MATHEMATICS

REPRESENTABILITY IN LAMBDA ALGEBRAS

BY

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SUMMARY

§ 1 is concerned with the term model of the λ-calculus. It is proved that Church's δ is not definable and that the definable functions into the numerals are constant. In § 2 it is proved that for several λ-algebras the range of a representable function is either a singleton or infinite. In § 3 it is examined in which λ-algebras the local representability of external functions implies the global representability.

INTRODUCTION

Let $\mathcal{M} = \langle M, \cdot \rangle$ be a λ-algebra (i.e. a model of the λ-calculus). Elements of M are thought of as functions. Arbitrary $f: M \to M$ are called external functions. Such a function is *representable* (by an element $a \in M$) if $\forall b \in M\ f(b) = a \cdot b$. A function f is *definable* in $\mathcal{M}$ if f is representable by $[\![F]\!]^{\mathcal{M}}$ for some closed term F. Here $[\![F]\!]^{\mathcal{M}}$ denotes the value of F in the model $\mathcal{M}$.

Other notations:

$x, y, \ldots$ denote variables of the λ-calculus.
$a, b, \ldots$ denote variables ranging over the elements of a λ-algebra.
$F, G, \ldots$ denote λ-terms.

The *numerals* $\underset{\sim}{0}, \underset{\sim}{1}, \underset{\sim}{2}, \ldots$ denote some adequate representation of the natural numbers as λ-terms e.g. those of Church: $\underset{\sim}{n} = \lambda fx.\ f^n(x)$.

If $\mathcal{M} = \langle M, \cdot \rangle$ is a λ-algebra, then $\mathcal{M}^0$ is the sub-λ-algebra $\langle M^0, \cdot \rangle$ where $M^0 = \{[\![F]\!]^{\mathcal{M}} \in M \mid F \text{ closed term}\}$.

If T is a consistent extension of the λ-calculus, $\mathcal{M}(T)$ is the term-model of T, i.e. the set of all λ-terms modulo provable equality in T. The closed term-model of T, notation $\mathcal{M}^0(T)$, is defined as $(\mathcal{M}(T))^0$. A λ-algebra $\mathcal{M}$ is *hard* if $\mathcal{M} = \mathcal{M}^0$. In such an $\mathcal{M}$ a function is representable iff it is definable.

For other terminology see Barendregt [1976].

The three sections of the paper treat different aspects of the notion of representability.

In § 1 attention is restricted to the standard extensional term model $\mathcal{M} = \mathcal{M}(\lambda\eta)$.

Church's δ is an external function satisfying

($\star$) $\delta MM = \underline{0}$ if M is a closed normal form (nf)
$\delta MM' = \underline{1}$ if M, M' are different closed nf's.

In Böhm [1972] it is proved that $\forall N_1 \ldots N_n$ different $\beta\eta$-nf's $\exists F \vdash FN_i = \underline{i}$. As a consequence it follows that for every finite set A of nf's there is a term δ satisfying ($\star$) for $M, M' \in A$.

At the Orléans logic conference (1972) the question was raised whether the general Church's δ is definable as a λ-term.

We will give a negative answer which was already established in Barendregt [1972] and independently in Wadsworth [1972] (see also Hindley and Mitschke [1975]). All three proofs of the non-existence of δ are different.

Furthermore it is proved that the only definable functions from the terms into the numerals are the constant functions.

In § 2 it will be proved that definable functions in various λ-algebras have a range of cardinality 1 or $\aleph_0$. For representable functions this is not true in D_∞ and $P\omega$.

Two external functions f and g on $\mathscr{M}$ are *dual*, notation $f \sim_{\mathscr{M}} g$, if $f(a)\cdot b = g(b)\cdot a$ for all $a, b \in \mathscr{M}$. In that case for each b the map $\boldsymbol{\lambda} a.\ f(a).b$ is representable and f is said to be locally representable, similarly for g.

A model $\mathscr{M}$ is *rich* if for all f, g:

$$f \sim_{\mathscr{M}} g \Rightarrow f \text{ and } g \text{ are representable in } \mathscr{M}.$$

The results of § 3 are: D_∞ and $\mathscr{M}(\lambda\eta)$ are rich; rich models are extensional; hard sensible models (e.g. the interior of D_∞) are not rich.

