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J.E. Fenstad, R.O. Gandy, G.E. Sacks (Eds.)
GENERALIZED RECURSION THEORY II

THE CONTINUOUS FUNCTIONALS AND ^2E

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We give an alternative definition of C (the continuous functionals) which expresses the fact that C is the maximal type-structure which does not lead to discontinuities at type-2 (given some elementary means of relative definability). Then we solve a problem of Grilliot [3] using arguments from recursion on the continuous functionals.

§ 1. Preliminaries

1.1. We start with a definition of the continuous functionals, of natural types, using associates. $CTp(0) = \omega$.

$CTp(1) = \omega \to \omega$. $f \in CTp(1)$ has itself as an associate only.

$\alpha^F \in \omega \to \omega$ is an associate of $F^{n+1} : CTp(n) \to \omega$ if the following holds:

For every G^n and associate α^G of G^n the following conditions are satisfied:

i) $\exists n \ \alpha^F(\overline{\alpha}^G(n)) > 0$,

ii) $\forall n[\alpha^F(\overline{\alpha}^G(n)) > 0 \Rightarrow \alpha^F(\overline{\alpha}^G(n)) = F(G) + 1]$.

$CTp(n+1) = \{F^{n+1} : CTp(n) \to \omega \mid F^{n+1}$ has an associate$\}$.

We write $\leq$ for 'is recursive in' in the sense of [4], and $\leq_{RC}$ for 'is recursively countable in'. ($F \leq_{RC} G$ if there exists a recursive operator which transforms any associate of G into an associate of F).

REC is the class of recursive functionals, RC the class of recursively countable functionals.

1.2. *Definition* For all $n \geq 3$ the partial continuous function ψ^n is defined by:

$$\psi^n(F^{n-1},p) \cong q \quad \text{if} \quad \forall\alpha^F [\{p\}(\alpha^F) \cong q] \ .$$

In [1] we considered, in a different formulation, recursion relative to the ψ^n. The following theorems are easy to prove.

1.3. *Theorem* If $F^n \leq G^m, \psi^k$ then $F^n \leq_{RC} G^m$.

1.4. *Theorem* For all n : $F^{n+1} \in CTp(n+1)$ iff for no $\alpha \in CTp(1)$ ${}^2E \leq F^{n+1}, \psi^n, \alpha$.

1.5. *Theorem* F^n is recursively countable iff $F^n \leq \psi^{n-1}$.

Theorem 1.4. gives a connection between ${}^2E$ and the continuous functionals.

Remark In [1] we motivated the use of partial recursively continuous functionals like ψ^n by the possibility to define systems of recursion between REC and RC. As an example we gave: χ^3, defined by $\chi(F,\alpha,\beta) = n$ if $\forall m > CM(F,\beta)[\alpha(m) = n]$.
Here, $CM(F,\beta)$ is the modulus of continuity of F along β. But it turns out that (for total ${}^nF$) :

1.6. *Theorem* ${}^nF \leq \chi^3 \Rightarrow {}^nF \in REC$

To see this one proves that there exists a partial recursive extension χ' of χ (Using a trick like the modulus of continuity functional in [2]). At first sight, however, it seems that to compute $\chi(F,\alpha,\beta)$ one needs an associate for F.

We will now return to theorem 1.4.. An unsatisfactory aspect from the computational point of view are the ψ^n. However, we can prove a similar theorem which avoids these functionals.

§ 2. An alternative definition of the continuous functionals

We will give a definition of the continuous functionals which uses as only topological notion the continuity of a functional of type 2. Further we use a notion of relative definability which is introduced by means of a typed λ-calculus.

We consider type structures $A = \langle A_0, A_1 \ldots\rangle$ where $A_0 = \omega$ and for all $i > 0$ $A_i \subseteq A_{i-1} \to A_0$.

2.1. *Definition* of a typed λ-calculus (term system).

'Syntax' x_j^i are variables for objects of type i.

$\langle , \rangle$ is a recursive pairing function for numerals (i.e. $\langle \underline{n},\underline{m}\rangle \geq \langle \underline{n,m}\rangle$ for $n, m \in \omega$).

$(\ )_0$ and $(\ )_1$ are the corresponding projection functions

$\lambda x^i \cdot \tau$, abstraction operator; (,) brackets.

