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INVERTIBLE TERMS IN THE LAMBDA CALCULUS

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1. Introduction

It is well-known that the set of λ-terms modulo $\beta\eta$-convertibility is a semi-group with I as identity element and composition $\circ$, defined by $M \circ N = BMN$, where $B \equiv \lambda xyz \, . \, x(yz)$. In [6, pp. 167, 168] the question is raised under what conditions an element in this semi-group has an inverse.

Dezani-Ciancaglini gave in [8] a characterization of (w.r.t. $\lambda\beta\eta$-calculus) invertible terms having a normal form as the 'finite hereditary permutators', and she conjectures that these are all the $\beta\eta$-invertible terms, i.e. a term without normal form cannot have an inverse.

In this paper we confirm her conjecture. Two proofs are given for this fact, of which the first is more direct. The second proof uses the in itself interesting fact that certain 'λ-trees' can be represented as Böhm-trees of λI-terms (in fact we prove something more), plus Hyland's characterization of the equality in the Graph model $P\omega$ (see [9, 1]).

The result on representation of λ-trees is further used to characterize the λ-terms invertible in D_∞, Scott's well-known lattice model (see [12]).

Since for this last result a slightly more general form of the main lemma in [8] is needed, we have included a new proof of that lemma.

2. Preliminaries

In this section we collect the ingredients necessary for the sequel, without the proofs which can be found in the literature. The basic definitions and facts about the λ-calculus are supposed to be known.

Notation 1. (i) Λ is the set of λ-terms, Λ^0 the set of closed λ-terms. Similarly Λ_I and Λ_I^0 for λI-terms. $\rightarrow_{\beta(\eta)}$ is one step $\beta(\eta)$-reduction, $\twoheadrightarrow_{\beta(\eta)}$ its transitive reflexive closure; $=_{\beta(\eta)}$ is the equality generated by $\rightarrow_{\beta(\eta)}$.

Abbreviations: $M\vec{N}$ for $MN_1 \cdots N_k$; $\lambda\vec{x}.M$ for $\lambda x_1 \cdots x_k.M$.

(ii) $\ulcorner \; \urcorner : \Lambda \rightarrow \mathbb{N}$ is some recursive coding of λ-terms.

(iii) For $n \in \mathbb{N}$, we define $\mathbf{n} \in \Lambda_I^0$ by $\mathbf{n} = \lambda xy.x^{n+1}y$, where $x^1y = xy$ and $x^{n+1}y = x(x^ny)$.

Remark. $\mathbf{n}II \twoheadrightarrow_\beta I$ for all n.

(iv) A finite sequence of natural numbers $(n_1, \ldots, n_k)$ will be coded as a natural number, notation: $\langle n_1, \ldots, n_k\rangle$. Let Seq be the set of these codes, called 'sequence numbers'. Elements of Seq are denoted by $\sigma, \tau, \rho, \ldots$. Concatenation of sequence numbers is denoted by $*$, the (code of) the empty sequence by $\langle\ \rangle$. If $\sigma = \langle n_1, \ldots, n_k\rangle$, $\text{lth}(\sigma) = k$. $\leqslant$ is the usual p.o. on Seq: $\sigma \leqslant \tau \Leftrightarrow \exists\rho\ \sigma * \rho = \tau$.

Lemma 1. *Let $M \in \Lambda$. Then*

$$M \textit{ has a } \beta\textit{-normal form} \Leftrightarrow M \textit{ has a } \beta\eta\textit{-normal form.}$$

Proof. See [7, p. 124] or [3, Section 6.14].

Lemma 2. *Let $M \in \Lambda_I$. Then M has a normal form iff all its subterms have a normal form.*

Proof. See [5, p. 27 Theorem 7 XXXII].

Theorem 1 (Second fixed point theorem).

$$\forall F \in \Lambda\ \exists M \in \Lambda\ \ M \twoheadrightarrow_\beta F\ulcorner M\urcorner.$$

Proof. See [1, Theorem 2.20].

Theorem 2 (Kleene). *There exists an enumerator for closed λ-terms. This is also true for the restriction to λI-calculus; more precisely:*

$$\exists E \in \Lambda_I^0\ \forall M \in \Lambda_I^0\ \ E\ulcorner M\urcorner \twoheadrightarrow_\beta M.$$

Proof. [5, Section 16] or [2].

Lemma 3. $\forall M, N \in \Lambda_I^0\ \exists F \in \Lambda_I^0\ F\mathbf{0} \twoheadrightarrow_\beta M$ *and* $F\mathbf{1} \twoheadrightarrow_\beta N$.

Proof. See [5, 14 I, p. 46].

Theorem 3. (Representability of recursive functions in λI-calculus). *Let $f: \mathbb{N} \rightarrow \mathbb{N}$ be recursive. Then $\exists F \in \Lambda_I^0\ \forall n \in \mathbb{N}\ F\mathbf{n} \twoheadrightarrow_\beta \mathbf{f(n)}$.*

Proof. See [5, Ch. III, p. 39].

2.1. *Böhm trees*

In the sequel the concept of 'Böhm tree' introduced in [1] will prove to be useful. A Böhm tree can be considered as a kind of 'infinite normal form'. See also [10].

Definition 1. (i) A tree T is a subset of Seq such that $\sigma * \langle n\rangle \in T \Rightarrow \sigma \in T$. (The branches may be infinite.)

(ii) If $\sigma \in T$, then $T_\sigma = \{\tau \in T \mid \tau \geqslant \sigma\}$.

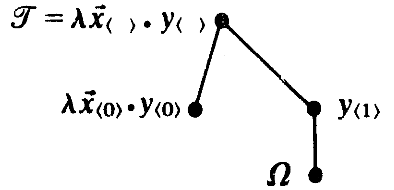
(iii) Let T be a tree. The function $l: T \to \mathbb{N}$ is defined by: $l(\sigma)$ = number of direct successors of σ in T.

(iv) The level of $\sigma \in T$ is $\text{th}(\sigma)$.

Definition 2. (i) An Ω-free λ-tree $\mathcal{T}$ is a tree T, notation $T = \text{Seq}(\mathcal{T})$, together with functions $\text{abs}: T \to \bigcup_{n\geqslant 0}(\text{Var})^n$ (the set of strings of variables, including the empty string) and $\text{occ}: T \to \text{Var}$. Instead of $\text{abs}(\sigma)$ we write $\lambda\vec{x}_\sigma$. Instead of $\text{occ}(\sigma)$ we write y_σ.

(ii) Further we will allow an extra symbol, Ω, as a label of nodes of $\mathcal{T}$—but only to terminal nodes. Moreover, if σ has label Ω, then there is no abstraction at σ, i.e. $\text{abs}(\sigma) = \emptyset$.

Example.



(iii) $\mathcal{T}_\sigma$ is $(\text{Seq}(\mathcal{T}))_\sigma$ plus corresponding labels.

(iv) $\mathcal{T}_1 =_n \mathcal{T}_2 \Leftrightarrow$ the levels $0, \ldots, n$ of $\mathcal{T}_1$, $\mathcal{T}_2$ are identical.

(v) *Notation.* $\vec{w}_\sigma$ is the list of variables abstracted before σ, i.e. if $\sigma = \langle n_0, \ldots, n_k\rangle$, then

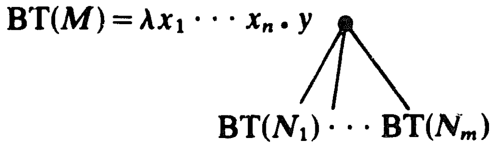
$$\vec{w}_\sigma = \vec{x}_{\langle\,\rangle}\vec{x}_{\langle n_0\rangle}\vec{x}_{\langle n_0,n_1\rangle}\cdots\vec{x}_{\langle n_0,\ldots,n_{k-1}\rangle}.$$

y_σ is bound iff $y_\sigma \in \vec{w}_\sigma \cup \vec{x}_\sigma$. $\text{FV}(\mathcal{T}_\sigma)$ is the set of free variables in $\mathcal{T}_\sigma$.

