

CHURCH-ROSSER STRATEGIES IN THE LAMBDA CALCULUS

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0. Introduction

In [1, Section 4] the concept of a reduction strategy was introduced. We repeat the main definitions.

(1) Λ is the set of λ -terms; \rightarrow_{β} is one step β -reduction, $\twoheadrightarrow_{\beta}$ its transitive reflexive closure; $=_{\beta}$ is convertibility (the equality generated by \rightarrow_{β}) and \equiv is syntactic equality. Sometimes we will forget the subscript β .

(2) A *reduction strategy* on Λ is a map $F: \Lambda \rightarrow \Lambda$ such that $M \twoheadrightarrow_{\beta} F(M)$. (Hence $F(M) \equiv M$ if M is in normal form.) A *one step reduction strategy* on Λ is a map $F: \Lambda \rightarrow \Lambda$ such that (i) $F(M) \equiv M$ if M is in normal form, (ii) $M \twoheadrightarrow_{\beta} F(M)$ otherwise.

A well-known example of a (more step) reduction strategy is the “Gross-reduction” strategy G where $G(M)$ is the result of a complete development of all redexes in M . It will be used in Section 2.

An example of a one step strategy is “normal” (or “leftmost”) reduction, in which every time the leftmost redex is contracted.

Note that “inside-out” reduction and “standard” reduction are no strategies in our sense.

(3) A strategy F is called a *Church–Rosser* (CR) strategy if for all $M, N \in \Lambda$:

$$M =_{\beta} N \Rightarrow \exists n, m \in \mathbb{N} \quad F^n(M) \equiv F^m(N).$$

This paper was motivated by the problem: does there exist a *recursive one step CR-strategy* on Λ ? Although we could not answer this question, we can give positive answers if the question is weakened as follows:

- by not insisting on *one step* strategies (Section 1).
- by considering *non-recursive* strategies (Section 2),
- by *restricting* Λ to the set Σ of *S*-terms (Section 3).

1. A recursive CR-strategy for λ -terms

We start with some general definitions and notations.

Definition 1.1. (i) Let V be an arbitrary set and $P \subseteq V \times V$ a binary relation on V . Write $a \xrightarrow{P} b$ instead of $P(a, b)$ or $(a, b) \in P$. (Example: $\beta \subseteq \Lambda \times \Lambda$) Let $X \subseteq V$. Then $\text{Cl}_P(X) :=$ the closure of X w.r.t. P , i.e. the least set $Y \supseteq X$ such that $a \in Y$ & $a \xrightarrow{P} b \Rightarrow b \in Y$.

(ii) Let V, P be as in (i). A sequence $a_0 \xrightarrow{P} a_1 \xrightarrow{P} \cdots \xrightarrow{P} a_n = a_0$ is called a *P-cycle*.

(iii) P is *CR on X* iff $\forall a, b \in X \exists c \in V a \xrightarrow{P} c \xleftarrow{P} b$. Here \xrightarrow{P} is the transitive reflexive closure of P . Note that c is not required to be in X .

Let $=_P$ be the equality generated by P . Then P is *CR* iff it is *CR* on all $=_P$ -equivalence classes.

(iv) $a \in V$ is a *minimal* point w.r.t. P iff $\forall b (a \xrightarrow{P} b \Rightarrow b \xrightarrow{P} a)$. E.g. in Λ the normal forms are minimal; also $(\lambda x.xx)(\lambda x.xx)$ is minimal.

Note that if P is *CR*, then a is *P-minimal* iff $\forall b (a =_P b \Rightarrow b \xrightarrow{P} a)$ (whence the name).

Note further that if \mathcal{C} is a *P-cycle* and some $a \in \mathcal{C}$ is *P-minimal*, then every $b \in \mathcal{C}$ is *P-minimal*. Hence we call \mathcal{C} *P-minimal* iff its points are *P-minimal*.

Definition 1.2. (i) Let $\{M_n \mid n \in \mathbb{N}\}$ be a recursive enumeration of Λ .

(ii) If $M \in \Lambda$, $|M| =$ number of occurrences of symbols in M + the sum of all the indices i of free variables x_i in M . (The last summand must ensure that $\{M \in \Lambda \mid |M| \leq n\}$ is finite.)

(iii) Let $M \xrightarrow{n} N$ mean that the reduct N can be reached in n or less reduction steps.

(iv) $(,) : \mathbb{N}^2 \rightarrow \mathbb{N}$ is a recursive, surjective pairing function; $(,)_0$ and $(,)_1$ are the corresponding projection functions.

