Convexity for invariant differential operators on semisimple symmetric spaces

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Introduction. Let \( X = G/H \) be a homogeneous space of a Lie group \( G \), and let \( D : C^\infty(X) \rightarrow C^\infty(X) \) be a non-trivial \( G \)-invariant differential operator. One of the natural questions one can ask for the operator \( D \) is whether it is solvable, in the sense that \( DC^\infty(X) = C^\infty(X) \). If \( G \) is the group of translations of \( X = \mathbb{R}^n \) and \( H \) is trivial, then \( D \) has constant coefficients, and it is a well known result of Ehrenpreis and Malgrange that hence \( D \) is solvable.

Assume for simplicity that \( G/H \) carries an invariant measure. This measure induces a bilinear pairing of \( C^\infty_c(X) \), the space of compactly supported smooth functions on \( X \), with itself. Let \( D^* \) denote the adjoint of \( D \) with respect to this pairing. The strategy employed by Ehrenpreis and Malgrange was essentially to use the following properties of \( D \):

(i) There exists a fundamental solution for \( D \), that is, \( \delta \in D D'(X)^H \), where \( \delta \) is the Dirac measure at the origin, and \( D'(X)^H \) is the space of left-\( H \)-invariant distributions on \( X \).

(ii) For each compact set \( \Omega \subset X \) there exists a compact set \( \Omega' \subset X \) such that

\[
\text{supp } D^* f \subset \Omega \Rightarrow \text{supp } f \subset \Omega'
\]

for all \( f \in C^\infty_c(X) \).

In fact, for \( X = \mathbb{R}^n \) one can take as \( \Omega' \) the convex hull of \( \Omega \). For this reason the support property (ii) has become known as the \( D \)-convexity of \( X \). It follows from (i)-(ii) that \( D \) is solvable.

The strategy has been applied in other cases as well, for example by Helgason in [14], where surjectivity is established for all non-trivial invariant differential operators on a Riemannian symmetric space. In a variant of the strategy (i) is replaced by the following weaker property (semi-global solvability):

(i') For each compact set \( \Omega \subset X \) and each function \( g \in C^\infty(X) \) there exists a function \( f \in C^\infty(X) \) such that \( Df = g \) on \( \Omega \).

The conjunction of (i') and (ii) is equivalent with the solvability of \( D \) (see Theorem 1 below). This is used by Rauch and Wigner in [19] where it is proved that the Casimir operator on a semisimple Lie group is solvable, and more generally by Chang in [5] where the Laplace-Beltrami operator on a semisimple symmetric space is shown to be solvable.

The purpose of the present paper is to give, also for a semisimple symmetric space \( X = G/H \), a sufficient condition on an invariant differential operator \( D \) to imply (ii), the \( D \)-convexity of \( X \). When \( G/H \) has rank one, our result follows from the above mentioned result of Chang, since the algebra \( D(G/H) \) of all invariant differential operators in this case is generated by the Laplace-Beltrami operator. In general this is not so, and our
result shows the $D$-convexity for a significantly larger class of operators $D$. In particular, when $G/H$ is split (that is, it has a vectorial Cartan subspace), all non-trivial elements of $D(G/H)$ satisfy our condition.

Though we do not consider the properties (i) or (i’) in this paper, we notice that in the above-mentioned references, an important step towards obtaining (i’) is to prove that $D^*$ acts injectively on, say $C^\infty_c(X)$ (see for example [5]). In fact the injectivity of $D^*$ is an immediate consequence of (i’). In the present case of a semisimple symmetric space, the sufficient condition that we give for (ii) is also sufficient for $D^*$ to be injective.

We also give a condition on $D$, which is necessary for both the $D$-convexity and the injectivity. In particular, when $G/H$ is not split, there exists a non-trivial operator $D \in D(G/H)$, which does not have these properties and hence is not solvable. This provides a wide class of spaces $G/H$ for which there exist non-solvable non-trivial operators. Unfortunately our necessary condition is weaker than the sufficient condition, and the complete classification (for non-split $G/H$) of all $D \in D(G/H)$ which have these properties remains open.

In the special case where the semisimple symmetric space is Riemannian (that is, when $H$ is compact), we have that $G/H$ is split and thus our condition reduces to the requirement that $D$ is non-trivial. In this case our result is part of the above-mentioned proof by Helgason that $D$ is surjective (see [14, p.473]). Helgason’s proof is based on his inversion formula and Paley-Wiener theorem for the Fourier transform on the Riemannian symmetric space $X$. These results are in turn based on the work of Harish-Chandra. Simplifications avoiding these strong tools were given by Chang, [7], and Dadok, [8]. In another special case, that of a semisimple Lie group considered as a symmetric space, our result was obtained by Duflot and Wigner, [9].