To each $M \in \Lambda$ a λ-tree $\text{BT}(M)$ (Böhm tree of M) is associated as follows:

(i) If M is unsolvable (equivalently: has no head normal form (hnf)), then $\text{BT}(M) = \Omega$,

(ii) Otherwise, $M =_\beta \lambda x_1 \cdots x_n\,.\,yN_1 \cdots N_m$ for some $n, m \geqslant 0$ and some N_i. Then



i.e. level 0 of $\text{BT}(M)$ is known. By iteration we find all levels.

Corresponding to the subtrees $(\mathrm{BT}(M))_\sigma$ as defined above, we define terms M_σ for $\sigma \in \mathrm{Seq}(\mathrm{BT}(M))$:

(i) $M_{\langle\,\rangle} \equiv M$,

(ii) Let M_σ be defined and let σ have a successor in $\mathrm{Seq}(\mathrm{BT}(M))$. Then M_σ has a hnf; say $M_\sigma =_\beta \lambda x_1 \cdots x_n \,.\, yN_1 \cdots N_m$. Define $M_{\sigma * \langle i \rangle} \equiv N_i$, $1 \leq i \leq m$.

Remark. M_σ is defined only modulo $=_\beta$.

To minimize the troubles with α-conversion (renaming of bound variables), we fix the following convention. All the λ-trees $\mathcal{T}$ in the sequel will be in the following 'α-normal form':

Let $\mathrm{Var} = \{v_{n,m} \mid n, m \in \mathbb{N}\}$ and let $\vec{x}_\sigma$ be an abstraction vector in $\mathcal{T}$ of length k. Then $\vec{x}_\sigma \equiv v_{\sigma,1} v_{\sigma,2} \cdots v_{\sigma,k}$.

Remark. This means that in taking a subtree $\mathcal{T}_\sigma$ of $\mathcal{T}$ we have to shift the indices of the abstracted and the bound variables: $v_{\sigma * \rho, i} \mapsto v_{\rho,i}$.

Proposition 1. $(\mathrm{BT}(M))_\sigma = \mathrm{BT}(M_\sigma)$.

Proof. The proof is left to the reader.

We will give an example as illustration of some concepts above. This example also illustrates a point in the sequel, *nl*, that the assignment $\sigma \mapsto \mathrm{FV}(\mathrm{BT}(M))_\sigma$ need not be recursive.

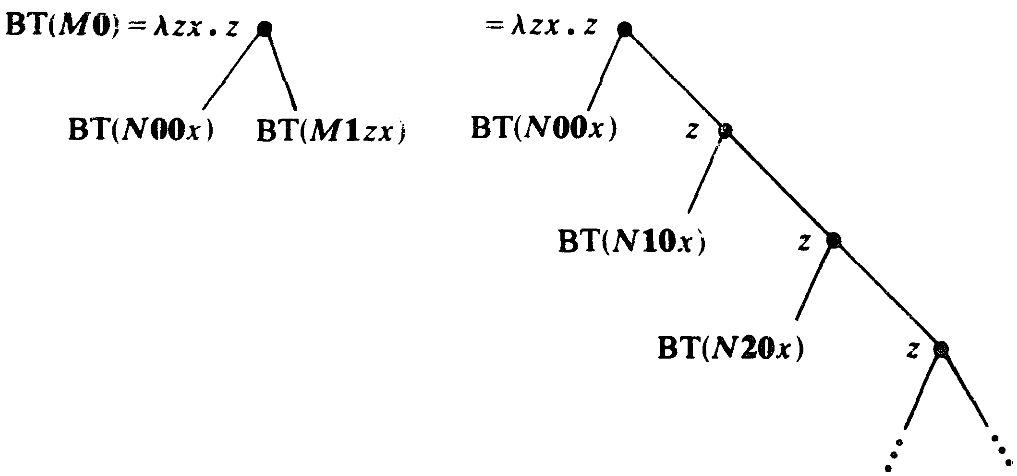
Example 1. Let $R(m, k)$ be a recursive binary predicate, and let F be a λI-term representing R, i.e. $Fmk = \mathbf{0}$ if $R(m, k)$ and $= \mathbf{1}$ else. Define a λI-term N such that

$$Nmk \twoheadrightarrow \begin{cases} \lambda ab \,.\, b(Nm\, k+\mathbf{1}\; a), & \text{if } R(m,k), \\ I, & \text{else.} \end{cases}$$

Further, define $M \in \Lambda_I$ such that

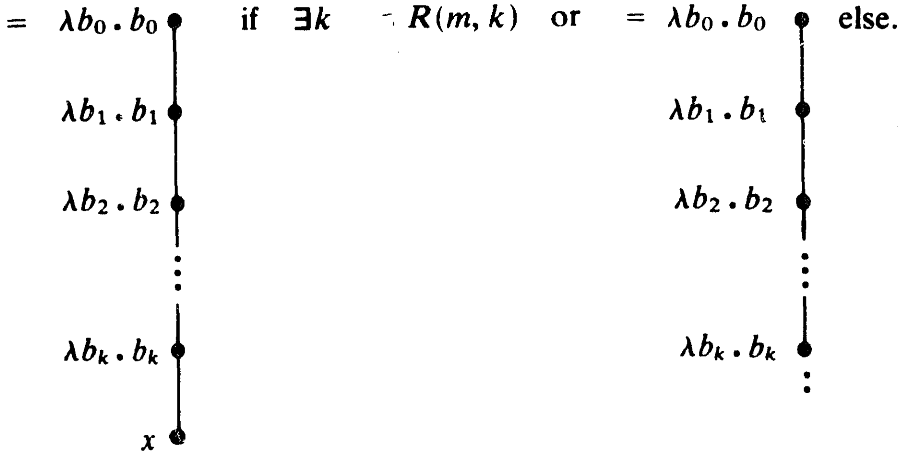
$$Mn \twoheadrightarrow \lambda zx \,.\, z(Nn\mathbf{0}x)(Mn+\mathbf{1}\; zx).$$

Now



where

$$(\mathrm{BT}(M\mathbf{0}))_{\underbrace{\langle 1,1,\ldots,1,0\rangle}_{m \text{ times}}} = BT(Nm\mathbf{0}x) =$$



So

$$\mathrm{FV}(\mathrm{BT}(M\mathbf{0}))_{\underbrace{\langle 1,1,\ldots,1,0\rangle}_{m \text{ times}}} = \begin{cases} \{x\}, & \text{if } \exists k \ \neg R(m,k), \\ \emptyset & \text{else.} \end{cases}$$

Since $\exists k \ \neg R(m, k)$ is not necessarily recursive, the assignment $\sigma \mapsto \mathrm{FV}(\mathrm{BT}(M\mathbf{0}))_\sigma$ is therefore not necessarily recursive.

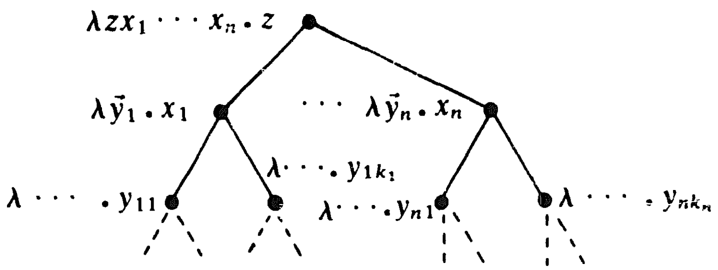
Theorem 4 (Hyland). *Let* $\mathrm{P}\omega$ *be the graph model* (see [11]). *Let* $M, N \in \Lambda^0$. *Then*

$$\mathrm{P}\omega \models M = N \Leftrightarrow \mathrm{BT}(M) = \mathrm{BT}(N).$$

Proof. See [9, 2].

3. Proof of the main lemma

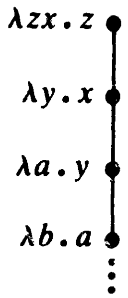
Lemma 4. $M =_{D_\infty} I$ iff $\mathrm{BT}(M)$ *has the form*



i.e. except for the head variable z, all abstracted variables occur exactly one, one level lower, in the same order. Here $n \geq 0$ *and the abstraction vectors* $\lambda\vec{y}_1, \ldots, \lambda\vec{y}_n$, *etc. may be empty.*

Proof. See [1] or [9].

