A residue calculus for root systems

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Abstract

Let V be a finite dimensional real vector space on which a root system Σ is given. Consider a meromorphic function φ on $V_{\mathbb{C}} = V + iV$, the singular locus of which is a locally finite union of hyperplanes of the form $\{\lambda \in V_{\mathbb{C}} \mid \langle \lambda, \alpha \rangle = s\}$, $\alpha \in \Sigma$, $s \in \mathbb{R}$. Assume φ is of suitable decay in the imaginary directions, so that integrals of the form $\int_{\eta + iV} \varphi(\lambda)\, d\lambda$ make sense for generic $\eta \in V$. A residue calculus is developed that allows shifting η. This residue calculus can be used to obtain Plancherel and Paley–Wiener theorems on semisimple symmetric spaces.

Introduction

In several fundamental papers on harmonic analysis related to symmetric spaces or Lie groups, a certain application of the Cauchy theorem plays an important role. In its simplest form, the idea is present already in the proof of the Paley–Wiener–Schwartz theorem for the Euclidean space (see, for example, [12, p. 182]), where the integral

$$\int_{i\mathbb{R}^n} e^{\langle x, \xi \rangle} \psi(\xi)\, d\xi \tag{0.1}$$

over the imaginary space is shifted in a real direction $\eta \in \mathbb{R}^n$ to an integral

$$\int_{\eta + i\mathbb{R}^n} e^{\langle x, \xi \rangle} \psi(\xi)\, d\xi \tag{0.2}$$

over a parallel space. Here, ψ is an entire function on $\mathbb{C}^n$ of exponential type, that is, it satisfies an estimate of the form

$$\sup_{\zeta \in \mathbb{C}^n} (1 + |\zeta|)^k e^{-R\,|\mathrm{Re}\,\zeta|} |\psi(\zeta)| < \infty$$

for some $R > 0$ and all $k \in \mathbb{N}$. It is the polynomial decay at infinity (in the imaginary directions), following from this estimate, that permits the use of Cauchy's theorem to shift the integral (0.1) to (0.2). The shifted integral allows an estimate that is used to show that the (inverse) Fourier–Laplace transform (0.1) of ψ has compact support. The use of such an argument in the context of more general symmetric spaces goes back to Helgason, [11]. Later, Helgason's result was successfully applied by Rosenberg, [16], to give a new proof of the Plancherel theorem for a Riemannian symmetric space. In [3], where we obtain the

Keywords: hyperplane configuration, residue operator, residue weight, root system
Subject classification: Primary: 22E30, Secondary: 32A20

most-continuous part of the Plancherel decomposition for a semisimple symmetric space, an analogous shift of integrals plays a key role.

In other situations in harmonic analysis, the same technique is used with a meromorphic function ψ. Then the shift of integrals results in the appearance of residues, which contribute to lower dimensional spectrum. This is for instance the case in the fundamental work of Selberg and Langlands on automorphic forms ([13], [14]; see also the exposition in [15], in particular Section V.1.5(c)). In the spirit of the classical proof, but with residues appearing, Paley–Wiener theorems are proven in various contexts in [9], [6], and [1]; the analysis in the former two papers is in one complex variable, whereas that of Arthur in the last mentioned paper is in several variables (like in Langlands' work on automorphic forms). In [10], Heckman and Opdam treat the Plancherel decomposition for graded Hecke algebras by a residue calculus in a similar multi-variable setting.

In [4] we employ a multi-variable calculus with residues to obtain an inversion formula for the Fourier transform related to a semisimple symmetric space. The results of [4] will be used in [5] to prove the Paley–Wiener and the Plancherel theorem for these spaces (see the introductions of [4] and [8] for more details, and for references to related work by other people).

In the present paper we prepare the ground for [4] and [5] by developing the necessary residue calculus. The basic tool is the one–variable residue theorem. In order to apply it in the multidimensional setting with root systems, some geometric and combinatorial problems have to be solved. It is the treatment of these problems that is the essential purpose of this paper. The calculus is formulated entirely in terms of root systems, without any reference to (analysis on) semisimple symmetric spaces, but the scope of theory is naturally directly motivated by the intended application. We believe there may be other applications than the one we have in mind, and that the calculus is therefore of independent interest. This is our motivation for presenting this part of the program [2], [3], [4], [5] in a separate paper.

The main result is stated in Theorem 3.16 and Corollary 3.18. In the application the left-hand side of the equation (3.26) in Corollary 3.18 corresponds to a so called pseudo wave packet. It is the formation of the pseudo wave packet that is shown in [4] to invert the Fourier transform. The terms in the right-hand side of (3.26) then constitute the contributions of the several generalized principal series to the Plancherel decomposition.

Besides Theorem 3.16, there are several features of the paper that are crucial for the application, and that also add new insight to the cases of the previously cited papers by Langlands, Arthur, Heckman and Opdam. First of all, the residues are obtained by operators that are defined independently of choices (Theorem 1.13). This was already observed by Heckman and Opdam in their case. These operators are naturally represented in a certain projective limit space (Section 1.3). Another noteworthy result is the support theorem (Theorem 3.15). The proof of this theorem demands some quite delicate combinatorial and geometric arguments (given in Section 2). The theorem is the key to the Plancherel theorem; as will be seen in [5] it follows from this support theorem that the individual contributions in (3.26) are of tempered behavior. The concept of a residue weight (which will be explained below) is introduced to facilitate some of the involved combinatorics. Together with the transitivity theorem (Theorem 3.14) it is motivated by the induction that takes place in [4]. The Weyl group invariance (Section 3.5) contributes to a proper understanding of the Maass–Selberg relations, as will be discussed in [5].

We shall end this introduction by giving an outline of the paper, at the same time further explaining some of the motivating ideas.

Throughout the paper, V is a finite dimensional real linear space, equipped with an inner product $\langle \cdot\,, \cdot \rangle$, and $V_{\mathbb{C}}$ denotes its complexification. We assume that a locally finite collection $\mathcal{H}$ of hyperplanes in V is given, and consider the space $\mathcal{M}(V, \mathcal{H})$ of meromorphic functions on $V_{\mathbb{C}}$ with singular locus contained in the union of the complex hyperplanes $H_{\mathbb{C}}$, $H \in \mathcal{H}$. Let $\mathcal{P}(V, \mathcal{H})$ be the subspace of functions $\varphi \in \mathcal{M}(V, \mathcal{H})$ having polynomial decay along the shifted imaginary space $\eta + iV$, for every η in $\mathrm{reg}(V, \mathcal{H})$, the complement in V of the union of the hyperplanes from $\mathcal{H}$. For $\varphi \in \mathcal{P}(V, \mathcal{H})$ and $\eta \in \mathrm{reg}(V, \mathcal{H})$ we consider the integral

$$\int_{\eta+iV} \varphi \, d\mu_V, \tag{0.3}$$

where $d\mu_V$ denotes the pull back of Lebesgue measure on (the real linear space) iV under the translation $v \mapsto v - \eta$. When η varies in a fixed initial component C of $\mathrm{reg}(V, \mathcal{H})$, the integral in (0.3) is independent of η, by Cauchy's theorem. We shall therefore also write it with $\mathrm{pt}(C)$ in place of η, to indicate that an arbitrary point of C may be taken, without changing the value of the integral. It is of interest to study the behavior of the integral when η is moved to a different component of $\mathrm{reg}(V, \mathcal{H})$.

If L is any affine subspace of V (i.e., a translate of a linear subspace), then by $c(L)$ we denote the central point of L, i.e., the point of L closest to the origin in V. We note that $L = c(L) + V_L$, with V_L a uniquely determined linear subspace of V. We shall call $c(L) + iV_L$ the tempered real form of $L_{\mathbb{C}}$, since in the applications this is the subspace of $L_{\mathbb{C}}$ where tempered spectrum is located.

For the applications it is now of particular interest to move the η in (0.3) as close to 0 (the central point of V) as possible, so that the domain $\eta + iV$ of integration comes close to the tempered real form iV of $V_{\mathbb{C}}$ (this idea is also central in the previously cited work of Langlands and Arthur). In general one cannot move η all the way to the origin 0, since 0 might be contained in $V \setminus \mathrm{reg}(V, \mathcal{H})$, hence in the singular locus of φ. The best one can do here is to move η to one of the (finitely many) central chambers, i.e., the components of $\mathrm{reg}(V, \mathcal{H})$ having the central point 0 in their closure. For the applications it is important not to discriminate between the (central) chambers. With this in mind we introduce, in Section 1.7, the concept of a residue weight. It prescribes for what part of the integral (0.3) the point η is moved to other components of $\mathrm{reg}(V, \mathcal{H})$. On the level of V a residue weight is a function $t\colon \mathrm{comp}(V, \mathcal{H}) \to [0, 1]$ with finite support, and such that $\sum_{C' \in \mathrm{comp}(V, \mathcal{H})} t(C') = 1$. The sum

$$\sum_{C' \in \mathrm{comp}(V, \mathcal{H})} t(C') \int_{\mathrm{pt}(C')+iV} \varphi \, d\mu_V \tag{0.4}$$

may be viewed as a redistribution of the integral (0.3) over the various components of $\mathrm{reg}(V, \mathcal{H})$. If t is supported by the central chambers (such a t is called central), then each non-zero term of the above sum involves a central chamber C'; the point $\mathrm{pt}(C')$ may be chosen arbitrarily close to the central point 0 of V, without changing the value of the corresponding integral. In (0.4) the domains of integration are thus brought as close as possible to the tempered real form iV of $V_{\mathbb{C}}$.