We would like to draw the proof of 3.6 to the reader's attention. There variables of the λ-calculus are not just used in the usual way, but also serve as separate entities.

§ 1. NON-DEFINABILITY RESULTS

The main tool in this section is the "Böhm out" technique 1.4. This result is also of use in § 2.

1.1. DEFINITION. Let $BT(M)$ be the Böhm tree of M, see Barendregt [1976], § 6. $x \in BT(M)$ iff $x \in FV(M^k)$ for some k, where M^k is the k^{th} approximate normal form of M.

1.2. DEFINITION. (i) A selector is a term of the form

$$U \equiv \lambda x_1 \ldots x_n \cdot x_i, \quad 1 \leqslant i \leqslant n.$$

A permutator is a term of the form

$$C \equiv \lambda x_1 \ldots x_n \cdot x_{\pi(1)} \ldots x_{\pi(n)}$$

for some permutation π.

(ii) Simple terms are inductively defined by: Any variable, selector or permutator is a simple term. If P, Q are simple terms, so is PQ.

1.3. LEMMA. Simple terms have a normal form (*nf*).

PROOF. Realize that each simple term is of the form $x\vec{P}$, $U\vec{P}$, $C\vec{P}$ with $\vec{P}$ simple, U a selector and C a permutator. Then it can be shown by induction on the term length that they have a *nf*. ■

1.4. THEOREM. Let $FV(M)=\{x\}$ and $x \in BT(M)$. Then
(i) For some $\vec{P}, \vec{Q}$, with $x \notin FV(\vec{P})$, $\lambda \vdash M\vec{P}=x\vec{Q}$ ("x is Böhmed out").
(ii) Moreover $\vec{P}$ can be chosen as a sequence of simple terms.

PROOF. Let x occur in $BT(M)$ at depth $k>0$. By a similar construction as in Barendregt [1976] 6.14, 6.15, for some Böhm-transformation π_1, x occurs in $BT(M^{\pi_1})$ at depth $k-1$. Iterating this leads to $M^{\pi_2}=\lambda\vec{y}\cdot x\vec{Q}$, hence $M^{\pi_2}\vec{y}=x\vec{Q}$, for a Böhm transformation π_2.
Checking the details of the construction of π_2 one verifies that

$$M^{\pi_2}\vec{y} \equiv M \ldots x_i \ldots [x_j/Cx_j] \ldots [x_k/Ux_k] \ldots \vec{y} \equiv M\vec{P}$$

for some simple terms $\vec{P}$ with $x \notin FV(\vec{P})$ (where C is a permutator and U a selector). ■

1.5. LEMMA. Let F be a closed λ-term such that F is not constant, i.e. $\lambda \nvdash FX_1=FX_2$ for some X_1, X_2, and suppose that for some closed λ-term M, FM has a *nf*. Then $x \in BT(Fx)$ for all x.

PROOF. Note that if P, P' have equal finite Ω-free Böhm-trees, then $\lambda \vdash P=P'$. Now suppose $x \notin BT(Fx)$ for some x. Then for all k, $x \notin FV((Fx)^k)$ (N^k is the k-th approximate normal form of N, cf. Barendregt [1976] 7.4 (iv)). Hence $(FM)^k \equiv (Fx)^k\,[x/M] \equiv (Fx)^k$ for all k, and it follows that $BT(FM)=BT(Fx)$. But since FM has a *nf*, $BT(FM)$ is finite and Ω-free and therefore $\lambda \vdash FM=Fx$. Since F, M are closed it follows that for all λ-terms N, $\lambda \vdash FN=FM$, i.e. F is constant, a contradiction. ■

REMARK. 1.5 also holds for F, M not necessarily closed.

1.6. DEFINITION. $\underline{0}=I$, $\underline{n+1}=K\ \underline{n}$.

1.7. LEMMA. The function *sg* is not λ-definable with respect to $\{\underline{n} | n \in \omega\}$, i.e. for no closed λ-term $F \vdash F\ \underline{0}=\underline{0}$, $\vdash F\ \underline{n+1}=\underline{1}$.

