

## DEGREES OF SENSIBLE LAMBDA THEORIES

HENK BARENDREGT, JAN BERGSTRA, JAN WILLEM KLOP AND HENRI VOLKEN

**Summary.** A  $\lambda$ -theory  $T$  is a consistent set of equations between  $\lambda$ -terms closed under derivability. The *degree* of  $T$  is the degree of the set of Gödel numbers of its elements.  $\mathcal{H}$  is the  $\lambda$ -theory axiomatized by the set  $\{M = N \mid M, N \text{ unsolvable}\}$ . A  $\lambda$ -theory is *sensible* iff  $T \supset \mathcal{H}$ ; for a motivation see [6] and [4].

In §1 it is proved that the theory  $\mathcal{H}$  is  $\Sigma_2^0$ -complete. We present Wadsworth's proof that its unique maximal consistent extension  $\mathcal{H}^*$  ( $= \text{Th}(D_\infty)$ ) is  $\Pi_2^0$ -complete. In §2 it is proved that  $\mathcal{H}\eta$  ( $= \lambda\eta$ -calculus +  $\mathcal{H}$ ) is not closed under the  $\omega$ -rule (see [1]). In §3 arguments are given to conjecture that  $\mathcal{H}\omega$  ( $= \lambda + \mathcal{H} + \omega$ -rule) is  $\Pi_1^0$ -complete. This is done by representing recursive sets of sequence numbers as  $\lambda$ -terms and by connecting wellfoundedness of trees with provability in  $\mathcal{H}\omega$ . In §4 an infinite set of equations independent over  $\mathcal{H}\eta$  will be constructed. From this it follows that there are  $2^{n_0}$  sensible theories  $T$  such that  $\mathcal{H} \subset T \subset \mathcal{H}^*$  and  $2^{n_0}$  sensible hard models of arbitrarily high degrees. In §5 some nonprovability results needed in §§1 and 2 are established. For this purpose one uses the theory  $\mathcal{H}\eta$  extended with a reduction relation for which the Church–Rosser theorem holds. The concept of Gross reduction is used in order to show that certain terms have no common reduct.

**Preliminaries.** Some frequently used  $\lambda$ -calculus notions are reviewed below. For a more detailed treatment see [2]. For notions about degrees see [8].

**Reductions.** If  $\rightarrow$  is a reduction relation,  $\rightarrow^*$  denotes its transitive reflexive closure.  $\rightarrow_\beta, \rightarrow_\eta$  are one step  $\beta$ - resp.  $\eta$ -reduction.  $\rightarrow_{\beta\eta} = \rightarrow_\beta \cup \rightarrow_\eta$ .

**Numerals.** It is immaterial how the numerals  $0, 1, 2, \dots$  are defined as  $\lambda$ -terms as long as the recursive functions are  $\lambda$ -definable w.r.t. this system.

**Truth values.**  $T = \lambda xy. x, F = \lambda xy. y$ .  $T, F$  represent true and false respectively.

**Conditional.** “If  $B$  then  $M$  else  $N$ ” is represented as  $BMN$ , since  $\lambda \vdash TMN = M$  and  $\lambda \vdash FMN = N$ .

**Pairing.**  $[x, y] = \lambda z. zxy, (x)_0 = xT, (x)_1 = xF$ .

**Fixed points.** Let  $Y \equiv \lambda f. (\lambda x. f(xx))(\lambda x. f(xx))$ . Then  $Yf$  is a *fixed point* of  $f$ . Often a term is implicitly defined as follows: let  $F$  be a term such that

$$(1) \quad Fxy = \dots x \dots y \dots F \dots$$

Equation (1) is satisfied by  $F \equiv Y(\lambda fxy. \dots x \dots y \dots f \dots)$ .

**Contexts.** A *context* is a term  $C[ ]$  with some holes in it.  $C[M]$  denotes the result of placing  $M$  in the holes of  $C[ ]$ . In this act free variables of  $M$  may become bound.

**Solvability.** A closed term  $M$  is *solvable* iff

$$\exists n \exists N_1 \dots N_n \lambda \vdash MN_1 \dots N_n = I.$$

An arbitrary term  $M$  is solvable iff its closure  $\lambda x. M$  is.  $\Omega$  denotes the term

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$(\lambda x. xx)(\lambda x. xx)$ . *Note.*  $\Omega$  is unsolvable.  $M$  is *solvably equivalent* to  $N$ , notation  $M \sim_s N$ , iff

$$\forall C[ ] (C[M] \text{ is solvable} \Leftrightarrow C[N] \text{ is solvable}).$$

$$\mathcal{H}^* = \{M = N \mid M \sim_s N\}.$$

*Head normal forms.* A *head normal form* (hnf) is a term of the form  $\lambda x_1 \cdots x_n. x_i M_1 \cdots M_m$ ,  $n, m \geq 0$ .  $M$  has a hnf  $N$  iff  $\lambda \vdash M = N \& N$  is hnf.

*Fact.*  $M$  is solvable iff  $M$  has a hnf.

*Böhm trees.* Let  $M$  be a  $\lambda$ -term. The *Böhm tree* of  $M$ ,  $BT(M)$ , is defined as follows. If  $M$  is unsolvable, then  $BT(M) = \Omega$  ( $\Omega$  is just some symbol). Else  $M$  has a hnf, say  $\lambda x_1 \cdots x_n. x_i M_1 \cdots M_m$ . Then

$$BT(M) = \lambda x_1 \cdots x_n. x_i \begin{array}{c} \swarrow \quad \downarrow \quad \searrow \\ BT(M_1) \cdots BT(M_m) \end{array}$$

*The model  $D_\infty$ .*  $D_\infty$  is Scott's lattice theoretic model for the  $\lambda$ -calculus over a complete lattice  $D$  (see [9]). Hyland and Wadsworth proved (see [2]):

$$D_\infty \vDash M = N \Leftrightarrow M = N \in \mathcal{H}^*,$$

$$BT(M) = BT(N) \Rightarrow D_\infty \vDash M = N.$$

*$\omega$ -rule.* The  *$\omega$ -rule* is:  $FZ = F'Z$  for all closed  $Z \Rightarrow F = F'$ . If  $T$  is a  $\lambda$ -theory, then  $T + \omega$  is the closure of  $T$  under the  $\omega$ -rule.

