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LORENTZIAN AFFINE HYPERSPHERES WITH CONSTANT AFFINE SECTIONAL CURVATURE

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Dedicated to the sixtieth birthday of Udo Simon

ABSTRACT. We study affine hyperspheres M with constant sectional curvature (with respect to the affine metric h). A conjecture by M. Magid and P. Ryan states that every such affine hypersphere with nonzero Pick invariant is affinely equivalent to either

$$(x_1^2 \pm x_2^2)(x_3^2 \pm x_4^2)\dots(x_{2m-1}^2 \pm x_{2m}^2) = 1$$

or

$$(x_1^2 \pm x_2^2)(x_3^2 \pm x_4^2)\dots(x_{2m-1}^2 \pm x_{2m}^2)x_{2m+1} = 1$$

where the dimension n satisfies $n = 2m - 1$ or $n = 2m$. Up to now, this conjecture was proved if M is positive definite or if M is a 3-dimensional Lorentz space. In this paper, we give an affirmative answer to this conjecture for arbitrary dimensional Lorentzian affine hyperspheres.

1. INTRODUCTION

In this paper, we study nondegenerate affine hypersurfaces M^n in $\mathbb{R}^{n+1}$. It is well known that on such hypersurfaces there exists a canonical transversal vector field ξ called the affine normal vector field. If for all $p \in M$, $\xi(p)$ passes through a fixed point (resp. is parallel), M^n is called a proper affine sphere (resp. improper affine sphere). The standard models of affine spheres are the quadrics. Unlike in Euclidean geometry, where the only umbilical submanifolds are the spheres and the linear subspaces, the class of all equiaffine spheres is simply too large to classify. Therefore, in order to better understand the geometry of affine spheres, it is necessary to impose an extra condition. This can either be a completeness assumption, which only works in the positive definite case, as studied by Blaschke, Calabi, Pogorelov, Cheng, Yau, Sasaki, Li, Ferrer, Martinez, Milan and others (see [LSZ]) or an additional assumption about the curvature, which is the approach we will follow here.

Here, we will focus on affine hyperspheres for which the affine metric has constant sectional curvature. In $\mathbb{R}^3$, affine spheres with constant curvature metric have been studied, amongst others, by J. Radon [R], A.M. Li and G. Penn [LP], M. Magid

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and P. Ryan [MR1] and by U. Simon [S]. They obtained a complete classification of all surfaces which are affine spheres and have constant curvature metric. As for higher dimensions, we have the following results by A.M. Li, U. Simon and the second author and by M. Magid and P. Ryan:

Theorem 1.1 ([VLS])**.** *Let M^n be a positive definite affine hypersphere in $\mathbb{R}^{n+1}$ with constant sectional curvature. Then M is an open part of a quadric or M is affinely equivalent to the hypersurface $x_1x_2\dots x_{n+1}=1$.*

Theorem 1.2 ([MR2])**.** *Let M^3 be a Lorentzian affine hypersphere in $\mathbb{R}^4$ with nonzero Pick invariant and with constant curvature a with respect to the affine metric h. Then $a=0$ and M is affinely equivalent to an open part of either*
(1) $(x_1^2+x_2^2)(x_3^2-x_4^2)=1$,
(2) $(x_1^2+x_2^2)(x_3^2+x_4^2)=1$.

Since a metric on a 3-dimensional manifold is either Lorentzian or positive definite, Theorem 1.1 together with Theorem 1.2 give a complete classification of the 3-dimensional affine hyperspheres with constant affine sectional curvature and nonzero Pick invariant J. The case of zero Pick invariant leads to many new examples and will be treated in a forthcoming paper [DMV].

Later, the following was conjectured by M. Magid and P. Ryan:

Conjecture. *Let M^n be an affine hypersphere in $\mathbb{R}^{n+1}$ with constant sectional curvature a and with nonzero Pick invariant J. Then $a=0$ and M^n is equivalent to*

$$(x_1^2\pm x_2^2)(x_3^2\pm x_4^2)\dots(x_{2m-1}^2\pm x_{2m}^2)=1,$$

if $n=2m-1$ or with

$$(x_1^2\pm x_2^2)(x_3^2\pm x_4^2)\dots(x_{2m-1}^2\pm x_{2m}^2)x_{2m+1}=1,$$

if $n=2m$.

The main theorem of this paper, in Section 4, will give an affirmative answer to this conjecture in the case that the affine metric is Lorentzian, thus generalizing Theorem 1.2 to arbitrary dimensions. A partial answer in this case, under the strong additional assumptions that M is flat, the difference tensor is parallel (with respect to the Levi Civita connection of the affine metric) and $J>0$ (after having chosen the affine normal ξ in such a way that the signature of the affine metric has signature 1) was given by Wang [W]. This last condition has not been stated explicitly but follows from Wang's conventions.

This paper is organised as follows. In Section 2, we recall the basic formulas for an affine hypersphere with constant sectional curvature. In Section 3, we then assume that M is Lorentzian and construct a special h-orthonormal frame at each point p of M. In Section 4, we then show that this frame can be extended to local vector fields satisfying the same properties at every point. This special choice of frame is then used to show that M is flat with respect to the induced metric h and to prove the main theorem.

2. Preliminaries

We briefly recall the basic formulas for affine differential geometry. For more details, we refer to [NS]. Let M^n be a connected differentiable n-dimensional hypersurface of the equiaffine space $\mathbb{R}^{n+1}$ equipped with its usual flat connection D

and a parallel volume element ω, given by the determinant. We allow M to be immersed by an immersion f, but we will not denote the immersion if there is no confusion possible. Let ξ be an arbitrary local transversal vector field to $f(M)$. For any vector fields $X, Y, X_1, \ldots, X_n$, we write

$$D_X Y = \nabla_X Y + h(X,Y)\xi,$$
$$\theta(X_1, \ldots, X_n) = \omega(X_1, \ldots, X_n, \xi),$$

thus defining an affine connection ∇, a symmetric (0,2)-type tensor h, called the second fundamental form, and a volume element θ. M is said to be nondegenerate if h is nondegenerate (and this condition is independent of the choice of transversal vector field ξ). If M is nondegenerate, it is known that there is a unique choice (up to sign) of transversal vector field such that the induced connection ∇, the induced second fundamental form h and the induced volume element θ satisfy the following conditions:

(i) $\nabla\theta = 0$,
(ii) $\theta = \omega_h$,

where ω_h is the metric volume element induced by h. ∇ is called the induced affine connection, ξ the affine normal and h the affine metric. If M is definite, we assume that ξ is chosen such that h is positive definite. Also, in the case that M is Lorentzian (the signature of M is either $n-1$ or 1), we assume that ξ is chosen such that the signature of M equals 1.

A nondegenerate immersion equipped with this special transversal vector field is called a Blaschke immersion. Through this paper, M is always assumed to be a Blaschke immersion. Condition (i) implies that $D_X\xi$ is tangent to M for any tangent vector X to M. Hence we can define a (1,1)-tensor field S, called the affine shape operator, by $D_X\xi = -SX$.

Let $\widehat{\nabla}$ denote the Levi Civita connection of the affine metric h. The difference tensor K is defined by $K(X,Y) = K_X Y = \nabla_X Y - \widehat{\nabla}_X Y$ for tangent vector fields X and Y. Notice that K is symmetric in X and Y. From (i) and (ii), we deduce

$$\operatorname{trace} K_X = 0 \text{ (apolarity condition)}.$$

If we define the cubic form C by $C(X,Y,Z) = (\nabla h)(X,Y,Z)$, then the Codazzi equation says that C is totally symmetric. Moreover, we have the following relation

$$h(K_X Y, Z) = -\tfrac{1}{2}C(X,Y,Z), \tag{2.1}$$

such that K_X is a symmetric operator w.r.t. h. The Pick invariant J is defined by $J = \frac{1}{n(n-1)}h(K,K)$. The curvature tensor $\widehat{R}$ of $\widehat{\nabla}$ is related to S and K by

$$\begin{aligned}\widehat{R}(X,Y)Z = \tfrac{1}{2}\big(&h(Y,Z)SX - h(X,Z)SY\\ &+h(SY,Z)X - h(SX,Z)Y\big) - [K_X, K_Y]\,Z.\end{aligned} \tag{2.2}$$

We also have

$$\begin{aligned}(\widehat{\nabla}_X K)(Y,Z) - (\widehat{\nabla}_Y K)(X,Z) = \tfrac{1}{2}(&h(Y,Z)SX - h(X,Z)SY\\ &-h(SY,Z)W + h(SX,Z)W).\end{aligned} \tag{2.3}$$

If $S = \lambda I$, then M is called an affine sphere with affine mean curvature λ. If M is an affine sphere and $n \geq 2$, then λ is constant. M is called a proper affine sphere if $\lambda \neq 0$ and an improper affine sphere if $\lambda = 0$. If M is a proper affine sphere, then the affine normal ξ satisfies $\xi = -\lambda(f - p)$, where f is the position vector and p is a fixed point in $\mathbb{R}^{n+1}$, called the center of M. If M is an improper affine sphere, so $S = 0$, then ∇ is flat and the affine normal ξ is constant. For an affine sphere with constant sectional curvature a, the equations (2.2) and (2.3) reduce to

$$[K_X, K_Y]Z = -J(h(Y,Z)X - h(X,Z)Y), \tag{2.4}$$

$$(\widehat{\nabla}_X K)(Y,Z) = (\widehat{\nabla}_Y K)(X,Z), \tag{2.5}$$

where $J = a - \lambda$ is the Pick invariant.