PROOF. Suppose F exists. Then by 1.5 $x \in BT(Fx)$. Hence by 1.4 $Fx\vec{P}=x\vec{Q}$ for some $\vec{P}, \vec{Q}=Q_1 \ldots Q_m$. But then for all $n>m$,

$$\vdash \underline{1}\vec{P}=F\ \underline{n}\ \vec{P}=\underline{n}\ Q_1 \ldots Q_m=\underline{n-m}$$

contradicting the Church-Rosser theorem since the k are different *nf*'s. ■

1.8. Definition. A system of terms $\{M_n | n \in \omega\}$ is an *adequate system of numerals* iff

(i) Each M_n has a *nf*.

(ii) Each recursive function can be λ-defined with respect to the M_n.

In Barendregt [1977] is shown that the second condition can be replaced by (ii'): The successor, predecessor and *sg* functions can be λ-defined with respect to the M_n.

The following corollary was proved independently by Barendregt [1972] and Wadsworth [1972].

1.9. Corollary.

(i) $\{\underline{\underline{n}} | n \in \omega\}$ is not an adequate system of numerals. (ii) Church's δ is not λ-definable.

Proof. (i) Immediate. (ii) If δ were λ-definable, then so would be *sg*, viz. by $\lambda x \cdot \delta x\ \underline{\underline{0}}\ \underline{\underline{0}}\ \underline{\underline{1}}$. ∎

Remark. (i) Although not definable, δ can consistently be added to the λ-calculus, see Church [1941].

(ii) Contrary to this, the corresponding δ for *open* λ-terms would be inconsistent at once. For let $x \not\equiv y$, then

$$(\lambda y \cdot \delta xy(KK)S)x = (\lambda y \cdot \underline{1}(KK)S)x = (\lambda y \cdot KKS)x = KKS = K$$

but also

$$(\lambda y \cdot \delta xy(KK)S)x = \delta xx(KK)S = \underline{0}(KK)S = S.$$

(iii) One could also consider the definability of a δ for *all* closed terms, i.e.: $\delta MM = \underline{0}$ for M closed

$\delta MN = \underline{1}$ for M, N closed such that $\nvdash M = N$.

But then the following version of the Russell paradox would result.

Define $\neg X = \delta X \underline{1}$. If $\nvdash \underline{0} = \underline{1}$ then $\nvdash X = \underline{1} \Leftrightarrow\ \vdash \neg X = \underline{1}$.

Now let $A = FP \neg$ (i.e. the fixed point of $\neg$: $\vdash A = \neg A$).

Then $\nvdash A = \underline{1} \Leftrightarrow\ \vdash A = \underline{1}$. Thus $\vdash \underline{0} = \underline{1}$.

To see the relation with the Russell paradox, note that $A = BB$ with $B = \lambda x.\ \neg (xx)$. (In illative combinatory logic MN is interpreted as $N \in M$ and $\lambda x \cdot P$ as $\{x | P\}$.)

1.10. Theorem. Let $\underline{\omega} = \{\underline{n} | n \in \omega\}$ be an adequate system of numerals and let f be a map into $\underline{\omega}$ definable by F. Then f is constant.

Proof. First assume $\underline{\omega}$ is Church's system of numerals.

Suppose f is not constant, then by 1.5 $x \in BT(Fx)$. Hence for some simple $\vec{P}$ and $\vec{Q}$, $\lambda \vdash Fx\vec{P} = x\vec{Q}$.

Hence $\lambda \vdash FM\vec{P} = M\vec{Q}$ for all M. But $M\vec{Q}$ can take arbitrary values and not $FM\vec{P}$, since $\underline{n}\ \vec{P} = P_1{}^n(P_2)P_3 \ldots P_k$ always has a *nf* by 1.3.

Now let $\underset{\sim}{\omega}$ be an arbitrary system of numerals. It is well-known how to define a term G such that $G\underset{\sim}{n}=\underline{n}$.

Suppose a non-constant f: terms $\rightarrow \underset{\sim}{\omega}$ would be definable, then $G \circ f$ were a definable non-constant mapping into $\underline{\omega}$. ■

First alternative proof (due to the referee).