Terms - x_j^0 are terms

- if τ,ρ are terms then so are:

$\langle\tau,\rho\rangle$,

$(\tau)_0, (\tau)_1$ and

$x_i^{n+2}(\lambda x_j^n \cdot \tau)$ for $n, i, j \in \omega$.

All terms have type 0. In a straigthforward manner one defines the interpretation $A, s \models \tau$ of a term τ for a given valuation $s : var \to A$ (provided that λ-abstraction does not lead outside A, otherwise $A, s \models \tau$ does not exist).

As A is extensional $(A_{i+1} \subseteq A_i \to A_0)$ there are no difficulties in the interpretation of abstraction-terms. If s assigns the functionals $\vec{F_i}$ to the free variables of τ then we write $\tau_A(\vec{F_i})$ instead of $A, s \models \tau$ and we delete the subscript A if no confusion arises.

2.2. *Definition* $F : A_n \to \omega$ is explicitly definable from $\vec{G_i}^{n_i}$ if for some term τ :

$$\forall H^n \in A_n [F(H) = \tau_A(H, \vec{G_i}^{n_i})] \ .$$

$$(\text{i.e. } F = \lambda H \cdot \tau(H, \vec{G_i}^{n_i})) \ .$$

Remark In this case F is primitive recursive in the G_i.

Notation We write $\leq_E$ for 'explicitly definable from'.

2.3. *Definition* $A = \langle A_0,\dots,A_n\rangle$ is closed under explicit definition if every $F : A_i \to \omega (i<n)$ which is explicitly definable from a sequence $\vec{G}_i^{n_i}$ of elements of A is in fact an element of A_{i+1}.

2.4. *Lemma* For all i,j there are terms $\tau_{i,j}$ such that we may write

$$Tr_i^j(x^i) = \begin{cases} \tau_{i,j}(x^i) & \text{if } j = 0 \\ \lambda y^{j-1}\cdot\tau_{i,j}(x^i,y^{j-1}) & \text{if } j > 0 \end{cases}$$

and such that the Tr_i have the usual properties of lifting up and pushing down operators.

Proof straightforward. ▫

2.5. *Lemma* Let $\langle B_0\dots B_n\rangle$ be closed under $\leq_E$. Then there exists a unique maximal set B_{n+1} (denoted by $UME(\langle B_0,\dots,B_n\rangle)$) such that $\langle B_0,\dots,B_n,B_{n+1}\rangle$ is again closed under $\leq_E$.

Proof We take for B_{n+1} the set of all $F^{n+1} : B_n \to B_0$ which satisfy the following condition:

If $H^m \leq_E F$, $\vec{G}_i^{n_i}$ with $m, n_i < n$ then $H_m \in B_m$.

The maximality of B_{n+1} is obvious. Remains to show that $\langle B_0,\dots,B_{n+1}\rangle$ is closed under $\leq_E$.

Now suppose that $F^{i+1} = \lambda x^i\cdot\tau(x^i,\vec{G}_i^{n_i})$ $(n_i < n+1)$. We must prove $F^{i+1} \in B_{i+1}$.

There are two cases:

i) $\underline{i = n}$. Suppose $F^{n+1} \notin B_{n+1}$. Then there exist $j < n$, ρ, $\vec{H}_i^{n_i}$ $(n_i \leq n)$ such that $T^j = \lambda x^{j-1}\cdot\rho(x^{j-1},F^{n+1},\vec{H}_i^{m_i})$ is not in B_j.
Using an easy normalisation argument one finds a term σ such that $T_j = \lambda x^{j-1}\cdot\sigma(x^{j-1},\vec{H}_i^{m_i},\vec{G}_i^{n_i})$.
This reduces the case to the next case.

ii) $\underline{i < n}$. We write $\vec{G}_i^{n_i} = \vec{H}_i^{n+1}, \vec{L}_i^{m_i}$ $m_i < n$. The fact that we delete arguments of type n is justified by the possibility that we code them into an object of type $n+1$. $\langle J_1^j,\dots,J_n^j\rangle$ denotes the functional $\lambda x^{j-1}\cdot\langle J_1^j(x),\dots J_n^j(x)\rangle$. (Where $\langle a_1^0,\dots,a_n^0\rangle =$

$\langle\dots\langle\langle a_1^0,a_2^0\rangle,a_3^0\dots\rangle\dots\rangle)$.