Remark. Note that such a BT may be infinite; an example is Wadsworth's term $J \twoheadrightarrow \lambda zx.z(Jx)$ (see [13]) which has the BT:



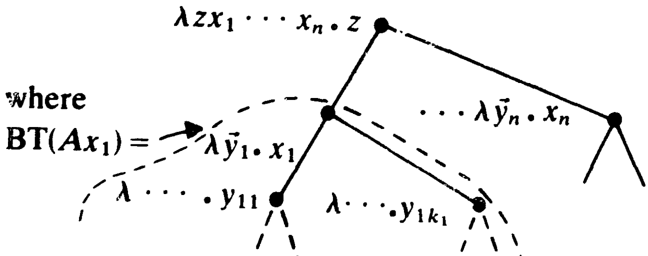
J is in some sense an 'infinite η-expansion of I', namely compare the following sequence of η-expansions:

$$\lambda z.z \underset{\eta}{\leftarrow} \lambda zx.zx \underset{\eta}{\leftarrow} \lambda zx.z(\lambda y.xy) \underset{\eta}{\leftarrow} \lambda zx.z(\lambda y.x(\lambda a.ya)) \underset{\eta}{\leftarrow} \cdots .$$

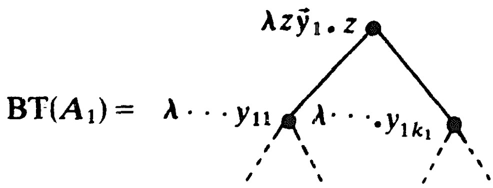
In a similar way every term M such that $M =_{D_\infty} I$ can be viewed as a possibly infinite η-expansion of I.

Lemma 5. *If $A_i (1 \leqslant i \leqslant n)$ are terms such that* $\mathrm{BT}(A_i)$ *is closed and $M \equiv \lambda zx_1 \cdots x_n . z(A_1x_1) \cdots (A_nx_n) =_{D_\infty} I$, then $A_i =_{D_\infty} I$ $(1 \leqslant i \leqslant n)$.*

Proof. By Lemma 4 BT(M) has the form described above:



Since BT(A_1) is closed, it is evident that



and this is again the BT of a possibly infinite η-expansion of I, i.e. $A_1 =_{D_\infty} I$.

Notation 2. (i) $M \sim_{\beta\eta} N := M \circ N =_{\beta\eta} I$ and $N \circ M =_{\beta\eta} I$. Analogous $M \sim_{D_\infty} N$. (M, N are each others inverse in $\lambda\beta\eta$-calculus resp. in D_∞.)

(ii) Let $\boldsymbol{z}$ be the set of terms M having a head normal form (hnf) with free head variable z, i.e. $M =_\beta \lambda\vec{x}\,.\,z\vec{N}$ for some $\vec{x}$ ($z \notin \vec{x}$) and some $\vec{N}$.

(iii) Let $\lambda z\,.\,\boldsymbol{z}$ be the set $\{\lambda z\,.\,M \mid M \in \boldsymbol{z}\}$. So $P \in \lambda z\,.\,\boldsymbol{z}$ iff $\exists \vec{x}, \vec{N}\, P =_\beta \lambda z\vec{x}\,.\,z\vec{N}$.

(iv) Let M be such that $z \notin \mathrm{FV}(M)$. The symbol $'$ will be reserved to denote special substitution instances of M, namely those where elements of $\boldsymbol{z}$ are substituted for some of the free variables of M. Notation: M', M'', M'''.

Definition 3. $M \sim'_{D_\infty} N$ (M, N are 'almost' each others inverse in D_∞) iff

(i) $M, N \in \lambda z\,.\,\boldsymbol{z}$ and

(ii) $M' \circ N' =_{D_\infty} I$ for some M', N' and

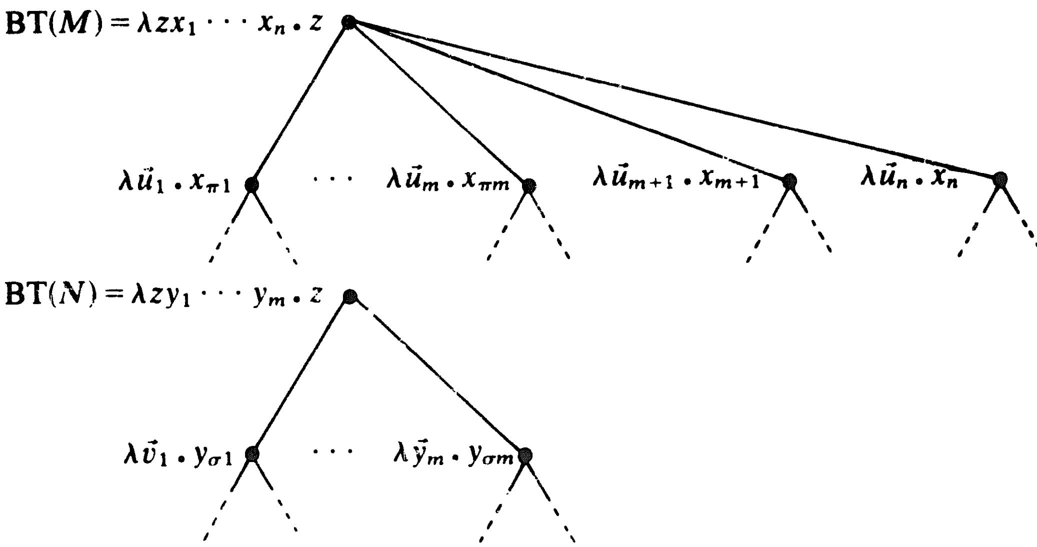
(iii) $N'' \circ M'' =_{D_\infty} I$ for some M'', N''.

Proposition 2. $M \sim_{D_\infty} N \Rightarrow M \sim'_{D_\infty} N$.

Proof. We only need to prove Definition 3(i). Suppose $M \sim_{D_\infty} N$, then $M \circ N =_{D_\infty} I$. Now M has a hnf, for otherwise $\mathrm{BT}(M \circ N) = \mathrm{BT}(\lambda z\,.\,M(Nz)) = \lambda z\,.\,\Omega$, which is in contradiction with Lemma 5. Also N has a hnf, for otherwise $\mathrm{BT}(M \circ N) = \mathrm{BT}(\lambda z\,.\,M\Omega)$ and now $z \notin \mathrm{FV}(M\Omega)$, which is in contradiction with Lemma 5.

Hence, $M =_\beta \lambda z\vec{x}\,.\,pM_1 \cdots M_m$ and $N =_\beta \lambda z\vec{y}\,.\,qN_1 \cdots N_n$ for some $\vec{x}$, $\vec{y}$ and $m \geq 0$, $n \geq 0$. Further $z \equiv p \equiv q$, since $M(Nz)$ has z as head variable. Hence $M, N \in \lambda z\,.\,\boldsymbol{z}$.

Lemma 6. (i) *Let* $M \sim'_{D_\infty} N$. *Then the first two levels of* $\mathrm{BT}(M)$ *and* $\mathrm{BT}(N)$ *have the form* (*for some* $n \geq 0$, $m \geq 0$, *say* $n \geq m$, *and some permutations* π, σ *of* $\{1, \cdots, m\}$):

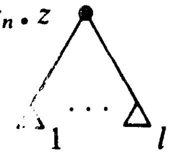


where the abstraction vectors $\lambda\vec{u}_i$ and $\lambda\vec{v}_j$ may be empty.

(ii) *Moreover, π and σ are each others inverse.*

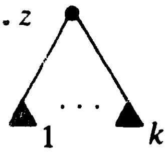
Proof. $M \sim'_{D_\infty} N$, hence $M, N \in \lambda z\,.\,z$. So

$$\mathrm{BT}(M) = \lambda z x_1 \cdots x_n\,.\,z$$



where $\triangle_i = \mathrm{BT}(P_i)$ and

$$\mathrm{BT}(N) = \lambda z y_1 \cdots y_m\,.\,z$$



where $\blacktriangle_j = \mathrm{BT}(Q_j)$.

