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AN ADDENDUM TO "PATH-LIFTING FOR GROTHENDIECK TOPOSES"

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The purpose of this note is to prove the conjecture in [M], viz. that if $\mathcal{F} \to \mathcal{E}$ is a connected and locally connected morphism between (Grothendieck) toposes then $\mathcal{F}^I \to \mathcal{E}^I$ is an open surjection. In [M], the following partial results were proved (cf. also Remark (ii) below):

(1) If $\mathcal{F} \to \mathcal{E}$ is a connected and locally connected, then $\mathcal{F}^I \to \mathcal{E}^I$ is a stable surjection (between toposes).
(2) If $Y \to X$ is a connected and locally connected map between locales, then $Y^I \to X^I$ is an open surjection (between locales).

Here, as in [M], I denotes the unit interval $[0,1]$, viewed as a locale or as a topos (viz. the topos of sheaves of sets on $[0,1]$).

I would like to gratefully acknowledge that my attention was drawn again to this issue in relation to my work with Bunge on the paths-fundamental group of a topos [BM], and with Kock on étendues. Indeed, it is the stronger result to be proved here which is needed in these applications. (Without going into details, the point is that this result implies that for a localic groupoid G with connected and locally connected source and target maps, one can now immediately obtain a groupoid representation for the path-space of the classifying topos of G, by the equivalence $\mathcal{B}(G)^I \simeq \mathcal{B}(G^I)$; I assume here that G is étale complete, of course.)

Consider for a topos $\mathcal{E}$ its "canonical" connected and locally connected cover

$$X_{\mathcal{E}} \to \mathcal{E}$$

by a locale, where $X_{\mathcal{E}} = \mathcal{E}[En(G)]$ is the topos of $\mathcal{E}$-internal sheaves on the locale $En(G)$ of infinite-to-one partial enumerations of a generating object G in $\mathcal{E}$ (see [JM]). Recall that this construction is "natural in $\mathcal{E}$" in the sense that it is the pullback of the generic case where $\mathcal{E}$ is the object classifier $\mathcal{S}[U]$ and G is the universal object U. I will show:

Proposition. *$(X_{\mathcal{S}[U]})^I \to \mathcal{S}[U]^I$ is an open surjection.*

From this proposition, together with (2) above, the desired result follows:

Corollary. *If $\mathcal{F} \to \mathcal{E}$ is connected and locally connected then $\mathcal{F}^I \to \mathcal{E}^I$ is an open surjection.*

Proof. In the proof of the corollary, we will use the following well-known stability properties of open (surjective) topos morphisms: these are closed under composition

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and pullback, they descend down pullback along an open surjection, and are stable under taking quotients (in the sense that if fg is an open surjection and g is surjective then f is an open surjection).

For the three "canonical" localic covers $X_{\mathcal{E}} \to \mathcal{E}, X_{\mathcal{F}} \to \mathcal{F}$ and $X_{\mathcal{S}[U]} \to \mathcal{S}[U]$, their path-topoi fit into a diagram

(1)
$$\begin{tikzcd}
(X_{\mathcal{S}[U]})^I \arrow[d] & {X_{\mathcal{E}}}^I \arrow[l] \arrow[d] & & (X_{\mathcal{E}} \times_{\mathcal{E}} X_{\mathcal{F}})^I \arrow[ll] \arrow[d] \\
\mathcal{S}[U]^I & \mathcal{E}^I \arrow[l] & \mathcal{F}^I \arrow[l] & X_{\mathcal{F}}^I \arrow[l]
\end{tikzcd}$$

Here the lower left map is induced by the classifying map $\mathcal{E} \to \mathcal{S}[U]$ for a generating object G in $\mathcal{E}$, and both squares are pullbacks. By the proposition, the left-hand vertical map is an open surjection. Since open surjections are pullback stable, the other two vertical maps as well as $(X_{\mathcal{F}})^I \to \mathcal{F}^I$ must also be open surjections. Next, by the assumption on the morphism $\mathcal{F} \to \mathcal{E}$ and fact (2) above, the upper right horizontal map is an open surjection. Thus, the composite $(X_{\mathcal{F}})^I \to \mathcal{E}^I$ must be an open surjection. Since $(X_{\mathcal{F}})^I \to \mathcal{F}^I$ is an open surjection, it thus follows that $\mathcal{F}^I \to \mathcal{E}^I$ is an open surjection.