With induction on the structure of the terms we can prove the following.

Lemma Let ρ be a term which has its free variables in $\vec{x}_i^{m_i}$ and $\vec{y}_i^{n+1}$, then there exists a functional $U_\rho \in B_n$ s.t. for all $\vec{K}_i^{m_i}$ $\rho(\vec{K}_i^{m_i},\vec{H}_i^{n+1}) = U_\rho(\langle \vec{Tr}_{m_i}^{n-1}(K_i^{m_i})\rangle)$.

Proof - If $\rho = x_j^0$ then U_ρ exists by closure under $\leq_E$.

- Let $\rho = \langle\rho_1,\rho_2\rangle$.
Then $U_\rho(x) = \langle U_{\rho_1}(x),U_{\rho_2}(x)\rangle$; similar with $(\;)_0$, $(\;)_1$.

- if $\rho = x^{j+1}(\lambda x^j\cdot\sigma(x^j,—))$ then there are again two cases: $j < n$, $j = n$.

i) $\underline{j < n}$. Then $\rho(\langle \vec{Tr}_{m_i}^{n-1}(\vec{x}_i^{m_i})\rangle)$ is found from U_σ and $\langle Tr\dots\rangle$ by means of explicit definition relative to functionals of type $\leq n$.

ii) $\underline{j = n}$. Then x^{j+1} must be one of the H_i^{n+1}.
In this case $U_\rho(A)$ is found by explicit definition relative to:
A of type $n-1$, U_σ of type n and H_i^{n+1}. This is possible by the definition of B_{n+1} ($H_i^{n+1} \in B_{n+1}$ of course) ▫

Using this lemma we see that $\lambda x^{j-1}\cdot\sigma(x^{j-1},\vec{H}_i^{m_i},\vec{G}_i^{n_i}) =$
$\lambda x^{j-1}\cdot U^n(\langle Tr_{j-1}^{n-1}(x^{j-1}),Tr_{m_i}^{n-1}(\vec{H}_i^{m_i})\rangle)$.
Hence it is in B_j as $\langle B_0,\dots,B_n\rangle$ is closed under $\leq_E$. ▫

2.6. *Definition* $A_0 = \omega = CTp(0)$

$A_1 = \omega \to \omega = CTp(1)$

$A_2 = \{F;A_1 \to A_0 | F$ is continuous$\} = CTp(2)$.

A_{n+3} $UME(\langle A_0,\dots,A_{n+2}\rangle)$ (= the unique maximal extension of $\langle A_0,\dots A_{n+2}\rangle$ which is closed under explicit definition and exists as a consequence of the previous lemma).

2.7. *Theorem* for all n $A_n = CTp(n)$.

Proof With induction on n . For $n = 0,1,2$ the situation is obvious. So we assume that

$$A_0 = CTp(0) \wedge \ldots \wedge A_{n+2} = CTp(n+2) .$$

$A_{n+3} \subseteq CTp(n+3)$: Given any $F^{n+3} \notin CTp(n+3)$ we show that $\langle A_0, \ldots A_{n+2}, B_{n+3} \rangle$ is not closed under explicit definition as soon as B_{n+3} contains F^{n+3} . To prove this we try to define an associate α^F of F .

$$\alpha^F(\sigma) = \begin{cases} k+1 & \text{if for all } G \in CTp(n+2)(=A_{n+2}) \text{ and all } \beta^G \text{ (associ-} \\ & \text{ate of } G \text{) the following implication holds:} \\ & \beta^G \text{ extends } \sigma \Rightarrow F(G) = k \\ 0 & \text{otherwise} \end{cases}$$

As F is not in $CTp(n+3)$ it has no associate. Therefore for some G^{n+2} with associate β^G we have

$$\forall m[\alpha^F(\overline{\beta}^G(m)) = 0] \quad \text{or}$$

$$\exists m[\alpha^F(\overline{\beta}^G(m)) > 0 \wedge \alpha^F(\overline{\beta}^G(m)) \neq F(G) + 1]$$

The second situation, however, is impossible therefore there exist associates β^{G_n} for functionals $G_n \in CTp(n+2)$ such that:

i) $\forall m[\overline{\beta}^{G_m}(m) = \overline{\beta}^G(m)]$ and

ii) $F(G_m) \neq F(G)$.