Proposition 1.3. Let $\{M_0, \dots, M_n\} \subseteq \Lambda$ be such that $M_i =_\beta M_j$ for all $i, j \leq n$. Then we can compute a common reduct L of the M_i ($i \leq n$).

Proof. Induction on n ; the basis step is trivial. Induction hypothesis: suppose the proposition is true for k ; and let L_k be a common reduct of all the M_i ($i \leq k$). Now let $M_{k+1} =_\beta L_k$. All we have to do is to compute a common reduct of L_k and M_{k+1} .

Let $[L_k]_i = \{L \mid L_k \xrightarrow{i} L\}$ and similar for M_{k+1} . Clearly, all the $[L_k]_i$ and $[M_{k+1}]_i$ ($i \geq 0$) are computable. Now compute alternately $[L_k]_1, [M_{k+1}]_1, [L_k]_2,$

$[M_{k+1}]_2, \dots$ meanwhile checking whether we have found a common reduct already. By the Church–Rosser theorem we are sure to find one.

Definition 1.4. $B(M, n)$ is the sphere with center M and radius n :

$$B(M, n) = \{N \in \Lambda \mid |N| \leq n \text{ \& \& } \exists L \underset{n}{M} \rightarrow L \underset{n}{\leftarrow} N\}.$$

Remark. $B(M, n)$ is finite and computable from M, n .

Proposition 1.5. Let $B_n = B(M_{(n)_1}, (n)_0)$ for all $n \in \mathbb{N}$. This sequence of spheres has the following property:

$$M_i =_\beta M_j \Rightarrow \exists n \ M_i, M_j \in B_n.$$

Proof. Suppose $M_i =_\beta M_j$. By the Church–Rosser theorem a common reduct L can be found. Say $M_i \underset{k}{\rightarrow} L$ and $M_j \underset{l}{\rightarrow} L$ for some k, l . Now take B_n such that $(n)_1 = i$ and $(n)_0 \geq \max(|M_i|, k, l)$.

Theorem 1.6. There exists a recursive CR-strategy on Λ .

Proof. We will construct an increasing sequence $F_0 \subseteq F_1 \subseteq \dots \subseteq F_n \subseteq \dots$ of partial functions $F_n : \Lambda \rightarrow \Lambda$. In fact, the F_n will be finite and computable from n . Hence $F = \bigcup_{n \in \mathbb{N}} F_n$ is a recursive function, and this will be the desired CR-strategy.

Simultaneously we will define $C_n \supseteq B_n$ for all $n \geq 1$ such that $\text{Dom}(F_n) = C_1 \cup \dots \cup C_n$ ($n \geq 1$). The C_n ($n \geq 1$) will also be finite and computable from n .

Basis step: $F_0 = \emptyset$.

Induction step: Suppose that F_n and C_1, \dots, C_n are defined.

Then define

$$C_{n+1} = \text{Cl}_{F_n}(B_{n+1}).$$

Now let $A_{n+1} = C_{n+1} - \text{Dom}(F_n)$. We have to define F_{n+1} on A_{n+1} . Before we do this, we state the following *induction hypothesis* IH_n :

- (i) each F_n -cycle is β -minimal, and
- (ii) let $\mathcal{C}, \mathcal{C}'$ be two F_n -cycles, $M \in \mathcal{C}, M' \in \mathcal{C}'$ and $M =_\beta M'$. Then $\mathcal{C} = \mathcal{C}'$.
- (iii) F_n is CR on C_1, \dots, C_n .

Remark. (i) and (ii) prevent the formation of F -cycles that would spoil the CR-property of F . Namely, as to (i): suppose an F -cycle \mathcal{C} (heavy lines in Fig. 1) is formed containing a non-minimal (w.r.t. β) term M . Then obviously the chains $M \rightarrow F(M) \rightarrow F^2(M) \rightarrow \dots$ and $N \rightarrow F(N) \rightarrow \dots$ will not intersect. Similarly for clause (ii), as Fig. 2 suggests.

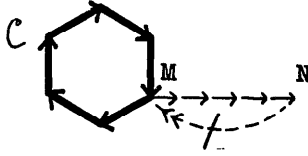


Fig. 1.

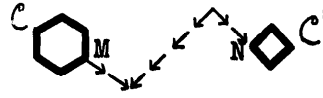
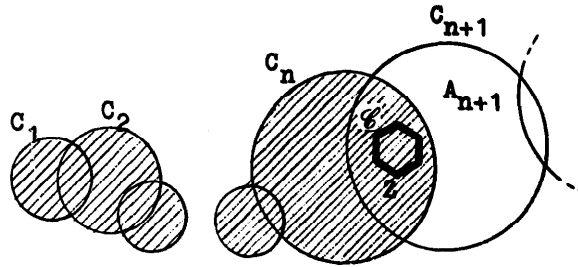


Fig. 2.