The difference of (0.3) with its weighted redistribution (0.4) can be written as the sum of the integrals $t(C')[\int_{\eta+iV} \varphi d\mu_V - \int_{\mathrm{pt}(C')+iV} \varphi d\mu_V]$. The expression in the square brackets may in turn be rewritten as a sum of residual integrals of the form:

$$\int_{\xi+iV_H} R(\varphi) d\mu_H. \tag{0.5}$$

Here $H \in \mathcal{H}$ is a hyperplane separating η from C'. Moreover, let $\mathcal{H}_H = \{H \cap H' \mid H' \in \mathcal{H},\ \emptyset \neq H \cap H' \subsetneq H\}$ be the hyperplane configuration in H induced by $\mathcal{H}$. Then ξ is a point in $\mathrm{reg}(H, \mathcal{H}_H)$. Finally, R is a linear operator from $\mathcal{P}(V, \mathcal{H})$ to $\mathcal{P}(H, \mathcal{H}_H)$, arising from taking a one variable residue in a variable transversal to H. The operator R is an example of what we call a Laurent operator, since it encodes the procedure of taking a coefficient in a Laurent series expansion transversal to H. Laurent operators are introduced and studied in Section 1.3.

The procedure of rewriting (0.3) as a sum of integrals is now continued as follows. Each of the residual integrals (0.5) is redistributed over chambers of H at the cost of codimension 2 residual integrals. The redistribution over the various chambers in H is prescribed by a residue weight on the level of H (relative to $\mathcal{H}_H$). The codimension 2 residual integrals are redistributed by a similar prescription, and we continue in this fashion until the final step, where point residues in finitely many points of V occur. (In the application these correspond to discrete spectrum.)

We thus end up with the formula of Theorem 1.13, which describes the original integral (0.3) as the following sum of residual integrals:

$$\int_{\mathrm{pt}(C)+iV} \varphi \, d\mu_V = \sum_{L \in \mathcal{L}} \sum_{C' \in \mathrm{comp}(L, \mathcal{H}_L)} t(C') \int_{\mathrm{pt}(C')+iV_L} \mathrm{Res}_L^{C,\,t} \varphi \, d\mu_L. \tag{0.6}$$

Here $\mathcal{L}$ denotes the collection of non-empty intersections of hyperplanes from $\mathcal{H}$; for each $L \in \mathcal{L}$ the induced collection of hyperplanes in L is denoted by $\mathcal{H}_L$, and the associated set of connected components of $\mathrm{reg}(L, \mathcal{H}_L)$ by $\mathrm{comp}(L, \mathcal{H}_L)$. Finally, $\mathrm{Res}_L^{C,t}$ is a Laurent operator from $\mathcal{P}(V, \mathcal{H})$ to $\mathcal{P}(L, \mathcal{H}_L)$. It is of crucial importance that the occurring Laurent operators $\mathrm{Res}_L^{C,t}$ are uniquely determined by the formula (0.6); as mentioned, this observation goes back to Heckman and Opdam [10]. We call these uniquely determined operators the residue operators associated with the initial data $\mathcal{H}, C$ and the residue weight t.

Thus we see that, as in the theory of automorphic forms, the residue operators essentially arise as compositions of one variable residues (in variables transversal to singular hyperplanes). However, since the characterization by (0.6) determines the residue operators uniquely, it is clear from the start that it is of no importance in which order the compositions are taken. This seems to distinguish the calculus of [10] and the present paper from that of Langlands [14] and Arthur [1]. It is the uniqueness of the residue operators that makes it possible to develop a full residual calculus. We end Section 1 with discussing properties of the residue operators needed in the later sections.

In Section 2 we study the residual support of an initial chamber $C \in \mathrm{comp}(V, \mathcal{H})$, i.e., the collection of $L \in \mathcal{L}$ such that the associated residual operator $\mathrm{Res}_L^{C,t}$ is non-zero. The purpose is to prepare for the support theorem, Theorem 3.15.

In Section 3 we specialize the theory developed so far to hyperplane configurations related to a root system Σ in V. Let $\mathcal{H}_\Sigma$ be the collection of all hyperplanes H in V with $V_H = \alpha^\perp$ for some $\alpha \in \Sigma$, and let $\mathcal{L}_\Sigma$ be the collection of all non-empty intersections of hyperplanes from $\mathcal{H}_\Sigma$. We now consider a locally finite affine hyperplane configuration $\mathcal{H}$ that is Σ-admissible, i.e., $\mathcal{H} \subset \mathcal{H}_\Sigma$. Moreover, we assume that a positive system Σ^+ is given and that $\mathcal{H}$ is bounded in the anti-dominant direction in the sense that the inner products $\langle \alpha, c(H) \rangle$, for $\alpha \in \Sigma^+$ and $H \in \mathcal{H}$, are uniformly bounded from below. Such $\mathcal{H}$ occur as sets of singular hyperplanes in the applications. Moreover, it is natural to choose as initial chamber C the unique component of $\mathrm{reg}(V, \mathcal{H})$ on which every positive root is unbounded from below.

Of particular interest is the hyperplane configuration $\mathcal{H}_\Sigma(0)$ consisting of the hyperplanes from $\mathcal{H}_\Sigma$ containing 0. In other words, $\mathcal{H}_\Sigma(0)$ is the collection of root hyperplanes. The associated collection $\mathcal{L}_\Sigma(0)$ of non-empty intersections is equal to the collection $\mathcal{R}$ of root spaces in V. Given $\mathfrak{b} \in \mathcal{R}$, let $\mathcal{P}(\mathfrak{b})$ be the collection of connected components of $\mathrm{reg}(\mathfrak{b}, \mathcal{H}_\Sigma(0))$. Then V is the disjoint union of the elements of

$$\mathcal{P} = \cup_{\mathfrak{b} \in \mathcal{R}} \mathcal{P}(\mathfrak{b}),$$

also called the Coxeter complex of Σ. (If Σ is the root system of a Cartan subalgebra in a semisimple algebra, then $\mathcal{P}$ is in bijective correspondence with the collection of parabolic subalgebras containing the Cartan subalgebra, whence the notation.) A residue weight on $\mathcal{P}$ is by definition a function $t \colon \mathcal{P} \to [0,1]$ such that $\sum_{Q \in \mathcal{P}(\mathfrak{b})} t(Q) = 1$ for every $\mathfrak{b} \in \mathcal{R}$. In Section 3.4, formula (3.6), we define a residue operator $\mathrm{Res}_L^{P,t}$ associated with data $P \in \mathcal{P}(V)$, t, $L \in \mathcal{L}_\Sigma$. It is universal in the following sense. The chamber P determines the positive system $\Sigma^+ = \Sigma^+(P)$ of roots positive on P. Let $\mathcal{H}$ be any Σ-admissible hyperplane configuration that is bounded relative to Σ^+. The residue weight t naturally induces a central residue weight $\omega(t)$ on $\mathcal{H}$. Proposition 3.6 now expresses that each of the residue operators in (0.6), associated with the data $\mathcal{H}$, C and $\omega(t)$ (where C is the initial chamber), is equal to one of the universal residue operators $\mathrm{Res}_L^{P,t}$.

An important feature of the universal residue operator is that it has transitivity properties reflecting parabolic induction. The main result in this direction, Theorem 3.14, essentially expresses that every residue operator equals a point residue operator associated with a sub root system of Σ. This transitivity is of crucial importance for the applications to analysis, since it allows induction as a method of proof.

In the main result of the present paper, Theorem 3.16, formula (0.6) is reformulated in terms of the universal residue operators. Via Weyl group conjugations Theorem 3.16 may be reformulated as Corollary 3.18. As mentioned above, this corollary is applied directly in [4] and [5]; it gives the Plancherel decomposition of a pseudo wave packet. The first summation in formula (3.26) extends over the subsets F of Δ, the collection of simple roots in Σ^+. Each subset F determines a so called standard σ-parabolic subgroup P_F. The sum of terms in (3.26) with F fixed corresponds with the contribution to the Plancherel decomposition of the generalized principal series associated with P_F.

1 The residue scheme

Let V be a finite dimensional real linear space, equipped with an inner product $\langle\,\cdot\,,\,\cdot\,\rangle$, and let $V_{\mathbb{C}}$ denote the complexification of V, equipped with the complex bilinear extension of $\langle\,\cdot\,,\,\cdot\,\rangle$. Let $i \in \mathbb{C}$ be the imaginary unit. We shall often regard $V_{\mathbb{C}}$ as the Cartesian product of its real subspaces V and iV.