Now we will define a continuous D^{n+2} and a function α such that ${}^2E$ is explicitly definable from F, G, D, τ . (Now ${}^2E \notin CTp(2)$ and we are done).

$$\alpha(x) = \begin{cases} 1 & \text{if } x = F(G) \\ 0 & \text{otherwise} \end{cases}$$

$$E(\delta) = \alpha(F(\lambda H^{n+1} \cdot \tau(\delta, D, H))) .$$

Here τ and D must be chosen such that

$$\forall m[\delta(m) \neq 0] \Rightarrow \lambda H^{n+1} \cdot \tau(\delta, D, H) = G \quad \text{and}$$

$$\exists m[\delta(m) = 0] \Rightarrow \exists m \ \lambda H^{n+1} \cdot \tau(\delta, D, H) = G_m .$$

Definition: $\langle\langle \delta, H^{n+1} \rangle\rangle =$

$$\lambda x^n \cdot \langle \delta((x^n(\lambda y^{n-2} \cdot 0))_0), H^{n+1}(\lambda y^{n-1} \cdot (x^n(y^{n-1}))_1) \rangle$$

(if $n < 2$ we need a slightly different definition).

For each L^{n+1} there exist δ_L and H_L^{n+1} such that $L^{n+1} = \langle\langle \delta_L, H_L^{n+1} \rangle\rangle$

Take $\delta_L = \lambda m \cdot (L(\lambda z^{n-1} \cdot m))_0$ and

$$H_L^{n+1} = \lambda U^A \cdot (L(\lambda y^{n-1} \cdot \langle 0, U^n(y^{n-1}) \rangle)) .$$

We put $\tau(\delta, D, H) = D(\langle\langle \delta, H \rangle\rangle)$.

Now we must find a countable D such that

*) $\forall m[\delta(m) \neq 0] \Rightarrow D(\langle\langle \delta, H \rangle\rangle) = G$ and

**) $\exists m[\delta(m) = 0] \Rightarrow D(\langle\langle \delta, H \rangle\rangle) = G_m$ for some m .

We show how to compute $D(\langle\langle \delta, H \rangle\rangle)$ from δ and an associate β^H of H .

As an oracle we need a function γ which encodes β and the $\beta_i (= \beta^{G_i})$

$$\text{Let} \quad \beta_{[\delta]}(x) = \begin{cases} \beta(x) & \text{if } \forall i \leq x\, [\delta(i) \neq 0] \\ \beta_i(x) & \text{if } i = \mu j[\delta(j) = 0] \text{ otherwise} \end{cases}$$

Now $D(\langle\langle \delta, H \rangle\rangle) = CAP(\beta_{[\delta]}, \beta^H) = \beta_{[\delta]}(\overline{\beta}^H(k)) - 1$ where $k = \mu l[\beta_{[\delta]}(\overline{\beta}^H(l)) > 0]$.

It is easy to see that *) and **) are satisfied.

This shows that we can find a satisfactory countable D in $CTp(n+2)$.

$A_{n+3} \supseteq CTp(n+3)$:

To show this we note that $\langle CTp(0), \ldots, CTp(n+3) \rangle$ is closed under explicit definition as it is closed under relative $S_1, \ldots, S_9$ recursion. ∎

2.8. *Corollary* Another definition of C is as follows:

$CTp(0) = \omega$, $CTp(1) = \omega \to \omega$, $CTp(2)$ is the set of continuous elements of $Tp(2)$.

$CTp(n+1)$ = the set of all F^{n+1} such that ${}^2E$ is not

$S_1, \ldots, S_9$ recursive in F^{n+1} and any $G^n \in CTp(n)$.

Proof Immediate. ⊠

§ 3. Some properties of semirecursive sets of functionals

3.1. *Introduction* We consider a type structure A :

$$A = \langle A_0, A_1, \ldots \rangle \text{ with}$$

$$A_0 = \omega$$

$$A_{n+1} \subseteq A_n \to A_0 .$$

A subset V of A_n is recursive in functionals $\vec{F}$ from A if for some $p \in \omega$:

$${}^nG \in V \iff \{p\}({}^nG, \vec{F}) \simeq 0 \text{ and}$$

$${}^nG \notin V \iff \{p\}({}^nG, \vec{F}) \simeq 1$$

V is semirecursive (in $\vec{F}$) if for some $p \in \omega$

$${}^nG \in V \iff \{p\}({}^nG, \vec{F}) \downarrow$$

Here $\{\cdot\}(\text{——}) \simeq$ denotes Kleene's computation relation as defined in [4].