**§1. Degrees of  $\mathcal{H}$ ,  $\mathcal{H}^*$  and  $\mathcal{H}\omega$ .** The  $\lambda$ -theory  $\mathcal{H}$  has a unique maximal consistent extension  $\mathcal{H}^*$  [2, §5]. Let  $\mathcal{H}\omega$  be the set of equations provable in  $\lambda + \mathcal{H} + \omega$ -rule. Then one has  $\mathcal{H} \subseteq \mathcal{H}\eta \subseteq \mathcal{H}\omega \subseteq \mathcal{H}^*$ . The first two inclusions are trivial; the last one follows from the fact that  $\mathcal{H}^* = \text{Th}(D_\infty)$  and  $D_\infty$  satisfies the  $\omega$ -rule (see [12]). Moreover the inclusions are proper.  $\mathcal{H} \neq \mathcal{H}\eta$  follows from the C-R property for  $\mathcal{H}\eta$  (see [3]).  $\mathcal{H}\eta \neq \mathcal{H}\omega$  is proved in 2.3.  $\mathcal{H}\omega \neq \mathcal{H}^*$  follows by an extension of the consistency proof in [1]: It can be proved that if  $\mathcal{H}\omega \vdash M = I$ , then  $\lambda \vdash M\mathbf{I} = I$  where  $\mathbf{I}$  is some sequence of  $I$ 's. If  $\mathcal{H}\omega = \mathcal{H}^*$ , then  $\mathcal{H}\omega \vdash J = I$ , where  $J$  is Wadsworth's term  $Y(\lambda jxy. x(jy))$ , since  $J$  and  $I$  have equivalent Böhm trees [2, 6.7]. So  $\lambda \vdash J\mathbf{I} = I$ , which leads as follows to a contradiction.  $J\mathbf{I}$  is a  $\lambda I$ -term, hence has a normal form (nf) iff all its subterms do.  $J$  has no nf, so  $J\mathbf{I}$  has no nf, contradiction.

It will be proved that  $\mathcal{H}(\eta)$  is  $\Sigma_2^0$ -complete and that  $\mathcal{H}^*$  is  $\Pi_2^0$ -complete. It is conjectured that  $\mathcal{H}\omega$  is  $\Pi_1^1$ -complete.

1.1. LEMMA. *Let  $R(\mathbf{x})$  be an r.e. predicate. Then for some term  $F$ ,*

$$\mathcal{H} \vdash F\mathbf{n} = I \quad \text{if } R(\mathbf{n}),$$

$$\mathcal{H} \vdash F\mathbf{n} = \Omega \quad \text{if } \neg R(\mathbf{n}).$$

PROOF. Let  $R(\mathbf{x}) \Leftrightarrow \exists y A(\mathbf{x}, y)$  with  $A$  recursive. Let  $F\mathbf{x} = G\mathbf{x}\mathbf{0}$  where  $G$  is defined by the fixed point theorem as

$$G\mathbf{x}y = \text{if } A(\mathbf{x}, y) \text{ then } I \text{ else } G\mathbf{x}(y + 1).$$

Clearly,  $\mathcal{H} \vdash F \underline{n} = I$  if  $R(\underline{n})$ . Now suppose  $\neg R(\underline{n})$ . Then it is easily checked that there is an infinite reduction sequence from  $F \underline{n}$  in which infinitely many times the head redex is contracted. By Theorem 8.5.22 in [5] the existence of such a reduction sequence implies that  $F \underline{n}$  is unsolvable. Hence  $F \underline{n} = \Omega$  in  $\mathcal{H}$ .  $\square$

1.2. DEFINITION. (i) *Ordered tuples are represented as terms as follows:*

$$\langle M_0 \rangle = M_0, \quad \langle M_0, \dots, M_{n+1} \rangle = [M_0, \langle M_1, \dots, M_{n+1} \rangle].$$

(ii) If  $(M_0, M_1, M_2, \dots) = (M_i)_{i \in \mathbb{N}}$  is a definable sequence of terms, i.e. for some  $M$ ,  $\lambda \vdash M \underline{i} = M_i$  for all  $i$ , then this infinite sequence is represented as a term  $A \underline{0}$ , where  $A$  is such that  $A \underline{n} \rightarrow_{\beta}^* [M_n, A \underline{n+1}]$ .  $A$  exists by the fixed point theorem. *Notation.*  $\langle M_0, M_1, M_2, \dots \rangle \equiv \langle M_i \rangle_{i \in \mathbb{N}} \equiv A \underline{0}$ .

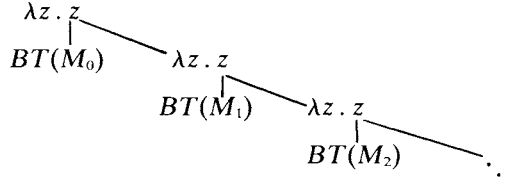
1.3. LEMMA. *There is a term  $\pi$  such that  $\lambda \vdash \pi \underline{n} \langle M_j \rangle = M_n$  (for definable sequences  $(M_j)$ ).*

PROOF. Define  $\pi i x =$  if  $i = 0$  then  $(x)_0$  else  $\pi i - 1((x)_i)$ .  $\square$

1.4. LEMMA. *If  $(M_i), (N_i)$  are definable sequences then*

$$\forall i \mathcal{H}^* \vdash M_i = N_i \Leftrightarrow \mathcal{H}^* \vdash \langle M_i \rangle_{i \in \mathbb{N}} = \langle N_i \rangle_{i \in \mathbb{N}}.$$

PROOF.  $\Rightarrow$ : The Böhm tree of  $\langle M_i \rangle$  is



and similarly for  $\langle N_i \rangle$ . By the theorem of Hyland and Wadsworth  $\mathcal{H}^* \vdash P = Q \Leftrightarrow BT(P) \sim_{\eta} BT(Q)$  (see [2, 7.1.(i)]). It follows that the mentioned trees are equivalent and the result follows.