1.1 The singular configuration. By an affine subspace of V we mean any translate of a real linear subspace of V. Thus, if A is an affine subspace, there exists a unique linear subspace $V_A \subset V$ such that $A = a + V_A$ for all $a \in A$. The unique point in A with minimal distance to the origin is called the central point of A and is denoted by $c(A)$. Note that we have $A = c(A) + V_A$ and $c(A) \perp V_A$. We agree to call $A_{\mathbb{C}} := c(A) + (V_A)_{\mathbb{C}} \subset V_{\mathbb{C}}$ the complexification of A. For $a \in V_{\mathbb{C}}$ let $T_a\colon V_{\mathbb{C}} \to V_{\mathbb{C}}$ be given by $\lambda \mapsto \lambda + a$, then $T_{c(A)}\colon (V_A)_{\mathbb{C}} \to A_{\mathbb{C}}$ is an affine isomorphism mapping V_A onto A. Via this isomorphism we equip A and $A_{\mathbb{C}}$ with the structure of a real, resp. complex, linear space. Moreover, we equip these spaces with the inner product obtained from the restriction of $\langle\,\cdot\,,\,\cdot\,\rangle$ to V_A, resp. $(V_A)_{\mathbb{C}}$. We denote by $\mathcal{A}$ the collection of affine subspaces of V.

An affine subspace A of V, such that the codimension of V_A in V is one, is called an affine hyperplane; a locally finite collection of affine hyperplanes is called an affine hyperplane configuration. Let such a configuration $\mathcal{H}$ be given. We shall assume that for every $H \in \mathcal{H}$ a non-zero vector α_H in the one-dimensional space $V_H^{\perp}$ is chosen. Moreover, we define the first degree polynomial $\ell_H\colon V_{\mathbb{C}} \to \mathbb{C}$, by

$$\ell_H(\lambda) = \langle \alpha_H, \lambda - c(H) \rangle; \tag{1.1}$$

then H and $H_{\mathbb{C}}$ are the null sets of ℓ_H in V and $V_{\mathbb{C}}$, respectively. We call the elements of the set $\mathrm{sing}(V_{\mathbb{C}}, \mathcal{H}) := \cup_{H\in\mathcal{H}} H_{\mathbb{C}}$ the singular elements; those of its complement $\mathrm{reg}(V_{\mathbb{C}}, \mathcal{H})$ in $V_{\mathbb{C}}$ are the regular elements. We define the subsets $\mathrm{sing}(V, \mathcal{H}), \mathrm{reg}(V, \mathcal{H}) \subset V$ similarly.

Let $\mathbb{N}^{\mathcal{H}}$ denote the space of maps $\mathcal{H} \to \mathbb{N} = \{0, 1, 2, \ldots\}$, and let $d \in \mathbb{N}^{\mathcal{H}}$. If $K \subset V$ is a compact subset we define a polynomial π_K on $V_{\mathbb{C}}$ by

$$\pi_K = \pi_{K,d} = \prod_{H\in\mathcal{H},\, H\cap K\neq\emptyset} \ell_H^{d(H)} \tag{1.2}$$

(if $H \cap K = \emptyset$ for all $H \in \mathcal{H}$ we let $\pi_K = 1$). We denote by $\mathcal{M}(V, \mathcal{H}, d)$ the space of meromorphic functions $\varphi\colon V_{\mathbb{C}} \to \mathbb{C}$ such that for every compact subset $K \subset V$ the function $\pi_{K,d}\varphi$ is holomorphic on an open neighborhood of $K \times iV$. Observe that $\mathcal{M}(V, \mathcal{H}, d)$ is independent of the choice of the normal vectors α_H, $H \in \mathcal{H}$. The functions in $\mathcal{M}(V, \mathcal{H}, d)$ are holomorphic on the open set $\mathrm{reg}(V_{\mathbb{C}}, \mathcal{H})$, which is connected and dense in $V_{\mathbb{C}}$.

We equip $\mathbb{N}^{\mathcal{H}}$ with the ordering $\preceq$ defined by $d \preceq d'$ if and only if $d(H) \leq d'(H)$ for all $H \in \mathcal{H}$. Then we have $\mathcal{M}(V, \mathcal{H}, d) \subset \mathcal{M}(V, \mathcal{H}, d')$ when $d \preceq d'$. We now define

$$\mathcal{M}(V, \mathcal{H}) = \cup_{d\in\mathbb{N}^{\mathcal{H}}} \mathcal{M}(V, \mathcal{H}, d).$$

Let $L \in \mathcal{A}$. We define

$$\mathcal{H}(L) = \{H \in \mathcal{H} \mid H \supset L\}, \tag{1.3}$$

and

$$\mathcal{H}_L = \{H \cap L \mid H \in \mathcal{H} \setminus \mathcal{H}(L),\ H \cap L \neq \emptyset\}. \tag{1.4}$$

These are affine hyperplane configurations in V and L, respectively, hence we may define the spaces $\mathcal{M}(V, \mathcal{H}(L))$ and $\mathcal{M}(L, \mathcal{H}_L)$ as above. Notice that $\mathcal{H}(L)$ is finite and that $\mathcal{M}(V, \mathcal{H}(L)) \subset \mathcal{M}(V, \mathcal{H})$.

For $d \in \mathbb{N}^{\mathcal{H}}$ (or $d \in \mathbb{N}^{\mathcal{H}(L)}$) let q_L be the polynomial on $V_{\mathbb{C}}$ defined by

$$q_L = q_{L,d} = \prod_{H \in \mathcal{H}(L)} \ell_H^{d(H)}. \tag{1.5}$$

In particular we have $q_V = 1$, and $q_H = \ell_H^{d(H)}$ for $H \in \mathcal{H}$. We observe that $q_L \varphi$ is holomorphic on a neighborhood of $\mathrm{reg}(L_{\mathbb{C}}, \mathcal{H}_L)$ for all $\varphi \in \mathcal{M}(V, \mathcal{H}, d)$, and that $\varphi \mapsto q_L \varphi$ maps the subspace $\mathcal{M}(V, \mathcal{H}(L), d|_{\mathcal{H}(L)})$ of $\mathcal{M}(V, \mathcal{H}, d)$ bijectively onto the space $\mathcal{O}(V_{\mathbb{C}})$ of entire functions on $V_{\mathbb{C}}$, for all $L \in \mathcal{A}$.

1.2 Residues. Let $V, \mathcal{H}$ be as above, and let $\varphi \in \mathcal{M}(V, \mathcal{H})$, $H \in \mathcal{H}$. For $\lambda \in \mathrm{reg}(H_{\mathbb{C}}, \mathcal{H}_H)$ let ψ_λ denote the meromorphic function $z \mapsto \varphi(\lambda + z\alpha_H/|\alpha_H|)$ on $\mathbb{C}$. We define the residue $\mathrm{Res}_H^V \varphi$ of φ along H to be the function $\mathrm{reg}(H_{\mathbb{C}}, \mathcal{H}_H) \to \mathbb{C}$ given by

$$\mathrm{Res}_H^V \varphi(\lambda) = 2\pi \,\mathrm{Res}_{z=0}\, \psi_\lambda(z) = \int_{C_\epsilon} \varphi(\lambda + z\frac{\alpha_H}{|\alpha_H|})\frac{dz}{i},$$

where C_ϵ is the positively oriented circle in $\mathbb{C}$ of center 0 and sufficiently small radius $\epsilon > 0$. Notice that the residue depends only on the normal vector α_H through its orientation: If the orientation is changed then $\mathrm{Res}_H^V \varphi$ changes by a factor -1.

Let $S(V)$ denote the symmetric algebra of $V_{\mathbb{C}}$. We shall view its elements as constant coefficient holomorphic differential operators on $V_{\mathbb{C}}$ in the usual fashion, that is, via the homomorphism induced by viewing the elements of $V_{\mathbb{C}}$ as constant vector fields on $V_{\mathbb{C}}$. The real subalgebra of $S(V)$ generated by V (and 1) is denoted $S_{\mathbb{R}}(V)$; its elements are called the real elements in $S(V)$.