We will prove:

(a) $l = n$, $k = m$,

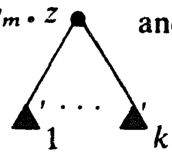
(b) each of the subtrees $\triangle_i$ has $x_{\pi i}$ as top variable, and each of the $\blacktriangle_j$ has $y_{\sigma j}$ as top variable,

(c) π is a permutation of $\{1, \ldots, n\}$, σ a permutation of $\{1, \ldots, m\}$,

(d) $\pi \circ \sigma = \sigma \circ \pi = id$. (Remark: π, σ do not in general have the same domain, since $m \neq n$, but it is obvious how to define $\pi \circ \sigma$ and $\sigma \circ \pi$ by a trivial extension of the permutation with least domain.)

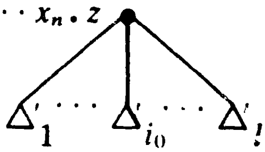
By $M \sim'_{D_\infty} N$ we have $M'' \circ N'' =_{D_\infty} I$ and $N' \circ M' =_{D_\infty} I$ for some M'', N'' and M', N'. Now suppose for some i_0, $\triangle_{i_0}$ has top Ω or $v \notin \{v_1, \ldots, x_n\}$. First we remark that

$$\mathrm{BT}(N') = \lambda z y_1 \cdots y_m\,.\,z$$



and

$$\mathrm{BT}(M') = \lambda z x_1 \cdots x_n\,.\,z$$



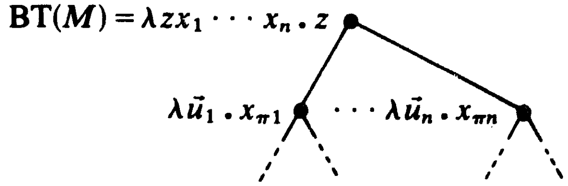
where $\triangle'_i = \mathrm{BT}(P'_i)$ and $\blacktriangle'_j = \mathrm{BT}(Q'_j)$. Now also $\triangle'_{i_0} = \Omega$ or has a top variable $w \notin \{x_1, \ldots, x_n\}$. For either there was no substitution for v, in which case $v \equiv w$, or some $P \in \mathbf{w}$ was substituted for v and hence w is top variable of $\triangle_{i_0}$. Evidently $w \notin \{x_1, \ldots, x_n\}$ since in substitutions free variables may not become bound.

Now consider $N' \circ M'$, a possibly infinite η-expansion of I as in Lemma 4. An easy computation of levels 0 and 1 of $\mathrm{BT}(N' \circ M')$ shows that the Ω or w in question again appears at level 1, contradiction with Lemma 4.

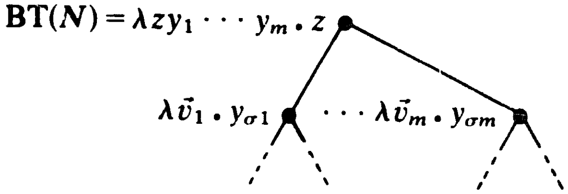
Hence every $\triangle_i$ has a top variable $\in \{x_1, \ldots, x_n\}$. By symmetry also every $\blacktriangle_j$ has a top $\in \{y_1, \ldots, y_m\}$.

Now suppose that some x_r occurs twice or does not occur at level 1 of BT(M). Then again a simple computation of the first two levels of BT($N' \circ M'$) gives a contradiction with Lemma 4.

So we have proved that



and



for some π, σ, $n \geqslant 0$, $m \geqslant 0$.

To simplify the computation in the final argument of the proof, we will suppose $n = m$. This is not an essential restriction. From BT(M) and BT(N) we see that

$$M =_\beta \lambda z x_1 \cdots x_n \,.\, z(\lambda \vec{u}_1 \,.\, x_{\pi 1}\vec{X}_1) \cdots (\lambda \vec{u}_n \,.\, x_{\pi n}\vec{X}_n)$$

and (1)

$$N =_\beta \lambda z y_1 \cdots y_n \,.\, z(\lambda \vec{v}_1 \,.\, y_{\sigma 1}\vec{Y}_1) \cdots (\lambda \vec{v}_n \,.\, y_{\sigma n}\vec{Y}_n)$$

for some maybe empty $\vec{X}_i$, $\vec{Y}_j$. For M', N' we have expressions similar to those for M, N with X_i, Y_j replaced by X'_i, Y'_j. Hence

$$\begin{aligned} M' \circ N' &=_\beta \lambda z \,.\, M'(N'z) \\ &=_\beta \lambda z x_1 \cdots x_n \,.\, N'z(\lambda \vec{u}_1 \,.\, x_{\pi 1}\vec{X}'_1) \cdots (\lambda \vec{u}_n \,.\, x_{\pi n}\vec{X}'_n) \\ &=_\beta \lambda z x_1 \cdots x_n \,.\, z(\lambda \vec{v}_1 \,.\, [\lambda \vec{u}_{\sigma 1} \,.\, x_{\pi(\sigma 1)}\vec{X}'_{\sigma 1}]\vec{Y}'_1) \cdots (\cdots) \\ &=_\beta \lambda z x_1 \cdots x_n \,.\, z(\lambda \vec{w}_1 \,.\, x_{\pi(\sigma 1)}\vec{Z}_1) \cdots (\lambda \vec{w}_n \,.\, x_{\pi(\sigma n)}\vec{Z}_n). \end{aligned}$$

Since $M' \circ N' =_{D_\infty} I$, we must have by Lemma 4: $\pi(\sigma j) = j$. By symmetry also $\sigma(\pi i) = i$.

Main lemma (M. Dezani-Ciancaglini). *Suppose $M \sim'_{D_\infty} N$. Then for some $n, m \geqslant 0$, say $n \geqslant m$, there are permutations π, σ and $M_i (1 \leqslant i \leqslant n)$, $N_j (1 \leqslant j \leqslant m)$ such that*

(i) $M =_\beta \lambda z x_1 \cdots x_n \,.\, z(M_1 x_{\pi 1}) \cdots (M_n x_{\pi n})$,

$N =_\beta \lambda z y_1 \cdots y_m \,.\, z(N_1 y_{\sigma 1}) \cdots (N_m y_{\sigma m})$,

$\pi \circ \sigma = \sigma \circ \pi = id$,

(ii) $N_i \sim'_{D_\infty} M_{\sigma i} (1 \leqslant i \leqslant m)$, $M_i \sim'_{D_\infty} I (m < i \leqslant n)$.

Proof. (i) Follows from Lemma 6 by taking in (1) $M_i \equiv \lambda z\vec{u}_i . z\vec{X}'_i$ and $N_j \equiv \lambda z\vec{v}_j . z\vec{Y}'_j$. Further

$$\begin{aligned} M' \circ N' &= \lambda z . M'(N'z) \\ &= \lambda z x_1 \cdots x_n . N'z(M''x_{\pi 1}) \cdots (M''_n x_{\pi n}) \\ &= \lambda z x_1 \cdots x_n . z[(N'_1 \circ M''_{\sigma 1})x_1] \cdots [(N'_m \circ M''_{\sigma m})x_m] \\ &\quad \cdots [M''_{m+1}x_{m+1}] \cdots [M''_n x_n] \end{aligned}$$

(here we used that $N'z \in \mathbf{z}$ and $M''_i x_{\pi i} \in x_{\pi i}$)

$$= _{D_\infty} I.$$

By Lemma 5 we are through if we can prove that the $\mathrm{BT}(N'_i \circ M''_{\sigma i})$ $(1 \leq i \leq m)$ and the $\mathrm{BT}(M''_i)$ $(m \leq i \leq n)$ are closed.

Consider $\mathrm{BT}(A_i)$, where $A_i \equiv N'_i \circ M'_{\sigma i}$. Inspection of $\mathrm{BT}(M' \circ N')$ shows that $\mathrm{BT}(A_i x_i)$ contains as only free variable x_i and x_i occurs only at the top. From this it follows directly that $\mathrm{FV}(\mathrm{BT}(A_i)) \subseteq \{x_i\}$.

Claim. $\mathrm{FV}(\mathrm{BT}(A_i)) = \emptyset$.