3. A pointwise frame for Lorentzian hyperspheres

From now on, we assume that M^n is a Lorentzian affine hypersphere with constant sectional curvature with respect to the affine metric. We also assume that $J \neq 0$, i.e. M^n satisfies all the conditions of the Magid-Ryan conjecture. Then, in the next lemma, we will gradually choose a special frame at a point p of M.

Lemma 3.1. *Let $p \in M$. There does not exist a null-vector $v \in T_pM$ such that $K(v,v)$ is a multiple of v.*

Proof. Assume that there exists a null vector v such that

$$K(v,v) = \mu v.$$

We now complexify the tangent space and assume that $\tilde{\mu}$ is a (possibly complex valued) eigenvalue of the symmetric operator K_v which is different from μ. Let $\tilde{v}$ be an eigenvector corresponding to the eigenvalue $\tilde{\mu}$. Then, we have

$$\tilde{\mu} h(\tilde{v}, v) = h(K_v \tilde{v}, v) = h(\tilde{v}, K_v v) = \mu h(\tilde{v}, v).$$

Hence $h(\tilde{v}, v) = 0$. But now it follows from (2.4) that

$$0 = [K_{\tilde{v}}, K_v]v = \tilde{\mu}(\mu - \tilde{\mu})\tilde{v}.$$

Hence 0 is the only possible eigenvalue different from μ and the apolarity condition $\operatorname{trace} K_v = 0$ implies $\mu = 0 = \tilde{\mu}$. Hence the operator K_v has only eigenvalue 0, in particular, $K_v v = 0$.

Since for any vector u we have

$$K_v K_v u = K_v K_u v = [K_v, K_u]v = -Jh(u,v)v,$$

there exists a vector u with $h(u,v) \neq 0$ such that $K_v u$ and v are linearly independent. Writing $w = K_v u$ we have $h(v,v) = 0$, $h(v,w) = h(v, K_v u) = h(K_v v, u) = 0$, and

$$h(w,w) = h(K_v u, K_v u) = h(K_v K_v u, u) = -Jh(u,v)^2.$$

Since $J \neq 0$, the above formulas imply that the space W spanned by u, v and w is nondegenerate and invariant under K_v and that the Pick invariant J is negative. Since K_v is a symmetric operator, $W^\perp$ is invariant under K_v also. Since h restricted to $W^\perp$ is definite and K_v only has a zero eigenvalue, we deduce that $K_v x = 0$ for $x \in W^\perp$. A short calculation shows that there are unique $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that $e_1 = v$, $e_2 = \alpha_1 u + \alpha_2 v + \alpha_3 w$, $e_3 = (-J)^{-\frac{1}{2}} K_v e_2$ span W and satisfy $h(e_1,e_1) = h(e_2,e_2) = h(e_1,e_3) = h(e_2,e_3) = 0$, $h(e_1,e_2) = 1$. We now choose $e_4, \dots, e_n$ as h-orthonormal vectors which span $W^{\perp}$. For simplicity we put $\alpha^2 = -J$. Then we have that $h(e_3, e_3) = 1$ and

$$\begin{aligned} K_{e_1} e_1 &= 0, \\ K_{e_1} e_2 &= \alpha e_3, \\ K_{e_1} e_3 &= \alpha e_1, \\ K_{e_1} e_i &= 0, \end{aligned}$$

for $i > 3$. Then, we have

$$-\alpha^2 e_i = [K_{e_2}, K_{e_i}] e_1 = -\alpha K_{e_i} e_3,$$

and

$$-\alpha^2 e_3 = [K_{e_2}, K_{e_3}] e_1 = K_{e_2}(\alpha e_1) - K_{e_3}(\alpha e_3) = \alpha^2 e_3 - \alpha K_{e_3} e_3.$$

So, we have that

$$\begin{aligned} h(K_{e_3} e_i, e_i) &= \alpha, \\ h(K_{e_3} e_1, e_2) &= \alpha, \\ h(K_{e_3} e_3, e_3) &= 2\alpha. \end{aligned}$$

Hence the apolarity condition for K_{e_3} yields $0 = (n+1)\alpha$. Hence $\alpha = 0$ and M has zero Pick invariant J in contradiction to our general assumption $J \neq 0$. $\square$

Lemma 3.2. *Let $p \in M$. There exist a null vector $v \in T_pM$ such that*

(1) *v and $K(v, v)$ span a nondegenerate plane σ,*
(2) *$h(K(v, v), v) \neq 0$,*
(3) *$h(K(v, v), K(v, u)) = 0$ for every vector u which is orthogonal to σ.*

Proof. The existence of v satisfying (1) is obvious from Lemma 3.1. In order to see that (2) and (3) can be satisfied as well, we have to consider different cases.

Case 1. Assume that $h(K(v, v), K(v, v)) \neq 0$ for every nonzero null vector v. Then, we define a function f on the set of all null-vectors by

$$f(v) = \frac{(h(K(v, v), v))^8}{(h(K(v, v), K(v, v)))^6}.$$

It is clear that f only depends on the direction of our null vector. Since the set of directions of null vectors is a compact set, there exists a vector e_1 such that f attains an absolute maximum at e_1. If the maximal value for f is equal to 0, then $h(K(v, v), v) = 0$ holds for every null vector. Then applying Lemma 3.1 of [NV] implies that K vanishes identically. This contradicts our assumption that $h(K(v, v), K(v, v))$ is never zero for a null vector. By rescaling e_1 if necessary, we may now assume that

$$h(K(e_1, e_1), e_1) = \tfrac{1}{2} h(K(e_1, e_1), K(e_1, e_1)) =: \alpha.$$

We then introduce e_2 as the null vector in the plane determined by e_1 and $K_{e_1} e_1$ such that $h(e_1, e_2) = 1$. So, we have that

$$K_{e_1} e_1 = e_1 + \alpha e_2. \tag{3.1}$$

Denote by σ the plane spanned by e_1 and e_2. Since K_{e_1} is a symmetric operator and h restricted to $\sigma^{\perp}$ is positive definite, we can choose an h-orthonormal basis

$e_3, \ldots, e_n$ of $\sigma^\perp$ such that $h(K_{e_1}e_i, e_j) = \lambda_i\delta_{ij}$, for $i, j > 2$. We now define a function $f(t)$ by

$$f(t) = \frac{f_1^8(v(t))}{f_2^6(v(t))},$$

where

$$\begin{aligned} v(t) &= e_1 - \tfrac{1}{2}t^2 e_2 + te_i, \\ f_1(w) &= h(K(w,w),w), \\ f_2(w) &= h(K(w,w),K(w,w)), \end{aligned}$$

where $i \geq 3$ and w is any null-vector. A straightforward computation, using (3.1) shows that

$$\begin{aligned} h(v(t),v(t)) &= 0, \\ f_1(v(t)) &= \alpha + o(t^2), \\ f_2(v(t)) &= 2\alpha + 4th(K(e_1,e_1),K(e_1,e_i)) + o(t^2). \end{aligned}$$

Since $f(t)$ has an absolute maximum at $t = 0$, we deduce that

$$0 = f'(0) = 8\frac{f_1'(0)}{f_1(0)}f(0) - 6\frac{f_2'(0)}{f_2(0)}f(0) = -24h(K(e_1,e_1),K(e_1,e_i))\frac{f(0)}{f_2(0)}.$$

Hence $h(K(e_1,e_1),K(e_1,e_i)) = 0$. This completes the proof of the lemma in this case.

Case 2. We denote by U_1 the set of null-vectors such that $K(v,v)$ is a timelike vector and we assume that $U_1 \neq \emptyset$. Denote by U_2 the set of null-vectors v such that $h(K(v,v),K(v,v)) = 0$. It follows from Lemma 3.1 that if $v \in U_2$, then $h(K(v,v),v) \neq 0$. Since on U_1 $K(v,v)$ is timelike, the same condition is also satisfied on U_1. The null-directions determined by U_1 and U_2 form a closed subset V (and thus compact) of the set of all null-directions. We define a function on V by

$$g(w) = \frac{f_2(w)^6}{f_1(w)^8},$$

where $w \in V$. Since V is compact, the function g attains an absolute maximum at a null vector v and since $U_1 \neq \emptyset$, we have that $v \in U_1$. This also implies that the function f, which we considered in the previous case, is well defined on a neighborhood of v and attains a relative extremum at v. We now proceed exactly as in Case 1 to complete the proof.