All of the references mentioned above, except [14], use the uniqueness theorem of Holmgren to derive the $D$-convexity of $X$, and so do we. The main difficulty in the present generalization lies in the handling of the more complicated geometry of $X$. Our main tool to overcome this difficulty is the convexity theorem of [1].

In [3] (see also [4]) the result of the present paper will be applied to obtain injectivity of the Fourier transform on $C^\infty_c(X)$. Our reasoning will thus be the opposite of the original reasoning of Helgason in the Riemannian case: we shall deduce properties of the Fourier transform from the $D$-convexity.

**Motivation.** As mentioned in the introduction the main motivation for studying $D$-convexity is the following theorem. Here $G$ is a Lie group (with at most countably many connected components) and $H$ is a closed subgroup, of which we only assume that $G/H$ carries an invariant measure (this assumption is only used for defining $D^*$).

**Theorem 1.** Let $D \in D(G/H)$ be an invariant differential operator. Then $D$ is solvable if and only if (i’) and (ii) hold.

**Proof:** This follows from [21, Ch.I, Thm.3.3], using regularization by $C^\infty_c(G)$ to prove the equivalence of our definition of $D$-convexity with that of [21, Ch.I, Def.3.1]. Note also the final remark of that section in loc.cit.  


Notation. From now on, let $G$ be a real reductive Lie group of Harish-Chandra’s class, $\tau$ an involution of $G$, and $H$ an open subgroup of the fixed point group $G^\tau$. Then $X = G/H$ is a reductive symmetric space of Harish-Chandra’s class (see [2]). Let $K$ be a $\tau$-stable maximal compact subgroup of $G$, and let $\theta$ be the associated Cartan involution. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q} = \mathfrak{k} + \mathfrak{p}$ be the eigen-decompositions of the Lie algebra $\mathfrak{g}$ induced by $\tau$ and $\theta$, then $\mathfrak{h}$ and $\mathfrak{k}$ are the Lie algebras of $H$ and $K$, respectively. Let $B$ be a non-degenerate, $G$- and $\tau$-invariant bilinear form on $\mathfrak{g}$ which extends the Killing form on $[\mathfrak{g}, \mathfrak{g}]$, and which is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. Then the above-mentioned eigen-decompositions are orthogonal with respect to $B$.

Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p} \cap \mathfrak{q}$, and a maximal abelian subspace (a Cartan subspace) $\mathfrak{a}_1$ of $\mathfrak{q}$, containing $\mathfrak{a}$. Then $\mathfrak{a} = \mathfrak{a}_1 \cap \mathfrak{p}$. Let $\mathfrak{m}$ be the orthocomplement (with respect to $B$) of $\mathfrak{a}$ in its centralizer $\mathfrak{g}^\mathfrak{a}$, and let $\mathfrak{a}_m = \mathfrak{a}_1 \cap \mathfrak{m}$. Via the orthogonal decomposition $\mathfrak{a}_1 = \mathfrak{a}_m + \mathfrak{a}$ we view $\mathfrak{a}_m^* \subset \mathfrak{a}^*$ as subspaces of $\mathfrak{a}^*_1$. Let $\Sigma$ and $\Sigma_1$ denote the root systems of $\mathfrak{a}$ and $\mathfrak{a}_1$ in $\mathfrak{g}_c$, respectively, then $\Sigma$ consists of the non-trivial restrictions to $\mathfrak{a}$ of the elements of $\Sigma_1$. Denote by $W$ and $W_1$ the Weyl groups of these two root systems, then $W$ is naturally isomorphic to $N_{W_1}(\mathfrak{a})/Z_{W_1}(\mathfrak{a})$, the normalizer modulo the centralizer of $\mathfrak{a}$ in $W_1$, and to $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$, the normalizer modulo the centralizer of $\mathfrak{a}$ in $K$. Let $W_{K\cap H}$ be the canonical image of $N_{K\cap H}(\mathfrak{a})$ in $W$.

Recall that $G = KAH$, and that if $g = kah$ according to this decomposition, then the orbit $W_{K\cap H} \log a$ is uniquely determined by $g$. For a $W_{K\cap H}$-invariant set $S \subset \mathfrak{a}$, we denote the subset $K \exp(S)H$ of $X$ by $X_S$. Then $S = \{ \log a \mid aH \in X_S \}$, and every $K$-invariant subset of $X$ is of the form $X_S$.

