbf{x}, t) + n_0(\mathbf{x}, t)) + \frac{1}{2} m \mathbf{v}_s^2(\mathbf{x}, t) \right). \quad (238)$$

Here, we used that $\langle \phi(\mathbf{x}) \rangle(t) = \sqrt{n_0(\mathbf{x}, t)} e^{i\theta(\mathbf{x}, t)}$, $\mathbf{v}_s(\mathbf{x}, t) = \hbar \nabla \theta(\mathbf{x}, t)/m$, and have applied the Thomas-Fermi approximation, which is appropriate in the strong-coupling limit. It will presumably not come as a surprise that these coupled equations for $N(\mathbf{x}, \mathbf{k}; t)$, $n_0(\mathbf{x}, t)$, and $\mathbf{v}_s(\mathbf{x}, t)$ can be shown to fulfill the Kohn theorem exactly. This is an important constraint on any theory for the collective modes of a trapped gas, as stressed recently by Zaremba, Griffin, and Nikuni.¹⁴⁶

As we have mentioned previously, in trapped atomic gases it is possible to cool the gas even to such low temperatures that $k_B T \ll n_0(\mathbf{x}; t) \times T^{(+)}(\mathbf{0}, \mathbf{0}; 0)$. In this regime it is no longer allowed to use the Hartree-Fock approximation and we must use the Bogoliubov approximation instead. This is achieved as follows. In the Hartree-Fock approximation we have taken for the probability distribution $P[\phi^*, \phi; t]$ the separable form $P_0[\phi_\delta^*, \phi_0; t] P_1[\phi'^*, \phi'; t]$. For such an ansatz the $U(1)$ invariance of the

theory requires in fact that $P[\phi^*, \phi; t]$ is invariant under independent phase transformations on ϕ_0 and ϕ' . Clearly this ansatz is therefore too symmetric in general, since it is only required that $P[\phi^*, \phi; t]$ is invariant under simultaneous phase transformations on ϕ_0 and ϕ' . For such more general solutions the probability distributions for the condensed and noncondensed parts of the gas are then given by $P_0[\phi_0^*, \phi_0; t] \equiv \int d[\phi'^*] d[\phi'] P[\phi^*, \phi; t]$ and $P_1[\phi'^*, \phi'; t] \equiv \int d[\phi_0^*] d[\phi_0] P[\phi^*, \phi; t]$, respectively. The Fokker-Planck equations for these probability distributions can thus be found from integrating the general Fokker-Planck equation in Eq. (223) first over $\phi'(\mathbf{x})$ and subsequently over $\phi_0(\mathbf{x})$. Taking care that we do not double count effects of the interaction that are already accounted for in the various selfenergies, we find after a gradient expansion that the Fokker-Planck equation for the condensate is identical to Eq. (231). The collisionless dynamics of the condensate is therefore again determined by the nonlinear Schrödinger equation in Eq. (236) or in the Thomas-Fermi approximation by Eqs. (237) and (238). The Hartree-Fock and Bogoliubov approximations are clearly identical in this respect. This is, however, not true if we consider the dynamics of the noncondensed part of the gas.

Integrating over $\phi_0(\mathbf{x})$ we now obtain in the collisionless limit that

$$\begin{aligned}
& i\hbar \frac{\partial}{\partial t} P_1[\phi'^*, \phi'; t] \\
&= - \int d\mathbf{x} \frac{\delta}{\delta\phi'(\mathbf{x})} \left\{ \left(-\frac{\hbar^2 \nabla^2}{2m} + V^{ex}(\mathbf{x}) - \mu + \frac{8\pi a\hbar^2}{m} n(\mathbf{x}, t) \right) \phi'(\mathbf{x}) \right. \\
&\quad \left. + \frac{4\pi a\hbar^2}{m} n_0(\mathbf{x}, t) e^{2i\theta(\mathbf{x}, t)} \phi'^*(\mathbf{x}) \right\} P_1[\phi'^*, \phi'; t] \\
&\quad + \int d\mathbf{x} \frac{\delta}{\delta\phi'^*(\mathbf{x})} \left\{ \left(-\frac{\hbar^2 \nabla^2}{2m} + V^{ex}(\mathbf{x}) - \mu + \frac{8\pi a\hbar^2}{m} n(\mathbf{x}, t) \right) \phi'^*(\mathbf{x}) \right. \\
&\quad \left. + \frac{4\pi a\hbar^2}{m} n_0(\mathbf{x}, t) e^{-2i\theta(\mathbf{x}, t)} \phi'(\mathbf{x}) \right\} P_1[\phi'^*, \phi'; t]. \tag{239}
\end{aligned}$$