Proof of Proposition. For any topos $\mathcal{T}$, consider the pullback $\mathcal{T}'$ along any map f,

(2)
$$\begin{tikzcd}
\mathcal{T}' \arrow[r] \arrow[d] & (X_{\mathcal{S}[U]})^I \arrow[d] \\
\mathcal{T} \arrow[r, "f"] & \mathcal{S}[U]^I
\end{tikzcd}$$

We will prove that $\mathcal{T}' \to \mathcal{T}$ is an open surjection. The map f classifies an internal sheaf $S \to I$ on the unit interval inside the topos $\mathcal{T}$. The topos $\mathcal{T}'$, as a topos over $\mathcal{T}$, classifies an (internal) partial enumeration in $Sh(I)$ of this sheaf S. To show that $\mathcal{T}' \to \mathcal{T}$ is an open surjection, it thus suffices to prove the following lemma, (and apply it inside $\mathcal{T}$).

Lemma. *Let S be a sheaf on I (in a base topos $\mathcal{S}$). There exists a classifying topos $\mathcal{E}$ for infinite-to-one partial enumerations*

(3) $$\triangle(\mathbb{N}) \hookleftarrow D \twoheadrightarrow S \qquad (\textit{inside sheaves on } I)$$

and this topos $\mathcal{E}$ is an open surjective topos over the base $\mathcal{S}$.

(In the statement of the lemma, $\mathbb{N}$ denotes the natural numbers object of $\mathcal{S}$, and $\triangle(\mathbb{N})$ the corresponding constant sheaf on I.)

Proof. (We use the language as if $\mathcal{S}$ = Sets, as usual.) Consider closed rational intervals $C = [p, q]$ where $0 \le p \le q \le 1$, and let $\mathcal{C}$ be the collection of all such. For $C \in \mathcal{C}$, denote by $\Gamma(C, S)$ the "set" (object of $\mathcal{S}$) of sections of S over C. Let $\Gamma(\mathcal{C}, S)$ be the union of all the sets $\Gamma(C, S)$. Now consider *finite* partial functions

(4) $$a : \mathcal{C} \times \mathbb{N} \to \Gamma(\mathcal{C}, S)$$

with the property that

(i) if $a(C, n)$ is defined then $a(C, n) \in \Gamma(C, S)$,
(ii) if $a(C, n)$ and $a(D, n)$ are defined then $a(C, n) \restriction C \cap D = a(D, n) \restriction C \cap D$.

Define a preorder on the set of such a, by setting $a \leq b$ iff whenever $b(C,n)$ is defined, there exists a $D \supseteq C$ so that $a(D,n)$ is defined and $a(D,n) \restriction C = b(C,n)$. (A partial function a as in (2) is meant to code up part of the partial enumeration (1); if $a \leq b$ then a contains "more information" than b.)

Now equip this preorder with the following covers:

(a) If $a(C,n)$ is defined and equals $s \in \Gamma(C,S)$, extend s to a section t on an interval D with $C \subseteq Int(D)$. Then a is covered by the set of all functions $a \cup \{(C', n, t \restriction C')\}$, where $C \subseteq Int(C')$ and $C' \subseteq D$.

(b) For any C and any $s \in \Gamma(C,S)$ and any $n \in \mathbb{N}$, a is covered by the set of all extensions of the form $a \cup \{(C_1, n_1, s \restriction C_1), \cdots, (C_k, n_k, s \restriction C_k)\}$, where $C = C_1 \cup \cdots \cup C_k$ and $n_i \geq n$.