Now the definition of F_{n+1} . There are two cases.

(1) $\text{Dom}(F_n) \cap C_{n+1}$ contains an F_n -cycle \mathcal{C} . See Fig. 3. Let Z be the first element of \mathcal{C} in the recursive enumeration of Λ we considered. By IH_n (i), Z is β -minimal, hence $M \twoheadrightarrow_{\beta} Z$ for all $M \in A_{n+1}$. So we take $F_{n+1}(M) \equiv Z$ for all $M \in A_{n+1}$, and $F_{n+1} \upharpoonright C_1 \cup \dots \cup C_n = F_n$.



$\text{Dom}(F_n)$: shaded area

Fig. 3.

We have to check IH_{n+1} : (i), (ii) follow from IH_n (i), (ii) and the fact that no new F_{n+1} -cycle is created. For, every new F_{n+1} -cycle obviously contains Z and all F_{n+1} -cycles containing Z are identical, since F_{n+1} is a function.

To check IH_{n+1} (iii) we have to show that F_{n+1} is CR on C_{n+1} . Claim:

$$\forall M \in C_{n+1} \exists k \quad F_{n+1}^k(M) \equiv Z.$$

For $M \in A_{n+1}$ this is obvious ($k = 1$). If $M \in C_{n+1} \cap \text{Dom}(F_n)$, consider the sequence $M \twoheadrightarrow F_n(M) \twoheadrightarrow F_n^2(M) \twoheadrightarrow \dots$ as far as defined. If this sequence contains a cycle (so that it is infinite), then by IH_n (ii) this must be the cycle \mathcal{C} containing Z , and the claim is true for M . If not, then the sequence must stop in A_{n+1} and we are through also.

(2) $\text{Dom}(F_n) \cap C_{n+1}$ contains no F_n -cycles. Let Y be a common reduct of all elements in A_{n+1} , computed as in Proposition 1.3.

Now it is tempting to put $F_{n+1}(M) \equiv Y$ for all $M \in A_{n+1}$. However, then there may arise an F_{n+1} -cycle which is not correct in the sense of IH_{n+1} (i), (ii). For example, if $Y \in A_{n+1}$ we could have an instantaneous loop (a cycle of length 1) $Y \xrightarrow{F_{n+1}} Y$ without being sure that Y is minimal w.r.t. β . Or, a situation as in Fig. 4 could occur, where defining $F_{n+1}(M) \equiv Y$ for all $M \in A_{n+1}$ (see the intermittent heavy line) would create a maybe dangerous cycle.

Therefore we want to decide whether Y has a reduct $Z \notin C_1 \cup \dots \cup C_{n+1}$ or not. This can be done as follows. Let $[Y]$ be the set of reducts of Y and start computing

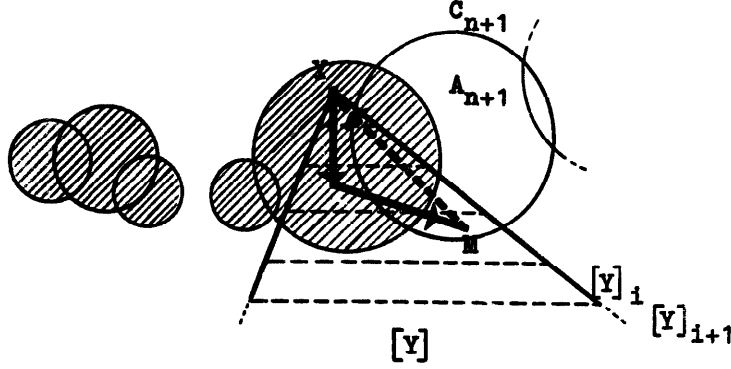


Fig. 4.

the consecutive “layers” of $[Y]$, in the notation of Proposition 1.3, $[Y]_1, [Y]_2, \dots$. If $[Y]$ is infinite, we are sure to find a $Z \in [Y]$ such that $Z \notin C_1 \cup \dots \cup C_{n+1}$ since the last set is finite. Then we stop computing and we are in case

(2.1) Y has a reduct $Z \notin C_1 \cup \dots \cup C_{n+1}$. Define $F_{n+1}(M) \equiv Z$ for all $M \in A_{n+1}$ and now it is easily checked that IH_{n+1} is satisfied.