In [3] Grilliot states the following three problems concerning semi-recursive sets:

- *The union problem*: Is the union of two s.r. sets again s.r.?
- *The negation problem*: If V and V are s.r. must V be recursive?
- *The reduction problem*: If V , W are s.r. do there exist s.r. V_1 , W_1 such that $V_1 \cap W_1 = \emptyset$, $V_1 \cup W_1 = V \cup W$, $V_1 \subseteq V$ and $W_1 \subseteq W$?

We add to these problems the following one:

- *The density problem*: If V , W are s.r. such that $V \subsetneq W$ does there exist an s.r. U such that $V \subsetneq U \subsetneq W$?

Of course these problems have relativised versions and can be posed in all kinds of type structures at every type.

3.2. *Semirecursive subsets of Tp(2) (unrelativised case)*

Platek [5] (page 131) solves the negation problem negatively.

Grilliot [3] (page 16) solves the union problem negatively. He leaves open the reduction problem which we solve negatively below. We leave the density question unanswered.

3.2.1. *Theorem* There are s.r. sets $A, B \subseteq Tp(1) \times Tp(1) \times Tp(2)$ such that for no part. rec. ψ_A , ψ_B the following holds:

i) $Dom(\psi_A) \subseteq A$, $Dom(\psi_B) \subseteq B$

ii) $Dom(\psi_A) \cup Dom(\psi_B) = A \cup B$

iii) $Dom(\psi_A) \cap Dom(\psi_B) = \emptyset$

Proof We begin with some preliminaries concerning trees of computation. The presentation will be rather informal. To any computation $\{p\}(\vec{\alpha}, F)$ (which possibly diverges) we can associate a tree $Tr(p, \vec{\alpha}, F)$ which is a prefix-closed set of sequence numbers. (We take only arguments $\vec{\alpha}$ and F for the sake of easy presentation, obviously one may define $Tr(p,-)$ for all sequences of arguments - .)

$Tr(p, \vec{\alpha}, F)$ has the following properties:

i) if $\{p\}(\vec{\alpha}, F)\downarrow$ then it is (uniformly) recursive in p, $\vec{\alpha}$, F .

ii) if $\{p\}(\vec{\alpha}, F)\uparrow$ then it is (uniformly) semi-recursive in p, $\vec{\alpha}$, F .

iii) $\{p\}(\vec{\alpha}, F)\downarrow$ iff $Tr(p, \vec{\alpha}, F)$ is well-founded.

iv) the σ in $Tr(\beta, \vec{\alpha}, F)$ label subcomputations:

notation: σ labels the computation.

$$\{p_\sigma\}(\vec{m}_\sigma, \vec{\alpha}, F)$$

It is not necessary that this computation converges for σ to be in $Tr(p,-)$.

p_σ and $\vec{m}_\sigma$ are coded in σ somehow.

v) in a computation $\{p\}(\vec{\alpha}, F)$ F is only evaluated at arguments β if for some σ p_σ is an S_8-index (application) and $\{p_\sigma\}(\vec{m}_\sigma \vec{\alpha}, F) = F(\beta)$

where $\beta = \lambda x\cdot\{q\}(x,\vec{m}_\sigma,\alpha,F)$. Here $q = p_{\sigma^x}$ where σ^x is the immediate successor of σ corresponding to the subcomputation with argument x .

notation: for β : $\theta_p(\sigma,\vec{\alpha},F)$.