Invariant differential operators. Let $\mathcal{D}(G/H)$ be the algebra of invariant differential operators on $G/H$. Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$, and $U(\mathfrak{g})^H$ the subalgebra of $H$-invariant elements, then there is a natural isomorphism of the quotient $U(\mathfrak{g})^H / (U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h}_c)$ with $\mathcal{D}(G/H)$, induced by the right action $R$ of $U(\mathfrak{g})$ on $C^\infty(G)$ (see [15, p. 285]).

Let $\Sigma_1^+$ be a positive system for $\Sigma_1$, and let $\mathfrak{n}_1$ be the sum of the corresponding positive root spaces $\mathfrak{g}_\alpha^\circ (\alpha \in \Sigma_1^+)$. We have the following direct sum decomposition

\begin{equation}
\mathfrak{g}_c = \mathfrak{n}_1 + \mathfrak{a}_1c + \mathfrak{h}_c.
\end{equation}

Using this decomposition and Poincare-Birkhoff-Witt, a map $\gamma : U(\mathfrak{g}) \rightarrow U(\mathfrak{a}_1)$ is defined by $u \equiv \gamma(u)$ modulo $\mathfrak{n}_1 U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{h}_c$. From this map an algebra isomorphism $\gamma$ of $\mathcal{D}(G/H) \cong U(\mathfrak{g})^H / (U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h}_c)$ onto $S(\mathfrak{a}_1)^{W_1}$, the set of $W_1$-invariant elements in the symmetric algebra of $\mathfrak{a}_1c$ (which is isomorphic to $U(\mathfrak{a}_1)$ because $\mathfrak{a}_1$ is abelian), is obtained by letting $\gamma(u)(\lambda) = \gamma(u)(\lambda + \rho_1)$ for $u \in U(\mathfrak{g})^H, \lambda \in \mathfrak{a}_1^*$ (see [11, p. 15, Thm. 3]). Here $\rho_1 \in \mathfrak{a}_1^*$ is given by half the trace of the adjoint action on $\mathfrak{n}_1$. Thus $\mathcal{D}(G/H)$ is identified as a polynomial algebra with dim $\mathfrak{a}_1$ independent generators.

Assume that $\Sigma_1^+$ is chosen to be compatible with $\mathfrak{a}$, that is, the set of nonzero restrictions to $\mathfrak{a}$ of elements from $\Sigma_1^+$ is a positive system $\Sigma^+$ for $\Sigma$. Let $\mathfrak{n}$ be the sum of the corresponding positive root spaces $\mathfrak{g}_\alpha^\circ (\alpha \in \Sigma^+)$, then we also have the following direct sum decomposition

\begin{equation}
\mathfrak{g} = \mathfrak{n} + \mathfrak{m} + \mathfrak{a} + \mathfrak{h}.
\end{equation}
Let $\rho \in a^*$ and $\rho_m \in a_{mc}$ be given by half the trace of the adjoint actions on $n$, and on $n_1 \cap m_c$, respectively.

Using the decomposition (2) a map $\gamma : U(g) \to U(a)$ is defined by $u \equiv \gamma(u)$ modulo $(n_c + m_c)U(g) + U(g)h_c$, and we obtain by restriction to $U(g)^H$ a homomorphism, also denoted $\gamma$, from $D(G/H) \approx U(g)^H/(U(g)^H \cap U(g)h_c)$ into $S(a)$. Let $\eta(D) \in S(a)$ be defined by $\eta(D)(\lambda) = \gamma(D)(\lambda + \rho)$. 

**Lemma 1.** We have

(3) \[ \eta(D)(\lambda) = \gamma(D)(\lambda \Leftrightarrow \rho_m) \]

for all $D \in D(G/H)$, $\lambda \in a^*$. Moreover $\eta(D) \in S(a)^W$, and $\eta(D)$ is independent of the choice of $\Sigma^+$. 

**Proof:** We first prove the following equation:

(4) \[ \rho_1 = \rho + \rho_m. \]

We have $\rho_1 = \frac{1}{2} \sum_{\alpha \in \Sigma_1^+} (\dim g_c^\alpha) \alpha$ and $\rho_m = \frac{1}{2} \sum_{\alpha \in \Sigma_1^+, \alpha|_a = 0} (\dim g_c^\alpha) \alpha$. Let \[ \bar{\rho} = \rho_1 \Leftrightarrow \rho_m = \frac{1}{2} \sum_{\alpha \in \Sigma_1^+, \alpha|_a \neq 0} (\dim g_c^\alpha) \alpha, \]

then it is clear that $\bar{\rho} = \rho$ on $a$. On the other hand, since the set of $\alpha \in \Sigma_1^+$ with $\alpha|_a \neq 0$ is $\sigma \theta$-invariant, we get that $\sigma \theta \bar{\rho} = \bar{\rho}$, and hence $\bar{\rho} = 0$ on $a_m$, so that in fact $\bar{\rho} = \rho$.