Case 3. The set $U_1 = \emptyset$ but $U_2 \neq \emptyset$. Let $u \in U_2$. From Lemma 3.1 it follows that we can rescale u such that $h(K(u,u),u) = 1$. We consider the function $\tilde{g}(v) = h(K(v,v),K(v,v))(h(K(v,v),v))^{-\frac{4}{3}}$. Since $U_1 = \emptyset$, f_2 attains an absolute minimum at the vector u. We put $e_1 = u$, $e_2 = K(u,u)$ and define $e_3, \ldots, e_n$ as before. We consider $v(t)$ as before. Then $\tilde{g}(v(t))$ has a relative extremum at $t = 0$ and thus we again get that

$$h(K(e_1,e_1),K(e_1,e_i)) = 0.$$

This completes the proof. ∎

Lemma 3.3. *Let $p \in M$ and let v be as in Lemma 3.2. Then $K(v,v)$ is not a null-vector, i.e. $h(K(v,v),K(v,v)) \neq 0$.*

Proof. In order to prove this lemma, it is sufficient to show that Case 3 of Lemma 3.2 cannot occur. Therefore, let $e_1, e_2, \ldots, e_n$ be the frame constructed in Case 3 of Lemma 3.2. Then, we have

$$\begin{aligned} K(e_1, e_1) &= e_2, \\ K(e_1, e_2) &= \beta e_1 + \sum_{i=3}^{n} \alpha_i e_i, \\ K(e_1, e_i) &= \alpha_i e_1 + \lambda_i e_i, \end{aligned}$$

where $i > 2$. Because of Lemma 3.2, we have that $\alpha_i = 0$. The apolarity condition for K_{e_1} now gives us that

$$\sum_{i=3}^{n} \lambda_i = 0. \tag{3.2}$$

Using (2.4), we then get that

$$0 = [K_{e_1}, K_{e_i}]e_1 = \lambda_i^2 e_i - K_{e_i} e_2.$$

Hence $K_{e_2} e_i = \lambda_i^2 e_i$. Since $h(K_{e_2} e_2, e_1) = h(K_{e_2} e_1, e_2) = \beta$, we find that the apolarity condition for K_{e_2} reduces to

$$2\beta + \sum_{i=3}^{n} \lambda_i^2 = 0. \tag{3.3}$$

Applying (2.4) once more, we find that

$$\begin{aligned} J e_i &= [K_{e_2}, K_{e_i}]e_1 \\ &= \lambda_i K_{e_2} e_i - \beta K_{e_i} e_1 \\ &= (\lambda_i^3 - \beta \lambda_i) e_i. \end{aligned}$$

Hence $\lambda_3, \ldots, \lambda_n$ are solutions of the third order equation:

$$x^3 - \beta x - J = 0. \tag{3.4}$$

Using the fact that the λ_i are soltutions of the above equation we find that

$$\sum_{i=3}^{n} \lambda_i^4 = \beta \sum_{i=3}^{n} \lambda_i^2 + J \sum_{i=3}^{n} \lambda_i = -2\beta^2.$$

Hence $\lambda_i = \beta = 0$ and a contradiction follows again from (3.4). This completes the proof of the lemma. ◻

Lemma 3.4. *Let $p \in M$. Then there exist vectors $e_1, e_2, \ldots, e_n$ with*

$$\begin{aligned} &h(e_1, e_1) = h(e_2, e_2) = h(e_1, e_i) = h(e_2, e_i) = 0, \\ &h(e_1, e_2) = 1, \\ &h(e_i, e_j) = \delta_{ij}, \end{aligned}$$

where $i, j > 2$ such that the difference tensor is given by

$$\begin{aligned}
K(e_1, e_1) &= e_1 - \tfrac{4n(n-1)}{(n-2)^3}\tfrac{1}{J}e_2,\\
K(e_1, e_2) &= \tfrac{(n-2)^2}{4(n-1)}Je_1 + e_2,\\
K(e_2, e_2) &= -\tfrac{(n-2)^3 nJ^2}{16(n-1)^2}e_1 + \tfrac{(n-2)^2}{4(n-1)}Je_2,\\
K(e_1, e_i) &= -\tfrac{2}{n-2}e_i,\\
K(e_2, e_i) &= -\tfrac{(n-2)}{2(n-1)}Je_i.
\end{aligned}$$

Proof. We may assume that Case 1 or Case 2 of Lemma 3.2 is satisfied. We proceed in the same way as in the previous lemma. Denote by $e_1, e_2, \dots, e_n$ be the frame constructed in Case 1 or Case 2 of Lemma 3.2. Then, we have

$$\begin{aligned}
K(e_1, e_1) &= e_1 + \alpha e_2,\\
K(e_1, e_2) &= \beta e_1 + e_2 + \sum_{i=3}^{n} \alpha_i e_i,\\
K(e_1, e_i) &= \alpha_i e_1 + \lambda_i e_i,
\end{aligned}$$

where $i > 2$ and α is a nonzero number. Because of Lemma 3.2, we have that $\alpha_i = 0$. The apolarity condition for K_{e_1} now gives us that

$$\sum_{i=3}^{n} \lambda_i = -2. \tag{3.9}$$

Using (2.4), we then get that

$$0 = [K_{e_1}, K_{e_i}]e_1 = \lambda_i^2 e_i - \lambda_i e_i - \alpha K_{e_i} e_2.$$

Hence $K_{e_2} e_i = \frac{\lambda_i^2 - \lambda_i}{\alpha} e_i$. Since $h(K_{e_2} e_2, e_1) = h(K_{e_2} e_1, e_2) = \beta$, we find that the apolarity condition for K_{e_2} reduces to

$$2\alpha\beta + \sum_{i=3}^{n} (\lambda_i^2 - \lambda_i) = 0.$$

Using (3.9), we can still rewrite this as

$$2(\alpha\beta + 1) + \sum_{i=3}^{n} \lambda_i^2 = 0. \tag{3.10}$$

Applying (2.4) once more, we find that

$$\begin{aligned}
Je_i &= [K_{e_2}, K_{e_i}]e_1\\
&= \lambda_i K_{e_2} e_i - \beta K_{e_i} e_1 - K_{e_i} e_2\\
&= \tfrac{1}{\alpha}(\lambda_i^3 - 2\lambda_i^2 + (1 - \alpha\beta)\lambda_i)e_i.
\end{aligned}$$

Hence $\lambda_3, \dots, \lambda_n$ are solutions of the third order equation:

$$x^3 - 2x^2 + (1 - \alpha\beta)x - \alpha J = 0. \tag{3.11}$$

Denote by μ_1, μ_2, μ_3 the three solutions of the above equations (of which 2 can be complex conjugate numbers). Then, it follows from (3.11) that

$$\mu_1 + \mu_2 + \mu_3 = 2, \tag{3.12}$$

$$\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 = 1 - \alpha\beta. \tag{3.13}$$

Combining (3.12) and (3.13) we deduce that

$$\mu_1^2+\mu_2^2+\mu_3^2=(\mu_1+\mu_2+\mu_3)^2-2+2\alpha\beta=2(1+\alpha\beta). \tag{3.14}$$

On the other hand, it follows from (3.9) and (3.10) that there exist natural numbers n_1, n_2 and n_3 such that

$$\begin{aligned} n_1\mu_1+n_2\mu_2+n_3\mu_3&=-2,\\ n_1\mu_1^2+n_2\mu_2^2+n_3\mu_3^2&=-2(1+\alpha\beta). \end{aligned} \tag{3.15}$$

If (3.11) has 3 real roots, then it follows from comparing (3.14) with (3.15) that $1+\alpha\beta=0=\mu_1=\mu_2=\mu_3$. Hence from (3.10) it then follows that $\alpha J=\mu_1\mu_2\mu_3=0$. Therefore, (3.11) has only 1 real root which we may assume to be μ_1. But then we have that $\lambda_i=\mu_1$ for all i, $n_1=n-2$ and $n_2=n_3=0$. Equation (3.9) then shows that $\lambda_i=-\frac{2}{n-2}$, whereas (3.10) implies that $(\alpha\beta+1)=-\frac{2}{(n-2)}$. Hence $\beta=-\frac{n}{\alpha(n-2)}$. Finally, it follows from (3.11) that

$$\alpha=-\tfrac{4n(n-1)}{(n-2)^3}\tfrac{1}{J}.$$

Finally, we write $K_{e_2}e_2=\gamma e_1+\beta e_2$ and use (2.4) once more to deduce that

$$\begin{aligned} -Je_1&=[K_{e_1},K_{e_2}]e_1\\ &=K_{e_1}(\beta e_1+e_2)-K_{e_2}(e_1+\alpha e_2)\\ &=\beta(e_1+\alpha e_2)-\alpha(\gamma e_1+\beta e_2)\\ &=(\beta-\alpha\gamma)e_1. \end{aligned}$$