$\Leftarrow$ : By applying  $\pi \underline{i}$  of 1.3.  $\square$

1.5. THEOREM (WADSWORTH [11]).  $\mathcal{H}^*$  is  $\Pi_2^0$ -complete.

PROOF. (i)  $\mathcal{H}^* \vdash M = N \Leftrightarrow M = N \in \mathcal{H}^* \Leftrightarrow \forall C [ ] [C[M]$  is solvable  $\Leftrightarrow C[N]$  is solvable] (see [2, §5]). The latter is clearly  $\Pi_2^0$ .

(ii) Let  $\forall a \exists b A(a, b, c)$  be any  $\Pi_2^0$  predicate with  $A$  recursive.

By 1.1 there is a term  $F$  such that

$$\mathcal{H} \vdash F \underline{c} \underline{a} = I, \quad \text{if } \exists b A(a, b, c),$$

$$\mathcal{H} \vdash F \underline{c} \underline{a} = \Omega, \quad \text{else.}$$

Let  $H \underline{c} = \langle F \underline{c} \underline{0}, F \underline{c} \underline{1}, \dots \rangle$ ,  $H' \underline{c} = \langle I, I, \dots \rangle$ . Now

$$\forall a \exists b A(a, b, c) \Leftrightarrow \forall a \mathcal{H} \vdash F \underline{c} \underline{a} = I$$

$$\Leftrightarrow \forall a \mathcal{H}^* \vdash F \underline{c} \underline{a} = I \quad \text{since } \mathcal{H} \subset \mathcal{H}^* \text{ and } \mathcal{H}^* \not\vdash I = \Omega$$

$$\Leftrightarrow \mathcal{H}^* \vdash H \underline{c} = H' \underline{c} \quad \text{by 1.4.}$$

Therefore each  $\Pi_2^0$  predicate can be reduced to provability in  $\mathcal{H}^*$ .  $\square$

1.6. THEOREM.  $\mathcal{H}(\eta)$  is  $\Sigma_2^0$ -complete.

PROOF. (i) The set of axioms  $\{M = N \mid M, N \text{ unsolvable}\}$  is clearly  $\Pi_1^0$ , therefore they generate a  $\Sigma_2^0$  theory.

(ii) Let  $\exists a \forall b A(a, b, c)$  be any  $\Sigma_2^0$  predicate with  $A$  recursive. By 1.1 there is a term  $F$  such that

$$\begin{aligned} \mathcal{H} \vdash F \underline{c} \underline{a} &= \Omega \quad \text{if } \forall b A(a, b, c), \\ &= I \quad \text{else.} \end{aligned}$$

Let  $Hcia \rightarrow_{\beta}^* [I, Fca(Hcia + 1)]$  by the fixed point theorem. Let  $x, y$  be different variables.

*Claim.*  $\exists a \forall b A(a, b, c) \Leftrightarrow \mathcal{H}(\eta) \vdash H \underline{c} x \underline{0} = H \underline{c} y \underline{0}$ .

$\Rightarrow$ : If  $\exists a \forall b A(a, b, c)$  then  $\mathcal{H} \vdash H \underline{c} x \underline{0} = [I, I, \dots, \Omega] = H \underline{c} y \underline{0}$ .

$\Leftarrow$ : If  $\neg \exists a \forall b A(a, b, c)$  then  $\forall a \mathcal{H} \vdash F \underline{c} \underline{a} = I$ , so  $H \underline{c} i \underline{n} \rightarrow_{\beta}^* [I, H \underline{c} i \underline{n} + 1]$  and  $H \underline{c} i \underline{0} \rightarrow_{\beta}^* [I, I, \dots, H \underline{c} i \underline{n}]$ . Then  $\mathcal{H} \eta \not\vdash H \underline{c} x \underline{0} = H \underline{c} y \underline{0}$  as is proved in §5.

So each  $\Sigma_2^0$  predicate can be reduced to provability in  $\mathcal{H}(\eta)$ .  $\square$

## §2. $\mathcal{H} \eta \not\vdash \omega$ .

2.1. DEFINITION. A  $\lambda$ -theory  $T$  is closed under the  $\omega$ -rule, notation  $T \vdash \omega$ , if for all closed  $F, F', T \vdash FZ = F'Z$  for all closed  $Z \Rightarrow T \vdash F = F'$ .

If  $\mathcal{M}$  is a model of the  $\lambda$ -calculus,  $\mathcal{M}^0$  denotes the interior of  $\mathcal{M}$ , i.e. the set of interpretations of closed  $\lambda$ -terms. Note that  $\text{Th}(\mathcal{M}) \vdash \omega$  iff  $\mathcal{M}^0$  is extensional.

In [7] it is shown that  $\lambda \eta \not\vdash \omega$ . Now two proofs will be given that  $\mathcal{H} \eta \not\vdash \omega$ . In the first proof the terms constructed play a symmetric role. Not so in the alternative one. There a term  $A$  is constructed which in  $\mathcal{H} \eta$  is constant on all closed terms, but not constant in general. Also in [7] a pseudoconstant term is used to prove  $\lambda \eta \not\vdash \omega$ . The construction is totally different however. See also [1].

2.2. LEMMA. Let  $FM^{-n} = FM \cdots M$  ( $M$   $n$  times). Then  $\forall Z$  closed  $\exists n \mathcal{H} \vdash Z \Omega^{-n} = \Omega$ .

PROOF. If  $Z$  is unsolvable, then  $\mathcal{H} \vdash Z = \Omega$ . Otherwise  $Z$  has a head normal form  $\lambda x_1 \cdots x_n. x_i N_1 \cdots N_m$ . Then  $\mathcal{H} \vdash Z \Omega^{-n} = \Omega$ .  $\square$

2.3. THEOREM.  $\mathcal{H} \eta \not\vdash \omega$ .

FIRST PROOF. Define a term  $O$  such that  $Oi \underline{n} \rightarrow_{\beta}^* \lambda y. y \Omega^{-n} (Oi \underline{n} + 1 y)$ .  $O$  can be constructed by the fixed point theorem and an  $F$  such that  $\lambda \vdash Fy \underline{n} = y \Omega^{-n}$ . Take e.g.  $Fyn = \text{if Zero } n \text{ then } y \text{ else } Fy(n-1)\Omega$ .