Since $m_c = m_c \cap n_1 + a_{mc} + m_c \cap h_c$ it follows from (1) and (2) that $\gamma(D)(\lambda) = \gamma(D)(\lambda)$. From this and (4) we get (3).

The proof will be completed by using the following observation: Every element $w \in W$ can be represented by an element $\bar{w} \in N_{W_1}(a)$; this element also normalizes $a_m$, and can be chosen so that $\bar{w} \rho_m = \rho_m$.

The $W$-invariance of $\eta(D)$ now follows from (3) and the $W_1$-invariance of $\gamma(D)$, in view of the above observation. By using this observation once more, it follows from (3) and the fact that $\gamma$ is independent of the choice of the positive system $\Sigma_1^+$, that $\eta$ is independent of the choice of $\Sigma^+$. \[ \blacksquare \]

Let $s : S(g) \to U(g)$ be the symmetrization map, then the restriction of $s$ to the set $S(q)^H$ of $H$-invariants in $S(q)$ gives rise to a linear bijection (also denoted by $s$) of $S(q)^H$ with $D(G/H)$ (see [15, p. 287, Thm. 4.9]). A differential operator $D \in D(G/H)$ is called **homogeneous** if it is the image of a homogeneous element of $S(q)^H$. For $P \in S(q)^H$ let $r(P) \in S(a)$ denote the restriction of $P$ to $a$. Here $P$ is identified with a polynomial on $q$ by means of the Killing form.

**Lemma 2.** Let $D \in D(G/H)$ be non-constant and let $D = s(P)$, $P \in S(q)^H$. Then

(5) \[ \deg(\eta(D) \Leftrightarrow r(P)) < \deg P = \text{order } D. \]
In particular, if $D$ is homogeneous then $\deg \eta(D) = \text{order } D$ if and only if $r(P) \neq 0$.

**Proof:** That order $D = \deg P$ follows from the explicit expression for $s(P)$ in [15, p. 287, Thm. 4.9]. Let $r_1(P)$ denote the restriction of $P$ to $a_1$, then it follows from [15, p. 305, eq. (38)] that

$$\deg(\gamma(D) \lhd r_1(P)) < \deg P.$$  

It follows from (3) that $\eta(D) \lhd r(P)$ and the restriction of $\gamma(D) \lhd r_1(P)$ to $a$ have the same degree, and hence (5) follows from (6). If $P$ is homogeneous, then either $\deg r(P) = \deg P$ or $r(P) = 0$, and the final statement follows from (5). $\blacksquare$

Notice that $r_1(P)$ has the same degree as $P$ (to see this, let $P$ be homogeneous, then $\deg r_1(P) = \deg P$ unless $r_1(P) = 0$. But $r_1(P) = 0$ implies $P = 0$ by the $H$-invariance, because $\text{Ad}(H)(a_1)$ contains an open subset of $a_1$. Hence it follows from (6) that also $\gamma(D)$ has this degree (which equals the order of $D$). Thus $\gamma$ is a degree preserving isomorphism of $D(G/H)$ onto $S(a_1)^{W_1}$.

However, a similar statement is not valid for $\eta(D)$; its degree can be strictly smaller than that of $D$. In fact $\eta$ is not injective in general: Since $D(G/H)$ and $S(a)^{W}$ are polynomial algebras in $\dim a_1$ and $\dim a$ algebraically independent generators, respectively, $\eta$ is not injective if $a \neq a_1$ (otherwise it would cause the existence of an injection of the quotient field of $D(G/H)$ into the quotient field of $S(a)^W$, which is impossible, since their transcendence degrees over $C$ are $\dim a_1$ and $\dim a$, respectively (see [22, Ch.II, §12])). On the other hand, if $a_1 = a$, in which case the symmetric space $G/H$ is called *split*, then $\eta$ is injective since it equals $\gamma$. Examples of split symmetric spaces are the Riemannian symmetric spaces and the symmetric spaces of $K_i$-type (see [18]). In the special case (the ‘group case’) of a semisimple Lie group $G'$ considered as a symmetric space, where $G$ is $G' \times G'$ and $H$ is the diagonal, the notion of split for the space $G/H$ coincides with the notion of split (also called a normal real form) for $G'$.