(2.2) If we cannot find such a Z , then $[Y]$ is finite ($[Y] \subseteq C_1 \cup \dots \cup C_{n+1}$) and we compute the whole $[Y]$. See Fig. 5. Now we have for the last time to distinguish two cases.

(2.2.1) $[Y]$ contains no F_n -cycle. Then compute a $Z \in [Y]$ which is β -minimal in $[Y]$ (i.e. such that $\forall M \in [Y] M \twoheadrightarrow_{\beta} Z$) and for definiteness, such that Z is the first term in the recursive enumeration of Λ with that property. (Since we have already computed the finite $[Y]$, this is clearly possible.)

Now let F_{n+1} map A_{n+1} on Z .

(2.2.2) $[Y]$ contains an F_n -cycle \mathcal{C} as in Fig. 5. Let Z be the first element of \mathcal{C} in the enumeration of Λ , and let again F_{n+1} map A_{n+1} on Z . Note that if a Z' as in case 2.2.1 is used, then a second cycle \mathcal{C}'' could be created in $[Y]$ as in the figure, which would violate $IH_{n+1}(ii)$.

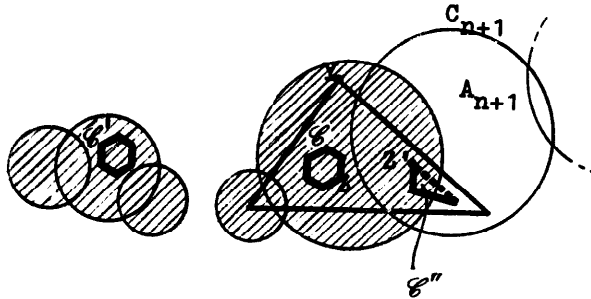


Fig. 5.

We check IH_{n+1} for the last two cases:

(i) for case 2.2.1. Suppose a new F_{n+1} -cycle \mathcal{C} is created; so $Z \in \mathcal{C}$. Now Z is β -minimal in $[Y]$, hence β -minimal in Λ . Therefore \mathcal{C} is β -minimal.

(ii) for case 2.2.1. Let $\mathcal{C}, \mathcal{C}'$ be two F_{n+1} -cycles and let $M \in \mathcal{C}, M' \in \mathcal{C}'$ and $M =_{\beta} M'$. If $\mathcal{C}, \mathcal{C}'$ are both “old” (i.e. F_n -cycles) there is nothing to prove by IH_n .

If both are new, they must both contain Z and hence $\mathcal{C} = \mathcal{C}'$. The case that just one of them is old, say \mathcal{C}' , cannot occur because by its minimality \mathcal{C}' should be in $[Y]$, contrary to the assumption for this case 2.2.1.

(i), (ii) for 2.2.2. Suppose a new cycle \mathcal{C}' is produced. Then $Z \in \mathcal{C}'$. We had also $Z \in \mathcal{C}$, hence $\mathcal{C} = \mathcal{C}'$. So no new cycles are produced and we are through by IH_n (i), (ii).

(iii) for both cases. Note that in all subcases of 2, all elements of $C_{n+1} \cap \text{Dom}(F_n)$ go by iterated application of F_n to A_{n+1} , since the former set is finite and contains no F_n -cycles. Further, F_{n+1} maps A_{n+1} on Z . Hence F_{n+1} is CR on C_{n+1} .

Finally we must prove that $F = \bigcup_{n \in \mathbb{N}} F_n$ is the desired strategy. Let $M \in \Lambda$ and compute n such that $M \equiv M_n$. So $M \in B_{(n,m)}$ for some m . Now compute $F_0, \dots, F_{(n,m)}$ and $F(M) = F_{(n,m)}(M)$ is found.

That F is a CR-strategy follows immediately from Proposition 1.3 and the fact that each F_n is CR on $C_n \supseteq B_n$.

Remark. Note that only in case 2.2.1 new cycles can be created. Further, in each $=_{\beta}$ -equivalence class $[M]_{=\beta}$ there will be at most one F -cycle.

The above procedure works clearly also for Combinatory Logic (CL) and other General Replacements Systems (in the sense of Rosen [3]) having the CR-property.