This implies that, in contrast to the Hartree-Fock treatment, the equation of motion for $\langle \phi'(\mathbf{x}) \phi'^*(\mathbf{x}') \rangle(t)$ is coupled to the equation of motion of the so-called anomalous average $\langle \phi'^*(\mathbf{x}) \phi'(\mathbf{x}') \rangle(t)$. In the framework of a gradient expansion, these equations can however be decoupled by performing a local Bogoliubov transformation.¹⁷ In this manner we can then

show that the Wigner distribution for the Bogoliubov quasiparticles obeys the expected collisionless Boltzmann equation

$$\frac{\partial}{\partial t} N(\mathbf{x}, \mathbf{k}; t) + \frac{\partial \omega(\mathbf{x}, \mathbf{k}; t)}{\partial \mathbf{k}} \cdot \frac{\partial}{\partial \mathbf{x}} N(\mathbf{x}, \mathbf{k}; t) - \frac{\partial \omega(\mathbf{x}, \mathbf{k}; t)}{\partial \mathbf{x}} \cdot \frac{\partial}{\partial \mathbf{k}} N(\mathbf{x}, \mathbf{k}; t) = 0, \quad (240)$$

with the dispersion given by

$$h\omega(\mathbf{x}, \mathbf{k}; t) = \sqrt{\varepsilon^2(\mathbf{k}) + \frac{8\pi a \hbar^2}{m} n_0(\mathbf{x}, t) \varepsilon(\mathbf{k}) + \hbar \mathbf{k} \cdot \mathbf{v}_s}. \quad (241)$$

Note that this shows that the distribution function $N(\mathbf{x}, \mathbf{k}; t)$ gives the number of Bogoliubov quasiparticles with momentum $\hbar \mathbf{k}$ at position \mathbf{x} in the frame in which the superfluid velocity vanishes.^{14, 89} Note also that the above derivation of the Bogoliubov theory is particle-number conserving since the probability distribution $P[\phi^*, \phi; t]$ is explicitly $U(1)$ invariant. It is therefore equivalent to the approach recently put forward by Gardiner,¹⁴⁹ and Castin and Dum.¹⁵⁰ More important for our purposes is that the coupled equations for $N(\mathbf{x}, \mathbf{k}; t)$, $n_0(\mathbf{x}, t)$, and $\mathbf{v}_s(\mathbf{x}, t)$ can again be shown to fulfill the Kohn theorem exactly. We therefore believe that they give an accurate description of the collisionless modes of a trapped Bose condensed gas, that may be able to explain the existing discrepancy between theory and experiment.

As a first step towards this goal we here consider the time-dependent Hartree-Fock theory for the collective modes of the Bose condensed gas, in which the collisionless Boltzmann equation in Eq. (230) is coupled to the nonlinear Schrödinger equation in Eq. (236). It turns out that a numerical determination of the eigenmodes of the collisionless Boltzmann equation is rather difficult. Therefore, we have recently proposed a variational approach to solve these coupled equations.¹⁵¹ Denoting the anisotropic harmonic trapping potential used in the experiments^{123, 124} by $V^{ex}(\mathbf{x}) = \sum_j m\omega_j^2 x_j^2/2$, we first of all take for the condensate wavefunction the gaussian *ansatz*

$$\langle \phi(\mathbf{x}) \rangle(t) = \sqrt{N_0} \prod_j \left(\frac{1}{\pi q_j^2(t)} \right)^{1/4} \exp \left\{ -\frac{x_j^2}{2q_j^2(t)} \left(1 - i \frac{m q_j(t)}{\hbar} \frac{dq_j(t)}{dt} \right) \right\}, \quad (242)$$

with the functions $q_j(t)$ describing the “breathing” of the condensate in three independent directions. Clearly, this is the anisotropic generalization of the gaussian *ansatz* that we used in Sec. V.B to study the stability of a condensate with negative scattering length. Although it is not immediately

obvious that such an *ansatz* is also appropriate in the strong-coupling regime of interest here, it has been shown by Perez-Garcia *et al.*¹¹⁸ that at zero temperature the assumption of a gaussian density profile nevertheless leads to rather accurate results for the frequencies of the collective modes. We therefore anticipate that at nonzero temperatures the same will be the case.