Notice also that $\eta$ in general is not surjective. This can be seen already in the group case mentioned above, where $D(G/H)$ is naturally isomorphic with $Z(g')$, the center of $U(g')$, and where $\eta$ by transference under a suitable isomorphism can be identified with the natural homomorphism of $Z(g')$ into $D(G'/K')$. It is known from [13, 16] that this homomorphism is surjective when $G'$ is classical, but not surjective for certain exceptional groups $G'$.

For $v \in S(a_1)$ or $v \in S(a)$ we define $v^*$ by $v^*(\nu) = v(\lhd \nu)$, where $\nu \in a_{1c}$ or $\nu \in a^*$. 

**Lemma 3.** Let $D \in D(G/H)$. Then $\gamma(D^*) = \gamma(D)^*$ and $\eta(D^*) = \eta(D)^*$.

**Proof:** Choose $u \in U(g)^H$ such that $D = R_u$, and let $v \mapsto \tilde{v}$ be the antiautomorphism of $U(g)$ determined by $\tilde{v} = \lhd v$ for $v \in g$. Using [15, Ch.I, Thm.1.9 and Lemma 1.10] it is easily seen that $D^* = R_u$. The equality for $\gamma$ will follow if we prove that $\gamma(\tilde{u}) = \gamma(u)^*$ for $u \in U(g)^H$. Using [11, p.16, Cor.4] it is now seen that it suffices to consider the case of a Riemannian symmetric space, that is, we may assume that $H$ is compact. In this special case, the statement is proved in [15, p.307]. This proves that $\gamma(D^*) = \gamma(D)^*$. From (3) we now get that

$$\eta(D^*)(\lambda) = \gamma(D^*)(\lambda \lhd p_m) = \gamma(D)(\lhd \lambda + p_m).$$
Using that there exists an element \( w \) in the Weyl group of the root system of \( a_m \) in \( m \) such that \( w \rho_m = \rho_m \), and that this Weyl group is a subgroup of \( W_1 \), we get that

\[
\gamma(D) (\rho + \rho_m) = \gamma(D) (\rho - \rho_m) = \eta(D) (\rho + \rho_m),
\]
proving the equality for \( \eta \).

In the final section of this paper we relate \( \eta(D) \) to the radial part of \( D \) with respect to the \( KAH \) decomposition. In particular we shall prove that the condition that \( \eta(D) = 0 \) has the following strong consequence:

**Lemma 4.** Let \( D \in \mathcal{D}(G/H) \) and assume that \( \eta(D) = 0 \). Then \( D = 0 \) for all \( K \)-invariant smooth functions \( f \) on \( G/H \).

**Convexity.** We are now ready to state our main theorem:

**Theorem 2.** Let \( D \in \mathcal{D}(G/H) \) be non-zero.

(i) If \( \deg \eta(D) = \text{order} \ D \) then

\[
\text{supp} \ f \subset X_S \Leftrightarrow \text{supp} \ Df \subset X_S \Leftrightarrow \text{supp} \ D^*f \subset X_S
\]

for all \( f \in C_c^\infty(X) \) and all convex, compact \( W_{K\cap H} \)-invariant sets \( S \subset a \). In particular, \( X \) is \( D \)-convex, and \( D^* \) is injective on \( C_c^\infty(X) \).

(ii) If \( \eta(D) = 0 \) there exists for each closed ball \( S \subset a \), centered at the origin, a function \( f \in C_c^\infty(X) \) such that \( D^*f = 0 \) and \( \text{supp} \ f = X_S \). In particular, \( X \) is not \( D \)-convex, and \( D^* \) is not injective on \( C_c^\infty(X) \).

**Proof:** We first prove (i). The implication of \( \text{supp} \ Df \subset X_S \) from \( \text{supp} \ f \subset X_S \) is obvious. Assume \( \text{supp} \ Df \subset X_S \). Expanding \( f \) as a sum of \( K \)-finite functions, we have, since \( X_S \) is \( K \)-invariant, that \( f \) is supported in \( X_S \) if and only if all the summands are supported in \( X_S \). Moreover, \( D \) can be applied termwise to the sum, and hence we see that we may assume \( f \) to be \( K \)-finite. Then the support of \( f \) is \( K \)-invariant, and it suffices to prove that \( \text{supp} \ f \cap \exp(S)H \subseteq \exp(S)H \).