Then, applying Lemma 5 we have $A_i =_{D_\infty} I$. By symmetry also $M_{\sigma i} \circ N''' =_{D_\infty} I$ for some $M_{\sigma i}$ and N'''. Hence $N_i \sim'_{D_\infty} M_{\sigma i}$.

Proof of the claim. Suppose $\mathrm{FV}(\mathrm{BT}(A_i)) = \{x_i\}$. Then x_i cannot occur below the top of $\mathrm{BT}(A_i)$ because otherwise it would occur at the same place below the top in $\mathrm{BT}(A_i x_i)$, contradiction. So

$$\mathrm{BT}(A_i) = \lambda y_1 \cdots y_p . x_i$$

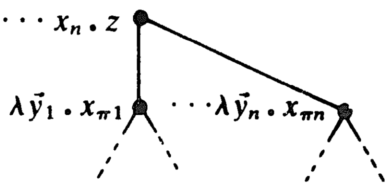
for some $p \geq 0$. If $p = 0$, then $\mathrm{BT}(A_i x_i)$ contains x_i twice, contradiction. So $p \geq 1$. Further, $y_1 \not\equiv x_i$ because x_i is a free variable in $\mathrm{BT}(A_i)$. But in $A_i \equiv N'_i \circ M''_{\sigma i} =_\beta \lambda z . N'_i(M''_{\sigma i}z)$ the z is head-variable (i.e. occurs at the top of $\mathrm{BT}(A_i)$) because $N'_i \equiv \lambda z\vec{v}_i . z\vec{Y}'_i$ and $M_{\sigma i} \equiv \lambda z\vec{u}_{\sigma i} . z\vec{X}_{\sigma i}$. Contradiction.

A similar but simpler argument shows that $M_i \sim'_{D_\infty} I$ for $m < i \leq n$.

4. Characterization of invertible terms in $\lambda\beta\eta$-calculus

Definition 4. (i) HP (hereditary permutators) is the set of λ-terms defined by:

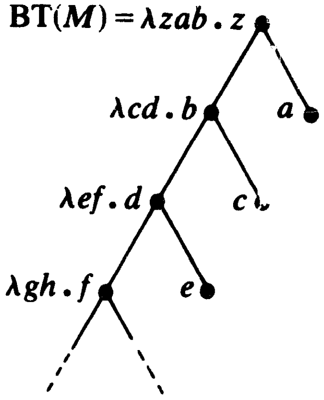
$$M \in \mathrm{HP} \Leftrightarrow \mathrm{BT}(M) = \lambda z x_1 \cdots x_n . z$$



i.e. every abstracted variable occurs only once and except for the head-variable z, every abstracted variable occurs one level lower.

Examples. (1) The possibly infinite η-expansions of I are in HP.

(2) Let M be such that $M \twoheadrightarrow_\beta \lambda zab.\, z(Mb)a$, then



and hence $M \in \mathrm{HP}$.

(ii) FHP (finite h.p.'s) is the subset of HP of terms having a β-nf, or equivalently, having a finite BT.

Definition 5. $M \in_n \mathrm{HP} \Leftrightarrow \exists N \in \mathrm{HP}\ \mathrm{BT}(M) =_n \mathrm{BT}(N)$. ($M$ is w.r.t. levels $0, \ldots, n$ in HP.) Note that $M \in_0 \mathrm{HP}$ iff $M \in \lambda z.z$.

Proposition 3. $M \sim'_{D_\infty} N \Rightarrow M, N \in_1 \mathrm{HP}$.

Proof. Directly from Lemma 5.

Corollary 1. $M \sim_{D_\infty} N \Rightarrow M, N \in \mathrm{HP}$.

Proof. Suppose $M \sim_{D_\infty} N$. Then by 2.4 $M \sim'_{D_\infty} N$. Let A_n be the sentence $\forall M, N$ $M \sim'_{D_\infty} N \Rightarrow M, N \in_n \mathrm{HP}$. We will prove $\forall n\, A_n$ by induction on n. A_1 is Proposition 3. Induction hypothesis: A_n. By the main lemma:

$$M \sim'_{D_\infty} N \;\Rightarrow\; N_i \sim'_{D_\infty} M_{\sigma i},$$

where N_i, σ are as in the main lemma.

By induction hypothesis N_i, $M_{\sigma i} \in_n \mathrm{HP}$. Further it is evident that

$$N_i, M_{\sigma i} \in_n \mathrm{HP} \;\Rightarrow\; N, M \in_{n+1} \mathrm{HP}.$$

Hence, $M \sim'_{D_\infty} N \Rightarrow N, M \in_{n+1} \mathrm{HP}$, i.e. A_{n+1}. So $\forall n\, A_n$. Therefore $M \sim'_{D_\infty} N \Rightarrow \forall n\, M, N \in_n \mathrm{HP} \Rightarrow M, N \in \mathrm{HP}$.

Definition 6. Let $\omega + 1 = \omega \cup \{\omega\}$ be the ordinal number with the usual ordering. Then $|\ |: \mathrm{HP} \to \omega + 1$ is a map defined as follows:

(i) If BT(M) is finite, $|M|$ = depth of BT(M), i.e.

$$|M| = \max\{\text{lth}(\sigma) \mid \sigma \in \text{Seq}(\text{BT}(M))\}.$$

(ii) Otherwise $|M| = \omega$.

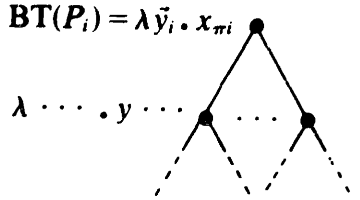
Corollary 2 (M. Dezani-Ciancaglini). *Suppose M has a β-*nf. *Then*:

$$\exists N M \sim_{\beta\eta} N \Leftrightarrow M \in \text{FHP}.$$

Proof. ($\Rightarrow$) $M \sim_{\beta\eta} N \Rightarrow M \sim_{D_\infty} N \Rightarrow M \in \text{HP}$ and since M has a β-nf, $M \in \text{FHP}$.
($\Leftarrow$) Induction on $|M|$.

Lemma 7. *Let $M \in$ HP. Then there are π, $n \geq 0$, $M_1, \ldots, M_n$ such that*
(i) $M_1, \ldots, M_n \in \text{HP}$ *and* $M =_\beta \lambda z x_1 \cdots x_n . z(M_1 x_{\pi 1}) \cdots (M_n x_{\pi n})$ *and*
(ii) $|M| \geq m+1 \Leftrightarrow \exists i\ (1 \leq i \leq n)\ |M_i| \geq m$.

Proof. (i) $M \in \text{HP} \Rightarrow M =_\beta \lambda z x_1 \cdots x_n . z P_1 \cdots P_n$, where



Now $P_i =_\beta (\lambda z . P_i[x_{\pi i} := z])x_{\pi i} \equiv M_i x_{\pi i}$ and clearly $M_i \in \text{HP}$.
(ii) Is obvious from the definitions.

Lemma 8. HP *and* FHP *are closed under composition.*

Proof. For FHP we can prove this by induction on $|M|$; this proof is routine. For HP we prove by induction on k that

$$M, N \in \text{HP} \Rightarrow M \circ N \in_k \text{HP}.$$

For $k = 1$ this is a simple computation. For the induction step we use Lemma 1(i) and an analogous argument as in the proof of Corollary 1.

Lemma 9. *Let $M, N \in$ HP. Then*

$$|M| \geq m \Rightarrow |M \circ N|,\quad |N \circ M| \geq m.$$

Proof. Induction on m: $m = 0$ is trivial. Suppose the proposition is proved for m, and suppose $M, N \in \text{HP}$, $|M| \geq m+1$. By Lemma 7(i)

$$M =_\beta \lambda z x_1 \cdots x_n . z(M_1 x_{\pi 1}) \cdots (M_n x_{\pi n})$$

and

$$N =_\beta \lambda z y_1 \cdots y_k . z(N_1 y_{\sigma 1}) \cdots (N_k y_{\sigma k})$$

for some $\pi, \sigma, M_1, \ldots, M_n$ and $N_1, \ldots, N_k \in$ HP. By Lemma 7(ii) $|M_i| \geq m$ for some $1 \leq i \leq n$.