Lemma 1.1. *Let $d \in \mathbb{N}^{\mathcal{H}}$, $\varphi \in \mathcal{M}(V, \mathcal{H}, d)$, and $H \in \mathcal{H}$. Then*

$$\mathrm{Res}_H^V \varphi(\lambda) = \frac{2\pi}{(d(H)-1)!\,|\alpha_H|^{2d(H)-1}} \alpha_H^{d(H)-1}(q_H\varphi)(\lambda), \qquad (\lambda \in \mathrm{reg}(H_{\mathbb{C}}, \mathcal{H}_H)).$$

Proof. Fix $\lambda \in \mathrm{reg}(H_{\mathbb{C}}, \mathcal{H}_H)$ and let $\psi_\lambda(z)$ be as above. We have $\ell_H(\lambda + z\alpha_H/|\alpha_H|) = z|\alpha_H|$, and hence

$$z^{d(H)}\psi_\lambda(z) = |\alpha_H|^{-d(H)}(q_H\varphi)(\lambda + z\frac{\alpha_H}{|\alpha_H|}).$$

Thus we see that ψ_λ has a pole of order at most $d(H)$ at 0, and hence

$$\begin{aligned}\mathrm{Res}_{z=0}\, \psi_\lambda(z) &= \frac{1}{(d(H)-1)!}\left(\frac{d}{dz}\right)^{d(H)-1}[z^{d(H)}\psi_\lambda(z)]_{z=0} \\ &= \frac{1}{(d(H)-1)!}|\alpha_H|^{-2d(H)+1}\alpha_H^{d(H)-1}(q_H\varphi)(\lambda),\end{aligned}$$

and the lemma follows. $\square$

Lemma 1.2. *Let $d \in \mathbb{N}^{\mathcal{H}}$, $L \in \mathcal{A}$, and $u \in S(V)$. There exists an element $d' \in \mathbb{N}^{\mathcal{H}_L}$ such that $u(q_L\varphi)|_{L_{\mathbb{C}}} \in \mathcal{M}(L, \mathcal{H}_L, d')$ for all $\varphi \in \mathcal{M}(V, \mathcal{H}, d)$.*

Proof. For each $H' \in \mathcal{H}_L$ we have

$$\mathcal{H}(H') = \{H \in \mathcal{H} \mid H \supset H'\} \supsetneq \mathcal{H}(L).$$

Let $n = \deg(u)$ and let $d' \in \mathbb{N}^{\mathcal{H}_L}$ be defined by

$$d'(H') = (n+1) \sum_{H \in \mathcal{H}(H') \setminus \mathcal{H}(L)} d(H), \qquad (H' \in \mathcal{H}_L). \tag{1.6}$$

We claim that the result holds for this d'.

We assume that for each $H' \in \mathcal{H}_L$ a normal vector in $V_L \cap V_{H'}^{\perp}$ has been chosen, and that a corresponding first order polynomial $\ell_{H'}^L \colon L_{\mathbb{C}} \to \mathbb{C}$ is defined (cf. (1.1)), such that $H'_{\mathbb{C}} = (\ell_{H'}^L)^{-1}(0)$. We observe that $\ell_{H'}^L$ is proportional to $\ell_H|_{L_{\mathbb{C}}}$ by a non-zero real constant, for every $H \in \mathcal{H}(H') \setminus \mathcal{H}(L)$. If $K \subset L$ is compact we define

$$\pi_K^L = \prod_{H' \in \mathcal{H}_L,\, H' \cap K \neq \emptyset} (\ell_{H'}^L)^{d'(H')} \colon L_{\mathbb{C}} \to \mathbb{C}; \tag{1.7}$$

our claim then amounts to $\pi_K^L u(q_L\varphi)$ being holomorphic on an open neighborhood of $K \times iV_L$ in $L_{\mathbb{C}}$ for all $\varphi \in \mathcal{M}(V, \mathcal{H}, d)$.

Let

$$p = \prod_{H \in \mathcal{H} \setminus \mathcal{H}(L),\, H \cap K \neq \emptyset} \ell_H^{d(H)} \colon V_{\mathbb{C}} \to \mathbb{C},$$

then $\pi_K^L = cp^{n+1}|_{L_{\mathbb{C}}}$ for some non-zero constant $c \in \mathbb{R}$. Moreover, if $K \neq \emptyset$ we have $\pi_K = p\, q_L$, where π_K and q_L are given in (1.2) and (1.5). On $\mathrm{reg}(L_{\mathbb{C}}, \mathcal{H}_L)$ we now have, by the Leibniz rule of differentiations,

$$u(q_L\varphi) = u(p^{-1}\pi_K\varphi) = p^{-(n+1)} \sum_j q_j u_j(\pi_K\varphi) \tag{1.8}$$

for some polynomials q_j on $L_{\mathbb{C}}$ and some $u_j \in S(V)$, and the claimed property of $\pi_K^L u(q_L\varphi)$ follows. $\square$

From Lemmas 1.1 and 1.2 we immediately obtain:

Corollary 1.3. *Let $d \in \mathbb{N}^{\mathcal{H}}$, $H \in \mathcal{H}$. There exists $d' \in \mathbb{N}^{\mathcal{H}_H}$ such that Res_H^V maps $\mathcal{M}(V, \mathcal{H}, d)$ into $\mathcal{M}(H, \mathcal{H}_H, d')$.*

1.3 Laurent operators. Let $L \in \mathcal{A}$. We call a linear map $R\colon \mathcal{M}(V, \mathcal{H}) \to \mathcal{M}(L, \mathcal{H}_L)$ a Laurent operator if there exists, for each $d \in \mathbb{N}^{\mathcal{H}}$, an element u_d in $S(V_L^{\perp})$ such that

$$R\varphi = u_d(q_{L,d}\,\varphi)|_{L_{\mathbb{C}}} \tag{1.9}$$

for all $\varphi \in \mathcal{M}(V, \mathcal{H}, d)$. Here $q_{L,d}$ is defined in (1.5). A Laurent operator is called real if it can be realized as above with u_d real for all d. In particular, if $V_{\mathbb{C}} = \mathbb{C}$ and L is a point, then a (real) Laurent operator is a map, that associates to a meromorphic function a finite (real) linear combination of the coefficients of its Laurent series at this point. We denote by $\mathrm{Laur}(V, L, \mathcal{H})$, resp. $\mathrm{Laur}_{\mathbb{R}}(V, L, \mathcal{H})$, the space of Laurent operators, resp. real Laurent operators, from $\mathcal{M}(V, \mathcal{H})$ to $\mathcal{M}(L, \mathcal{H}_L)$. Notice that $\mathrm{Laur}(V, V, \mathcal{H}) = \mathbb{C}$; the only Laurent operators from $\mathcal{M}(V, \mathcal{H})$ to itself are the constants times the identity operator. It follows from Lemma 1.1 and Corollary 1.3 that $\mathrm{Res}_H^V \in \mathrm{Laur}_{\mathbb{R}}(V, H, \mathcal{H})$, for $H \in \mathcal{H}$.

Notice that the notion of a Laurent operator is independent of the choice of the normal vectors α_H for $H \in \mathcal{H}$. We now fix such a choice. Then the following lemma shows that for a given Laurent operator R, the elements $u_d \in S(V_L^{\perp})$ in (1.9) are unique. Moreover, u_d only depends on d through its restriction to $\mathcal{H}(L)$. We denote by u_R the family $(u_d)_{d \in \mathbb{N}^{\mathcal{H}(L)}}$ of elements from $S(V_L^{\perp})$.

Lemma 1.4. *Let $d \in \mathbb{N}^{\mathcal{H}(L)}$ and $u \in S(V_L^{\perp})$ be given. If $u(q_{L,d}\,\varphi)|_{L_{\mathbb{C}}} = 0$ for all $\varphi \in \mathcal{M}(V, \mathcal{H}(L), d)$ then $u = 0$.*

Proof. Since $q_{L,d}^{-1}\psi \in \mathcal{M}(V, \mathcal{H}(L), d)$ for $\psi \in \mathcal{O}(V_{\mathbb{C}})$, we have $u\psi|_{L_{\mathbb{C}}} = 0$ for all such functions ψ. The space $\mathcal{O}(V_{\mathbb{C}})$ is translation invariant, and so is the differential operator u, hence we conclude that $u\psi = 0$ on $V_{\mathbb{C}}$, for all $\psi \in \mathcal{O}(V_{\mathbb{C}})$. This implies $u = 0$. $\square$

It will be useful to have identified exactly those families $u = (u_d)_{d \in \mathbb{N}^{\mathcal{H}(L)}}$ of elements from $S(V_L^{\perp})$ that occur as u_R for some Laurent operator $R \in \mathrm{Laur}(V, L, \mathcal{H})$ (clearly, u_R determines R). For this purpose we need the following definitions.

Let V be a real linear space, and let X be a finite (possibly empty) collection of complex non-zero linear functionals on V. For $d \in \mathbb{N}^X$ we define the homogeneous polynomial function $\varpi_{X,d}\colon V_{\mathbb{C}} \to \mathbb{C}$ by

$$\varpi_{X,d} = \prod_{\xi \in X} \xi^{d(\xi)}$$

(if $X = \emptyset$ we let $\varpi_{X,d} = 1$). Let $\preceq$ be the partial ordering on $\mathbb{N}^X$ defined by $d' \preceq d$ if and only if $d'(\xi) \leq d(\xi)$ for all $\xi \in X$. For d', d with $d' \preceq d$ we define $d - d' \in \mathbb{N}^X$ componentwise by differences as suggested by the notation. Then $\varpi_{X,d} = \varpi_{X,d'}\varpi_{X,d-d'}$. It follows from the Leibniz rule that given $u \in S(V)$ there exists an element $u' \in S(V)$ such that

$$u(\varpi_{X,d-d'}\varphi)(0) = u'\varphi(0)$$

for all germs φ of holomorphic functions at 0 on $V_{\mathbb{C}}$. Clearly u' is unique; we denote it by $j_{d',d}(u)$ (in fact, it only depends on $d - d'$). It is also clear that $j_{d'',d'} \circ j_{d',d} = j_{d'',d}$ if $d'' \preceq d' \preceq d$. We now define the space $S_{\leftarrow}(V, X)$ as the projective limit

$$S_{\leftarrow}(V, X) = \varprojlim\, (S(V), j_{\cdot}). \tag{1.10}$$

By definition, this is the space of all families $(u_d)_{d\in\mathbb{N}^X}$ of elements in $S(V)$, that are directed with respect to the maps $j_{d',d}$, that is, satisfy

$$u_{d'} = j_{d',d}(u_d) \tag{1.11}$$

for all d', d with $d' \preceq d$.