In fact we will only consider computations with F_0 as an argument where

$$F_0 = \lambda\ \alpha\cdot 0\ .$$

For computations of this kind we have the following continuity property:

Suppose 1) $\{p\}(\vec{\alpha},F_0)\downarrow$

2) $\sigma \in Tr(p,\vec{\alpha},F_0)$

3) p_σ is an S_8-index

4) $n \in \omega$

Then for some $l \in \omega$ the following holds:

<u>If</u> a) the α'_i extend $\bar{\alpha}_i(l)$ for $i < lth(\vec{\alpha})$ and

b) $\sigma \in Tr(p,\vec{\alpha}',F_0)$ and

c) $\theta(\sigma,\vec{\alpha}',F_0)$ is total

<u>Then</u> for all $j \leq n$

$\theta_p(\sigma,\vec{\alpha},F_0)(j) = \theta_p(\sigma,\vec{\alpha}',F_0)(j)$

(Note that we do not require $\{p\}(\vec{\alpha}',F_0)\downarrow$. The meaning of this property is that we can fix initial segments of functions occurring in $\{p\}(\vec{\alpha}',F_0)$ by fixing initial segments of $\vec{\alpha}$. The proof of it is easy and only uses the continuity of F_0).

Now we can prove the theorem.

For $\alpha \in Tp(1)$ we define $\phi(\alpha) : \omega \xrightarrow{\text{part}} \omega$ by

$$\phi(\alpha)(x) = \mu y[\alpha(\langle x,y\rangle) \neq 0]\ .$$

We extend ϕ to sequence numbers by

$$\phi(\sigma)(x) = \mu y[\langle x,y\rangle < lth(\sigma) \text{ and } \sigma_{\langle x,y\rangle} \neq 0]\ .$$

Now we define A and B by:

$$(\alpha,\beta,F) \in A \text{ iff } F(\phi(\alpha)) = 0 \text{ and}$$

$$(\alpha,\beta,F) \in B \text{ iff } F(\phi(\beta)) = 0\ .$$

Suppose that ψ_A and ψ_B satisfy the properties i) and ii) (in the formulation of the theorem).

We will construct α and β such that

$$\psi_A(\alpha,\beta,F_0)\downarrow \text{ and } \psi_B(\alpha,\beta,F_0)\downarrow\ .$$

This will contradict requirement iii).

Let a, b be indices of ψ_A, ψ_B .

3.2.2. *Lemma* There are α, β such that:

i) $\phi(\alpha)$ and $\phi(\beta)$ are total and unequal,

ii) $\phi(\alpha)$ does not occur in the comp. $\{b\}(\alpha,\beta,F_0)$,

iii) $\phi(\beta)$ does not occur in the comp. $\{a\}(\alpha,\beta,F_0)$.

The lemma is sufficient to prove the theorem. To see this consider $\psi_A(\alpha,\beta,F_0)$ where α, β are as in the lemma. If $\psi_A(\alpha,\beta,F_0)\uparrow$ then $\{a\}(\alpha,\beta,F_0)\uparrow$. Define F_0^γ by

$$F_0^\gamma(\delta) = \begin{cases} 1 & \text{if } \gamma = \delta \\ 0 & \text{otherwise} \end{cases}$$

As $\phi(\beta)$ does not occur in the comp. $\{a\}(\alpha,\beta,F_0)$ we know $\{a\}(\alpha,\beta,F_0^{\phi(\beta)})\uparrow$. But $(\alpha,\beta,F_0^{\phi(\beta)}) \notin B$. Therefore $\psi_A(\alpha,\beta,F_0^{\phi(\beta)})$ must converge as $F_0^{\phi(\beta)}(\alpha) = 0$ (by the difference of $\phi(\alpha)$ and $\phi(\beta)$. This however implies $\psi_A(\alpha,\beta,F_0)\downarrow$. Similarly one proves $\psi_B(\alpha,\beta,F_0)\downarrow$.

Proof of the lemma We use a spoiling construction to find extending sequences σ_n^α and σ_n^β of sequence numbers and take $\alpha = \lim_n \sigma_n^\alpha$ and $\beta = \lim_n \sigma_n^\beta$. We want to ensure that:

i) $\phi(\alpha)$ and $\phi(\beta)$ are total. (This is possible by the fact that each σ has an extension γ such that $\sigma(\gamma)$ is total; in fact we want $\phi(\sigma_{2n}^\alpha)(n)$ and $\phi(\sigma_{2n}^\beta)(n)$ to be defined for all n .)

ii) $\phi(\alpha) \neq \phi(\beta)$. (This is possible by choosing σ_0^α and σ_0^β such that $\phi(\sigma_0^\alpha)$ and $\phi(\sigma_0^\beta)$ are incompatible.)

iii) Let $|\sigma|$ be a code for σ in ω (In fact we may take $|\sigma| = \sigma$ but this seems notationally confusing). Then we want:

1. <u>if</u> $\tau \in Tr(a,\alpha',\beta',F_0)$ for some α',β' which extend $\sigma^{\alpha}_{2|\tau|+1}$ (and $\sigma^{\beta}_{2|\tau|+1}$) and $\theta_a(\tau,\alpha',\beta',F_0)$ is total <u>then</u> $\phi(\beta') \neq \theta_a(\tau,\alpha',\beta',F_0)$

2. a similar condition concerning $\phi(\alpha')$ and $\theta_b(\tau,\alpha',\beta',F_0)$.