Let \( m = \text{order} \ D \), then \( m = \deg \eta(D) \) by the assumption on \( D \). Let \( u_0 \) denote the homogeneous part of \( \eta(D) \) of degree \( m \), then \( u_0 \neq 0 \). Notice that \( u_0 \) is also the homogeneous part of \( \eta(D) \) of degree \( m = \deg \eta(D) \) for any choice of \( \Sigma^+ \).

Assume that \( \text{supp} \ f \cap \exp(S)H \not\subseteq \exp(S)H \), and write

\[
\text{supp}_a f = \{ Y \in a \mid \exp(Y)H \in \text{supp} \ f \}.
\]

Then \( \text{supp}_a f \) is compact and not contained in \( S \). By the convexity of \( S \) there exists a non-empty open set of linear forms \( \lambda \in a^* \) with the property that

\[
0 < \max_{Y \in S} \lambda(Y) < \max_{Y \in \text{supp}_a f} \lambda(Y).
\]

Since \( u_0 \neq 0 \) there exists a \( \lambda \in a^* \) with \( u_0(\lambda) \neq 0 \), and satisfying (7). Let \( Y_0 \in \text{supp}_a f \) be a point where the value on the right side of (7) is attained. Then \( Y_0 \not\in S \) and we have that

\[
\lambda(Y_0) \leq \lambda(Y_0), \quad (Y \in \text{supp}_a f).
\]
Let \( a_0 = \exp Y_0 \), then

\[ a_0 H \not\in \text{supp } D f \]

by the assumption on \( \text{supp } D f \), and

\[ a_0 H \in \text{supp } f. \]

Choose a positive system \( \Sigma^+ \) such that \( \lambda \) is antidominant, and let \( n \) and \( N \) be given correspondingly. Let \( \Omega \) denote the open (see [20, Prop. 7.1.8]) subset \( \Omega = N M A H \) of \( X = G/H \), and define \( g \in C^\infty(\Omega) \) by \( g(nmaH) = \lambda(\log a) \) for \( n \in N, m \in M, a \in A \). We claim that

\[ f = 0 \text{ on } \{ x \in \Omega \mid g(x) > g(a_0) \}. \]

To prove (11) let \( x = nmaH \in \Omega \cap \text{supp } f \). Then we must show that \( g(x) \leq g(a_0) \), or equivalently, that \( \lambda(\log a) \leq \lambda(Y_0) \). To see that this holds, write

\[ nma = k \exp(Z)h, \quad (k \in K, Z \in a, h \in H_e) \]

according to the \( G = KAH_e \) decomposition; here \( H_e \) denotes the identity component of \( H \). Then

\[ \exp(Z)h \in KNMa = KMaN, \]

and by the convexity theorem of [1, Thm. 3.8] it follows that \( \log a = U + V \), where \( U \) is contained in the convex hull of \( W_{K \cap H}Z \), and \( V \) belongs a certain subcone of the closed convex cone \( \{ V \in a \mid \langle V, Y \rangle \geq 0, Y \in a^+ \} \), which is dual to the positive Weyl chamber \( a^+ \). In particular, \( \lambda(V) \leq 0 \) by the antidominance of \( \lambda \), and hence

\[ \lambda(\log a) \leq \lambda(U) \leq \max_{w \in W_{K \cap H}} \lambda(wZ). \]

Now \( \exp(wZ)H = w \exp(Z)H = wk^{-1}xH \) for \( w \in W_{K \cap H} \), and from \( x \in \text{supp } f \) and the \( K \)-invariance of the support we then see that \( \exp(wZ)H \in \text{supp } f \). Hence \( wZ \in \text{supp}_a f \), and we conclude by (8) that

\[ \lambda(\log a) \leq \lambda(Y_0). \]

This implies (11).

Let \( \sigma(D) \) be the principal symbol of \( D \). We have

\[ \sigma(D)(dg(a_0)) = \frac{1}{m!} D((g \Leftrightarrow g(a_0))^m)(a_0). \]

It follows immediately from the definition of \( g \) that \( R_ug = 0 \) for \( u \in U(g)h_e \). Moreover, since \( g \) is left \( NM \)-invariant, and since \( n \) and \( m \) are normalized by \( A \), we also have that \( R_ug(a) = 0 \) for \( a \in A, u \in (n + m)_cU(g) \). Hence \( Dg(a) = R_{\eta(D)}g(a) \). Applying the same reasoning to the function \( (g \Leftrightarrow g(a_0))^m \) we obtain that

\[ D((g \Leftrightarrow g(a_0))^m)(a) = R_{\eta(D)}(g \Leftrightarrow g(a_0))^m(a) = m! a_0(\lambda). \]
Combining (12) and (13) we obtain that $\sigma(D)(dg(a_0)) = u_0(\lambda)$ and hence

\begin{equation}
\sigma(D)(dg(a_0)) \neq 0 \tag{14}
\end{equation}

by the assumption on $\lambda$.