Next we also need an *ansatz* for the distribution function $N(\mathbf{x}, \mathbf{k}; t)$ of the thermal cloud. In Ref. 151 we considered the gaussian

$$N(\mathbf{x}, \mathbf{k}; t) = N' \prod_j \left(\frac{\hbar\omega_j}{k_B T \langle \alpha_j \rangle^2} \right) \prod_j \exp \left\{ -\frac{1}{k_B T \langle \alpha_j \rangle^2} \times \left[\frac{\hbar^2 \alpha_j^2(t)}{2m} \left(k_j - \frac{m x_j}{\hbar \alpha_j(t)} \frac{d\alpha_j(t)}{dt} \right)^2 + \frac{m \omega_j^2}{2} \frac{x_j^2}{\alpha_j^2(t)} \right] \right\}, \quad (243)$$

where $\alpha_j(t)$ are again three scaling parameters, $N' = N - N_0$ is the number of noncondensed particles, and $\langle \alpha_j \rangle$ denotes the equilibrium value of $\alpha_j(t)$ that in general is greater than 1 due to the repulsive interactions between the atoms. Here, we however also consider the more accurate Bose distribution

$$N(\mathbf{x}, \mathbf{k}; t) = N' \frac{1}{\zeta(3)} \prod_j \left(\frac{\hbar\omega_j}{k_B T \langle \alpha_j \rangle^2} \right) \left(\exp \left\{ \sum_j \frac{1}{k_B T \langle \alpha_j \rangle^2} \times \left[\frac{\hbar^2 \alpha_j^2(t)}{2m} \left(k_j - \frac{m x_j}{\hbar \alpha_j(t)} \frac{d\alpha_j(t)}{dt} \right)^2 + \frac{m \omega_j^2}{2} \frac{x_j^2}{\alpha_j^2(t)} \right] \right\} - 1 \right)^{-1}, \quad (244)$$

with $\zeta(3) \simeq 1.202$. In both cases the functional form of the distribution function is physically motivated by the following reasons. First, by integrating over the momentum $\hbar \mathbf{k}$ we see that the time-dependent density profile of the noncondensed cloud $n'(\mathbf{x}, t)$ is a time-independent function of $x_j/\alpha_j(t)$, as desired. Second, for such a density profile we must have that the local velocity $\hbar \langle k_j \rangle(\mathbf{x}, t)/m$ is given by $(x_j/\alpha_j(t)) d\alpha_j(t)/dt$. Third, we want the equilibrium distribution to be only anisotropic in coordinate space, and not in momentum space. This explains the appearance of the factor $1/\langle \alpha_j \rangle^2$ in the exponent.

Using the ideal gas result for the number of condensate particles and substituting the above *ansatz* for the condensate wavefunction and the Wigner distribution function into the nonlinear Schrödinger and collisionless Boltzmann equations, it is straightforward but somewhat tedious to derive the six coupled equations of motion for the “breathing” parameters $q_j(t)$ and $\alpha_j(t)$. Linearizing these equations around the equilibrium solutions $\langle q_j \rangle$ and $\langle \alpha_j \rangle$, respectively, we can then also obtain the desired

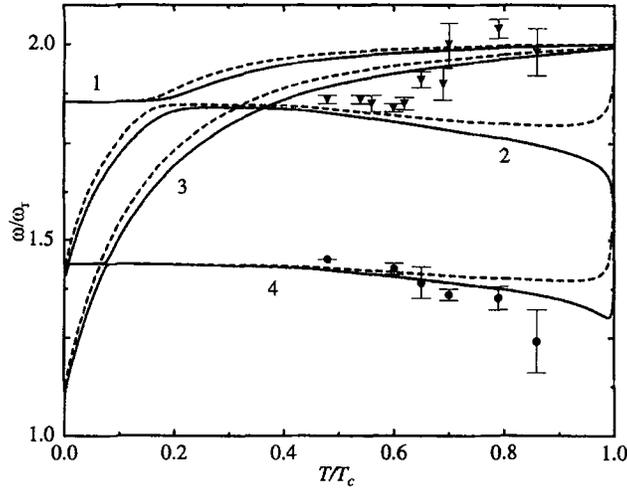


Fig. 5. Collisionless modes in a Bose condensed ^{87}Rb gas. The dashed lines give the results with a gaussian *ansatz* for the distribution function of the thermal cloud and the full lines for a Bose distribution *ansatz*. Curves 1 and 2 correspond to the $m=0$ in and out-of-phase modes, respectively. Similarly, curves 3 and 4 give the frequencies of the $m=2$ in and out-of-phase modes. The experimental data is also shown. Triangles are for a $m=0$ mode and circles for a $m=2$ mode.

eigenmodes and eigenfrequencies. The results of this calculation for the trap parameters of Jin *et al.*¹²³ are shown in Fig. 5 together with the experimental data. Note that the trap has a disk geometry with $\omega_x = \omega_y \equiv \omega_r$ and $\omega_z = \sqrt{8} \omega_r$. Due to the axial symmetry, the modes of the gas can be labeled by the usual quantum number m that describes the angular dependence around the z -axis. On the whole there appears to be a quite reasonable agreement between this simple theory and experiment. The only exceptions are the two data point almost halfway in between the in and out-of-phase $m=0$ modes. A possible explanation for this discrepancy is that experimentally both modes are excited simultaneously.¹⁵² To make sure of this, however, we need to determine the oscillator strength of the various modes. Work in this direction is in progress and will be presented in a future publication.