For CL, based on the combinators S, K, I we note the peculiar fact that the procedure gives no F -cycles at all. This is a consequence of the above procedure and the following theorem in [2]:

Theorem. For every term $Y \in \text{CL}_{S,K,I}$, if the set of reducts $[Y] = \{Z \mid Y \rightarrow Z\}$ contains a reduction cycle, then $[Y]$ is infinite. In fact such an $[Y]$ contains infinitely many cycles, as follows:

$$Z_1 \rightleftharpoons Z_2 \rightleftharpoons Z_3 \rightleftharpoons \dots \quad (Z_i \neq Z_j \text{ for } i \neq j).$$

2. A (non-recursive) one step CR-strategy for λ -terms

First we define a kind of lexicographic ordering on the set of finite reduction sequences.

Definition 2.1. (i) Count the head- λ 's of the redexes in M from left to right and let $n(R, M)$ be the natural number thus associated to the redex R in M .

Let $\mathcal{R}^+ = M_0 \xrightarrow{R_0} M_1 \xrightarrow{R_1} \dots \xrightarrow{R_{n-1}} M_n$ be a finite reduction sequence with specification of the contracted redexes. Then we can describe \mathcal{R}^+ as the pair

$(M_0, \sigma(\mathcal{R}^+))$ where $\sigma(\mathcal{R}^+)$ is the sequence number $\langle n(R_0, M_0), \dots, n(R_{n-1}, M_{n-1}) \rangle$.

(ii) On the set of sequence numbers we define the following well-ordering \triangleleft :

$$\langle n_0, \dots, n_k \rangle \triangleleft \langle m_0, \dots, m_l \rangle \Leftrightarrow k < l \quad \text{or if } k = l:$$

$$n_0 = m_0, \dots, n_{i-1} = m_{i-1} \quad \text{and} \quad n_i < m_i \quad \text{for some } i \leq k.$$

(iii) If $\mathcal{R} = M_0 \rightarrow \dots \rightarrow M_n$ is a finite reduction sequence without specification of the contracted redexes, we define

$$\sigma(\mathcal{R}) = \text{minimum w.r.t. } \triangleleft \text{ of } \{\sigma(\mathcal{R}^+) \mid \mathcal{R}^+ \text{ is } \mathcal{R} \text{ plus some specification of contracted redexes}\}.$$

Example. If $\mathcal{R} = (\lambda x. [(\lambda x. A)(IB)])B \rightarrow \lambda x. [(\lambda x. A)B]B \rightarrow (\lambda x. A)B \rightarrow A$, then $\sigma(\mathcal{R}) = \min\{\langle 3, 0, 0 \rangle, \langle 3, 1, 0 \rangle\} = \langle 3, 0, 0 \rangle$.

(iv) Let $\text{Red}(M)$ be the set of all finite reductions (without specification of contracted redexes) starting with M . Then we define a well-ordering $<$ on $\text{Red}(M)$ by:

$$\mathcal{R}_1 < \mathcal{R}_2 \Leftrightarrow \sigma(\mathcal{R}_1) \triangleleft \sigma(\mathcal{R}_2).$$

Remark 2.2. If $M_0 \rightarrow M_1 \rightarrow \dots \rightarrow M_n$ is the minimal (w.r.t. $<$) reduction path from M_0 to M_n , then $M_1 \rightarrow \dots \rightarrow M_n$ is the minimal path from M_1 to M_n .

Definition 2.3. Let X be a set of λ -terms. The sequence $\{M_n \mid n \in \mathbb{N}\}$, $M_n \in X$, is *cofinal in X* iff $\forall N \in X \exists n \in \mathbb{N} N \twoheadrightarrow M_n$.

Lemma 2.4. Let $X = \{M \in \Lambda \mid M =_\beta M_0\}$. Then there is a reduction sequence $M_0 \rightarrow M_0^1 \rightarrow M_0^2 \rightarrow \dots$ which is cofinal in X .

Proof. Define for $N \in \Lambda$, $G(N)$ as the Gross-reduct of N , i.e. the complete development of N w.r.t. all its redexes, as in [1, Section 3]. Then $\{G^n(N) \mid n \in \mathbb{N}\}$ is cofinal in the set of reducts of N (see [1, 3.1.3] for a proof) and hence, by the Church–Rosser theorem, also cofinal in the set of terms convertible with N .

To get a cofinal (one step) reduction sequence, interpolate between $G^n(M_0)$ and $G^{n+1}(M_0)$ the minimal reduction sequence. Call the resulting reduction sequence $\mathcal{G}(M_0)$.

Remark 2.5. Let X be as in Lemma 2.4. If X contains a normal form N , then $\mathcal{G}(M_0)$ stops in N .

Now we show that there is a *one step* CR-strategy on Λ . However, it is not recursive.