Hence $\gamma=\frac{(\beta+J)}{\alpha}$. □

We note at this point that all the formulas of Lemma 3.4 remain valid if we choose a different orthonormal basis for $\sigma^\perp$, where σ is the plane spanned by e_1 and e_2. Therefore, we still have the freedom to rotate $e_3,\ldots,e_n$ in this space. In order to do so in a consistent way, we follow the approach of [VLS]. We introduce a function g on $\sigma^\perp$ by

$$g(x)=h(K(x,x),x), \tag{3.16}$$

where $x\in\sigma^\perp$. We now denote $\sigma^\perp=\sigma^1$ and we define an operator K^1 on $\sigma^\perp$ by

$$K_u^1v=K_uv-h(K_uv,e_1)e_2-h(K_uv,e_2)e_1=K_uv+h(u,v)(\tfrac{(n-2)J}{2(n-1)}e_1+\tfrac{2}{n-2}e_2),$$

where $u,v\in\sigma^\perp$. Then, we have that K^1 is a symmetric operator from $\sigma^1\times\sigma^1$ to σ^1. We also have that $\operatorname{trace}K^1=0$ and it follows from (2.4) that for $u,v,w\in\sigma^1$, we have that

$$\begin{aligned} [K_u^1,K_v^1]w&=K_u^1K_v^1w-K_v^1K_u^1w\\ &=K_uK_v^1w+h(u,K_v^1w)(\tfrac{(n-2)}{2(n-1)}Je_1+\tfrac{2}{n-2}e_2)\\ &\quad-K_vK_u^1w-h(v,K_u^1w)(\tfrac{(n-2)}{2(n-1)}Je_1+\tfrac{2}{n-2}e_2)\\ &=[K_u,K_v]w-\tfrac{2}{n-1}J(h(v,w)u-h(u,w)v)\\ &=-\tfrac{n+1}{n-1}J(h(v,w)u-h(u,w)v). \end{aligned} \tag{3.17}$$

Now, we first state the following lemma of [VLS], together with some general lemmas about mappings satisfying (3.17) on positive definite spaces:

Lemma 3.5. *Let h be a positive definite metric on $\mathbb{R}^k$, $k > 1$, and let $T^1 : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ be a mapping such that $h(T^1(u,v),w)$ is totally symmetric. Denote by $S^{k-1} = \{v \in \mathbb{R}^k | h(v,v) = 1\}$ and consider the function $g(v) = h(T^1_v v, v)$ on S^{k-1}. Let u be a vector at which g attains an extremal value and let $w \in S^{k-1}$ such that $h(u,w) = 0$. Then $h(T^1_u u, w) = 0$. Moreover, if g attains a relative maximum at u, then $h(T^1_u u, u) - 2h(T^1_w w, u) \geq 0$.*

Lemma 3.6. *Let T^1, h and g be as in Lemma 3.5. Assume also that*

$$\operatorname{trace} T^1_u = 0, \tag{3.18}$$

$$[T^1_u, T^1_v]w = -\tilde{J}(h(v,w)u - h(u,w)v), \tag{3.19}$$

where $u, v, w \in \mathbb{R}^k$. Denote by A the set of critical values of the function g. Then A is finite. Moreover, if g attains an absolute maximum at a vector e_1, with $h(e_1, e_1) = 1$, then

$$\begin{aligned} T^1_{e_1} e_1 &= (k-1)\lambda_1 e_1, \\ T^1_{e_1} u &= -\lambda_1 u, \end{aligned}$$

for all $u \perp e_1$, where $\lambda_1 \geq 0$ and $\lambda_1^2 = \frac{\tilde{J}}{k}$. Finally, the map T^2 from $\mathbb{R}^{k-1} \times \mathbb{R}^{k-1} \to \mathbb{R}^{k-1}$, where $\mathbb{R}^{k-1}$ is the linear space orthogonal to e_1, defined by

$$T^2_u v = T^1_u v + \lambda_1 h(u,v) e_1,$$

for $u, v \in \mathbb{R}^{k-1}$ also satisfies

$$\begin{aligned} &h(T^2_\cdot \cdot, \cdot) \text{ is totally symmetric}, \\ &\operatorname{trace} T^2_u = 0, \\ &[T^2_u, T^2_v]w = -(k+1)(\lambda_1)^2(h(v,w)u - h(v,u)w). \end{aligned} \tag{3.20}$$

Proof. Let v be a vector in which the function g attains a relative extremum. Applying Lemma 3.5, we see that v is an eigenvector of T^1_v with eigenvalue, say, μ_1. We put $v = e_1$. Then since $T^1_{e_1}$ is a symmetric operator on a positive definite space, there exists h-orthonormal vectors $e_2, \dots, e_k$ and numbers $\mu_2, \dots, \mu_k$ such that

$$T_{e_1} e_j = \mu_j e_j,$$

where $j = 2, \dots, k$. Applying (3.19) then gives that

$$\begin{aligned} \tilde{J} e_i &= [T^1_{e_1}, T^1_{e_i}] e_1 \\ &= (\mu_i^2 - \mu_i \mu_1) e_i. \end{aligned}$$

Hence $\mu_2, \dots, \mu_k$ are solutions of the equation $y^2 - \mu_1 y - \tilde{J} = 0$. Hence there exist an integer r, $0 \leq r \leq (k-1)$ such that if necessary after renumbering the basis, we have

$$\begin{aligned} &T^1_{e_1} e_1 = \mu_1 e_1, \\ &T^1_{e_1} e_s = \tfrac{1}{2}(\mu_1 + \sqrt{\mu_1^2 + 4\tilde{J}}) e_s, \qquad 1 < s \leq r+1, \\ &T^1_{e_1} e_s = \tfrac{1}{2}(\mu_1 - \sqrt{\mu_1^2 + 4\tilde{J}}) e_s, \qquad r+1 < s \leq k. \end{aligned} \tag{3.21}$$

Now using (3.18), we deduce that

$$0 = \tfrac{1}{2}(k+1)\mu_1 + \tfrac{1}{2}(2r - k + 1)\sqrt{\mu_1^2 + 4\tilde{J}},$$

which is equivalent to

$$\mu_1^2 = \tfrac{(k-2r-1)^2}{(k-r)(r+1)}\tilde{J}. \tag{3.22}$$

Since r is a bounded integer this shows that the set of critical values is finite.

Let us assume now that g attains a relative maximum at a vector e_1. It then follows from (3.21) and Lemma 3.5 that $r = 0$. Putting $\mu_1 = (k-1)\lambda_1$, equation (3.22) then implies that $\lambda_1^2 = \frac{\tilde{J}}{k}$. Since $r = 0$, it then follows that $\mu_2 = \cdots = \mu_k = -\lambda_1$.

Finally, we restrict ourselves to $\mathbb{R}^{k-1}$, the linear space orthogonal to e_1 and define the map T^2 as in the lemma. The first assertion in (3.20) is clear from the definition of T^2, and the third assertion follows from

$$\begin{aligned}
[T_u^2, T_v^2]w &= T_u^1(T_v^2 w) + \lambda_1 h(u, T_v^2 w)e_1 - T_v^1(T_u^2 w) - \lambda_1 h(v, T_u^2 w)e_1 \\
&= [T_u^1, T_v^1]w - \lambda_1^2(h(v,w)u - h(u,w)v) \\
&= -(\tilde{J} + (\lambda_1)^2)(h(v,w)u - h(u,w)v) \\
&= -(k+1)(\lambda_1)^2(h(v,w)u - h(u,w)v).
\end{aligned}$$

The second one follows immediately from the fact that $h(T_u^1 e_1, e_1) = h(T_{e_1}^1 e_1, u) = 0$, for $u \in \mathbb{R}^{k-1}$. □

Applying the above lemma to our situation, we get

Lemma 3.7. *Let M^n be a Lorentzian affine sphere in $\mathbb{R}^{n+1}$ with constant sectional curvature a and nonzero Pick invariant J. Then for any $p \in M$, there exist a basis $\{e_1, e_2, \ldots, e_n\}$ of T_pM and numbers $\lambda_1, \ldots, \lambda_{n-2}$ with*

$$\begin{aligned}
&h(e_1,e_1) = h(e_2,e_2) = 0 \qquad && h(e_i,e_j) = \delta_{ij}, \\
&h(e_1,e_2) = 1 && h(e_1,e_i) = h(e_2,e_i) = 0,
\end{aligned}$$

where $i, j = 3, \ldots, n$, such that

$$\begin{aligned}
&K_{e_1}e_1 = e_1 - \tfrac{4(n-1)n}{(n-2)^3}\tfrac{1}{J}e_2, \qquad K_{e_1}e_2 = \tfrac{(n-2)^2 J}{4(n-1)}e_1 + e_2, \\
&K_{e_1}e_i = -\tfrac{2}{n-2}e_i, \qquad K_{e_2}e_i = -\tfrac{(n-2)}{2(n-1)}Je_i, \\
&K_{e_2}e_2 = -\tfrac{(n-2)^3 nJ^2}{16(n-1)^2}e_1 + \tfrac{(n-2)^2 J}{4(n-1)}e_2, \qquad K_{e_i}e_j = -\lambda_{i-2}e_j, \\
&K_{e_i}e_i = -\tfrac{(n-2)}{2(n-1)}Je_1 - \tfrac{2}{n-2}e_2 - \sum_{k=3}^{i-1}\lambda_{k-2}e_k + (n-i)\lambda_{i-2}e_i,
\end{aligned}$$

where $j, i = 3, \ldots, n$ with $j > i$. If $n > 3$ we have $J > 0$ and $\lambda_1, \ldots, \lambda_{n-2}$ are inductively defined by $\lambda_{i+1} = \sqrt{\frac{(n-i)}{n-i-2}}\lambda_i$, $\lambda_1 = \sqrt{\frac{(n+1)J}{(n-1)(n-2)}}$.