VI. DISCUSSION AND CONCLUSIONS

The purpose of this paper has been twofold. First, we have derived a single Fokker-Planck equation for the nonequilibrium dynamics of a weakly interacting Bose gas. This Fokker-Planck equation is capable of

describing simultaneously the coherent and incoherent processes taking place in the gas and can be used for homogeneous as well as inhomogeneous situations. It can be derived in a very natural way in the Schwinger-Keldysh formalism, to which we have therefore also tried to give an introduction. The Fokker-Planck equation is here primarily used to study the dynamics of Bose-Einstein condensation in homogeneous and inhomogeneous atomic Bose gases. In particular, by treating both the weak-coupling and strong-coupling limits of the theory, we have been able to consider not only trapped atomic Bose gases with effectively attractive but also with effectively repulsive interactions, which leads to a better understanding of the fundamental difference between the physics in these two cases. Although we have only applied the Schwinger-Keldysh formalism to atomic Bose gases here, it can also be used to study nonequilibrium problems in trapped Fermi gases. An example is the study of the timescale for the formation of Cooper pairs in atomic ${}^6\text{Li}$,¹⁵³ which due to the anomalously large and negative value of the interatomic scattering length for this species, is of considerable interest at present.

Second, and maybe most importantly, with our Fokker-Planck equation we have been able to make some theoretical progress on three topics in the field of Bose condensed atomic gases where there exists at present a discrepancy between theory and experiment. We refer here to the experiments on the formation of the condensate, on the spectrum of the collisionless modes and on the collapse of a condensate with negative scattering length. The main difficulty with the first topic is that the experiments are performed in the strong-coupling limit. In the language of quantum optics this implies that we are dealing with a multi-mode situation. Although this complicates the theory considerably, it also makes the physics involved much richer because coherent interaction effects now compete with the incoherent collisions. In this regime we may therefore hope to study in some detail the formation of topological defects in the wake of a phase transition, which turns out to be an important problem in cosmology as well.¹⁵⁴ Moreover, recent experiments have shown that by a sinusoidal modulation of the trapping potential it is possible to adiabatically cycle through the phase transition.¹⁵⁵ A theoretical description of this experiment is easily obtained within our strong-coupling formalism. We only have to replace $V^{ex}(\mathbf{x})$ by $V^{ex}(\mathbf{x}, t)$.

The second discrepancy between theory and experiment is associated with the eigenfrequencies of the collisionless modes of a Bose condensed gas at nonzero temperatures, when also a considerable density of non-condensed atoms is present in the trap. The most accurate theory that has recently been applied to this problem, is an approximate many-body T-matrix calculation in which the noncondensate part of the gas is assumed

to have no dynamics.¹⁵⁶ Due to this last feature the modes of the gas are essentially the modes of a condensate in the effective potential $V^{ex}(\mathbf{x}) + 2n'(\mathbf{x}) T^{(+)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0)$, which near the center of the trap has effectively smaller harmonic oscillator frequencies than the external trapping potential. Therefore, the Kohn theorem is not satisfied. As we have shown, this problem can be overcome if the dynamics of the noncondensed part of the gas is determined by an appropriate collisionless Boltzmann equation. Furthermore, by including the collision terms we can also find the damping of the long-wavelength modes of interest in a similar way as Kavoulakis, Pethick, and Smith have done for the normal state of the gas.¹⁵⁷

The last theoretical challenge that we have touched upon is an accurate determination of the number of atoms that remain after a single collapse of a condensate with negative scattering length. We have argued that this in principle requires the solution of the Gross-Pitaevskii equation in combination with a rate equation for the number of condensate particles. In addition, we have indicated how we can find an approximate description of the dynamics of the collapse that systematically improves upon our gaussian results. As in the two previous problems, work is in progress to numerically investigate if the theory presented in this paper is indeed sufficiently accurate to resolve the discrepancies with experiment. Although we cannot be sure about the outcome of these studies at present, we believe that the many-body T-matrix approximation used throughout this paper is an excellent starting point in all three cases, and that our Fokker-Planck approach to the dynamics of weakly interacting Bose gases presents a convenient framework in which to derive in a straightforward way also the necessary corrections if they turn out not to be negligible.

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