Theorem 2.6. *There exists a one step CR-strategy on Λ .*

Proof. Clearly it suffices to define the strategy F on all equivalence classes (w.r.t. convertibility) of λ -terms separately. Let us consider such a class, say $X = \{M \in \Lambda \mid M =_{\beta} M_0\}$. There are two cases.

(i) $\exists N \in X \forall M \in X M \rightarrow N$. Choose such an N . Let for each $M \in X$, \mathcal{R}_M be the minimal reduction path from M to N , say $\mathcal{R}_M = M \rightarrow M' \rightarrow M'' \rightarrow \dots \rightarrow N$. Now let $F(M) \equiv M'$.

(ii) Not (i). Let $\{M_0^i \mid i \in \mathbb{N}\}$ be the reduction sequence $\mathcal{G}(M_0)$. Now for all i , M_0^i occurs only finitely many times in $\{M_0^i \mid i \in \mathbb{N}\}$. For suppose not, then choose j such that M_0^j occurs infinitely many times in $\{M_0^i\}$. Clearly, by the cofinality of $\{M_0^i\}$ we have: $M \in X \Rightarrow M \rightarrow M_0^j$. But then we are in case (i), contradiction.

We have to cut away the cycles in $\{M_0^i\}$. Therefore we define a new cofinal reduction sequence $\{A_i \mid i \in \mathbb{N}\}$ as follows.

Basis: $A_0 \equiv M_0$.

Induction step: suppose $A_n \in \{M_0^i\}$ is defined. Let m be the largest number such that $M_0^m \equiv A_n$. Then $A_{n+1} \equiv M_0^{m+1}$. Clearly $\{A_i \mid i \in \mathbb{N}\}$ contains no repetitions.

Now we can define the strategy F . Let $F(A_i) \equiv A_{i+1}$ for all $i \in \mathbb{N}$ and for $M \notin \{A_i \mid i \in \mathbb{N}\}$ let F send M to the cofinal sequence $\{A_i \mid i \in \mathbb{N}\}$ via the minimal reduction path to that sequence. See Fig. 6.

In both cases it is obvious that F is a one step CR-strategy. (Here Remark 2.2 is used.)

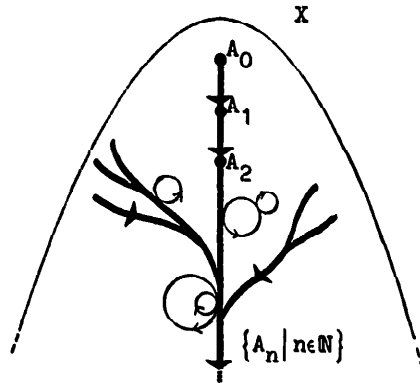


Fig. 6. \rightarrow : reduction paths, \rightarrow : F -paths.

3. A recursive one step CR-strategy for S -terms

In this section we restrict the set of terms to the set Σ of S -terms defined as follows:

$$S \in \Sigma,$$

$$A, B \in \Sigma \Rightarrow (AB) \in \Sigma.$$

Here S is the usual combinator from Combinatory Logic with the reduction rule $SABC \rightarrow AC(BC)$.

The interest of Σ is that it is a rather simple system: there are no reduction cycles and the relation \rightarrow (transitive reflexive closure of \rightarrow) is decidable. On the other hand Σ is a non trivial system; there are non-normalizing S -terms (see [1, Section 6]) and it is an open problem whether the property of having a normal form is decidable and whether the equality (= generated by \rightarrow) is decidable.

Definition 3.1. Let $M \in \Sigma$. Then

(i) $|M|$ is the *length* of M , defined inductively:

$$|S| = 1,$$

$$|(AB)| = |A| + |B|.$$

(ii) $\|M\|$ is the *weight* of M , defined inductively:

$$\|S\| = 1,$$

$$\|(AB)\| = 2\|A\| + \|B\|.$$

Proposition 3.2. (i) If $M \rightarrow N$, then $|M| \leq |N|$.

(ii) If $M \rightarrow N$ and $|M| = |N|$, then the “ S -redex” $SABC$ contracted in M , is in fact $SABS$.

(iii) Let $C[]$ be an S -context (i.e. an S -term containing one hole). Then

$$\|M\| > \|N\| \Rightarrow \|C[M]\| > \|C[N]\|.$$

Proof. Routine.

Lemma 3.3. There are no reduction cycles in Σ .