Let us now return to the situation that $\mathcal{H}$ is a hyperplane configuration in V and $L \in \mathcal{A}$. Let

$$X(L) = \{\alpha_H \mid H \in \mathcal{H}(L)\}.$$

Via the inner product on V we identify the elements of $X(L)$ with linear functionals on $V_L^\perp$, and via the bijection $\mathcal{H}(L) \to X(L)$ we identify $\mathbb{N}^{X(L)}$ with $\mathbb{N}^{\mathcal{H}(L)}$. Then

$$q_{L,d}(\lambda + \nu) = \varpi_{X(L),d}(\nu), \tag{1.12}$$

for all $\lambda \in L_{\mathbb{C}}$, $\nu \in V_{L\mathbb{C}}^\perp$, $d \in \mathbb{N}^{\mathcal{H}(L)}$. Hence for $u \in S(V_L^\perp)$ and $d' \preceq d$ we have

$$u(q_{L,d}\varphi)|_{L_{\mathbb{C}}} = j_{d',d}(u)(q_{L,d'}\varphi)|_{L_{\mathbb{C}}} \tag{1.13}$$

for all functions φ, that are defined and meromorphic on a neighborhood of $L_{\mathbb{C}}$ and for which $q_{L,d'}\varphi$ is regular on $\mathrm{reg}(L_{\mathbb{C}}, \mathcal{H}_L)$. In particular, if $R\colon \mathcal{M}(V, \mathcal{H}) \to \mathcal{M}(L, \mathcal{H}_L)$ is a Laurent operator, then the family $u_R = (u_d)_{d\in\mathbb{N}^{\mathcal{H}(L)}}$ satisfies (1.9), hence

$$u_d(q_{L,d}\varphi)|_{L_{\mathbb{C}}} = u_{d'}(q_{L,d'}\varphi)|_{L_{\mathbb{C}}}$$

for $d' \preceq d$ and $\varphi \in \mathcal{M}(V, \mathcal{H}, d')$, and we conclude from (1.13) and Lemma 1.4 that (1.11) holds. Hence $u_R \in S_{\leftarrow}(V_L^\perp, X(L))$.

Lemma 1.5. *The map $R \mapsto u_R$ is a linear isomorphism* $\mathrm{Laur}(V, L, \mathcal{H}) \xrightarrow{\sim} S_{\leftarrow}(V_L^\perp, X(L))$.

Proof. Only the surjectivity remains to be seen. Let $u = (u_d)_{d\in\mathbb{N}^{\mathcal{H}(L)}} \in S_{\leftarrow}(V_L^\perp, X(L))$. For each $d \in \mathbb{N}^{\mathcal{H}}$ we let $u_d := u_{d|_{\mathcal{H}(L)}}$ and define $R = R_u\colon \mathcal{M}(V, \mathcal{H}, d) \to \mathcal{M}(L, \mathcal{H}_L)$ by (1.9) for $\varphi \in \mathcal{M}(V, \mathcal{H}, d)$ (cf. Lemma 1.2). It follows easily from (1.11) and (1.13) that R is well defined on $\mathcal{M}(V, \mathcal{H})$. That R belongs to $\mathrm{Laur}(V, L, \mathcal{H})$ and satifies $u = u_R$ is then obvious. $\square$

We call $S_{\leftarrow}(V_L^\perp, X(L))$ the projective limit model for $\mathrm{Laur}(V, L, \mathcal{H})$. In what follows we shall sometimes identify objects in $\mathrm{Laur}(V, L, \mathcal{H})$ and its model by means of the isomorphism in Lemma 1.5. In particular, since $S_{\leftarrow}(V_L^\perp, X(L))$ only depends on $\mathcal{H}$ through $\mathcal{H}(L)$, it follows that $\mathrm{Laur}(V, L, \mathcal{H}) \simeq \mathrm{Laur}(V, L, \mathcal{H}(L))$.

Example 1.6. Let $V \simeq \mathbb{R}$, let $\xi \in V \setminus \{0\}$, let $x \in V^*$ be defined by $x(\xi) = 1$, and finally let $X = \{x\}$. We use the canonical identification $S(V) \simeq \mathbb{C}[\xi]$. It is easily seen that the map $j_{d,d+1}\colon S(V) \to S(V)$ for each $d \in \mathbb{N}$ is the map $u \mapsto u'$ that maps a polynomial $u \in \mathbb{C}[\xi]$ to its derivative. Hence $S_{\leftarrow}(V, X)$ is the space of all sequences $(u_d)_{d\in\mathbb{N}}$ of polynomials $u_d \in \mathbb{C}[\xi]$, for which $u'_{d+1} = u_d$ for all $d \in \mathbb{N}$. For example, let $l \in \mathbb{Z}$ be fixed, then $S_{\leftarrow}(V, X)$ contains the sequence $r^l = (r^l_d)_{d\in\mathbb{N}}$ defined by $r^l_d = (d-l)!^{-1}\xi^{d-l}$ for $d \geq \max(0, l)$ and $r^l_d = 0$ for $\max(0, l) > d \geq 0$.

Let $q \in V$, $L = \{q\}$, and assume that L belongs to the hyperplane configuration $\mathcal{H}$. The Laurent operator $R_u \in \mathrm{Laur}(V, L, \mathcal{H})$ corresponding to a sequence $(u_d)_{d\in\mathbb{N}} \in S_{\leftarrow}(V, X)$ is given by $\varphi \mapsto u_d(\partial_x)((x-q)^d\varphi)(q)$ for $\varphi \in \mathcal{M}(V, \mathcal{H})$ and d sufficiently large. For example, the Laurent operator that corresponds to the sequence r^l just defined is given by $\varphi \mapsto (d-l)!^{-1}(\partial_x)^{d-l}((x-q)^d\varphi)(q)$ for d sufficiently large, which is the operator that maps φ to the coefficient of $(x-q)^{-l}$ in its Laurent expansion at q.

On the other hand, if $L = \{q\} \notin \mathcal{H}$, then $\mathcal{H}(L) = \emptyset$ and $\mathbb{N}^{\mathcal{H}(L)}$ has just one element. Hence $S_{\leftarrow}(V, X(L)) = S(V)$, and the Laurent operator that corresponds to a polynomial $u \in \mathbb{C}[\xi]$ is given by $\varphi \mapsto u(\partial_x)(\varphi)(q)$.

The family of Laurent operators is relatively large. This is illustrated by the previous example as well as the following lemma.

Lemma 1.7. *Let $d' \in \mathbb{N}^X$.*

(i) *The map $j_{d',d}\colon S(V) \to S(V)$ is surjective for all $d \succeq d'$.*

(ii) *The canonical map $u \mapsto u_{d'}$ from $S_{\leftarrow}(V, X)$ to $S(V)$ is surjective.*

Proof. (i) Since $j_{d'',d'} \circ j_{d',d} = j_{d'',d}$ if $d'' \preceq d' \preceq d$, it suffices to prove the surjectivity for the case when $d(\xi) = d'(\xi)$ for all elements $\xi \in X$ except a given one, for which $d(\xi) = d'(\xi) + 1$. Assume that this is the case, and let $\xi \in X$ be this given element. Then $\varpi_{X,d-d'} = \xi$. Furthermore, let $u' \in S(V)$ be given. By linearity of $j_{d',d}$ we may assume that u' is of the form $u' = u''\alpha_\xi^k$ with $k \in \mathbb{N}$ and $u'' \in S(\xi^\perp)$, where $\alpha_\xi \in V$ is determined by $\xi = \langle \alpha_\xi, \cdot\rangle$. It is then seen from the Leibniz rule that $(u'\alpha_\xi)(\xi\varphi)(0) = (k+1)|\xi|^2 u'(\varphi)(0)$. Hence $j_{d',d}(u'\alpha_\xi) = (k+1)|\xi|^2 u'$, from which the asserted surjectivity of $j_{d',d}$ follows.