Suppose again $n = k$, then

$$M \circ N =_\beta \lambda z x_1 \cdots x_n . z[\cdots] \cdots [(M_i \circ N_{\sigma i}) x_{\pi\sigma i}] \cdots [\cdots].$$

By induction hypothesis $|M_i \circ N_{\sigma i}| \geq m$. Hence by Lemma 7(ii) $|M \circ N| \geq m+1$. Similarly $|N \circ M| \geq m+1$.

Corollary 3. $|M| = \omega \Rightarrow \forall N \in \mathrm{HP}\ |M \circ N| = |N \circ M| = \omega$.

Proof. $|M| = \omega \Rightarrow \forall m < \omega\ |M| \geq m \Rightarrow \forall m\ |M \circ N|, |N \circ M| \geq m$
$\Rightarrow |M \circ N|, |N \circ M| = \omega$.

Theorem 5. *M is invertible in the $\lambda\beta\eta$-calculus* $\Leftrightarrow M \in$ FHP.

Proof. ($\Leftarrow$) Follows from Corollary 2.

($\Rightarrow$) Suppose $\exists N M \sim_{\beta\eta} N$. Then $M \sim_{D_\infty} N$. By the main lemma $M \in$ HP. Suppose $M \notin$ FHP, so $|M| = \omega$. Then $|M \circ N| = \omega$. However, $M \sim_{\beta\eta} N \Rightarrow M \circ N =_{\beta\eta} I \Rightarrow M \circ N$ has a $\beta\eta$-normal form, hence, by Lemma 1, $M \circ N$ has a β-normal form. So BT($M \circ N$) is finite, i.e. $|M \circ N| < \omega$, contradiction.

5. Representation of λ-trees and characterization of the invertible terms in $\lambda\beta\eta$-calculus (second proof)

In this section we will restrict our attention to Ω-free and closed λ-trees. However, this restriction is not essential.

Given a term M, we can 'develop' M to its Böhm tree BT(M). Evidently this is a recursive λ-tree, i.e. the underlying set of sequence numbers σ is recursive and the assignment of the labels to the nodes σ is recursive. Vice versa, given a recursive λ-tree $\mathcal{T}$, it is not hard to find a representing term M for $\mathcal{T}$, i.e. a term M such that BT(M) $= \mathcal{T}$.

For the main purpose of this section we want to find a *λI-term M* representing some given recursive λ-tree $\mathcal{T}$. Of course this is not always possible for such a $\mathcal{T}$, as we will see now.

Definition 7. Let $\mathcal{T}$ be a λ-tree. $\mathcal{T}$ is a λI-like tree iff

$$\forall \sigma \in \mathrm{Seq}(\mathcal{T})\ \forall v \in \vec{x}_\sigma\ (v \notin \mathcal{T}_\sigma \Rightarrow \mathcal{T}_\sigma \text{ is infinite}).$$

($\vec{x}_\sigma$ is the string of variables abstracted at σ, see Definition 2.)

Proposition 4. *If M is a λI-term, then* $\mathrm{BT}(M)$ *is λI-like.*

Proof. Note that for all $\sigma \in \mathrm{Seq}(\mathrm{BT}(M))$:

$$v \in \mathrm{FV}(M_\sigma) \& v \not\equiv y_\sigma \Rightarrow \sigma \text{ has a successor,} \tag{2}$$

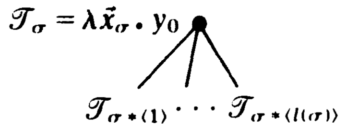
i.e.

$$\exists i\, \sigma * \langle i \rangle \in \mathrm{Seq}(\mathrm{BT}(M)).$$

This follows from the fact that M and hence M_σ is a λI-term, and that $M_\sigma \twoheadrightarrow \lambda \vec{x}_\sigma . y_\sigma M_{\sigma*\langle 1\rangle} \cdots M_{\sigma * \langle l(\sigma)\rangle}$.

Now suppose $\sigma \in \mathrm{Seq}(\mathrm{BT}(M))$, $v \in \vec{x}_\sigma$ and $v \notin \mathrm{BT}(M)_\sigma$. Therefore $v \not\equiv y_\sigma$. Hence, $\exists i\, \sigma * \langle i\rangle \in \mathrm{Seq}(\mathrm{BT}(M)) \& v \in \mathrm{FV}(M_{\sigma*\langle 1\rangle})$. Moreover $v \not\equiv y_{\sigma*\langle i\rangle}$. Now by (2) $\sigma * \langle i\rangle$ has a successor. Iteration gives an infinite branch in $(\mathrm{BT}(M))_\sigma$.

Let $\mathcal{T}$ be the Ω-free, recursive, λI-like closed λ-tree which we want to represent as $\mathrm{BT}(M)$ for some λI-term M. $\mathcal{T}$ is given by a recursive set of sequence numbers $\mathrm{Seq}(\mathcal{T})$ and recursive functions $\sigma \mapsto \vec{x}_\sigma$, $\sigma \mapsto y_\sigma$ and l (= number of successors of σ). So we have



To represent $\mathcal{T}$ we want to find terms M_σ such that

$$M_\sigma \twoheadrightarrow \lambda \vec{x}_\sigma . y_\sigma M_{\sigma*\langle 1\rangle} \cdots M_{\sigma*\langle l(\sigma)\rangle}.$$

The M_σ will contain in general free variables, viz. some of the $\vec{w}_\sigma$ (i.e. variables abstracted before σ, see Section 2.1). Since there is no enumerator E for *open* terms (see how E is used in the construction below), we must provide the necessary free variables on our own. So we put $M_\sigma = \boldsymbol{A\sigma}\vec{z}_\sigma$, where $\boldsymbol{A\sigma}$ is closed and $\vec{z}_\sigma \subseteq \vec{w}_\sigma$:

$$\boldsymbol{A\sigma}\vec{z}_\sigma \twoheadrightarrow \lambda \vec{x}_\sigma . y_\sigma\, (\boldsymbol{A\sigma * \langle 1\rangle}\vec{z}_{\sigma*\langle 1\rangle}) \cdots (\boldsymbol{A\sigma * \langle l(\sigma)\rangle}\vec{z}_{\sigma*\langle l(\sigma)\rangle}). \tag{3}$$

In this way the necessary free variables are passed down. Note that the order of the variables in $\vec{z}_\sigma$ is irrelevant. Note also that taking $\vec{z}_\sigma = \mathrm{FV}(\mathcal{T}_\sigma)$ does not work, since even if $\mathcal{T}$ is representable, the assignment $\sigma \mapsto \mathrm{FV}(\mathcal{T}_\sigma)$ need not be recursive; see Example 1.

Since we work in λI-calculus, it follows from (3) that

$$\vec{z}_\sigma \cup \vec{x}_\sigma = \{y_\sigma\} \cup \vec{z}_{\sigma*\langle 1\rangle} \cup \cdots \cup \vec{z}_{\sigma*\langle l(\sigma)\rangle} \tag{4}$$

and this requirement will prove to be sufficient.

Theorem 6. *Let $\mathcal{T}$ be an Ω-free, closed λ-tree. Then there is a λI-term M such that* $\mathrm{BT}(M) = \mathcal{T} \Leftrightarrow \mathcal{T}$ *is recursive, λI-like and there is a recursive free variable assignment* $\sigma \mapsto \vec{z}_\sigma$ $(\sigma \in \mathrm{Seq}(\mathcal{T}))$ *such that* $\vec{z}_\sigma \subseteq \vec{w}_\sigma$ *and* (4) *is satisfied.*

Proof. (⇒) If M is a λI-term, BT(M) is recursive, and by Proposition 4 λI-like. A recursive free variable assignment is given by $\vec{z}_\sigma = \mathrm{FV}(M_\sigma)$ and since M_σ is a λI-term, it evidently satisfies (4).