Suppose F is not constant, i.e. let $n_1 \neq n_2 \in Ra(F)$. Define G as the λ-defining term of the recursive function

$$g(x)=\begin{cases} \mathit{0} & \text{if } x=n_1, \\ \mathit{1} & \text{else.} \end{cases}$$

Then the range of $G \circ F$ is $\{\underline{0}, \underline{1}\}$ contrary to 2.3. ■

Second alternative proof. By Barendregt's lemma in de Boer [1975] it follows that if Ω is unsolvable and N a nf, then $F\Omega=N \Rightarrow Fx=N$ for all x. (General genericity lemma; see also Barendregt [1977a] for a proof.) Now if the values of F are numerals it follows that $F\Omega$ has a nf, i.e. F is constant. ■

1.11. Corollary. There is no F such that

$$FM = \underline{0} \text{ if } M \text{ is a numeral (i.e. } \vdash M=\underline{n} \text{ for some } n)$$
$$\underline{1} \text{ else}$$

for any adequate system.

1.12. Question. Is there a term F such that

FM has a nf (is solvable) if M is a numeral
has no nf (is unsolvable) else.

§ 2. THE RANGE PROPERTY

2.1. Definition. Let $\mathscr{M}=\langle M, \cdot\rangle$ be a λ-algebra. For each $f \in M$, we define $Ra^{\mathscr{M}}(f)$, the *range* of f in $\mathscr{M}$, as follows:

$$Ra^{\mathscr{M}}(f)=\{f\cdot x \mid x \in M\}.$$

Notation. $Ra^{\mathscr{M}}(F)=Ra^{\mathscr{M}}(\llbracket F \rrbracket^{\mathscr{M}})$ for terms F.

When possible, the superscript $\mathscr{M}$ will be dropped in $Ra^{\mathscr{M}}$.

2.2. Definition. A λ-algebra $\mathscr{M}$ satisfies the *range property* if for all $f \in M$, the cardinality of $Ra^{\mathscr{M}}(f)$ is 1 or $\aleph_0$.

2.3. Range theorem. (Barendregt; Myhill). Let T be a r.e. λ-theory. Then $\mathscr{M}(T)$ (and also $\mathscr{M}^0(T)$) has the range property.

Proof. Suppose $f \in M$ and $Ra(f)=\{m_0, \ldots, m_k\}$, $k>0$. Define

$$N_i=\{x \mid f\cdot x=m_i\} \subset M.$$

Every such N_i is r.e. Therefore $N = \bigcup_1^k N_i$, the complement of N_0 is also r.e. Hence N_0 is recursive.

On the other hand N_0 is non-trivial and closed under equality, which contradicts Scott's theorem (Barendregt [1976] 2.21).

The proof for $\mathcal{M}^0(T)$ is the same. ■

2.4. COROLLARY. $\mathcal{M}(\lambda)$, $\mathcal{M}^0(\lambda)$, $\mathcal{M}(\lambda\eta)$ and $\mathcal{M}^0(\lambda\eta)$ have the range property.

The range property, however, is not satisfied in every λ-algebra.

2.5. THEOREM. $P\omega$ and D_∞ do not satisfy the range property.

PROOF. Since the proof is similar in both cases, let $\mathcal{S} = (S, \leqslant)$ denote either $(P_\infty, \subseteq)$ or $(D_\infty, \sqsubseteq)$. We define the following function $f\colon S \to S$ by $f(x) = \top$ if $x \neq \bot$ else $\bot$ ($\top$ and $\bot$ are the largest respectively smallest element of S.)

Claim: f is continuous. Then by Scott [1972], [1975] f is representable and since f has range of cardinality two we are done.

For open O in S one has: $x \in O$ and $x \leqslant y \Rightarrow y \in O$.

See Scott [1972], [1975] for definition of the topologies involved.

Hence for open O, $\bot \in O \Rightarrow O = S$, and $O \neq \emptyset \Rightarrow \top \in O$.

Now for every open set O, $f^{-1}(O)$ is open:

Case 1. $\bot \in O$. Then $O = S$ so $f^{-1}(S) = S$ which is open.

Case 2. $\bot \notin O$. If $O = \emptyset$, then we are done. Else $\top \in O$ and hence $f^{-1}(O) = S - \{\bot\} = \{x \mid x \nleqslant \bot\} \underset{\text{def}}{=} U_\bot$.

$U_\bot$ is open in D_∞, see e.g. Barendregt [1976] 1.2.

$U_\bot$ is open in $P\omega$: Let $O_k = \{x \mid e_k \subseteq x\}$. Note $e_0 = \emptyset = \bot$ and that the O_k form a base for the topology on $P\omega$.