*Claim 1.*  $\forall Z$  closed  $\mathcal{H} \vdash Ox \underline{0} Z = Oy \underline{0} Z$ .

*Claim 2.*  $\mathcal{H} \eta \not\vdash Ox \underline{0} = Oy \underline{0}$ .

As to Claim 1,

$$\begin{aligned} \mathcal{H} \vdash Ox \underline{0} Z &= Z \Omega(Ox \underline{1} Z) = \cdots \\ &= Z \Omega(Z \Omega \Omega(\cdots (Z \Omega^{-n}(Ox \underline{n} + 1 Z)) \cdots)) = \cdots \end{aligned}$$

Hence by 2.2 there exists an  $n$  such that

$$\mathcal{H} \vdash Ox \underline{0} Z = Z \Omega(\cdots (Z \Omega^{-n-1}) \cdots) = Oy \underline{0} Z.$$

As to Claim 2, it is proved in §5.

By the claims  $\mathcal{H}\eta \not\prec \omega$ .  $\square$

ALTERNATIVE PROOF. Define a term  $A$  such that  $Az \rightarrow_{\beta}^* \lambda y. y(A(z\Omega))$ . Then

$$\begin{aligned} \mathcal{H} \vdash AI &= \lambda y. y(A\Omega) = \lambda y. y(\lambda y. y(A\Omega^{-2})) = \dots \\ &= C_{*}^n(A\Omega^{-n}) = \dots, \quad \text{where } C_{*}a = \lambda y. ya. \end{aligned}$$

Claim.  $\forall Z$  closed  $\mathcal{H} \vdash AZ = AI$ . Indeed,

$$\begin{aligned} AZ &\xrightarrow{\beta}^* \lambda y. y(A(Z\Omega)) \xrightarrow{\beta}^* C_{*}^2(A(Z\Omega^{-2})) \\ &\xrightarrow{\beta}^* C_{*}^n(A(Z\Omega^{-n})) \xrightarrow{\beta}^* C_{*}^n(A\Omega) \xleftarrow{\beta}^* AI \end{aligned}$$

for  $n$  large enough by 2.2. Hence  $\forall$  closed  $Z$ ,  $\mathcal{H} \vdash AZ = K(AI)Z$ . But  $\mathcal{H}\eta \not\prec A = K(AI)$  since  $\mathcal{H}\eta \not\prec Ax = AI$  as is proved in §5.

**§3. Conjecture:**  $\mathcal{H}\omega$  is  $\Pi_1^1$ -complete. We will give a strong argument to conjecture that  $\mathcal{H}\omega$ , the  $\lambda$ -calculus extended by the axioms of  $\mathcal{H}$  and the  $\omega$ -rule, is  $\Pi_1^1$ -complete.

Given a recursive set  $\mathfrak{X}$  of sequence numbers, two terms  $B_0, B_1$  can be constructed such that one can prove by bar induction

$$\mathfrak{X} \text{ is wellfounded} \Rightarrow \mathcal{H}\omega \vdash B_0 = B_1;$$

and the converse is probably true.

First we construct a term that “codes” the Baire space  $\mathbf{N}^{\mathbf{N}}$ . For simplicity we assume that the codes of finite sequences  $\langle n_0, \dots, n_k \rangle$  is such that every natural number is a sequence number.  $s * s'$  denotes the code of the sequence obtained by concatenating the sequences with codes respectively  $s, s'$ .  $s < s'$  denotes that the sequence with code  $s$  is an initial segment of the sequence with code  $s'$ . As is well known, the coding of the finite sequences can be done in such a way that  $*$  and  $<$  are recursive. For  $\alpha \in \mathbf{N}^{\mathbf{N}}$  and  $n \in \mathbf{N}$ ,  $\bar{\alpha}(n)$  is the sequence number  $\langle \alpha(0), \dots, \alpha(n-1) \rangle$ . For each  $\alpha$ ,  $\bar{\alpha}(0) = \langle \rangle$ , the code of the empty sequence.

3.1. LEMMA. *There exists a term  $B$  such that*

$$\lambda \vdash B \underline{s} \xrightarrow{\beta}^* \langle B \underline{s} * 0, B \underline{s} * 1, B \underline{s} * 2, \dots \rangle.$$

PROOF. Since the function  $*$  is recursive, there is a term  $\underline{*}$ , such that  $\lambda \vdash \underline{*} \underline{s} \underline{s}' \rightarrow_{\beta}^* \underline{s} * \underline{s}'$ . Hence, by 1.2(ii), the sequence  $\langle B \underline{s} * 0, B \underline{s} * 1, \dots \rangle$  can be represented. Then by the fixed point theorem  $B$  can be constructed.  $\square$

Now  $B \langle \underline{\quad} \rangle$  can be thought as representing the Baire space: if  $B \langle n_1, \dots, n_k \rangle$  is abbreviated as  $Bn_1 \dots n_k$ , then

$$\begin{aligned} B \langle \underline{\quad} \rangle &= \langle B0, B1, \dots \rangle \\ &= \langle \langle B00, B01, \dots \rangle, \langle B10, B11, \dots \rangle, \dots \rangle = \dots \end{aligned}$$

Let  $P(n) \equiv \forall \alpha \exists m R(\bar{\alpha}(m), n)$ , with  $R$  recursive, be a  $\Pi_1^1$  predicate. A sequence number  $s$  is  $n$ -secured iff  $\exists s' < s R(s', n)$ , otherwise  $n$ -unsecured. Then  $P(n)$  holds iff the  $n$ -unsecured sequence numbers are wellfounded (i.e.

not  $s_1 < s_2 < s_3 \cdots$  for some infinite sequence of  $n$ -unsecured sequence numbers).

Now the construction in 3.1 will be modified in order to code the set of  $n$ -unsecured sequence numbers.