(⇐) Let $\mathscr{T}$ and $\vec{z}_\sigma$ be given as stated in the theorem. By Theorem 3 a λI-term T can be defined such that

$$T\boldsymbol{\sigma} = \begin{cases} \mathbf{0}, & \text{if } \sigma \in \mathrm{Seq}(\mathscr{T})\ \&\ \sigma \text{ is terminal node of } \mathscr{T}, \\ \mathbf{1}, & \text{else.} \end{cases}$$

Let R_σ be the term (for all $\sigma \in \mathrm{Seq}(\mathscr{T})$)

$$\lambda a\vec{z}_\sigma\vec{x}_\sigma \,.\, y_\sigma(Ea\boldsymbol{\sigma} * \langle \mathbf{1} \rangle \vec{z}_{\sigma * \langle 1 \rangle}) \cdots (Ea\boldsymbol{\sigma} * \langle \boldsymbol{l(\sigma)} \rangle \vec{z}_{\sigma * \langle l(\sigma) \rangle}).$$

Note that $\sigma \mapsto R_\sigma$ is recursive, since $\sigma \mapsto \vec{z}_\sigma$ is recursive, and that R_σ is a closed λI-term by (4).

Let f be the recursive function defined by

$$f(\sigma) = \begin{cases} \ulcorner R_\sigma \urcorner, & \text{if } \sigma \text{ is not terminal,} \\ \ulcorner I \urcorner, & \text{else} \end{cases}$$

and let F be the λI-term representing f.

By Lemma 3 there is a $Q \in \Lambda_I^0$ such that $Q\mathbf{0} \twoheadrightarrow \lambda ab\,.\,bIa$ and $Q\mathbf{1} \twoheadrightarrow I$. Now for all A:

(i) if $T\boldsymbol{\sigma} = \mathbf{0}$, then $Q(T\boldsymbol{\sigma})I(E(F\boldsymbol{\sigma})\ulcorner \boldsymbol{A} \urcorner) \twoheadrightarrow Q\mathbf{0}I(E(\ulcorner \boldsymbol{I} \urcorner)\ulcorner \boldsymbol{A} \urcorner) \twoheadrightarrow Q\mathbf{0}I(\ulcorner \boldsymbol{A} \urcorner) \twoheadrightarrow \ulcorner \boldsymbol{A} \urcorner II \twoheadrightarrow I$, and

(ii) if $T\boldsymbol{\sigma} = \mathbf{1}$, then $Q(T\boldsymbol{\sigma})I(E(F\boldsymbol{\sigma})\ulcorner \boldsymbol{A} \urcorner) \twoheadrightarrow Q\mathbf{1}I(E(F\boldsymbol{\sigma})\ulcorner \boldsymbol{A} \urcorner) \twoheadrightarrow E(F\boldsymbol{\sigma})\ulcorner \boldsymbol{A} \urcorner)$.

By the second fixed point theorem (Theorem 1) there is an $A \in \Lambda_I^0$ such that:

$$A \twoheadrightarrow \lambda s\,.\,Q(Ts)I(E(Fs)\ulcorner \boldsymbol{A} \urcorner)$$

and hence

$$A\boldsymbol{\sigma} \twoheadrightarrow Q(T\boldsymbol{\sigma})I(E(F\boldsymbol{\sigma})\ulcorner \boldsymbol{A} \urcorner).$$

Claim. BT($A\langle\ \rangle$) = $\mathscr{T}$.

Proof of the claim. First we show that

$$(A\langle\ \rangle)_\sigma = A\boldsymbol{\sigma}\vec{z}_\sigma. \tag{5}$$

If $\sigma = \langle\ \rangle$, then indeed $(A\langle\ \rangle)_{\langle\rangle} = A\langle\ \rangle \vec{z}_{\langle\rangle}$ since $\vec{z}_{\langle\rangle} \subseteq \vec{w}_{\langle\rangle} = \emptyset$ and since $(A\langle\ \rangle)_{\langle\rangle} = A\langle\ \rangle$ by definition.

Suppose (5) is proved for σ. If σ is not terminal, we have

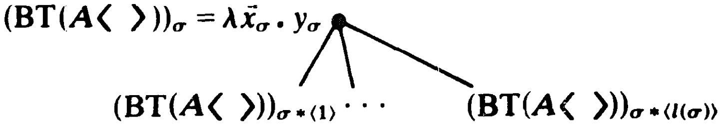
$$\begin{aligned} A\boldsymbol{\sigma}\vec{z}_\sigma &\twoheadrightarrow E(F\boldsymbol{\sigma})\ulcorner \boldsymbol{A} \urcorner \vec{z}\sigma \twoheadrightarrow E\ulcorner \boldsymbol{R_\sigma} \urcorner \ulcorner \boldsymbol{A} \urcorner \vec{z}_\sigma \twoheadrightarrow R_\sigma \ulcorner \boldsymbol{A} \urcorner \vec{z}\sigma \\ &\twoheadrightarrow \lambda \vec{x}_\sigma \,.\, y_\sigma(A\boldsymbol{\sigma} * \langle \mathbf{1} \rangle \vec{z}_{\sigma * \langle 1 \rangle}) \cdots (A\boldsymbol{\sigma} * \langle \boldsymbol{l(\sigma)} \rangle \vec{z}_{\sigma * \langle l(\sigma) \rangle}) \end{aligned}$$

and hence

$$(A\langle\ \rangle)_{\sigma * \langle i \rangle} = ((A\langle\ \rangle)_\sigma)_{\langle i \rangle} = (A\boldsymbol{\sigma}\vec{z}_\sigma)_{\langle i \rangle} = A\boldsymbol{\sigma} * \langle \boldsymbol{i} \rangle \vec{z}_{\sigma * \langle i \rangle}$$

for $1 \leq i \leq l(\sigma)$, which proves (5) for all $\sigma \in \mathrm{Seq}(\mathscr{T})$.

Now we will prove that

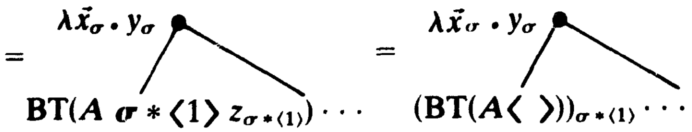


(6)

Since $\mathcal{T}$ has the same 'recursive definition', (6) proves the claim.

Case 1. σ is not terminal, i.e. $l(\sigma) > 0$. Then

$$(\mathrm{BT}(A\langle\ \rangle))_\sigma = \mathrm{BT}(A\langle\ \rangle)_\sigma = \mathrm{BT}(A\sigma\vec{z}_\sigma)$$



Case 2. σ is terminal. So (i) $\mathcal{T}_\sigma = \lambda y_\sigma \cdot y_\sigma$ ($\vec{x}_\sigma = y_\sigma$) or (ii) $\mathcal{T}_\sigma = y_\sigma$ ($\vec{x}_\sigma = \emptyset$), since $\mathcal{T}$ is λI-like.

In case (i) $\vec{z}_\sigma = \emptyset$, in case (ii) $\vec{z}_\sigma = y_\sigma$ by (4).

(i) $\mathrm{BT}(A\langle\ \rangle)_\sigma = \mathrm{BT}(A\sigma) = \mathrm{BT}(I) = \lambda y \cdot y$,

(ii) $\mathrm{BT}(A\langle\ \rangle)_\sigma = \mathrm{BT}(A\sigma y_\sigma) = \mathrm{BT}(I y_\sigma) = y_\sigma$,

This concludes the proof of (6).

Remark. Using terms E_n which enumerate all terms M such that $\mathrm{FV}(M) \subseteq \{v_1, \ldots, v_n\}$ we can deal with the case that $\mathcal{T}$ contains finitely many free variables. Also it is not hard to modify the construction above in such a way that a λ-tree $\mathcal{T}$ containing Ω's can be represented if $\mathcal{T}$ is recursive. Note however that the reverse is no longer true: $\mathrm{BT}(M)$ is recursively enumerable and no longer recursive if it contains Ω's. See also [2, Theorem 10.1.25].

Now we can give the second proof of

$$M \sim_{\beta\eta} N \Rightarrow M, N \text{ have a normal form.} \tag{7}$$

Lemma 10. *Let $M \sim_{\beta\eta} N$. Then* $\mathrm{BT}(M)$ *and* $\mathrm{BT}(N)$ *are Ω-free, recursive λI-like trees having free variable assignments $\vec{z}_\sigma$ satisfying* (4).