Observe that each of these covers is non-empty ("inhabited"): for (a), use that S is a sheaf; for (b), use that if n_i is chosen large enough, the compatibility condition (ii) is automatically met. Furthermore, observe that this system of covers is stable, in the sense that if $\{a_i\}$ covers a and $b \leq a$ then there is a cover $\{b_j\}$ of b so that each $b_j \leq$ some a_i. (This is easily verified and I omit the details.)

The preorder of all these finite partial functions (2), with this covering system, is a site, and the topos $\mathcal{E}$ of sheaves on this site clearly classifies partial enumerations of the form (1). Indeed, the covers of type (a) ensure that this partial enumeration is defined on an open subset of $\triangle(\mathbb{N})$, i.e. on a *subsheaf*, while the covers of type (b) ensure that this partial enumeration is infinite-to-one. Furthermore, $\mathcal{E}$ is an open surjective topos over $\mathcal{S}$, precisely by the fact that it is defined by a site consisting of inhabited covers only (see [JT]).

This completes the proof of the lemma. $\square$

Remarks. (i) The canonical cover $X_{\mathcal{E}} \to \mathcal{E}$, used above, is constructed in [JM] using the locale $E(G)$ of infinite-to-one enumerations, to ensure that the cover has contractible fibers. Here, we only need that $X_{\mathcal{E}} \to \mathcal{E}$ is connected and locally connected, and for this one can replace $En(G)$ by the locale of partial enumerations of G. (This would have allowed for a slightly simpler lemma.)

(ii) The statement of the original conjecture in [M] is slightly stronger than the statement of the Corollary, and asserts openness of the map $\mathcal{F}^I \to (\mathcal{F} \times \mathcal{F}) \times_{(\mathcal{E} \times \mathcal{E})} \mathcal{E}^I$; but the extra complications are only notational and technical, which is why I have chosen the simpler formulation here. The stronger statement, which concerns "path-lifting with fixed end-points", can be proved in exactly the same way, using an analogous lemma for the classifying topos of partial (infinite-to-one) enumerations of S as in (1) which extend given such enumerations of the stalks S_0 and S_1 at the endpoints.

(iii) In the proof of the lemma, I have used for a sheaf S on the unit interval, its sections on closed intervals C. Note that such a sheaf S can be completely described in terms of such sections by the following geometric theory: For each closed interval C there is a sort $\Gamma(C,S)$, and for each inclusion $D \subseteq C$ there is a function symbol $\rho = \rho_{D,C} : \Gamma(C,S) \to \Gamma(D,S)$. The axioms are:

(i) $\rho_{E,D} \circ \rho_{D,C} = \rho_{E,C}$, $\rho_{C,C} = id$ (for $E \subseteq D \subseteq C$).

(ii) $\forall s \in \Gamma(C,S) \ \bigvee_D \ \exists t \in \Gamma(D,S) \ \ \rho(t) = s$, where D ranges over intervals with $C \subseteq Int(D)$,

(iii) $\forall s,t \in \Gamma(C,S) \ \ [\rho_{D,C}(s) = \rho_{D,C}(t) \ \Rightarrow \ \bigvee_E \rho_{E,C}(s) = \rho_{E,C}(t)]$, where $D \subseteq Int(C)$, and E ranges over all intervals $E \subseteq C$ with $D \subseteq Int(E)$.

(iv) $\forall s \in \Gamma(C,S) \forall t \in \Gamma(D,S)\ [\rho_{C\cap D,C}(s) = \rho_{C\cap D,D}(t) \Rightarrow \exists! r \in \Gamma(E,S)(\rho_{C,E}(r) = s \wedge \rho_{D,E}(r) = t)]$, where $E = C \cup D$.

The fact that models of this geometric theory correspond to sheaves on the unit interval provides a construction of the exponential topos $\mathcal{E}^I$ (i.e. $\mathcal{E}^{Sh(I)}$), different from the one in [JJ].

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