3.2. LEMMA. *There exists a term  $\square$  such that*

$$\begin{aligned} \mathcal{H} \vdash \square \underline{n} \underline{s} &= I \quad \text{if } s \text{ is } n\text{-unsecured,} \\ &= \Omega \quad \text{else.} \end{aligned}$$

PROOF. The notion “ $s$  is an  $n$ -unsecured sequence number” is recursive and hence r.e. Therefore 1.1 applies.  $\square$

*Notation.*  $\square^*$  denotes  $\square x$  and similarly for terms derived from  $\square$ .

3.3. LEMMA. *There is a term  $B$  such that*

$$B^* \underline{s} \xrightarrow[\beta]{*} \square^* \underline{s} \langle B^* \underline{s} * 0, B^* \underline{s} * 1, \dots \rangle.$$

PROOF. As for 3.1.  $\square$

Thus  $B^* \langle \rangle$  represents the  $n$ -unsecured sequence numbers:

$$\begin{aligned} B^* \langle \rangle &= \langle B^* 0, B^* 1, \dots \rangle \\ &= \langle \langle B^* 00, \Omega, B^* 01, \dots \rangle, \langle \Omega, B^* 11, \dots \rangle \cdots \rangle = \cdots \end{aligned}$$

if e.g.  $\langle 0, 1 \rangle$  and  $\langle 1, 0 \rangle$  are  $n$ -secured.

Now it will be shown in a uniform way that for a sensible  $\lambda$ -theory  $T$ , the degree of  $T + \omega$  is closed under universal quantification. This is done by using the idea in the first proof of 2.3.

3.4. LEMMA. *There is a term  $\Pi$  such that for all sensible  $\lambda$ -theories  $T$*

$$\forall n \, T \vdash F \underline{n} = G \underline{n} \Rightarrow T + \omega \vdash \Pi F = \Pi G.$$

PROOF. By the fixed point theorem let  $A$  be a term such that

$$A f \underline{n} y \xrightarrow[\beta]{*} [f \underline{n}, y \Omega^{-n} (A f \underline{n} + 1 y)].$$

Then define  $\Pi = \lambda f. A f \underline{0}$ .

Suppose  $\forall n \, T \vdash F \underline{n} = G \underline{n}$ .

*Claim.* For all closed  $Z$ ,  $T \vdash \Pi F Z = \Pi G Z$ .

Indeed, by 2.2, for some  $n$ ,  $\mathcal{H} \vdash Z \Omega^{-n} = \Omega$ . Take the least such  $n$ . Then

$$\begin{aligned} T \vdash \Pi F Z &= A F \underline{0} Z = [F \underline{0}, Z(A F \underline{1} Z)] \\ &= [F \underline{0}, Z[F \underline{1}, Z \Omega \cdots [F \underline{n}, \Omega] \cdots]] \\ &= [G \underline{0}, Z[G \underline{1}, Z \Omega \cdots [G \underline{n}, \Omega] \cdots]] \\ &= A G \underline{0} Z = \Pi G Z. \end{aligned}$$

Therefore by the  $\omega$ -rule  $T + \omega \vdash \Pi F = \Pi G$ .  $\square$

REMARK. By the technique of Böhm (see [2, §6]) one can even prove that for sensible  $T$ ,

$$\forall n T + \omega \vdash F\underline{n} = G\underline{n} \Leftrightarrow T + \omega \vdash \Pi F = \Pi G.$$

Finally the main construction will be given.

3.5. DEFINITION. Let  $P(n) \equiv \forall \alpha \exists m R(\bar{\alpha}(m), n)$ , with  $R$  recursive, be an arbitrary  $\Pi_1^1$  predicate. Let  $\square$  be the term constructed in 3.2.

By the fixed point theorem there exists a term  $B$  such that

$$Bixs \xrightarrow[\beta]{*} \square^x s \Pi(\lambda a. Bix(* sa)).$$

Notation.  $B_i^x = B \underline{i} x$ ,  $s \underline{*} a = \underline{*} sa$ . Then

$$\lambda \vdash B_i^x s \xrightarrow[\beta]{*} \square^x s \Pi(\lambda a. B_i^x(s \underline{*} a)).$$

3.6. THEOREM. Let  $P, B$  be as in 3.5. Then

$$P(n) \Rightarrow \mathcal{H}\omega \vdash B_0^n \langle \underline{\quad} \rangle = B_1^n \langle \underline{\quad} \rangle.$$

PROOF. Claim. (i) If  $s$  is  $n$ -secured then  $\mathcal{H}\omega \vdash B_0^n \underline{s} = B_1^n \underline{s}$ .

(ii) If  $\forall a \mathcal{H}\omega \vdash B_0^n \langle \underline{s * a} \rangle = B_1^n \langle \underline{s * a} \rangle$ , then  $\mathcal{H}\omega \vdash B_0^n \underline{s} = B_1^n \underline{s}$ . As to (i), indeed

$$R(s, n) \Rightarrow \square^n \underline{s} = \Omega \Rightarrow B_0^n \underline{s} = \Omega = B_1^n \underline{s}.$$

As to (ii), indeed if  $s$  is  $n$ -secured, we are done by (i). Else  $\square^n \underline{s} = I$  and hence

$$\begin{aligned} \mathcal{H}\omega \vdash B_0^n \underline{s} &= \Pi(\lambda a. B_0^n \langle \underline{s \underline{*} a} \rangle), \\ \mathcal{H}\omega \vdash B_1^n \underline{s} &= \Pi(\lambda a. B_1^n \langle \underline{s \underline{*} a} \rangle), \end{aligned}$$

and the result follows from 3.4.

Now suppose  $P(n)$ . Then the set of  $n$ -unsecured sequence numbers is wellfounded. Therefore it follows by bar induction and the claim that  $\mathcal{H}\omega \vdash B_0^n \langle \underline{\quad} \rangle = B_1^n \langle \underline{\quad} \rangle$ .  $\square$

For the converse of 3.6, which establishes that  $\mathcal{H}\omega$  is  $\Pi_1^1$ -complete, a proof-theoretic analysis of  $\mathcal{H}\omega$  is needed.

§4.  $2^{\aleph_0}$  sensible hard models. Let  $T$  be a  $\lambda$ -theory. A set  $S$  of equations between  $\lambda$ -terms is *independent over*  $T$  if for  $M = N \in S$ ,  $T + S - \{M = N\} \not\vdash M = N$ . A set of terms  $X$  is independent over  $T$  if  $S_X = \{M = N \mid M, N \in X, M \neq N\}$  is independent over  $T$ .