Claim If the sequences σ^{α}_n and σ^{β}_n satisfy i), ii) and iii) they provide the example required by the lemma.

Proof The only non-trivial point is to verify conditions ii) and iii) of the lemma. Consider condition ii).

Suppose $\phi(\alpha)$ occurs in $\{b\}(\alpha,\beta,F_0)$. Let τ be in $Tr(b,\alpha,\beta,F_0)$ such that $\phi(\alpha) = \theta_b(\sigma,\alpha,\beta,F_0)$. Now α,β extend $\sigma^{\alpha}_{2|\tau|+1},\sigma^{\beta}_{2|\tau|+1}$. It is easy to see that this leads to a contradiction.

Finally we describe the definition of σ^{α}_n and σ^{β}_n . The steps to satisfy conditions i) and ii) above are obvious (and done in the easiest way).

We describe how to satisfy the first part of condition iii). The second part is handled similarly.

Let $n = 2|\tau| + 1$. Suppose $\sigma^{\alpha} = \sigma^{\alpha}_{n-1}$ and $\sigma^{\beta} = \sigma^{\beta}_{n-1}$ are defined.

We define two functions $D_0(\sigma^{\beta})$ and $D_1(\sigma^{\alpha})$ by

$$D_0(\sigma)(x) = \begin{cases} 0 & \text{if } \sigma(x) \text{ is undefined} \\ \sigma(x) & \text{otherwise.} \end{cases}$$

$$D_1(\sigma)(x) = \begin{cases} 1 & \text{if } \sigma(x) \text{ is undefined} \\ \sigma(x) & \text{otherwise} \end{cases}$$

$(\sigma(x) = \sigma_x)$.

Clearly $D_0(\sigma^{\beta})$ is non-total and $D_1(\sigma^{\alpha})$ is total.

Therefore $(D_1(\sigma^{\alpha}),D_0(\sigma^{\beta}),F_0) \in A\backslash B$ and hence $\{a\}(D_1(\sigma^{\alpha}),D_0(\sigma^{\beta}),F_0)\downarrow$

We consider two cases.

I) $\tau \notin Tr(a,D_1(\sigma^{\alpha}),D_0(\sigma^{\beta}),F_0)$.

In this case we choose extensions ρ^{α} and ρ^{β} of σ^{α}, σ^{β} , such that $\rho^{\alpha} < D_1(\sigma^{\alpha})$ and $\rho^{\beta} < D_0(\sigma^{\beta})$, for which the following holds:

if $\alpha' > \rho^{\alpha}$ and $\beta' > \rho^{\beta}$ then $\tau \notin Tr(a,\alpha',\beta',F_0)$.

To see that this is possible note the following:

if $\tau \in Tr(a,\alpha',\beta',F.)$ this is caused by the fact that some values computed as $\{\rho_{\tau'}\}(\vec{m}_{\tau'},\alpha',\beta',F_0)$ for τ' "before" τ lead to the computation belonging to τ . Now we can fix all values computed in subcomputations which can "lead to τ " by fixing initial segments of $D_1(\sigma^{\alpha})$, $D_0(\alpha^{\beta})$. Then we are sure that τ will not occur in computations with extensions of these initial segments. (However, this non-occurrence may as well have the reason that some subparts of the computation are divergent.)

Now we take $\sigma^{\alpha}_n = \rho^{\sigma} * \langle 0\rangle$ (to ensure that its length increases). Further we choose σ^{β}_n as an extension of ρ^{β} .

It is now obvious that the first part of condition iii) in the lemma is satisfied.

II) $\tau \in Tr(a,D_1(\sigma^{\alpha}),D_0(\sigma^{\beta}),F_0)$

Now choose $1 > lth(\sigma^{\beta})$.