(ii) Let $u' \in S(V)$ be given. For $k \in \mathbb{N}$ let $d_k \in \mathbb{N}^X$ be given by $d_k(\xi) = d'(\xi) + k$ for each $\xi \in X$. By (i) we can successively choose elements $u_0, u_1, \ldots \in S(V)$, such that $u_0 = u'$ and $j_{d_{k-1},d_k}(u_k) = u_{k-1}$ for all $k \geq 1$. For arbitrary $d \in \mathbb{N}^X$ we now define $u_d \in S(V)$ as follows. For k sufficiently large we have $d \preceq d_k$. Let $u_d = j_{d,d_k}(u_k)$. It is easily seen that u_d is well-defined and that the string $(u_d)_{d\in\mathbb{N}^X}$ belongs to $S_{\leftarrow}(V, X)$. The surjectivity follows, since $u_{d'} = u_{d_0} = u_0 = u'$. $\square$

1.4 Composition of Laurent operators. Let $L, L' \in \mathcal{A}$ with $L' \subset L$. It is easily seen that $\mathcal{H}_{L'} = (\mathcal{H}_L)_{L'}$.

Lemma 1.8. *Let $R\colon \mathcal{M}(V, \mathcal{H}) \to \mathcal{M}(L, \mathcal{H}_L)$ and $R'\colon \mathcal{M}(L, \mathcal{H}_L) \to \mathcal{M}(L', \mathcal{H}_{L'})$ be Laurent operators. Then $R' \circ R\colon \mathcal{M}(V, \mathcal{H}) \to \mathcal{M}(L', \mathcal{H}_{L'})$ is also a Laurent operator. If R and R' are real, then so is $R' \circ R$.*

Proof. Let $d \in \mathbb{N}^{\mathcal{H}}$. Since R is a Laurent operator there exists $u = u_d \in S(V_L^\perp)$ such that (1.9) holds for $\varphi \in \mathcal{M}(V, \mathcal{H}, d)$. According to Lemma 1.2 and its proof we have $R\varphi \in \mathcal{M}(L, \mathcal{H}_L, d')$ where $d' \in \mathbb{N}^{\mathcal{H}_L}$ is given by (1.6) with n equal to the degree of u. Similarly, since R' is a Laurent operator there exists $u' \in S(V_L \cap V_{L'}^\perp)$ such that $R'\varphi' = u'(q^L_{L',d'}\varphi')|_{L'_\mathbb{C}}$ for $\varphi' \in \mathcal{M}(L, \mathcal{H}_L, d')$, where

$$q^L_{L',d'} = \prod_{H' \in \mathcal{H}_L(L')} (\ell^L_{H'})^{d'(H')}\colon L_\mathbb{C} \to \mathbb{C} \tag{1.14}$$

is the analogue of (1.5), for L' inside L (see also the explanation leading up to (1.7)). Thus we have

$$(R' \circ R)\varphi = u'(q^L_{L',d'}[u(q_{L,d}\,\varphi)]|_{L_\mathbb{C}})|_{L'_\mathbb{C}},$$

and the claim is that there exists $u'' \in S(V_{L'}^\perp)$ such that this equals $u''(q_{L',d}\,\varphi)|_{L'_\mathbb{C}}$

Let $H' \in \mathcal{H}_L$. As mentioned in the proof of Lemma 1.2 we have that $\ell^L_{H'}$ is proportional to $\ell_H|_{L_\mathbb{C}}$ by a non-zero real constant, for all $H \in \mathcal{H}(H') \setminus \mathcal{H}(L)$. It then follows from (1.14) and (1.6) that

$$q^L_{L',d'} = c\Big[\prod_{H \in \mathcal{H}(L') \setminus \mathcal{H}(L)} (\ell_H)^{d(H)}\Big]^{n+1}|_{L_\mathbb{C}}$$

with $c \in \mathbb{R} \setminus \{0\}$. Let $p\colon V_\mathbb{C} \to \mathbb{C}$ denote the polynomial inside the square brackets, and observe that $q_{L',d} = p\,q_{L,d}$. We now have

$$u'(q^L_{L',d'}[u(q_{L,d}\,\varphi)]|_{L_\mathbb{C}})|_{L'_\mathbb{C}} = c\,u'([p^{n+1}u(p^{-1}q_{L',d}\,\varphi)]|_{L_\mathbb{C}})|_{L'_\mathbb{C}} = u''(q_{L',d}\,\varphi)|_{L'_\mathbb{C}}, \tag{1.15}$$

where

$$u'' = c\,u' \circ p^{n+1} \circ u \circ p^{-1}.$$

The latter is a differential operator on $V_\mathbb{C}$ whose coefficients are holomorphic (by the Leibniz rule). Moreover they are invariant under translations in directions of $V_{L'}$, because p is invariant under such translations. Since we take restrictions to $L'_\mathbb{C}$ in (1.15) we can replace u'' in (1.15) by the constant coefficient operator obtained from it by evaluation of its coefficients in any point of $L_\mathbb{C}{}'$, and since u and u' both belong to $S(V_{L'}^\perp)$, so does then u''. It is also seen that if u and u' are real, then so is u''. This completes the proof. □

1.5 Functions with polynomial decay. Let $\mathcal{H}$ be an affine hyperplane configuration, and let $d \in \mathbb{N}^\mathcal{H}$. We denote by $\mathcal{P}(V, \mathcal{H}, d)$ the subspace of $\mathcal{M}(V, \mathcal{H}, d)$ consisting of those functions φ for which

$$\sup_{\lambda \in K \times iV} (1 + |\lambda|)^n |(\pi_{K,d}\varphi)(\lambda)| < \infty \tag{1.16}$$

for every compact subset K of V, and every $n \in \mathbb{N}$ (with $\pi_{K,d}$ defined by (1.2)). Endowed with the collection of seminorms $\nu_{K,n}$ given by the left-hand side of (1.16), the space $\mathcal{P}(V, \mathcal{H}, d)$ becomes a Fréchet space.

Let $\mathcal{F}_V : C_c^\infty(V) \to \mathcal{O}(V_\mathbb{C})$ be the Fourier–Laplace transform, defined by

$$\mathcal{F}_V f(\lambda) = \int_V e^{-\langle \lambda, \nu \rangle} f(\nu)\, d\mu_V(\nu),$$

where $d\mu_V$ is Lebesgue measure on V. This is an isomorphism onto the Paley–Wiener space $\mathrm{PW}(V)$, consisting of all the functions $\psi \in \mathcal{O}(V_c)$ of exponential type, i.e., for which there exists $A > 0$ such that

$$\sup_{\lambda \in V_\mathbb{C}} (1 + |\lambda|)^n e^{-A\,|\mathrm{Re}(\lambda)|} |\psi(\lambda)|$$

is finite for all $n \in \mathbb{N}$. Notice that if $\psi \in \mathrm{PW}(V)$ then the functions $q_{L,d}^{-1}\psi$ belong to $\mathcal{P}(V, \mathcal{H}, d)$, for all $L \in \mathcal{A}$. Exploiting this observation, as in the proof of Lemma 1.4, we can improve that lemma as follows:

Lemma 1.9. *Let $d \in \mathbb{N}^{\mathcal{H}(L)}$ and $u \in S(V_L^\perp)$ be given. If $u(q_{L,d}\,\varphi)|_{L_\mathbb{C}} = 0$ for all $\varphi \in \mathcal{P}(V, \mathcal{H}(L), d)$ then $u = 0$.*

Observe that if $d \preceq d'$ in $\mathbb{N}^{\mathcal{H}}$ then $\mathcal{P}(V, \mathcal{H}, d) \subset \mathcal{P}(V, \mathcal{H}, d')$ with continuous inclusion map. We now define $\mathcal{P}(V, \mathcal{H}) = \cup_{d \in \mathbb{N}^{\mathcal{H}}} \mathcal{P}(V, \mathcal{H}, d)$ and endow this space with the inductive limit of the topologies. We define the spaces $\mathcal{P}(L, \mathcal{H}_L)$ similarly for all $L \in \mathcal{A}$. It follows from Lemma 1.9 that a Laurent operator $R\colon \mathcal{M}(V, \mathcal{H}) \to \mathcal{M}(L, \mathcal{H}_L)$ is uniquely determined by its restriction to $\mathcal{P}(V, \mathcal{H}(L))$.

Lemma 1.10. *Let $L \in \mathcal{A}$ and let $R\colon \mathcal{M}(V, \mathcal{H}) \to \mathcal{M}(L, \mathcal{H}_L)$ be a Laurent operator. Then R maps $\mathcal{P}(V, \mathcal{H})$ continuously into $\mathcal{P}(L, \mathcal{H}_L)$.*

Proof. Let $d \in \mathbb{N}^{\mathcal{H}}$ and let $u = u_d \in S(V_L^\perp)$ be such that (1.9) holds. Then we know from Lemma 1.2 that there exists $d' \in \mathbb{N}^{\mathcal{H}_L}$ such that R maps $\mathcal{M}(V, \mathcal{H}, d)$ into $\mathcal{M}(L, \mathcal{H}_L, d')$. We claim that it maps $\mathcal{P}(V, \mathcal{H}, d)$ continuously into $\mathcal{P}(L, \mathcal{H}_L, d')$. Let $\varphi \in \mathcal{P}(V, \mathcal{H}, d)$, let $K \subset L$ be compact and let π_K^L be given by (1.7). The required estimate for $\pi_K^L u(q_{L,d}\,\varphi)$ now follows from (1.8) and Lemma 1.11, to be proved next. The lemma follows immediately. □

Lemma 1.11. *Let $K \subset V$ be compact. Then there exists a compact set $K' \subset V$, and for every $u \in S(V)$ and $n \in \mathbb{Z}$ a constant $C > 0$ such that*

$$\sup_{\lambda \in K \times iV} (1 + |\lambda|)^n |(u\pi_{K,d}\psi)(\lambda)| \leq C \sup_{\lambda \in K' \times iV} (1 + |\lambda|)^n |(\pi_{K',d}\psi)(\lambda)|$$

for all $d \in \mathbb{N}^{\mathcal{H}}$ and $\psi \in \mathcal{M}(V, \mathcal{H}, d)$.