We will construct a countable set of closed terms  $\{B \underline{0}, B \underline{1}, \dots\}$  independent over  $\mathcal{H}\eta$ . Hence the theories  $\mathcal{H}\eta + T_A$ , where  $T_A = \{B \underline{n} = B \underline{0} \mid n \in A\}$ , with  $A \subset \mathbb{N} - \{0\}$ , are all different.

Since an equation is provable in a  $\lambda$ -theory iff it is true in its term model, it follows that the closed term models of  $\mathcal{H} + T_A$  are  $2^{\aleph_0}$  sensible hard models. By taking the open term models of  $\mathcal{H}\eta + T_A$ ,  $2^{\aleph_0}$  sensible extensional models are obtained.

A relation  $\rightarrow$  between terms has the Church-Rosser (CR) property iff

$$(M \rightarrow N \ \& \ M \rightarrow L) \Rightarrow \exists P (N \rightarrow P \ \& \ L \rightarrow P).$$

4.1. DEFINITION. Let  $B$  be a term such that  $Bx \rightarrow_{\beta}^* \lambda z . z (Bx)$ . To be explicit take  $B = \omega\omega$  with  $\omega = \lambda bxz . z (bbx)$ .

It will be proved that  $\{B\mathbf{0}, B\mathbf{1}, \dots\}$  is an independent set over  $\mathcal{H}\eta$ . In order to do this we introduce a reduction relation satisfying the Church–Rosser theorem, which generates the equality in the theory  $\mathcal{H}\eta A = \mathcal{H}\eta + \{B\mathbf{n} = B\mathbf{0} \mid n \in A\}$  for  $A \subset \mathbf{N} - \{0\}$ .

4.2. DEFINITION. (i)  $\Omega$ -reduction  $\rightarrow_{\Omega}$  is defined by

- (1)  $H \rightarrow_{\Omega} \Omega$  for all unsolvables  $H$ .
- (2)  $M \rightarrow_{\Omega} N \Rightarrow MZ \rightarrow_{\Omega} NZ, ZM \rightarrow_{\Omega} ZN, \lambda x . M \rightarrow_{\Omega} \lambda x . N$ , for all  $Z$ .
- (3)  $M \rightarrow_{\Omega} M$ .

(ii)  $\rightarrow_{\beta\eta\Omega} = \rightarrow_{\beta\eta} \cup \rightarrow_{\Omega}$ .

Clearly  $\rightarrow_{\beta\eta\Omega}$  generates the equality in  $\mathcal{H}\eta$ .

4.3. LEMMA.  $\rightarrow_{\beta\eta\Omega}^*$  has the CR property.

PROOF. See [3, 2.30].  $\square$

4.4. DEFINITION. (i)  $\text{Red}(Bx) = \{C(x) \mid Bx \rightarrow_{\beta}^* C(x)\}$ .

(ii) The reduction relation  $\rightarrow_A$  is defined by

- (1)  $C(\mathbf{n}) \rightarrow_A C(\mathbf{0})$  for all  $n \in A$  and  $C(x) \in \text{Red}(B)$ .
- (2)  $M \rightarrow_A N \Rightarrow MZ \rightarrow_A NZ, ZM \rightarrow_A ZN, \lambda x . M \rightarrow_A \lambda x . N$  (all  $Z$ ).
- (3)  $M \rightarrow_A M$ .

(iii)  $\rightarrow_{\beta\eta\Omega A} = \rightarrow_{\beta\eta\Omega} \cup \rightarrow_A$ .

Clearly  $\rightarrow_{\beta\eta\Omega A}$  generates the equality of  $\mathcal{H}\eta A$ .

The following notation is used in order to facilitate the computation of the reduction tree of  $Bx$ .

4.5. DEFINITION.  $\square := Bx \equiv \omega\omega x$ . If  $\Delta$  is a term, then  $\Delta^1 := ((\lambda xz . z\Delta)x)$  and  $\Delta^0 := (\lambda z . z\Delta)$ .

4.6. LEMMA.

$$(*) \quad Bx \xrightarrow{\beta}^* C(x) \Leftrightarrow C(x) \text{ has the form } \Delta^{i_1 \dots i_n}, \quad i_1, \dots, i_n \in \{0, 1\}.$$

PROOF. Note that

- (i) Each one step  $\beta$ -reduct of  $\Delta^0$  is  $O^0$  where  $O$  is a one step  $\beta$ -reduct of  $\Delta$ .
- (ii) Each one step  $\beta$ -reduct of  $\Delta^1$  is  $\Delta^0$  or  $O^1$  where  $O$  is a one step  $\beta$ -reduct of  $\Delta$ .

(iii) The only one step  $\beta$ -reduct of  $\square$  is  $\square^1$ .

From (i)–(iii) it follows that all possible  $\beta$ -reducts of  $\square$  are of the form (\*). Moreover all terms of the form (\*) are reducts of  $\square$ .  $\square$

4.7. COROLLARY. Let  $Bx \rightarrow_{\beta}^* C(x)$ . Then

- (i)  $C(x)$  has no  $\eta$ - or  $\Omega$ -redexes.
- (ii) The only free variable in  $C(x)$  occurs at the end.
- (iii)  $C(\mathbf{n}) \rightarrow_{\beta}^* Z \Rightarrow Z \equiv C'(\mathbf{n})$  with  $Bx \rightarrow_{\beta}^* C'(x)$ .
- (iv)  $C(x) \equiv \lambda c . P \Rightarrow P \equiv cQ$  and  $Bx \rightarrow_{\beta}^* Q$ .

PROOF. Immediate.  $\square$

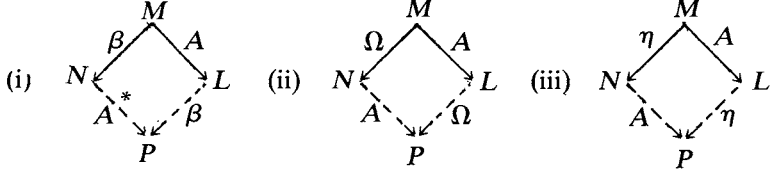
4.8. LEMMA.  $\rightarrow_A$  has the CR property.