Proof. First note that if $n = 3$, Lemma 3.7 is a direct consequence of Lemma 3.4 and the apolarity condition for K_{e_3}. Therefore, we may assume that $n > 3$. By Lemma 3.4, we already have e_1 and e_2. Applying Lemma 3.6 to the positive definite space σ^1 equipped with the totally symmetric operator K^1 introduced above, gives us a vector e_3 such that

$$K_{e_3}e_3 = -\tfrac{(n-2)}{2(n-1)}Je_1 - \tfrac{2}{n-2}e_2 + (n-3)\lambda_1 e_3,$$

and

$$K_{e_3}v = -\lambda_1 v,$$

for every vector v which belongs to σ^1 and is orthogonal to e_3. Equation (3.22) implies that J is positive. We now define as σ^2 the linear space orthogonal to e_3 and define K^2 from K^1 as in Lemma 3.6.

We now proceed by induction. Suppose that we have defined $e_1, ..., e_r$, $3 \leq r < (n-1)$, and an operator K^{r-1} on the positive definite space σ^{r-1} which is orthogonal to $e_1, \dots, e_r$ such that for $i, j \leq r$ Lemma 3.7 is satisfied and such that for $j > r$ we have $K_{e_i} e_j = \lambda_{i-2} e_j$, $i = 3, \dots, r$, $K_{e_1} e_j = -\frac{2}{(n-2)} e_j$ and $K_{e_2} e_j = -\frac{(n-2)}{2(n-1)} J e_j$. Applying Lemma 3.6 then completes the induction hypothesis. Hence we may assume that $\{e_1, \dots, e_{n-1}\}$ are chosen appropriately. This determines e_n uniquely. The proof is now completed by applying the apolarity condition for K_{e_n}. $\square$

4. The proof of the conjecture in the Lorentz case

We first show that we can extend the pointwise choice of basis made in Lemma 3.7 to a local frame satisfying the same properties at every point.

Lemma 4.1. *Let M be a Lorentzian affine hypersphere with constant sectional curvature c and nonzero Pick invariant J. Let $p \in M$. Then there exist local vector field $E_1, \dots, E_n$ near p such that*

$$\begin{aligned} &h(E_1, E_1) = h(E_2, E_2) = 0 \qquad && h(E_i, E_j) = \delta_{ij}, \\ &h(E_1, E_2) = 1 && h(E_1, E_i) = h(E_2, E_i) = 0, \end{aligned}$$

where $i, j = 3 \dots n$, such that

$$\begin{aligned} &K_{E_1} E_1 = E_1 - \tfrac{4(n-1)n}{(n-2)^3} \tfrac{1}{J} E_2, \qquad K_{E_1} E_2 = \tfrac{(n-2)^2 J}{4(n-1)} E_1 + E_2, \\ &K_{E_1} E_i = -\tfrac{2}{n-2} E_i, \qquad K_{E_2} E_i = -\tfrac{(n-2)}{2(n-1)} J E_i, \\ &K_{E_2} E_2 = -\tfrac{(n-2)^3 n J^2}{16(n-1)^2} E_1 + \tfrac{(n-2)^2 J}{4(n-1)} E_2, \qquad K_{E_i} E_j = -\lambda_{i-2} E_j, \\ &K_{E_i} E_i = -\tfrac{(n-2)}{2(n-1)} J E_1 - \tfrac{2}{n-2} E_2 - \sum_{k=3}^{i-1} \lambda_{k-2} E_k + (n-i) \lambda_{i-2} E_i, \end{aligned}$$

where $j, i = 3, \dots, n$ with $j > i$ and $\lambda_1, \dots, \lambda_{n-2}$ are as defined in Lemma 3.7.

Proof. First, we show that we can define E_1 and E_2 differentiably. We take a point $p \in M$ and we take the frame constructed at p in Lemma 3.7. We can extend this to local vector fields $F_1, \dots, F_n$ on a neighborhood of p such that $F_i(p) = e_i$ and

$$\begin{aligned} &h(F_1, F_1) = h(F_2, F_2) = 0 \qquad && h(F_i, F_j) = \delta_{ij}, \\ &h(F_1, F_2) = 1 && h(F_1, F_i) = h(F_2, F_i) = 0, \end{aligned}$$

where $i, j = 3 \dots n$. Then, the null-directions V, different from F_2, can be parameterized by

$$V(q, (y_1, \dots, y_n)) = F_1(q) - \tfrac{1}{2} \sum_{i=3}^{n} y_i^2 F_2(q) + \sum_{i=3}^{n} y_i F_i(q).$$

Let f, f_1, f_2 be defined as in Lemma 3.2. Since Case 3 of Lemma 3.2 could not occur, we have that $f \circ V$ is a well defined function on a neighbourhood of $(p, (0, \dots, 0))$. Since

$$f_1(V(p, (y_3, \dots, y_n)) = -\tfrac{4(n-1)n}{(n-2)^3} \tfrac{1}{J} - \tfrac{3}{2} \sum_{i=3}^{n} y_i^2 - \tfrac{6}{n-2} \sum_{i=3}^{n} y_i^2 + o(|y|^2),$$

and

$$f_2(V(p,(y_3,\dots,y_n))=-2\tfrac{4(n-1)n}{(n-2)^3}\tfrac{1}{J}+\tfrac{4}{n-2}\sum_{i=3}^n y_i^2+\tfrac{4n}{(n-2)^2}\sum_{i=3}^n y_i^2+\tfrac{16}{(n-2)^2}\sum_{i=1}^n y_i^2 - \tfrac{4}{n-2}\sum_{i=3}^n y_i^2+o(|y|^2).$$

By the proof of this lemma $f|_{\{p\}\times\mathbb{R}^{n-2}}$ has a relative extremum at $(p,(0,\dots,0))$, so $\frac{\partial f}{\partial y_j}=0$ and

$$\tfrac{\partial^2}{\partial y_j\partial y_i}f|_{(p,(0,\dots,0))}=\tfrac{3}{8}\tfrac{4(n-1)n}{(n-2)^3}\tfrac{n(n+3)}{(n-2)^2}\tfrac{1}{J}\delta_{ij}.$$

Hence the implicit function theorem shows that we can find local functions $y_3,\dots,y_n$ on M such that $f|_{\{q\}\times\mathbb{R}^{n-2}}$ attains an extremal value at every point q in a neighborhood of p. Repeating now the procedure of Lemma 3.4, we get the desired vector fields E_1 and E_2.

Now, suppose that we already have constructed the vector fields $E_1,\dots,E_r$, $r\geq 2$. Let

$$V=a_1F_{r+1}+\cdots+a_{n-r}F_n$$

and consider the system of equations

$$b_j(a_1,\dots,a_{n-r})=h(F_{r+j},K(V,V)-(n-r-1)\lambda_{r-1}V)\quad (j=1,\dots,n-r),$$

where λ_r is defined in Lemma 3.7. Since

$$\begin{aligned}(\frac{\partial b_j}{\partial a_k})|_{(1,0,\dots,0)}&=2h(F_{r+j},K(F_{r+k},F_{r+1})-(n-r-1)\lambda_{r-1}F_{r+k})\\ &=-(n-r+1)\lambda_{r-1}\delta_{kj},\quad j\neq 1\quad\text{or}\quad k\neq 1\\ &=(n-r-1)\lambda_{r-1},\quad k=j=1\end{aligned}$$

there exist local, (locally unique) functions $a_1,\dots,a_{n-r}$ in a neighbourhood of p such that $V(p)=e_{r+1}$ and $K(V,V)\equiv(n-r-1)\lambda_{r-1}V$ modulo $E_1,\dots,E_k$. By uniqueness of this construction and the corresponding pointwise result of Lemma 3.7, V is a unit vector field and we can set $E_{r+1}=V$.

Repeating the above procedure we get differentiable vector fields $E_1,\dots,E_{n-1}$. Taking E_n orthogonal to the above vector fields and applying the apolarity condition for K_{E_n} now completes the proof. □

Now, we use that by (2.5), the tensor K defines a Codazzi tensor. This allows us to prove the following lemma.

Lemma 4.2. *Let M be as in Lemma 4.1, let $p\in M$ and let $\{E_1,\dots,E_n\}$ be the local frame constructed in Lemma 4.1. Then, we have that*

$$\widehat{\nabla}_{E_i}E_j=0.$$

In particular, $a=0$ and M is flat.