Proof. Fix the compact set K' such that its interior contains K, and such that it meets only those hyperplanes from $\mathcal{H}$ that already meet K. Then $\pi_{K',d} = \pi_{K,d}$ for all $d \in \mathbb{N}^{\mathcal{H}}$.

Fix linear coordinates $\lambda^1, \ldots, \lambda^m$ on $V_\mathbb{C}$ that are real on V; for $\lambda_0 \in V_\mathbb{C}$, $\epsilon > 0$, let $D(\lambda_0, \epsilon)$ denote the polydisc $\{\lambda \in V_\mathbb{C} \mid \forall j : |\lambda^j - \lambda_0^j| < \epsilon\}$. Fix $\epsilon > 0$ such that $D(\lambda_0, \epsilon) \cap V \subset K'$ for all $\lambda_0 \in K$. Then $D(\lambda_0, \epsilon) \subset K' \times iV$ for every $\lambda_0 \in K \times iV$.

By a standard application of the Cauchy integral formula, we obtain the estimate

$$|(u\pi_{K,d}\psi)(\lambda_0)| \leq C' \sup_{\lambda \in D(\lambda_0, \epsilon)} |(\pi_{K,d}\psi)(\lambda)|$$

for all $d \in \mathbb{N}^{\mathcal{H}}$, $\psi \in \mathcal{M}(V, \mathcal{H}, d)$ and $\lambda_0 \in K \times iV$, with $C' > 0$ a constant depending only on u. On the other hand, there exists a constant $C_0 > 0$, such that for all $\lambda_0 \in V_\mathbb{C}$ and all $\lambda \in D(\lambda_0, \epsilon)$, we have $C_0^{-1} \leq (1 + |\lambda_0|)(1 + |\lambda|)^{-1} \leq C_0$. Combining this estimate with the former one, we obtain

$$(1 + |\lambda_0|)^n |(u\pi_{K,d}\psi)(\lambda_0)| \leq C \sup_{\lambda \in D(\lambda_0, \epsilon)} (1 + |\lambda|)^n |(\pi_{K,d}\psi)(\lambda)|,$$

with $C > 0$ depending only on u, n. The proof is now completed by using that $D(\lambda_0, \epsilon) \subset K' \times iV$ for every $\lambda_0 \in K \times iV$. □

1.6 The residue operator for adjacent chambers. Let $\mathcal{H}$ be as above. We call the connected components of $\mathrm{reg}(V,\mathcal{H})$ the chambers of V (with respect to $\mathcal{H}$), and denote the set of these by $\mathrm{comp}(V,\mathcal{H})$. The chambers are convex sets. Let C be a chamber in V, and let $\overline{C}$ denote its closure. If $H \in \mathcal{H}$ and the intersection $H \cap \overline{C}$ has a non-empty interior in H we call this interior a face of C. It is easily seen that the face equals $\overline{C} \cap \mathrm{reg}(H,\mathcal{H}_H)$, and that it is a chamber of H with respect to $\mathcal{H}_H$.

If C is a chamber of V we denote by $\mathrm{pt}(C)$ a point in C, arbitrarily chosen. We shall use this symbol only when it makes no difference if a different choice had been made.

Two chambers C_1 and C_2 of V are called adjacent if they are separated by precisely one hyperplane $H \in \mathcal{H}$ (i.e., there is a path from $\mathrm{pt}(C_1)$ to $\mathrm{pt}(C_2)$ passing through $\cup\mathcal{H}$ only in $\mathrm{reg}(H,\mathcal{H}_H)$). Notice that this is precisely the case when C_1 and C_2 have a unique face in common; we denote this face by $C_1 \wedge C_2$. If C_1 and C_2 are adjacent with the separating hyperplane $H \in \mathcal{H}$ we say that the pair (C_1, C_2) is positively ordered if the chosen normal vector α_H points in the direction from C_1 to C_2.

Let $d\mu_V$ denote Lebesgue measure on V, normalized with respect to the inner product. If φ is a measurable function defined on the set $\eta + iV \subset V_{\mathbb{C}}$ for some point $\eta \in V$ we denote by $\int_{\eta+iV} \varphi \, d\mu_V$ the integral $\int_V \varphi(\eta + i\nu)\, d\mu_V(\nu)$, if it exists. In particular, if $\varphi \in \mathcal{P}(V,\mathcal{H})$, then it follows from (1.16) that this integral exists for all $\eta \in \mathrm{reg}(V,\mathcal{H})$. Moreover, it follows easily from Cauchy's theorem together with (1.16) that the value of the integral only depends on η through the chamber $C \in \mathrm{comp}(V,\mathcal{H})$ to which η belongs. We therefore write it as

$$\int_{\mathrm{pt}(C)+iV} \varphi \, d\mu_V.$$

If $L \in \mathcal{A}$ we define $\int_{\mathrm{pt}(C)+iV_L} \varphi \, d\mu_L$ similarly, for $C \in \mathrm{comp}(L,\mathcal{H}_L)$ and $\varphi \in \mathcal{P}(L,\mathcal{H}_L)$. In particular, if L is a point, $L = \{\lambda_0\}$, then $C = L$ is the only chamber in L, and $\int_{\mathrm{pt}(C)+iV_L} \varphi \, d\mu_L$ is the evaluation of φ in λ_0.

Proposition 1.12. *Let $C_1, C_2 \in \mathrm{comp}(V,\mathcal{H})$ be adjacent chambers, with the common face $C_1 \wedge C_2 \subset H$, $H \in \mathcal{H}$. Then*

$$\int_{\mathrm{pt}(C_2)+iV} \varphi \, d\mu_V - \int_{\mathrm{pt}(C_1)+iV} \varphi \, d\mu_V = \epsilon \int_{\mathrm{pt}(C_1 \wedge C_2)+iV_H} \mathrm{Res}_H^V \varphi \, d\mu_H \tag{1.17}$$

for all $\varphi \in \mathcal{P}(V,\mathcal{H})$, where $\epsilon = 1$ if (C_1, C_2) is positively ordered, and $\epsilon = -1$ otherwise.

Proof. Notice that both sides of (1.17) are independent of the choice of α_H. Hence we may assume that (C_1, C_2) is positively ordered. Fix points $\eta_j \in C_j$, $j = 1, 2$. We may assume that $\eta_2 - \eta_1 \in V_H^{\perp}$; this vector then points in the same direction as α_H. Moreover, we may assume that the line segment from η_1 to η_2 passes $\cup\mathcal{H}$ in exactly one point, $p \in \mathrm{reg}(H,\mathcal{H}_H)$. Then

$$\eta_j = p + x_j \frac{\alpha_H}{|\alpha_H|}, \quad (j = 1, 2)$$

for suitable real numbers x_1 and x_2 with $x_1 < 0 < x_2$.

When evaluating the integrals along V we shall be using the diffeomorphism $\Phi\colon V_H \times \mathbb{R} \to V$ given by

$$\Phi(\lambda, y) = \lambda + y\frac{\alpha_H}{|\alpha_H|}.$$

Obviously the Jacobian of this map is 1. We now have

$$\begin{aligned}
&\int_{\eta_2+iV} \varphi\, d\mu_V - \int_{\eta_1+iV} \varphi\, d\mu_V = \int_V [\varphi(\eta_2 + i\nu) - \varphi(\eta_1 + i\nu)]\, d\mu_V(\nu) \\
&\quad = \int_{V_H} \int_{\mathbb{R}} [\varphi(\eta_2 + i\Phi(\lambda, y)) - \varphi(\eta_1 + i\Phi(\lambda, y))]\, dy\, d\mu_H(\lambda) \\
&\quad = \int_{V_H} [\int_{\mathbb{R}} \varphi(p + i\lambda + (x_2 + iy)\frac{\alpha_H}{|\alpha_H|})\, dy - \int_{\mathbb{R}} \varphi(p + i\lambda + (x_1 + iy)\frac{\alpha_H}{|\alpha_H|})\, dy]\, d\mu_H(\lambda).
\end{aligned}$$