Now:

$$x \in U_\bot \Leftrightarrow x \nsubseteq O \Leftrightarrow \underset{k \neq 0}{\exists k} \; e_k \subseteq x \Leftrightarrow x \in \bigcup_{k \neq 0} O_k$$

which is, as a union of elements of a base, indeed open. ■

The following theorem was announced in Wadsworth [1973] for the D_∞ case.

2.6. THEOREM. Let $\mathcal{S}$ be D_∞^0 or $P^0\omega$. Then $\mathcal{S}$ satisfies the range property.

PROOF. Let F be a closed term. Consider $BT(Fx)$.

Case 1. $x \notin BT(Fx)$. Then $BT(FM) = BT(FM')$ for all M, M'. Since terms with equal Böhm trees are equal in $\mathcal{S}$ (see Barendregt [1976], Hyland [1976]), it follows that $Ra^{\mathcal{S}}(F)$ has cardinality 1.

Case 2. $x \in BT(Fx)$. Then by 1.4 $\lambda \vdash Fx\vec{P} = x\vec{Q}$.

Since $[\![N\vec{Q}]\!]^{\mathcal{S}}$ can take arbitrary values in $\mathcal{S}$ when N ranges over the closed terms, $Ra^{\mathcal{S}}(F)$ is infinite. ■

2.7. Conjecture. $\mathscr{M}(\mathscr{H})$ satisfies the range property.

2.8. Question. Does every hard λ-algebra $\mathscr{M}$ (i.e. $\mathscr{M}=\mathscr{M}^0$) satisfy the range theorem?

§ 3. DUALITY

3.1. Definition. Let f, g be two external functions on a λ-algebra $\mathscr{M}=\langle M, \cdot\rangle$.

f, g are *dual* iff $\forall a, b \in M$: $f(a)\cdot b=g(b)\cdot a$. Notation $f\sim_{\mathscr{M}} g$, or simply $f\sim g$.

3.2. Definition. $\mathscr{M}$ is *rich* iff all dual functions on $\mathscr{M}$ are representable in $\mathscr{M}$.

Remarks. (i) Let f be an external function on $\mathscr{M}$. f is *locally representable* iff for each $b\in M$ the function h defined by $h(a)=f(a)\cdot b$ is representable. Then f is locally representable iff f has a dual. A model is rich iff all locally representable functions are representable.

(ii) If f is representable (by $f_0\in M$, say), then f has a dual g which is also representable (by $g_0=\lambda ab\cdot f_0 ba$).

(iii) Let $\mathscr{M}$ be extensional. Then f has at most one dual. Hence if $f\sim_{\mathscr{M}} g$ and f is representable, then by (ii) g is representable.

3.3. Theorem. If $\mathscr{M}$ is rich, then $\mathscr{M}$ is extensional.

Proof. Suppose $\mathscr{M}$ is not extensional. Then there exist $b, b'\in M$ such that for all $c\in M$ $b\cdot c=b'\cdot c$ and $b\neq b'$.

Define

$$f(a)=\begin{cases} b' & \text{if } a=b\\ b & \text{else.}\end{cases}$$

and

$$g=[\![\lambda y\cdot K(\underline{b}y)]\!]^{\mathscr{M}},$$

then for all $a, a'\in M$: $f(a)\cdot a'=b\cdot a'=g(a')\cdot a$, hence $f\sim g$. But f cannot be representable since it has no fixed point. Thus $\mathscr{M}$ is not rich. ■

3.4. Corollary. The following λ-algebras are not rich: $P\omega$; $P^0\omega$; $\mathscr{M}(\lambda)$; $\mathscr{M}^0(\lambda)$; $\mathscr{M}^0(\lambda\eta)$.

Proof.

1. $P\omega$ is not extensional:
Take for example $a=\{(0, 0)\}$ and $b=\{(0, 0), (1, 0)\}$.
Then $\forall c\in P\omega$ $a\cdot c=b\cdot c$ but $a\neq b$.

2. $P^0\omega$ is not extensional: Let $1=\lambda xy\cdot xy$, then $P^0\omega\models Ixy=1xy$, but $P^0\omega\not\models I=1$ for otherwise $P\omega\models I=1$, so $P\omega\models\forall x\ x=\lambda y\cdot xy$ which implies that $P\omega$ were extensional.