Clearly $\phi(D_0(\sigma^{\beta}))(1)$ is undefined.

Now choose ρ^{α} and ρ^{β} between σ^{α} and $D_1(\sigma^{\alpha})$ (resp. between σ^{β} and $D_0(\sigma^{\beta})$) such that the following holds:

whenever $\alpha' > \rho^{\alpha}$, $\beta' > \rho^{\beta}$, $\tau \in Tr(b,\alpha',\beta',F_0)$

and $\theta_b(\tau,\alpha',\beta',F_0)$ is total then $\theta_b(\tau,\alpha',\beta',F_0)(1) \simeq h(1) \underset{\text{def.}}{\simeq} \theta_b(\tau,D_1(\sigma^\alpha),D_0(\sigma^\beta),F_0)(1)$

(This again uses the continuity property of computations with continuous arguments.)

Now we want to take σ_n^α and σ_n^β as extensions of ρ^α and ρ^β such that $\phi(\sigma_n^\alpha) \neq h$. (This gives $\phi(\alpha) \neq \phi_b(\tau,\alpha',\beta',F)$ when even both functions are total and $\alpha' > \sigma_n^\alpha$, $\beta' > \sigma_n^\beta$.
To reach this we take σ_n^α to be an extension of ρ^α for which $\phi(\sigma_n^\alpha)(1){\downarrow}$ and $\phi(\sigma_n^\alpha)(1) > h(1)$. This is easily possible as $\phi(\rho^\alpha)(1){\uparrow}$.

This ends the description of a construction of functions α and β satisfying the conditions of the lemma. ⊠

3.3. *Semi-recursive subsets of $CTp(3)$*

In [1] we proved that in $CTp(3)$ the predicate

$\{p\}(^3F){\downarrow}$ has the quantifier form $\forall\alpha\exists n\ P(p,\alpha,n,h^F)$ with recursive P .

Here h^F is the graph of F on the primitive recursive functionals. On the other hand all predicates of this form are semi-recursive. From these facts it follows that we have the reduction and union property in this case. As Platek's counterexample to the negation property already works in $CTp(2)$ the negation property does not hold now.

3.4. *Subsets of $Tp(2)$ which are semi-recursive in $^3 0, {}^2F$*

3.4.1. *Theorem* The negation problem in the type-2 case relativised to $^3 0$ and E_1 has a negative solution.

Proof Take $V = \{(p,{}^2F) \mid \{p\}({}^2F,{}^2E_1){\downarrow}\}$

V is obviously not recursive in E_1 .

But: i) V is semi-recursive in $^3 0$ and E_1 .

and ii) $\overline{V}$ is also semi-recursive $^3 0$ and E_1 .

To prove this we must do some work.

Let $R_1(p,\alpha,F)$ denote:

" α codes a locally correct computation tree for $\{p\}(F,E_1)$".

R_1 is a predicate recursive in E_1 .

Then we have:

$\{p\}(F,E_1){\uparrow} \Leftrightarrow \forall\alpha[R_1(p,\alpha,F) \Rightarrow IB(\alpha)]$

(here $IB(\alpha)$ means α has an infinite branch, and α is considered as a tree, IB is recursive in E_1)

$\Leftrightarrow 0^3(\lambda\alpha\cdot\mu n[R_1(p,\alpha,F) \Rightarrow IB(\alpha)]){\downarrow}$. ⊠

3.4.2. *Theorem* The union property holds in the case relativised to any ^2F and $^3 0$.

Proof The predicate $\{p\}({}^2G,{}^2F,{}^3 0){\downarrow}$ has the quantifier form $\forall\alpha\exists n\ R(\alpha,n,{}^2G,{}^2F)$ with recursive R .
To see this note that $\{p\}({}^2G,{}^2F,{}^3 0){\uparrow}$ if it has an infinite branch (in its computation tree) which can be coded in a function. As local correctness is a matter of (defined) values only one does not need actual applications of $^3 0$ to check it.
Therefore $\{p\}({}^2G,{}^2F,{}^3 0){\uparrow}$ is Σ_1^1 .
From the quantifier form it follows that the disjunction of two s.r. (Π_1^1) predicates is again Π_1^1 and hence s.r. (in $^2F,{}^3 0$) . ⊠

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