Proof. Suppose such a cycle $\mathcal{C} = M_0 \rightarrow M_1 \rightarrow \dots \rightarrow M_n \equiv M_0$ ($n \geq 1$) exists. Then $|M_0| = |M_n|$, hence by Proposition 3.2(i) $|M_0| = |M_1| = \dots = |M_n|$. By 3.2(ii) every S -redex contracted in \mathcal{C} is therefore of the form $SABS$. However:

$$\|SABS\| = 8\|S\| + 4\|A\| + 2\|B\| + \|S\| = 4\|A\| + 2\|B\| + 9$$

and

$$\|AS(BS)\| = 4\|A\| + 2\|S\| + 2\|B\| + \|S\| = 4\|A\| + 2\|B\| + 3$$

hence by Proposition 3.2(iii)

$$\|M_0\| > \|M_1\| > \dots > \|M_n\| = \|M_0\|,$$

contradiction.

Proposition 3.4. *Let $\mathcal{R} = M_0 \rightarrow M_1 \rightarrow \dots$ be an infinite reduction sequence in Σ . Then $\forall n \in \mathbb{N} \exists m \in \mathbb{N} |M_m| > n$.*

Proof. Suppose not, then after some number of steps in \mathcal{R} the length of the terms in \mathcal{R} would remain constant. Since there are only finitely many S -terms of a given constant length, \mathcal{R} would therefore contain a cycle, in contradiction with Lemma 3.3.

Lemma 3.5. (i) \rightarrow is a decidable relation.

(ii) Let $\mathcal{R} = M_0 \rightarrow M_1 \rightarrow \dots$ be a given infinite reduction sequence. Then the (unary) relation $M \in \mathcal{R}$ is decidable.

Proof. (i) Let $M, N \in \Sigma$. Whether $M \rightarrow N$ or not, can be decided as follows. Write down all reduction sequences starting with M until the length of the terms in it exceeds that of N (by Proposition 3.4 this will indeed happen). Clearly this gives only finitely many reducts of M and we can check if N is among them.

(ii) Immediate by Proposition 3.4.

Definition 3.6. (i) Analogous to the preceding section, we define the cofinal one step reduction $\mathcal{G}(M)$ and the minimal reduction sequence among a set of finite coinitial reduction sequences. As in Remark 2.5, $\mathcal{G}(M)$ finds the normal form of M if it exists.

(ii) Further we define a partial recursive function f on $\Sigma \times \Sigma$, as follows. First decide if $M \in \mathcal{G}(N)$, which is possible by Lemma 3.5(ii). If so, $f(M, N)$ is the successor of M in $\mathcal{G}(N)$, unless M is in normal form, then $f(M, N) \equiv M$. Otherwise: let $\mathcal{G}(N)$ be $\{N_n \mid n \in \mathbb{N}\}$ and decide for every N_k whether $M \rightarrow N_k$ or not. Let N_{k_0} be the first term such that $M \rightarrow N_{k_0}$ if it exists. Let \mathcal{R}_{\min} be the minimal reduction sequence from M to N_{k_0} , and let $M \rightarrow f(M, N)$ be the first step in \mathcal{R}_{\min} , as in Fig. 7.

Remark that $f(M, N)$ is defined iff $M = N$, and that $f(M, N) \equiv M$ iff M is in normal form.

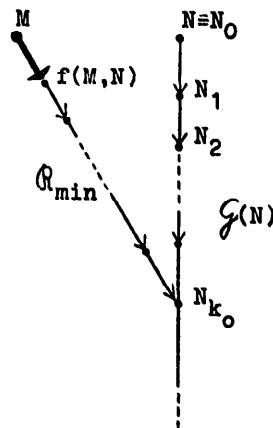


Fig. 7.

Definition 3.7. (i) Let some recursive enumeration of Σ be given, $\Sigma = \{M_n \mid n \in \mathbb{N}\}$, such that $n \geq m \Rightarrow |M_n| \geq |M_m|$.

(ii) Let $M, N \in \Sigma$. Then $M =_n N$ iff there is an equality proof of length $\leq n$, i.e. a conversion

$$M \equiv M_1 \leftrightarrow M_2 \leftrightarrow \cdots \leftrightarrow M_k \equiv N \quad \text{for some } k \leq n.$$

(\leftrightarrow is: \rightarrow or \leftarrow .)

(iii) Let $M \in \Sigma$. Then

$$A(M) = \{N \in \Sigma \mid N \rightarrow M \text{ \& } \neg \exists N' \neq N N' \rightarrow N\},$$

$$B(M) = \{N' \in \Sigma \mid \exists N \in A(M) N =_{|M|} N'\}.$$

(iv) If $M \in \Sigma$, then M^* is the first term in $B(M)$, in the sense of the enumeration in (i).