3. By the Church Rosser property $\lambda\not\vdash I=1$. So $\mathscr{M}(\lambda)$, $\mathscr{M}^0(\lambda)$ are not extensional.

4. $\mathcal{M}^0(\lambda\eta)$ is not extensional because the λ-calculus is ω-incomplete, see Plotkin [1974].

3.5. THEOREM. D_∞ is rich.

PROOF. Suppose that f, g are dual i.e.:

$$\forall a, b \in D_\infty \colon\ f(a)\cdot b = g(b)\cdot a.$$

We have to show that f, g are representable.

It is sufficient to show that f, g are continuous. Take a directed $X \subset D_\infty$. For all $b \in D_\infty$

$$f(\bigsqcup X)\cdot b = g(b)\cdot \bigsqcup X = \bigsqcup \{g(b)\cdot a \mid a \in X\} =$$

$$= \bigsqcup \{f(a)\cdot b \mid a \in X\} = \bigsqcup \{f(a) \mid a \in X\}\cdot b$$

by the duality condition and the continuity of application.

Thus by extensionality in D_∞: for all directed X $f(\bigsqcup X) = \bigsqcup \{f(a) \mid a \in X\}$ i.e. f is continuous. The proof for g is dual. ■

3.6. THEOREM. $\mathcal{M}(\lambda\eta)$ is rich.

PROOF. Define

$$M =_{\lambda\eta} N \text{ iff } \lambda\eta \vdash M = N,$$

$$x \in_{\lambda\eta} M \text{ iff for all } M' =_{\lambda\eta} M \text{ one has } x \in FV(M').$$

Let f, g be dual functions on $\mathcal{M}(\lambda\eta)$.

3.6.0. LEMMA. (i) $x \in_{\lambda\eta} M \Leftrightarrow \forall N[\lambda\eta \vdash M \to N \Rightarrow x \in FV(N)]$.

(ii) Let $M' \equiv M[z/y]$ and $\lambda \vdash M' \to N'$. Then $\exists N\ \lambda \vdash M \to N$ and $N' \equiv N[z/y]$.

$$\begin{array}{ccc} M & \dashrightarrow & N \\ {\scriptstyle [z/y]}\Big| & & \vdots\,{\scriptstyle [z/y]} \\ M' & \longrightarrow & N' \end{array}$$

(iii) $x \in_{\lambda\eta} M \Rightarrow x \in_{\lambda\eta} M[z/y]$, for $z \not\equiv x$.

PROOF. (i) $\Rightarrow$ Trivial. $\Leftarrow$ Suppose $M =_{\lambda\eta} M'$. By the Church-Rosser theorem $\lambda\eta \vdash M \to N$, $M' \to N'$ for some N. By assumption $x \in FV(N)$. But then $x \in FV(M')$.

(ii) Induction on the length of proof of $M' \to N'$. In the case that $M' \equiv (\lambda a \cdot P)Q$, $N' \equiv P[a/Q]$ it may be assumed that $a \not\equiv z, y$. Therefore one can apply the well-known substitution lemma

$$A[u/B][v/C] = A[v/C][u/B[v/C]] \text{ if } u \not\equiv v \text{ and } u \notin FV(C).$$

(iii) Suppose $\lambda\eta \vdash P[z/y] \to R'$. By (ii) for some R $\lambda\eta \vdash P \to R$ and $R' \equiv R[z/y]$. By assumption and (i), $x \in FV(R)$. Since $x \not\equiv z$ also $x \in FV(R')$. Therefore by (i) $x \in_{\lambda\eta} P[z/y]$. ■3.6.0

3.6.1. Lemma. (i) If $x \in_{\lambda\eta} \lambda y \cdot P$ then $x \in_{\lambda\eta} P$ and $x \not\equiv y$.
(ii) If $x \not\equiv y$, then $x \in_{\lambda\eta} M \Leftrightarrow x \in_{\lambda\eta} My$.