PROOF. Let two terms be obtained from some term  $M$  by replacing some  $\mathbf{n}$  by  $\mathbf{0}$ . Hence a common reduct  $P$  can be found by making both changes in  $M$ .  $\square$



4.9. LEMMA.  $\rightarrow_{\beta\eta\Omega}^*$  is CR.

PROOF. By 4.8 and 4.3,  $\rightarrow_A$  and  $\rightarrow_{\beta\eta\Omega}^*$  are CR. So by the lemma of Hindley–Rosen, [10, (1.2)], it is sufficient to prove that their transitive closures commute. For this it is sufficient to prove



(i) Let  $R \equiv (\lambda z. V)W$  be the  $\beta$ -redex contracted in  $M \rightarrow_{\beta} N$  and  $C(\underline{n})$  the “A-redex” in  $M \rightarrow_A L$ .

Case 1.  $R \cap C(\underline{n}) = \emptyset$  is trivial.

Case 2.  $R \subseteq C(\underline{n})$ . By 4.7(iii) we are done.

Case 3.  $C(\underline{n}) \subset R$ . 3.1:  $C(\underline{n}) \subseteq W$ , is easy. 3.2:  $C(\underline{n}) \subseteq V$ : since  $C(\underline{n})$  is closed this case is trivial. 3.3:  $C(\underline{n}) \equiv \lambda z. V$ . By 4.7(iv),  $C(\underline{n}) \equiv \lambda z. zC'(\underline{n})$  where  $C'(x) \in \text{Red}(Bx)$ ; hence  $N \equiv \dots WC'(\underline{n}) \dots$ ,  $L \equiv \dots C(\underline{0})W \dots \equiv \dots (\lambda z. zC'(\underline{0}))W \dots$ . Take  $P \equiv \dots WC'(\underline{0}) \dots$ .

(ii) Let  $H$  be the  $\Omega$ -redex and  $C(\underline{n})$  the A-redex in  $M$ .

Case 1.  $H \cap C(\underline{n}) = \emptyset$  is trivial.

Case 2.  $H \subseteq C(\underline{n})$  does not occur, by 4.7(i).

Case 3.  $C(\underline{n}) \subset H$ ;  $H \equiv H'[C(\underline{n})]$ ,  $M \equiv \dots H \dots$ ,  $N \equiv \dots \Omega \dots$ ,  $L \equiv \dots H'[C(\underline{0})] \dots$ .

Claim.  $H'[C(\underline{0})]$  is unsolvable. So take  $P \equiv N$  to complete the diagram.

PROOF OF CLAIM.  $C(\underline{n})$  and  $C(\underline{0})$  have the same Böhm-tree, hence are solvably equivalent, i.e. for every context  $D[ ]$  we have:  $D[C(\underline{n})]$  is unsolvable  $\Leftrightarrow D[C(\underline{0})]$  is unsolvable. Now take  $D[ ] \equiv H'[ ]$ .

(iii) Let  $E \equiv \lambda x. Fx$  be the  $\eta$ -redex and  $C(\underline{n})$  be the A-redex in  $M$ .

Case 1.  $E \cap C(\underline{n}) = \emptyset$  is trivial.

Case 2.  $E \subseteq C(\underline{n})$  does not occur, by 4.7(i).

Case 3.  $C(\underline{n}) \subset E$ ; 3.1:  $C(\underline{n}) \equiv Fx$  cannot occur by 4.7(ii). 3.2:  $C(\underline{n}) \subseteq F$ : easy.  $\square$

4.10. LEMMA. For  $n \notin A$ ,  $n \neq 0$ ,  $\mathcal{H}\eta A \not\vdash B\underline{n} = B\underline{0}$ .

PROOF. If the equation were provable there would be a term  $Z$  such that  $B\underline{n} \rightarrow_{\beta\eta\Omega}^* Z$  and  $B\underline{0} \rightarrow_{\beta\eta\Omega}^* Z$ . By 4.7(ii) it would follow that  $\underline{n}$  and  $\underline{0}$  would occur at the same place in  $Z$ .  $\square$

4.11. COROLLARY. Let  $0 \notin A$ ,  $A' \subset N$  and  $A \neq A'$ . Then  $\mathcal{H}\eta A \neq \mathcal{H}\eta A'$ .

PROOF. Let  $n \in A$  but  $n \notin A'$ , say. Then  $\mathcal{H}\eta A \vdash B\underline{n} = B\underline{0}$  and  $\mathcal{H}\eta A' \not\vdash B\underline{n} = B\underline{0}$  by 4.10.  $\square$

4.12. THEOREM. There are  $2^{\aleph_0}$  theories between  $\mathcal{H}\eta$  and  $\mathcal{H}^*$ .

PROOF. By 4.10 each  $\mathcal{H}\eta A$  is consistent, hence  $\subset \mathcal{H}^*$  by [2, 5.8]. The result follows from 4.11.  $\square$

4.13. COROLLARY. (i) There are  $2^{\aleph_0}$  sensible hard models.

(ii) There are  $2^{\aleph_0}$  sensible extensional models.

PROOF. Note that for  $\lambda$ -theories  $T, T'$ ,  $\mathcal{M}^{(0)}(T) = \mathcal{M}^{(0)}(T') \Leftrightarrow T = T'$ . The results follow by taking closed, respectively open, term models.  $\square$

**§5. Applications of Gross-reduction.** In the preceding paragraphs we have postponed some technicalities, viz. the proofs of

(1)  $\mathcal{H}\eta \not\vdash H\underline{c}x\underline{0} = H\underline{c}y\underline{0}$  where  $H$  is a term such that  $H\underline{c}x\underline{n} \rightarrow_{\beta}^* [I, \underline{n} (H\underline{c}x\underline{n+1})]$  and  $\underline{n} \rightarrow_{\beta}^* I$  for all  $n$ .