Proof. We use the fact that the difference tensor K is a Codazzi tensor with respect to the connection $\widehat{\nabla}$. First, from $(\widehat{\nabla}_{E_2}K)(E_1,E_1)=(\widehat{\nabla}_{E_1}K)(E_2,E_1)$ we get that

$$\begin{aligned}&\widehat{\nabla}_{E_2}(E_1-\tfrac{4(n-1)n}{(n-2)^3}\tfrac{1}{J}E_2)-2K(\widehat{\nabla}_{E_2}E_1,E_1)\\ &\quad=\widehat{\nabla}_{E_1}(\tfrac{(n-2)^2J}{4(n-1)}E_1+E_2)-K(E_2,\widehat{\nabla}_{E_1}E_1)-K(\widehat{\nabla}_{E_1}E_2,E_1).\end{aligned}\tag{4.1}$$

Looking at the E_1 and E_2 components of the above equation, we respectively find that

$$\begin{aligned} h(\widehat{\nabla}_{E_2}E_1, E_2) &= \tfrac{(n-2)^2 J}{4(n-1)} h(\widehat{\nabla}_{E_1}E_2, E_1), \\ -3\tfrac{4(n-1)n}{(n-2)^3}\tfrac{1}{J} h(\widehat{\nabla}_{E_2}E_2, E_1) &= h(\widehat{\nabla}_{E_1}E_2, E_1), \end{aligned}$$

from which we deduce that $h(\widehat{\nabla}_{E_2}E_2, E_1) = h(\widehat{\nabla}_{E_1}E_2, E_1) = 0$ and in turn

$$h(E_A, \widehat{\nabla}_{E_B}E_C) = 0$$

for all $A, B, C \in \{1, 2\}$. Therefore, (4.1) reduces to

$$\tfrac{n+2}{n-2}\widehat{\nabla}_{E_2}E_1 - \tfrac{4n(n-1)}{J(n-2)^3}\widehat{\nabla}_{E_2}E_2 = \tfrac{n}{n-2}\widehat{\nabla}_{E_1}E_2 + \tfrac{n(n-2)J}{4(n-1)}\widehat{\nabla}_{E_1}E_1. \tag{4.2}$$

In the same way, the Codazzi equation $(\widehat{\nabla}_{E_1}K)(E_2, E_2) = (\widehat{\nabla}_{E_2}K)(E_1, E_2)$ implies that

$$\tfrac{(n^2-4)J}{4(n-1)}\widehat{\nabla}_{E_1}E_2 - \tfrac{(n-2)^3 nJ^2}{16(n-1)^2}\widehat{\nabla}_{E_1}E_1 = \tfrac{n}{n-2}\widehat{\nabla}_{E_2}E_2 + \tfrac{(n-2)nJ}{4(n-1)}\widehat{\nabla}_{E_2}E_1. \tag{4.3}$$

Computing the E_1 and E_2 components of the Codazzi equations $(\widehat{\nabla}_{E_i}K)(E_1, E_1) = (\widehat{\nabla}_{E_1}K)(E_i, E_1)$ and $(\widehat{\nabla}_{E_i}K)(E_1, E_2) = (\widehat{\nabla}_{E_1}K)(E_i, E_2)$ we obtain the following systems of equations:

$$\begin{aligned} &\tfrac{12(n-1)n}{(n-2)^3 J} h(\widehat{\nabla}_{E_i}E_2, E_1) = -\tfrac{n+2}{n-2} h(\widehat{\nabla}_{E_1}E_1, E_i) + \tfrac{4n(n-1)}{(n-2)^3 J} h(\widehat{\nabla}_{E_1}E_2, E_i), \\ &h(\widehat{\nabla}_{E_i}E_2, E_1) = \tfrac{n}{n-2} h(\widehat{\nabla}_{E_1}E_2, E_i) + \tfrac{n(n-2)J}{4(n-1)} h(\widehat{\nabla}_{E_1}E_1, E_i), \\ &\tfrac{(n-2)^2 J}{4(n-1)} h(\widehat{\nabla}_{E_i}E_1, E_2) = \tfrac{(n^2-4)J}{4(n-1)} h(\widehat{\nabla}_{E_1}E_2, E_i) - \tfrac{(n-2)^3 nJ^2}{16(n-1)^2} h(\widehat{\nabla}_{E_1}E_1, E_i). \end{aligned}$$

Hence $h(\widehat{\nabla}_{E_i}E_1, E_2) = h(\widehat{\nabla}_{E_1}E_1, E_i) = h(\widehat{\nabla}_{E_1}E_2, E_i) = 0$. Since we already know that $h(E_A, \widehat{\nabla}_{E_B}E_C) = 0$ for $A, B, C \in \{1, 2\}$ we deduce that $\widehat{\nabla}_{E_1}E_1 = \widehat{\nabla}_{E_1}E_2 = 0$. From (4.2) it then follows that $\widehat{\nabla}_{E_2}E_1 = \widehat{\nabla}_{E_2}E_2 = 0$. Using this, we see that $\widehat{\nabla}_{E_1}E_i$ and $\widehat{\nabla}_{E_2}E_i$ are orthogonal to both E_1 and E_2. Therefore, we get that

$$(\widehat{\nabla}_{E_1}K)(E_i, E_1) = -\tfrac{2}{n-2}\widehat{\nabla}_{E_1}E_i - K(\widehat{\nabla}_{E_1}E_i, E_1) = 0.$$

Similarly, we also get that $(\widehat{\nabla}_{E_1}K)(E_i, E_2) = 0$. It now follows from the Codazzi equations that

$$\begin{aligned} 0 &= (\widehat{\nabla}_{E_i}K)(E_1, E_1) \\ &= \widehat{\nabla}_{E_i}E_1 - \tfrac{4n(n-1)}{(n-2)^3 J}\widehat{\nabla}_{E_i}E_2 - 2K(\widehat{\nabla}_{E_i}E_1, E_1) \\ &= \tfrac{(n+2)}{(n-2)}\widehat{\nabla}_{E_i}E_1 - \tfrac{4n(n-1)}{(n-2)^3 J}\widehat{\nabla}_{E_i}E_2, \\ 0 &= (\widehat{\nabla}_{E_i}K)(E_1, E_2) \\ &= \tfrac{(n-2)^2 J}{4(n-1)}\widehat{\nabla}_{E_i}E_1 + \widehat{\nabla}_{E_i}E_2 - K(\widehat{\nabla}_{E_i}E_1, E_2) - K(E_1, \widehat{\nabla}_{E_i}E_2) \\ &= \tfrac{(n-2)nJ}{4(n-1)}\widehat{\nabla}_{E_i}E_1 + \tfrac{n}{n-2}\widehat{\nabla}_{E_i}E_2. \end{aligned}$$

Therefore, we deduce that $\widehat{\nabla}_{E_i}E_1 = \widehat{\nabla}_{E_i}E_2 = 0$.

In order to show that $\widehat{\nabla}_{E_j}E_i = 0$ for all $j \geq i$ we use an induction argument. Suppose that we know that $\widehat{\nabla}_{E_j}E_k = 0$ for all $j \geq k$ and for all $k \leq i-1$. Then, for $j > i$ it follows from $(\widehat{\nabla}_{E_j}K)(E_i,E_i) = (\widehat{\nabla}_{E_i}K)(E_j,E_i)$ and our induction hypothesis that

$$\begin{aligned}(4.4)\qquad &\widehat{\nabla}_{E_j}(n-i)\lambda_{i-2}E_i - 2K(\widehat{\nabla}_{E_j}E_i,E_i)\\ &\quad= \widehat{\nabla}_{E_i}(-\lambda_{i-2}E_j) - K(E_i,\widehat{\nabla}_{E_i}E_j) - K(\widehat{\nabla}_{E_i}E_i,E_j),\end{aligned}$$

where $j > i$. From our induction hypothesis and since $\widehat{\nabla}$ is the Levi Civita connection of h, we know that $\widehat{\nabla}_{E_j}E_i$ is orthogonal to $E_1,\dots,E_i$. This together with Lemma 4.1 implies that the left side of (4.4) is orthogonal to E_i. Therefore, by taking the E_i component of (4.4), we find that

$$0 = 2\lambda_{i-2}h(\widehat{\nabla}_{E_i}E_i,E_j) - h(\widehat{\nabla}_{E_i}E_j,K(E_i,E_i)).$$

Since by our induction hypothesis, we have that $\widehat{\nabla}_{E_i}E_j$ is orthogonal to $E_1,\dots,$ E_{i-1}, we deduce from the above equation that

$$(n-i+2)\lambda_{i-2}h(\widehat{\nabla}_{E_i}E_i,E_j) = 0.$$

Again, using our induction hypothesis, we deduce that $\widehat{\nabla}_{E_i}E_i = 0$. If $l < i$ we have

$$h((\widehat{\nabla}_{E_i}K)(E_j,E_i),E_l) = -h(K(E_l,E_i),\widehat{\nabla}_{E_i}E_j) = -\lambda_{l-2}h(E_i,\widehat{\nabla}_{E_i}E_j) = 0$$

and for $l > i$ we have

$$h((\widehat{\nabla}_{E_i}K)(E_j,E_i),E_l) = -\lambda_{i-2}h(\widehat{\nabla}E_iE_j,E_l) - h(\widehat{\nabla}E_iE_j,-\lambda_{i-2}E_l) = 0.$$

Since we have already shown that $h((\widehat{\nabla}_{E_i}K)(E_j,E_i),E_i) = 0$ it follows that

$$(\widehat{\nabla}_{E_i}K)(E_j,E_i) = -\lambda_{i-2}\widehat{\nabla}_{E_i}E_j - K(\widehat{\nabla}_{E_i}E_j,E_i) = 0,$$

since $\widehat{\nabla}_{E_i}E_j$ is orthogonal to $E_1,\dots,E_i$. Again, using the Codazzi equation (4.4), we deduce that $\widehat{\nabla}_{E_j}E_i = 0$ for $j > i$.