Proof. (i) Since $x \in FV(\lambda y \cdot P)$ clearly $x \not\equiv y$. Suppose $P =_{\lambda\eta} N$, then $\lambda y \cdot P =_{\lambda\eta} \lambda y \cdot N$. By assumption $x \in FV(\lambda y \cdot N) \subset FV(N)$. Thus $x \in_{\lambda\eta} P$.
(ii) $\Rightarrow$ Suppose $\lambda\eta \vdash My \to N$ in order to prove $x \in FV(N)$.
Case 1. $N = M'y$ with $\lambda\eta \vdash M \to M'$. Since $x \in_{\lambda\eta} M$, also $x \in FV(M') \subset \subset FV(N)$.
Case 2. $M \to \lambda z \cdot M_1$ and $\lambda\eta \vdash My \to (\lambda z \cdot M_1)y \to M_1[z/y] \to N$.
Since $x \in_{\lambda\eta} M$, also $x \in_{\lambda\eta} \lambda z \cdot M_1$ and by (i) $x \in_{\lambda\eta} M_1$ and $z \not\equiv x$, so by 3.6.0. (iii) $x \in_{\lambda\eta} M_1[z/y]$. Therefore $x \in FV(N)$. ■

3.6.2. Lemma. If $\exists y \not\equiv x \; x \in_{\lambda\eta} f(y)$, then $\forall y \not\equiv x \; x \in_{\lambda\eta} g(y)$ (and hence $\forall y \not\equiv x \; x \in_{\lambda\eta} f(y)$).

Proof. Suppose $x \in_{\lambda\eta} f(y)$, $y \not\equiv x$. Let $y' \not\equiv x$. Then by 3.6.1. (ii) $x \in_{\lambda\eta} f(y) \cdot y' =_{\lambda\eta} g(y') \cdot y$. Hence, by 3.6.1. (ii), $x \in_{\lambda\eta} g(y')$. (The rest follows by applying the statement to $x \in_{\lambda\eta} g(y)$). ■3.6.2

3.6.3. Main lemma. There is a variable x such that for all terms $M: f(x)[x/M] = f(M)$.

Proof. Let v be any variable. Choose $x \not\equiv v$ such that $x \notin_{\lambda\eta} f(v)$. Then $x \notin_{\lambda\eta} g(z)$ for all $z \not\equiv x$, by the dual of 3.6.2.
Given M, one can find a y such that $y \notin_{\lambda\eta} M$, $f(M)$, x, $f(x)$. Hence $x \notin_{\lambda\eta} g(y)$. Now since $y \not\equiv x$ and

$$x \notin_{\lambda\eta} g(y),\; (f(x)[x/M]) \cdot y = (f(x) \cdot y)[x/M] = (g(y) \cdot x)[x/M] = g(y) \cdot M = f(M) \cdot y.$$

Since $y \notin f(x)$, M, $f(M)$, extensionality yields $f(x)[x/M] = f(M)$. ■3.6.3

Now it follows by 3.6.3. that f can be represented by the term $\lambda x \cdot f(x)$ and similary for g. ■3.6

The following construction is needed for the proof of 3.10.

3.7. Definition. Let $\#$ be a Gödel numbering of terms. $\ulcorner M \urcorner$ is the numeral $\# M$. A sequence of terms M_n is recursive if $\boldsymbol{\lambda} n \cdot \# M_n$ is a recursive function.

3.8. Lemma. (Coding of infinite sequences). Let $\{M_n\}$ be a recursive sequence of terms such that $FV(M_n) \subseteq \{x\}$ for all n. Then there exists a term X such that $p\underline{i}X = M_i$, for all i, where p is some fixed closed term. Par abus de langage we write $\langle M_n \rangle_{n \in \omega}$ for X.

Proof.
(1) As in Curry et al. [1972], 13 B3 there is a term E which enumerates all terms with x as only free variable:

$$E(\ulcorner M \urcorner) = M, \text{ for } M \text{ with } FV(M) = \{x\}.$$

(2) Let $[M, N]$ be a pairing of terms defined by $\lambda z \cdot zMN$. Then $[M, N]K = M$ and $[M, N](KI) = N$. Define ordered tuples as follows:

$$[M] = M, \ [M_1, \ldots, M_{n+1}] = [M_1, [M_2, \ldots, M_{n+1}]].$$

(3) Let M_n with $FV(M_n) \subseteq \{x\}$ be a recursive sequence of terms. We want to code the sequence $\{M_n\}$ as a λ-term. Let S^+ be such that $S^+ \underline{n} \xrightarrow[\beta]{*} \underline{n+1}$ and let $b \equiv \lambda xy \cdot [E(Fy), (x(S^+y))]$, where F λ-defines f, and $B \equiv FP\ b$ (i.e. the fixed point of b). Then

$$B\underline{n} \xrightarrow[\beta]{*} bB\underline{n} \xrightarrow[\beta]{*} [E(F\underline{n}), B\underline{n+1}] \xrightarrow[\beta]{*} [M_n, B\underline{n+1}].$$

So $B\underline{0} = [M_0, B\underline{1}] = [M_0, M_1, B\underline{2}] = \ldots$. Hence by setting $\langle M_n \rangle_{n \in \omega} = B\underline{0}$ we have a coding for infinite sequences of terms with one fixed free variable.