From (9), (11) and (14) it follows by Holmgren’s uniqueness theorem ([17, Thm. 5.3.1]) that $f = 0$ on a neighbourhood of $a_0H$, contradicting (10). This completes the proof of the first biimplication in (i). From Lemma 3 we get that $D^*$ also satisfies the assumption of (i), and hence the remaining statements in (i) follow.

We now prove (ii). Let $S$ be the ball of radius $R$ centered at the origin, and let $\varphi \in C^\infty(\mathbb{R})$ be positive on $[0; R^2]$ and zero on $[R^2; \infty[$. Define $f(kaH) = \varphi(\|a\|^2)$ for $k \in K$, $a \in A$. Then $f \in C^\infty(X)$ by [10, Thm. 4.1], and we clearly have supp $f = X_S$. Now (ii) follows from Lemma 4. $\blacksquare$

**Corollary 1.**

(i) If $X = G/H$ is split, then $X$ is $D$-convex and $D$ is injective on $C_c^\infty(X)$ for all non-trivial invariant differential operators $D$.

(ii) If $X$ is not split there exists a non-trivial invariant differential operator $D$, such that $X$ is not $D$-convex and which is not injective on $C_c^\infty(X)$.

**Remark 1.** By regularization it follows that the statements of Theorem 2 and its corollary hold with $C_c^\infty(X)$ replaced by the space of compactly supported distributions on $X$.

**Remark 2.** An explicit example of an operator $D$ as in part (ii) of Theorem 2 and its corollary is given in [6], where it is shown that the "imaginary part" $C_i$ of the Casimir operator on a complex semisimple Lie group $G'$ is not solvable. Viewing $G'$ as a symmetric space for $G' \times G'$ it is actually easily seen that $\eta(C_i) = 0$ (see also loc. cit. p. ?).

**The radial part.** Let $D \in \mathcal{D}(G/H)$. Choose a positive system $\Sigma^+$ and let $A^+ \subseteq A$ be the corresponding open chamber. Via the canonical map from $G$ to $G/H$ we identify $A^+$ with a submanifold of $X$. According to [15, p. 259] there exists a unique differential operator $\Pi(D)$ on $A^+$ such that $(Df)|_{A^+} = \Pi(D)(f)|_{A^+}$ for all $K$-invariant smooth functions $f$ on $X$. $\Pi(D)$ is called the radial part of $D$. The following result establishes a connection between $\Pi(D)$ and $\eta(D)$. It is a generalization of [12, p. 267, Lemma 26] (see also [15, p. 308, Prop. 5.23]).

Let $\mathcal{R}^+$ denote the ring of analytic functions $\varphi$ on $A^+$ which can be expanded in an absolutely convergent series on $A^+$ with zero constant term:

$$\varphi = \sum_{\nu \in \Lambda} c_\nu e^{-\nu}, \quad c_\nu \in \mathbb{C}, c_0 = 0$$

where the sum is over the set $\Lambda = \mathbb{N} \Sigma^+$ and where $e^{-\nu}$ is defined by $e^{-\nu}(a) = e^{-\nu(\log a)}$.

**Proposition 1.** Let $D \in \mathcal{D}(G/H)$. There exist a finite number of elements $v_i \in S(\mathfrak{a})$ and functions $g_i \in \mathcal{R}^+$ such that

\begin{equation}
\Pi(D) = e^{-\rho} R_{\eta(D)} \circ e^\rho + \sum_i g_i R_{v_i} \tag{15}
\end{equation}
on $A^+$. Moreover the order $m$ of $\Pi(D)$ equals the degree of $\eta(D)$, and we can select the $v_i$ such that

$$\deg v_i \leq m \Leftrightarrow 1$$

for all $i$ (where a negative degree of $v_i$ means that $v_i = 0$). In particular, $\Pi(D) = 0$ if and only if $\eta(D) = 0$.

**Proof:** The existence of the $v_i$ and $g_i$ such that (15) holds follows from [2, Lemma 3.9]. It remains to prove (16) (from the lemma of loc. cit. we only get that $\deg v_i < \text{order}(D)$, which is not sharp enough to conclude (16), because the order of $\Pi(D)$ in general may be smaller than that of $D$).