Proposition 3.8. (i) $A(M)$ and $B(M)$ are finite and recursive in M .

(ii) The function $*$: $\Sigma \rightarrow \Sigma$ is recursive.

Proof. Easy.

Theorem 3.9. There exists a recursive one step CR-strategy on Σ .

Proof. Define for $M \in \Sigma$: $F(M) = f(M, M^*)$. Since $M = M^*$, F is always defined and by Proposition 3.8(ii) F is recursive. Clearly F is a one step strategy. It remains to prove that F is CR.

Let $[M] = \{N \in \Sigma \mid M = N\}$ be some equivalence class w.r.t. convertibility. Let L be the first element of $[M]$ in the sense of Definition 3.7(i). Note that we do not claim that L can be found recursively from M .

Now we claim that

$$\forall N \in [M] \exists n \in \mathbb{N} \quad F^n(N) \in \mathcal{G}(L).$$

For, let $N \in [M]$, and consider the sequence $\alpha = \{F^n(N) \mid n \in \mathbb{N}\}$. There are two cases.

(i) For some k : $F^k(N) \equiv F^{k+1}(N) \equiv F^{k+2}(N) \equiv \cdots$. Then by the remark in Definition 3.6 $F^k(N)$ is a normal form. Since also $\mathcal{G}(L)$ reaches this normal form, which is unique in $[M]$ by the Church–Rosser theorem for Σ , the claim is proved for this case.

(ii) α does not stop. Let P be some term in $A(N)$ and let $P =_m L$ for some m . (See Fig. 8.) Note that $A(N) \subseteq A(F(N)) \subseteq A(F^2(N)) \subseteq \cdots$. By Proposition 3.4 we have: $\exists k \mid F^k(N) \mid > m$. Therefore $L \in B(F^k(N))$. Since L is the first element of $[M]$, it is also the first element of $B(F^k(N))$, i.e. $(F^k(N))^* \equiv L$. Remark that for all $r \in \mathbb{N}$: $(F^{k+r}(N))^* \equiv L$.

So F sends $F^k(N)$ along the minimal reduction sequence from $F^k(N)$ to $\mathcal{G}(L)$, which proves the claim.

Hence every maximal F -path intersects $\mathcal{G}(L)$, and from the point of intersection downwards, coincides with $\mathcal{G}(L)$. Hence F is CR on $[M]$. See also Fig. 8; a bar indicates where the situation “stabilizes”. For every Q below the bar we have $Q^* \equiv L$, and hence every Q below the bar is sent by F on a course to $\mathcal{G}(L)$.

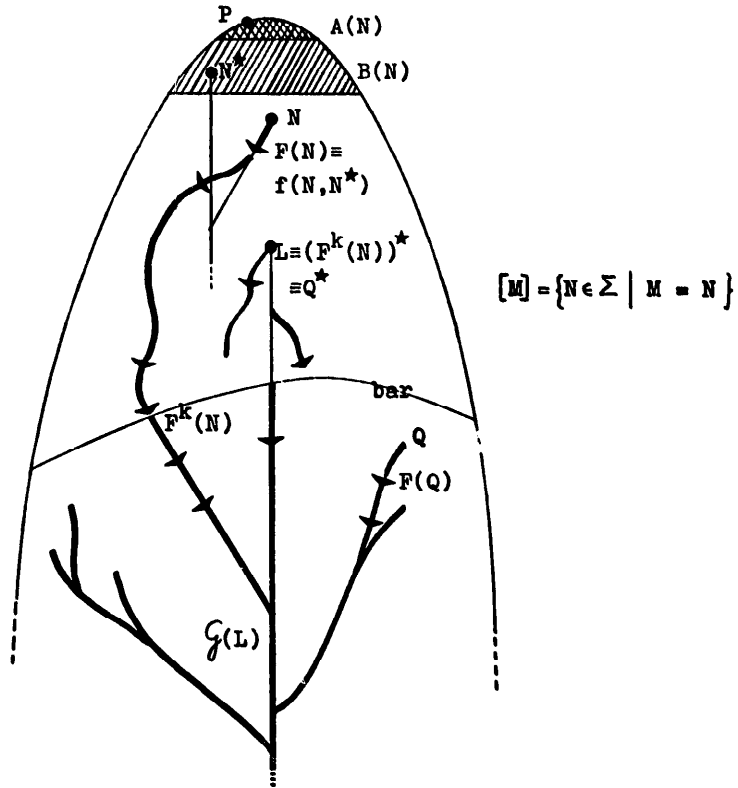


Fig. 8.

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