Proof. By Definition 4 and Corollary 1 it follows at once that the trees are Ω-free, recursive and λI-like. Taking $\vec{z}_{\langle\ \rangle} = \emptyset$ and $\vec{z}_\sigma = y_\sigma$ ($\sigma \neq \langle\ \rangle$) one verifies easily that (4) holds.

We will prove (7) by contraposition.

(1) Suppose that say M has no normal form.

(2) $\exists M^* \in \Lambda_I\ \exists N^* \in \Lambda_I\ \mathrm{BT}(M^*) = \mathrm{BT}(M)$ and $\mathrm{BT}(N^*) = \mathrm{BT}(N)$, by Lemma 10.

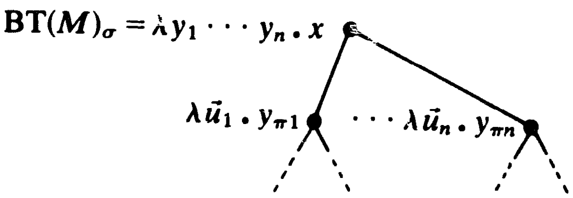
(3) $\mathrm{BT}(M) = \mathrm{BT}(M^*)$ is infinite, by (1).[1]
(4) M^* has no normal form, by (3).
(5) $M^* \circ N^* \equiv BM^*N^*$ is a λI-term.
(6) $M^* \circ N^*$ has no normal form, by (4), (5) and Lemma 2.
(7) $\mathrm{BT}(M^* \circ N^*)$ is infinite, by (6).
(8) $M^* =_{\mathrm{P}\omega} M$ and $N^* =_{\mathrm{P}\omega} N$ by (2) and Theorem 4.
(9) $M^* \circ N^* =_{\mathrm{P}\omega} M \circ N$.
(10) $\mathrm{BT}(M^* \circ N^*) = \mathrm{BT}(M \circ N)$ by (9) and Theorem 4.
(11) $\mathrm{BT}(M \circ N)$ is infinite, by (7) and (10).
(12) $M \circ N$ has no β-normal form by (11).
(13) $M \circ N$ has no $\beta\eta$-normal form by (12) and Lemma 1.
(14) $M \sim_{\beta\eta} N$, hence $M \circ N =_{\beta\eta} I$, hence $M \circ N$ has a $\beta\eta$-normal form.
Contradiction with (13).

6. Characterization of invertible terms in D_∞

We will prove that HP is precisely the set of λ-terms invertible in D_∞. To this end we construct a 'formal inverse' M^* of an $M \in \mathrm{HP}$.

Definition 8. (i) A *permutation tree* $\langle \Sigma, \Pi \rangle$ is a tree $\Sigma \subseteq \mathrm{Seq}$ together with a map Π from Σ to the set of finite permutations, such that for $\sigma \in \Sigma$, $\mathrm{dom}\, \Pi(\sigma) = \{i \mid \sigma * \langle i \rangle \in \Sigma\}$. *Notation*: $\Pi(\sigma) = \pi_\sigma$.

(ii) If $M \in \mathrm{HP}$ we can identify $\mathrm{BT}(M)$ with a permutation tree $\langle \Sigma_M, \Pi \rangle$, where $\Sigma_M = \mathrm{Seq}(\mathrm{BT}(M))$ and Π is defined as follows: if



then $\pi_\sigma = \pi$. This identification will be denoted by $\cong$.

Definition 9. Let $M \in \mathrm{HP}$, $\mathrm{BT}(M) \cong \langle \Sigma_M, \Pi \rangle$.

(i) A map $* : \Sigma_M \to \mathrm{Seq}$ is defined by induction on the length of $\sigma \in \Sigma_M$:

$$\langle\ \rangle^* = \langle\ \rangle, \qquad (\sigma * \langle i \rangle)^* = \sigma^* * \langle \pi_\sigma i \rangle.$$

(ii) To each $\sigma^* \in \Sigma_M^*$, the range of $*$, a permutation π_{σ^*} is associated:

$$\pi_{\sigma^*} = (\pi_\sigma)^{-1}.$$

In this way we have constructed a permutation tree $\langle \Sigma_M^*, \Pi^* \rangle$.

Now let $M^* \in \mathrm{HP}$ be a term such that $\mathrm{BT}(M^*) \cong \langle \Sigma_M^*, \Pi^* \rangle$; by Theorem 6 such an M^* exists.

[1] The numbers in parentheses refer to this proof.

Then M^* is called a *formal inverse* of M (w.r.t. D_∞). *Notation*: $M^* \sim^{\mathrm{f}}_{D_\infty} M$. Note that $\sim^{\mathrm{f}}_{D_\infty}$ is a symmetric relation, and that if $M \sim^{\mathrm{f}}_{D_\infty} N$, M and N have the same arity. Here M is said to have arity n iff $M =_\beta \lambda x_1 \cdots x_n . y\vec{N}$ for some $y, \vec{N}$.

Lemma 11. *Let $M \in \mathrm{HP}$ have arity n. Then*

$$M \sim^{\mathrm{f}}_{D_\infty} M^* \Rightarrow (M^*)_i \sim^{\mathrm{f}}_{D_\infty} (M)_{\pi^{-1}_{\langle\rangle} i} \quad (i = 1, \ldots, n)$$

Proof. For all $i \in \{1, \ldots, n\}$ the map $*$ induces in an obvious way (by leaving out the first coordinate of all $\sigma \in \Sigma_M$) a map $*_i$ which formally inverts $(M)_{\pi^{-1}_{\langle\rangle} i}$ to $(M^*)_i$.

Notation. (i) $[I]_{D_\infty} = \{M \mid M =_{D_\infty} I\}$.

(ii) $M \in_n [I]_{D_\infty} \Leftrightarrow \exists N \in [I]_{D_\infty}\ \mathrm{BT}(M) =_n \mathrm{BT}(N)$. ($M$ is up to the first $n+1$ levels of its Böhm tree equal to I in D_∞.)

Lemma 12. $M \sim^{\mathrm{f}}_{D_\infty} M^* \Rightarrow M \circ M^* \in_1 [I]_{D_\infty}$.

Proof. Simple.

Theorem 7. $M \in \mathrm{HP} \Leftrightarrow \exists M^* \in \mathrm{HP}\ M \sim_{D_\infty} M^*$.

Proof. ($\Leftarrow$) Is Corollary 1.

($\Rightarrow$) Construct a formal inverse M^*. Then by an analogous proof as that of Corollary 3.4 we have, using Lemma 11 and 12

$$M \sim^{\mathrm{f}}_{D_\infty} M^* \Rightarrow \forall n\, M \circ M^* \in_n [I]_{D_\infty}, \quad \text{i.e. } M \circ M^* =_{D_\infty} I.$$

By symmetry of $\sim^{\mathrm{f}}_{D_\infty}$ also $M^* \circ M =_{D_\infty} I$, and hence $M \sim_{D_\infty} M^*$.

7. Concluding remarks

The above results characterize the groups of invertible elements in resp. $\mathcal{M}(\Lambda^0_{\beta\eta})$, the closed term model corresponding to $\lambda\beta\eta$-calculus and in D^0_∞, the interior of D_∞ (after dividing out $=_{\beta\eta}$ resp. $=_{D_\infty}$).

For $\mathcal{M}(\Lambda^0_\beta)$ the group is $\{I\}$; see [4] for a characterization of normal forms possessing a left or right inverse in $\mathcal{M}(\Lambda^0_\beta)$.

In D_∞ one can find a larger group than $\mathrm{HP}/=_{D_\infty}$, by allowing permutation trees that are non-recursive. Question: what does the group of all invertible elements of D_∞ look like?

Another question is whether the following equivalence holds in $\mathcal{M}(\Lambda^0_{\beta\eta})$ or D^0_∞:

$$M \text{ is invertible} \Leftrightarrow M \text{ is bijective,}$$

where 'bijective' is meant in the obvious sense analogous to set-theoretic functions.

It is not hard to prove that this equivalence is indeed valid for $\mathcal{M}(\Lambda^0_\beta)$.

In [2] the group in question is determined for some other λ-calculus models, such as $P\omega$ and $\mathcal{M}(\mathcal{H})$. Also [2] gives information on the group-theoretic structure of the groups we considered.

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