(2)  $\mathcal{H}\eta \not\vdash O\underline{x}\underline{0} = O\underline{y}\underline{0}$  where  $O$  is such that  $O\underline{x}\underline{n} \rightarrow_{\beta}^* \lambda z. z \Omega^{-n} (O\underline{x}\underline{n+1})$ .

(3)  $\mathcal{H}\eta \not\vdash A\underline{x} = A\underline{I}$  where  $A$  is such that  $A\underline{x} \rightarrow_{\beta}^* \lambda z. z (A(\underline{x}\Omega))$ .

In all three cases the proof is similar: if an equation were provable, the terms would have a common reduct by the Church–Rosser theorem for  $\mathcal{H}\eta$ . In order to prove that this is impossible one wants to show that the first term has in each reduct the free variable  $x$  (and it is clear that for no reduct of the second term  $x$  occurs freely in it). The verification of the last statement is still quite intricate, since the reduction trees of the terms involved are quite complicated due to many detour reductions. To overcome this difficulty we use the concept of a (deterministic) Gross-reduction chain which is cofinal in the reduction tree, w.r.t. the relation  $\rightarrow^*$ . This cofinality enables us to reduce properties of the whole reduction tree to the more easily computable Gross-reduction chain.

Gross-reduction can be compared with Kleene’s ‘full computation rule’ for recursive function calculi.

5.1. DEFINITION. The *Gross-contraction* of a term  $M$ , notation  $M^*$ , is the complete reduction of  $M$  w.r.t. all of its redexes.

In [3] it is shown that  $M^*$  is uniquely defined in the sense of being independent of the order chosen for complete reduction of  $M$  w.r.t. all of its redexes.

The Gross-reduction-chain of  $M$  is the sequence  $(M)_0 = M, (M)_{n+1} = (M)_n^*$ .

5.2. LEMMA. For  $\mathcal{H}\eta$  the Gross-reduction-chain of  $M$  is cofinal in the reduction tree of  $M$ .

PROOF. See [3].  $\square$

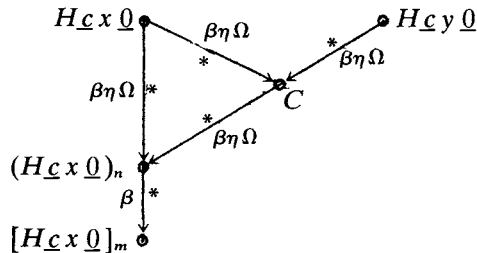
5.3. PROOF OF (1), (2), (3). (1) Define

$$[H\underline{c}x\underline{0}]_n = \underbrace{[I, [I, \dots [I, H\underline{c}x\underline{n}] \dots ]]}_{n \text{ times}}.$$

Simple but tedious calculation shows:

$$(*) \quad (H\underline{c}x\underline{0})_n \xrightarrow{\beta}^* [H\underline{c}x\underline{0}]_m \quad \text{for some } m.$$

Now  $\lambda\beta\eta\Omega \not\vdash H\underline{c}x\underline{0} = H\underline{c}y\underline{0}$ . Suppose not, then by (5.2), (5.3), (\*) we have



for some  $n, m$ . But  $x \in FV([H_{\mathcal{C}}x \underline{0}]_m)$ , hence  $x \in FV(H_{\mathcal{C}}y \underline{0})$ , contradiction.

(2) Define  $[Ox \underline{0}]_n = \lambda y. y(y\Omega(y\Omega(y\Omega(\cdots(y\Omega^{-n}(Ox \underline{n+1})))\cdots)))$ . Then  $(Ox \underline{0})_n \rightarrow_{\beta}^* [Ox \underline{0}]_n$  as direct computation shows. The rest of the proof is entirely analogous to that of (1).

(3)  $Ax \equiv \omega\omega x$ ,  $\omega \equiv \lambda axz. z(aa(x\Omega))$ . Define

$$[Ax]_n = \underbrace{\lambda z. z(\lambda z. z(\cdots(\lambda z. z(\omega\omega(x\Omega^{-n})))\cdots))}_{n \text{ times}}$$

A simple calculation shows  $(Ax)_n \rightarrow_{\beta}^* [Ax]_n$ . The rest of the proof is (almost) analogous to that of (1).  $\square$

## REFERENCES

- [1] H. P. BARENDREGT, *Combinatory logic and the  $\omega$ -rule*, *Fundamenta Mathematicae*, vol. 82 (1974), pp. 199–215.
- [2] ———, *The type free lambda calculus*, *Handbook of mathematical logic*, North-Holland, Amsterdam, 1977 (to appear).
- [3] H. P. BARENDREGT, J. A. BERGSTRA, J. W. KLOP and H. VOLKEN, *Some notes on lambda-reduction*, preprint 22, Department of Mathematics, University of Utrecht.
- [4] H. P. BARENDREGT, *Solvability in  $\lambda$ -calculi*, *Proceedings of the Logic Conference, Clermont-Ferrand, 1975* (M. Guillaume, Editor).
- [5] ———, *The lambda calculus*, North-Holland, Amsterdam (to appear).
- [6] J. M. E. HYLAND, *A survey of some useful partial order relations on terms of the lambda calculus,  $\lambda$ -calculus and computer science theory*, *Springer Lecture Notes in Computer Science*, No. 37 (1975), pp. 83–93.
- [7] G. PLOTKIN, *The  $\lambda$ -calculus is  $\omega$ -incomplete*, this JOURNAL, vol. 39 (1974), pp. 313–317.
- [8] H. ROGERS, *Theory of recursive functions and effective computability*, McGraw-Hill, New York, 1967.
- [9] D. SCOTT, *Continuous lattices*, *Springer Lecture Notes in Mathematics*, No. 274, pp. 97–136.
- [10] J. STAPLES, *Church–Rosser theorems for replacement systems*, *Algebra and logic*, *Springer Lecture Notes in Mathematics*, No. 450 (1975), pp. 291–307.
- [11] C. WADSWORTH, *Degrees associated with  $\lambda$ -calculus concepts*, handwritten note, 1973.
- [12] ———, *The relation between lambda-expressions and their denotations*, *Siam Journal of Computing*, vol. 5 (1976), pp. 488–521.