Let

$$\Pi(D) = \sum_{\nu \in \Lambda} e^{-\nu} R_{\bar{v}_\nu}$$

be the expansion of $\Pi(D)$ derived from (15), where $v_\nu \in S(\mathfrak{a})$ and where $v_0$ is given by $v_0(\lambda) = \eta(D)(\lambda + \rho)$. We claim that

$$\deg v_\nu \leq \deg v_0 \Leftrightarrow 1 \quad \text{for all} \quad \nu \neq 0,$$

from which both the statement that order $\Pi(D) = \deg \eta(D)$ and (16) follow. We shall obtain (18) by means of a recursion formula for the $v_\nu$, derived from the relation $L_X D = DL_X$, where $L_X$ is the Laplace-Beltrami operator on $X$ given in terms of the Casimir operator $\omega \in U(\mathfrak{g})^H$ by $L_X = R_\omega$.

The radial part of $L_X$ is easily computed (see [10, eq. (4.12)]):

$$\Pi(L_X) = J^{-1/2} (L_A \circ J^{1/2} \Leftrightarrow L_A(J^{1/2}))$$

where $L_A$ is the Laplacian on $A$, and $J = \prod_{\alpha \in \Sigma^+} (e^\alpha \Leftrightarrow e^{-\alpha}) p_\alpha (e^\alpha + e^{-\alpha}) q_\alpha$. Here $p_\alpha$ and $q_\alpha$ are certain integers given by root space dimensions, see [20, Thm 8.1.1].

Put $\Pi(D) = J^{1/2} \Pi(D) \circ J^{-1/2}$, then it follows from the commutation relation $[L_X, D] = 0$ and (19) that $\Pi(D)$ commutes with $L_A \Leftrightarrow d$, where $d$ is the function $J^{-1/2} L_A(J^{1/2})$. Expanding $d$ in a power series $d(a) = \sum_{\gamma \in \Lambda} d_\gamma a^{-\gamma}$ on $A^+$ and expanding $\Pi(D)$ in analogy with (17) as

$$\Pi(D) = \sum_{\nu \in \Lambda} e^{-\nu} R_{\bar{v}_\nu}$$

we obtain the following expression

$$\sum_{\nu, \gamma \in \Lambda} ([L_A, e^{-\nu}] R_{\bar{v}_\nu} \Leftrightarrow d_\gamma e^{-\nu} [e^{-\gamma}, R_{\bar{v}_\nu}]) = 0.$$

Comparing coefficients to $e^{-\nu}$ we get

$$[L_A, e^{-\nu}] R_{\bar{v}_\nu} = \sum_{\gamma \in \Lambda, \nu - \gamma \in \Lambda} d_\gamma e^{-(\nu - \gamma)} [e^{-\gamma}, R_{\bar{v}_{\nu - \gamma}}].$$
where the sum is finite. In this equation, if \( \nu \neq 0 \) and \( \tilde{v}_\nu \neq 0 \), the left side is a differential operator on \( A^+ \) of order \( 1 + \deg \tilde{v}_\nu \), whereas the order of the operator on the other side is less than the maximum of the degrees of all \( \tilde{v}_{\nu - \gamma} \), \( \gamma \in \Lambda \setminus \{0\} \). In particular, it follows by an easy induction that \( \deg \tilde{v}_\nu \leq \deg \tilde{v}_0 \Leftrightarrow 2 \) for \( \nu \neq 0 \).

In the series
\[
\Pi(D) = J^{-1/2} \Pi(D) \circ J^{1/2} = J^{-1/2} \sum_{\nu \in \Lambda} e^{-\nu} R_{\tilde{v}_\nu} \circ J^{1/2}
\]

it is seen that the only contribution in degree \( \deg \tilde{v}_0 \) is obtained in the \( \epsilon^0 \) term. Hence \( \tilde{v}_0 \) and \( \tilde{v}_\nu \) have the same degree (in fact it is easily seen that \( \tilde{v}_0 = \eta(D) \)), and \( \nu_\nu \) has a lower degree for all other \( \nu \). From this the claimed property (18) of the \( v_\nu \) follows.

The final statement of the proposition follows from the previous statements. \( \blacksquare \)

**Proof of Lemma 4**: Assume \( \eta(D) = 0 \) and let \( f \) be smooth and \( K \)-invariant. It follows from the final statement of Proposition 1 that \( Df = 0 \) on \( A^+ \). Since \( \Sigma^+ \) was arbitrary we conclude that \( Df = 0 \) on an open dense subset of the submanifold \( AH \) of \( X \). By \( G = KAH \) and the \( K \)-invariance of \( f \) we conclude that \( Df = 0 \). \( \blacksquare \)

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**References.**


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