(4) It is easy to construct a term p such that $p\underline{m}\langle M_n \rangle_{n \in \omega} = M_m$, (take e.g. $pxa =$ if zero x then aK else $p(x - \underline{1})(a(KI))$, using the fixed point theorem). ∎

3.9. Lemma. For all closed Z there is an n such that $Z\Omega^n =_{\mathscr{H}} \Omega$. ($Z\Omega^n$ is short for $Z\underbrace{\Omega\Omega \ldots \Omega}_{n \text{ times}}$).

Proof.

Case 1. Z is unsolvable; then $Z =_{\mathscr{H}} \Omega$, so $n = 0$.

Case 2. Z is solvable; then Z has a hnf, $Z = \lambda \vec{x} \cdot x_i A_1 \ldots A_m$ $(x_i \in \vec{x})$. Take $n = i$, so $Z\Omega^i = \lambda \vec{x}' \cdot \Omega A_1 \ldots A_m =_{\mathscr{H}} \Omega$. ∎

3.10. Theorem. If $\mathscr{M}$ is hard and sensible, then $\mathscr{M}$ is not rich.

Proof. If $\mathscr{M}$ is hard, then $\mathscr{M}$ is isomorphic to $\mathscr{M}^0(T)$, where $T = Th(\mathscr{M})$. We reason in $\mathscr{M}^0(T)$. Since $\mathscr{M}$ is sensible, $\mathscr{H} \subseteq T$.

Let $h: \underline{\omega} \to \underline{\omega}$ be a function not definable in $\mathscr{M}$. Such an h exists since a hard model is countable.

Let $A_n(x, y)$ be the term $x\Omega^n(y\Omega^n(h\underline{n}))$, $n \in \omega$. For closed M the sequence $A_0(M, y)$, $A_1(M, y)$, ... is by 3.9

$M(y(h\underline{0}))$, $M\Omega(y\Omega(h\underline{1}))$, ..., $M\Omega^n(y\Omega^n(h\underline{n}))$, Ω, Ω, ...,

where n is such that $M\Omega^{n+1} = \Omega$. Thus $\boldsymbol{\lambda} n \cdot A_n(M, y)$ is up to convertibility a recursive sequence containing one fixed free variable and hence representable as a term. Define $f(M) = \lambda y \cdot \langle A_n(M, y) \rangle_{n \in \omega}$. Similarly for closed N $\boldsymbol{\lambda} n \cdot A_n(x, N)$ is recursive and it is possible to define $g(N) = \lambda x \cdot \langle A_n(x, N) \rangle_{n \in \omega}$. Then for all closed M, N: $f(M)$ and $g(N)$ are well defined and $f(M) \cdot N = g(N) \cdot M = \langle A_n(M, N) \rangle_{n \in \omega}$ by construction. So f and g are dual.

Suppose now that $\mathscr{M}$ is rich, i.e. f were representable by some closed F. Then for all closed M, N: $FMN = f(M)N = \langle A_n(M, N) \rangle_{n \in \omega}$.

But then $p\underline{n}(F(K^n I)(K^n I)) = p\underline{n}\langle h(\underline{n}) \rangle_{n \in \omega} = h(\underline{n})$, hence h were definable, contradiction. Thus $\mathscr{M}$ is not rich. ∎

3.11. Corollary. D_∞^0 and $\mathscr{M}^0(T)$ for $T \supset \mathscr{H}$ are not rich.

3.12. Questions. (i) Is every extensional term model $\mathscr{M}(T)$ rich? (ii) Is $\mathscr{M}^0(\lambda\omega)$ rich?

Here $\lambda\omega$ is the λ-theory obtained by adding the ω-rule to the theory, see Barendregt [1974].

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