We still have to show that $\widehat{\nabla}_{E_i}E_j = 0$ for all $j > i$. Suppose that $\widehat{\nabla}_{E_i}E_k = 0$, where $j > 2$, $i < k$ and $k < j$. Using an analogous argument as before, it follows that for any number $j > i$, we have that

$$(\widehat{\nabla}_{E_j}K)(E_i,E_j) = 0.$$

On the other hand, we have that

$$\begin{aligned}(\widehat{\nabla}_{E_i}K)(E_j,E_j) &= \widehat{\nabla}_{E_i}K(E_j,E_j) - 2K(\widehat{\nabla}_{E_i}E_j,E_j)\\ &= (n-j)\lambda_{j-2}\widehat{\nabla}_{E_i}E_j - 2K(\widehat{\nabla}_{E_i}E_j,E_j)\\ &= (n-j+2)\lambda_{j-2}\widehat{\nabla}_{E_i}E_j.\end{aligned}$$

This completes the proof of the lemma. □

Now, we can formulate the main theorem.

Main Theorem. *Let M^n, $n \geq 3$, be a Lorentzian affine sphere with constant curvature a and nonzero Pick invariant J. Then a is zero and M is affinely equivalent to either*

$$(x_1^2+x_2^2)(x_3^2-x_4^2) = 1,$$
$$(x_1^2+x_2^2)(x_3^2+x_4^2) = 1,$$

if $n = 3$ or M is affinely equivalent to

$$(x_1^2+x_2^2)x_3\dots x_{n+1} = 1,$$

if $n > 3$.

Proof. We denote the immersion of M^n into $\mathbb{R}^{n+1}$ by f. We take the local orthonormal basis constructed in Lemma 4.2. Since $\widehat{\nabla}_{E_i}E_j = 0$, the sectional curvature a vanishes and there exist local coordinates $u_1, \dots, u_n$ such that $f_{u_i} = E_i$. From Lemma 4.1 and $D_V W = K_V W + \widehat{\nabla}_V W + h(V,W)Jf$ it follows that the immersion f is determined by the following system of differential equations:

$$(*)\qquad \begin{aligned} f_{u_1u_1} &= f_{u_1} - \tfrac{4(n-1)n}{(n-2)^3}\tfrac{1}{J} f_{u_2}, \\ f_{u_1u_2} &= \tfrac{(n-2)^2J}{4(n-1)} f_{u_1} + f_{u_2} + Jf, \\ f_{u_2u_2} &= -\tfrac{(n-2)^3nJ^2}{16(n-1)^2} f_{u_1} + \tfrac{(n-2)^2J}{4(n-1)} f_{u_2}, \\ f_{u_1u_i} &= -\tfrac{2}{n-2} f_{u_i}, \\ f_{u_2u_i} &= -\tfrac{(n-2)}{2(n-1)} J f_{u_i}, \\ f_{u_iu_j} &= -\lambda_{i-2} f_{u_j}, \\ f_{u_iu_i} &= -\tfrac{(n-2)}{2(n-1)} J f_{u_1} - \tfrac{2}{n-2} f_{u_2} - \sum_{k=3}^{i-1} \lambda_{k-2} f_{u_k} + (n-i)\lambda_{i-2} f_{u_i} + Jf, \end{aligned}$$

where $j, i = 3, \dots, n$ with $j > i$, and $\lambda_1, \dots, \lambda_{n-2}$ are inductively defined by

$$\lambda_1 = \sqrt{\tfrac{(n+1)|J|}{(n-1)(n-2)}}, \quad \lambda_{i+1} = \sqrt{\tfrac{n-i}{n-i-2}}\,\lambda_i.$$

Now our system of differential equations implies

$$\begin{aligned} f_{u_1u_1u_1} &= f_{u_1u_1} - \tfrac{4(n-1)n}{(n-2)^3}\tfrac{1}{J}\left(\tfrac{(n-2)^2J}{4(n-1)} f_{u_1} + f_{u_2} + Jf\right) \\ &= f_{u_1u_1} - \tfrac{n}{n-2} f_{u_1} + (f_{u_1u_1} - f_{u_1}) - \tfrac{4(n-1)n}{(n-2)^3} f \\ &= 2f_{u_1u_1} - \tfrac{2n-2}{n-2} f_{u_1} - \tfrac{4(n-1)n}{(n-2)^3} f. \end{aligned}$$

Any solution of this ordinary differential equation which satisfies $f_{u_1u_i} = -\frac{2}{n-2} f_{u_i}$ is given by

$$\begin{aligned} f =& A^1(u_2, u_3, \dots, u_n) e^{-\frac{2}{n-2}u_1} + B^1(u_2) e^{\frac{(n-1)}{(n-2)}u_1} \cos\left(\tfrac{\sqrt{n^2-1}}{n-2}u_1\right) \\ &+ B^2(u_2) e^{\frac{(n-1)}{(n-2)}u_1} \sin\left(\tfrac{\sqrt{n^2-1}}{n-2}u_1\right), \end{aligned}$$

where A^1, B^1, B^2 are arbitrary functions. Using $f_{u_2} = -\frac{(n-2)^3J}{4(n-1)n}(f_{u_1u_1} - f_{u_1})$ and the fact that $\left\{e^{-\frac{2}{n-2}u_1}, e^{\frac{(n-1)}{(n-2)}u_1}\cos(\frac{\sqrt{n^2-1}}{n-2}u_1), e^{\frac{(n-1)}{(n-2)}u_1}\sin(\frac{\sqrt{n^2-1}}{n-2}u_1)\right\}$ are linearly independent functions we obtain

$$\begin{aligned} A^1_{u_2} &= -\tfrac{(n-2)^3J}{4(n-1)n}\left(\tfrac{4}{(n-2)^2} + \tfrac{2}{n-2}\right)A^1 \\ &= -\tfrac{(n-2)J}{2(n-1)} A^1, \\ B^1_{u_2} &= -\tfrac{(n-2)^3J}{4(n-1)n}\left(\left(\tfrac{(n-1)^2}{(n-2)^2} - \tfrac{n^2-1}{(n-2)^2} - \tfrac{n-1}{n-2}\right)B^1 + \tfrac{\sqrt{n^2-1}}{n-2}\left(\tfrac{2(n-1)}{n-2} - 1\right)B^2\right) \\ &= \tfrac{(n-2)J}{4} B^1 - \tfrac{\sqrt{n^2-1}(n-2)J}{4(n-1)} B^2, \\ B^2_{u_2} &= \tfrac{\sqrt{n^2-1}(n-2)J}{4(n-1)} B^1 + \tfrac{(n-2)J}{4} B^2. \end{aligned}$$

Solving this linear system of differential equations and taking into account $f_{u_2u_i} = -\frac{(n-2)}{2(n-1)}Jf_{u_i}$, we get

$$\begin{aligned} f =&A^2(u_3,\dots,u_n)e^{-\frac{2}{n-1}\left(\frac{n-1}{n-2}u_1+\frac{(n-2)J}{4}u_2\right)} \\ &+C_1e^{\left(\frac{(n-1)}{(n-2)}u_1+\frac{(n-2)J}{4}u_2\right)}\cos\left(\tfrac{\sqrt{n^2-1}}{n-1}\left(\tfrac{n-1}{n-2}u_1-\tfrac{(n-2)J}{4}u_2\right)\right) \\ &+C_2e^{\left(\frac{(n-1)}{(n-2)}u_1+\frac{(n-2)J}{4}u_2\right)}\sin\left(\tfrac{\sqrt{n^2-1}}{n-1}\left(\tfrac{n-1}{n-2}u_1-\tfrac{(n-2)J}{4}u_2\right)\right). \end{aligned}$$

If $n=3$, it follows from

$$f_{u_3u_3} = Jf - \tfrac{(n-2)}{2(n-1)}Jf_{u_1} - \tfrac{2}{(n-2)}f_{u_2},$$

that

$$A^2_{u_3u_3} = 2JA^2.$$

Now, we have to consider two cases. If $J>0$, we obtain

$$A^2(u_3) = C_4e^{\sqrt{2J}u_3} + C_3e^{-\sqrt{2J}u_3}.$$

Hence in this case f is given by

$$\begin{aligned} f(u_1,u_2,u_3) =&(C_4e^{\sqrt{2J}u_3}+C_3e^{-\sqrt{2J}u_3})e^{-2u_1-\frac{1}{4}Ju_2} \\ &+C_1e^{2u_1+\frac{1}{4}Ju_2}\cos 2\sqrt{2}(u_1-\tfrac{1}{8}Ju_2) \\ &+C_2e^{2u_1+\frac{1}{4}Ju_2}\sin 2\sqrt{2}(u_1-\tfrac{1}{8}Ju_2). \end{aligned}$$

It follows that M is affinely equivalent to the hypersurface given by

$$x_1x_2(x_3^2+x_4^2) = 1.$$

If $J<0$, we proceed similarly to find $A^2 = C^4\cos(\sqrt{-2J}u_3) + C_3\sin(\sqrt{-2J}u_3)$ and therefore that M is affinely equivalent to $(x_1^2+x_2^2)(x_3^2+x_4^2)=1$.

If $n>3$, the last equation $(*)$ in our system of differential equations gives for $i=3$,

$$A^2_{u_3u_3} = (n-3)\lambda_1A^2_{u_3} + (n-2)(\lambda_1)^2A^2,$$

and therefore the existence of a function A^3 and a constant C^3 such that

$$A^2(u_3,\dots,u_n) = A^3(u_4,\dots,u_n)e^{-\lambda_1u_3} + C_3e^{(n-2)\lambda_1u_3}.$$

Inserting this equation back into $(*)$ we get

$$\begin{aligned} Jf - \tfrac{(n-2)}{2(n-1)}Jf_{u_1} - \tfrac{2}{n-2}f_{u_2} - \lambda_1f_{u_3}& \\ = (n-3)(\lambda_2)^2A^3(u_4,\dots,u_n)&e^{-\lambda_1u_3-\frac{2}{n-1}\left(\frac{n-1}{n-2}u_1+\frac{(n-2)J}{4}u_2\right)}. \end{aligned}$$

We will now show that there exists maps $A^3,\dots,A^{n-2}$ and constant vectors C_3, $\dots$, C_{n-1} such that

$$\begin{aligned} A^2(u_3,\dots,u_n) &= A^3(u_4,\dots,u_n)e^{-\lambda_1u_3} + C_3e^{(n-2)\lambda_1u_3} \\ A^3(u_4,\dots,u_n) &= A^4(u_5,\dots,u_n)e^{-\lambda_2u_4} + C_4e^{(n-3)\lambda_2u_4} \\ &\vdots \\ A^{n-2}(u_{n-1},u_n) &= A^{n-1}(u_n)e^{-\lambda_{n-3}u_{n-1}} + C_{n-1}e^{2\lambda_{n-3}u_{n-1}}. \end{aligned}$$

Suppose that for $j = 3, \dots, r < n-1$ there exists maps A_j and constant vectors C_j with

$$A^{j-1}(u_j, \dots, u_n) = A^j(u_{j+1}, \dots, u_n)e^{-\lambda_{j-2}u_j} + C_j e^{(n-j+1)\lambda_{j-2}u_j}$$

and that

$$(**)\qquad \begin{aligned} &Jf - \tfrac{(n-2)}{2(n-1)}Jf_{u_1} - \tfrac{2}{n-2}f_{u_2} - \sum_{i=1}^{r-2}\lambda_i f_{u_{i+2}} \\ &\quad = (n-r)(\lambda_{r-1})^2 A^r(u_{r+1}, \dots, u_n)e^{-\sum_{i=1}^{r-2}\lambda_i u_{i+2} - \frac{2}{n-1}\left(\frac{n-1}{n-2}u_1 + \frac{(n-2)J}{4}u_2\right)} \end{aligned}$$

holds. From the definition of $A^3, \dots, A^r$ we get

$$f_{u_{r+1}u_{r+1}} = A^r_{u_{r+1}u_{r+1}} e^{-\sum_{i=1}^{r-2}\lambda_i u_{i+2} - \frac{2}{n-1}\left(\frac{n-1}{n-2}u_1 + \frac{(n-2)J}{4}u_2\right)}.$$

On the other hand, equations $(*)$ and $(**)$ imply that

$$\begin{aligned} f_{u_{r+1}u_{r+1}} &= Jf - \tfrac{(n-2)}{2(n-1)}Jf_{u_1} - \tfrac{2}{n-2}f_{u_2} - \sum_{i=1}^{r-2}\lambda_i f_{u_{i+2}} + (n-r-1)\lambda_{r-1}f_{u_{r+1}} \\ &= ((n-r)(\lambda_{r-1})^2 A^r + (n-r-1)\lambda_{r-1}A^r_{u_{r+1}}) \\ &\quad e^{-\sum_{i=1}^{r-2}\lambda_i u_{i+2} - \frac{2}{n-1}\left(\frac{n-1}{n-2}u_1 + \frac{(n-2)J}{4}u_2\right)} \end{aligned}$$

and therefore

$$(***)\qquad A^r_{u_{r+1}u_{r+1}} = (n-r)(\lambda_{r-1})^2 A^r + (n-r-1)\lambda_{r-1}A^r_{u_{r+1}}.$$

Since $f_{u_{r+1}u_j} = -\lambda_{r-1}f_{u_j}$, for $j > r+1$ implies that

$$A^r(u_{r+1}, \dots, u_n) = A^{r+1}(u_{r+2}, \dots, u_n)e^{-\lambda_{r-1}u_{r+1}} + \tilde{C}_{r+1}(u_{r+1});$$

we can solve $(***)$ to get

$$A^r(u_{r+1}, \dots, u_n) = A^{r+1}(u_{r+2}, \dots, u_n)e^{-\lambda_{r-1}u_{r+1}} + C_{r+1}e^{(n-r)\lambda_{r-1}u_{r+1}}.$$

Now we only need to verify equation $(**)$ with r being replaced by $r+1$. But this follows from

$$\begin{aligned} &Jf - \tfrac{(n-2)}{2(n-1)}Jf_{u_1} - \tfrac{2}{n-2}f_{u_2} - \sum_{i=1}^{r-1}\lambda_i f_{u_{i+2}} \\ &\quad = f_{u_{r+1}u_{r+1}} - (n-r-1)\lambda_{r-1}f_{u_{r+1}} - \lambda_{r-1}f_{u_{r+1}} \\ &\quad = \Big(A^r_{u_{r+1}u_{r+1}} - (n-r-1)\lambda_{r-1}A^r_{u_{r+1}} - \lambda_{r-1}A^r_{u_{r+1}}\Big) \\ &\qquad \times e^{-\sum_{i=1}^{r-2}\lambda_i u_{i+2} - \frac{2}{n-1}\left(\frac{n-1}{n-2}u_1 + \frac{(n-2)J}{4}u_2\right)} \\ &\quad = \Big((n-r)(\lambda_{r-1})^2\big(A^{r+1}e^{-\lambda_{r-1}u_{r+1}} + C_{r+1}e^{(n-r)\lambda_{r-1}u_{r+1}}\big) \\ &\qquad - \lambda_{r-1}\big(-\lambda_{r-1}A^{r+1}e^{-\lambda_{r-1}u_{r+1}} + (n-r)\lambda_{r-1}C_{r+1}e^{(n-r)\lambda_{r-1}u_{r+1}}\big)\Big) \\ &\qquad \times e^{-\sum_{i=1}^{r-2}\lambda_i u_{i+2} - \frac{2}{n-1}\left(\frac{n-1}{n-2}u_1 + \frac{(n-2)J}{4}u_2\right)} \\ &\quad = (n-r-1)(\lambda_r)^2 A^{r+1}e^{-\lambda_{r-1}u_{r+1} - \sum_{i=1}^{r-2}\lambda_i u_{i+2} - \frac{2}{n-1}\left(\frac{n-1}{n-2}u_1 + \frac{(n-2)J}{4}u_2\right)}. \end{aligned}$$

Note that equation $(***)$ gives for $r = n-1$ the equation

$$A^{n-1}_{u_n u_n} = (\lambda_{n-2})^2 A^{n-1}$$

and therefore $A^{n-1}(u_n) = C_{n+1}e^{\lambda_{n-2}u_n} + C_n e^{-\lambda_{n-2}u_n}$. We can now successively solve for all A^j and obtain a solution which only depends on $n+1$ constant vectors $C_1, \dots, C_{n+1} \in \mathbb{R}^{n+1}$. The equiaffine transformation defined by

$$\begin{aligned} C_{n+1} &= (1, 0, \dots, 0), \\ C_n &= (0, 1, 0, \dots, 0), \\ &\vdots \\ C_1 &= (0, \dots, 0, 1), \end{aligned}$$

maps, therefore, our solution onto

$$\{x \in \mathbb{R}^n | x_1 x_2 x_3 \dots x_{n-1}(x_n^2 + x_{n+1}^2) = 1\}.$$

This completes the proof. □

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