The function $\psi_{p+i\lambda}\colon z \mapsto \varphi(p + i\lambda + z\alpha_H/|\alpha_H|)$ on $\mathbb{C}$ is meromorphic, and its only possible singularity in $[x_1; x_2] + i\mathbb{R}$ occurs at $z = 0$. It now follows from the residue theorem and the estimates in (1.16) that the difference between the two inner integrals in the expression above equals

$$2\pi \operatorname{Res}_{z=0} \psi_{p+i\lambda}(z) = \operatorname{Res}^V_H \varphi(p + i\lambda),$$

and the result is proved. □

1.7 Residue weights. Let

$$\mathcal{L} = \mathcal{L}_{\mathcal{H}} := \{H_1 \cap \ldots \cap H_k \neq \emptyset \mid H_i \in \mathcal{H}, k > 0\} \cup \{V\} \subset \mathcal{A}$$

be the collection of all the non-empty intersections of hyperplanes from $\mathcal{H}$, together with the full space V. We order $\mathcal{L}$ by inclusion. Let $\operatorname{comp}(\mathcal{H}) = \cup_{L\in\mathcal{L}} \operatorname{comp}(L, \mathcal{H}_L)$ denote the collection of all chambers of all the subspaces $L \in \mathcal{L}$. By a residue weight associated to $\mathcal{H}$ we mean a function $t\colon \operatorname{comp}(\mathcal{H}) \to [0; 1]$ such that for each $L \in \mathcal{L}$:

(a) $t|_{\operatorname{comp}(L,\mathcal{H}_L)}$ has finite support, i.e., the set $\{C \in \operatorname{comp}(L, \mathcal{H}_L) \mid t(C) \neq 0\}$ is finite,
(b) $\sum_{C\in\operatorname{comp}(L,\mathcal{H}_L)} t(C) = 1$.

For example, if a distinguished non-empty finite set of chambers, $\mathcal{C}(L) \subset \operatorname{comp}(L, \mathcal{H}_L)$, has been chosen for each $L \in \mathcal{L}$, then we obtain a residue weight by letting $t(C) = 1/|\mathcal{C}(L)|$ if $C \in \mathcal{C}(L)$ for some $L \in \mathcal{L}$ and $t(C) = 0$ otherwise. Here $|\mathcal{C}(L)|$ denotes the number of elements in $\mathcal{C}(L)$.

The set of residue weights associated to $\mathcal{H}$ is denoted $\mathrm{WT}(\mathcal{H})$. Observe that if $t \in \mathrm{WT}(\mathcal{H})$ and $L \in \mathcal{L}$ then the restriction t_L of t to $\operatorname{comp}(\mathcal{H}_L) = \cup_{L'\in\mathcal{L}_L} \operatorname{comp}(L', \mathcal{H}_{L'})$ belongs to $\mathrm{WT}(\mathcal{H}_L)$. Here $\mathcal{L}_L := \{L' \in \mathcal{L} \mid L' \subset L\}$.

Theorem 1.13. *Let $\mathcal{H}$ be an affine hyperplane configuration in V and let $t \in \mathrm{WT}(\mathcal{H})$. Then for every chamber $C \in \operatorname{comp}(V, \mathcal{H})$ there exists a unique family of Laurent operators $\operatorname{Res}^{C,t}_L\colon \mathcal{M}(V, \mathcal{H}) \to \mathcal{M}(L, \mathcal{H}_L)$, $L \in \mathcal{L}$, such that*

(a) $\{L \in \mathcal{L} \mid \operatorname{Res}^{C,t}_L \neq 0\}$ *is finite,*

(b) *for all $\varphi \in \mathcal{P}(V, \mathcal{H})$ we have*

$$\int_{\mathrm{pt}(C)+iV} \varphi \, d\mu_V = \sum_{L \in \mathcal{L}} \sum_{C' \in \mathrm{comp}(L, \mathcal{H}_L)} t(C') \int_{\mathrm{pt}(C')+iV_L} \mathrm{Res}_L^{C,\, t} \varphi \, d\mu_L. \tag{1.18}$$

Moreover, the operators $\mathrm{Res}_L^{C,\, t}$ are real and we have $\mathrm{Res}_V^{C,\, t} = \mathrm{I}$, the identity operator. For $H \in \mathcal{H}$, the operator $\mathrm{Res}_H^{C,\, t}$ is a real multiple of Res_H^V.

The proof of this result (inspired by [10, Lemma 3.1]) will be given in the following two subsections. Based on the theorem we define the residual support of $C \in \mathrm{comp}(V, \mathcal{H})$, relative to t, as the finite subset of $\mathcal{L}$ given in item (a). It is denoted $\mathrm{ressupp}(C, t)$. The expression (1.18) gives the motivation for the phrase 'residue weight'. Notice in particular, that the term in (1.18) corresponding to $L = V$ reads

$$\sum_{C' \in \mathrm{comp}(V, \mathcal{H})} t(C') \int_{\mathrm{pt}(C')+iV} \varphi \, d\mu_V,$$

that is, it is a weighted sum of shifted integrals.

1.8 The existence of the residue operators. We first prove the existence of the operators $\mathrm{Res}_L^{C,\, t}$ in Theorem 1.13. The proof is carried out by induction on the dimension of V. Thus let $m \in \mathbb{N}$ and assume that the existence of the residue operators has been established for all pairs $(V, \mathcal{H})$ with $\dim V < m$ and all residue weights $t \in \mathrm{WT}(\mathcal{H})$ (if $m = 0$ this is certainly all right, as there are no such pairs). Let a space V of dimension m and a chamber $C \in \mathrm{comp}(V, \mathcal{H})$ be given. We rewrite the left-hand side of (1.18) as follows:

$$\begin{aligned} \int_{\mathrm{pt}(C)+iV} \varphi \, d\mu_V = & \sum_{C' \in \mathrm{comp}(V, \mathcal{H})} t(C') \int_{\mathrm{pt}(C')+iV} \varphi \, d\mu_V \\ & + \sum_{C' \in \mathrm{comp}(V, \mathcal{H})} t(C') \left[\int_{\mathrm{pt}(C)+iV} \varphi \, d\mu_V - \int_{\mathrm{pt}(C')+iV} \varphi \, d\mu_V \right]. \end{aligned}$$

The first sum on the right-hand side is going to represent the part of (1.18) where $L = V$, with $\mathrm{Res}_V^{C,\, t} = \mathrm{I}$. In the second sum the expression in the square brackets can be written as a sum of terms

$$\int_{\mathrm{pt}(C_1)+iV} \varphi \, d\mu_V - \int_{\mathrm{pt}(C_2)+iV} \varphi \, d\mu_V$$

with adjacent chambers $C_1, C_2 \in \mathrm{comp}(V, \mathcal{H})$. Using Proposition 1.12 we can write each of these terms as

$$\pm \int_{\mathrm{pt}(C_1 \wedge C_2)+iV_H} \mathrm{Res}_H^V \varphi \, d\mu_H,$$

where $H \in \mathcal{H}$ is the separating hyperplane. By the induction hypothesis applied to $(H, \mathcal{H}_H)$ and the restriction t_H of t to $\mathrm{comp}(\mathcal{H}_H)$, the latter expression can be written as

$$\pm \sum_{L \in \mathcal{L}_H} \sum_{C'' \in \mathrm{comp}(L, \mathcal{H}_L)} t(C'') \int_{\mathrm{pt}(C'')+iV_L} \mathrm{Res}_L^{C_1 \wedge C_2,\, t_H} \mathrm{Res}_H^V \varphi \, d\mu_L$$

with real Laurent operators $\mathrm{Res}_L^{C_1\wedge C_2,\,t_H}\colon \mathcal{M}(H,\mathcal{H}_H)\to\mathcal{M}(L,\mathcal{H}_L)$. By Lemma 1.8 the operator

$$R=\mathrm{Res}_L^{C_1\wedge C_2,\,t_H}\circ\mathrm{Res}_H^V$$

is a real Laurent operator. The existence of the operator $\mathrm{Res}_L^{C,\,t}$ now follows; it is a real linear combination of operators of the form R, with $H\in\mathcal{H}(L)$.

1.9 The uniqueness of the residue operators. We shall now establish the uniqueness part of Theorem 1.13. Let $t\in\mathrm{WT}(\mathcal{H})$ and $C\in\mathrm{comp}(V,\mathcal{H})$ be given. If we have two families of operators satisfying (a) and (b) in Theorem 1.13, we obtain by subtraction a family of Laurent operators $R_L\colon\mathcal{M}(V,\mathcal{H})\to\mathcal{M}(L,\mathcal{H}_L)$, $L\in\mathcal{L}_{\mathcal{H}}$, satisfying (a) and

$$0=\sum_{L\in\mathcal{L}}\;\sum_{C'\in\mathrm{comp}(L,\mathcal{H}_L)}t(C')\int_{\mathrm{pt}(C')+iV_L}R_L\varphi\,d\mu_L$$

for all $\varphi\in\mathcal{P}(V,\mathcal{H})$. In order to obtain the desired result we must prove that $R_L=0$ for all $L\in\mathcal{L}$. This results immediately from the following proposition.

Proposition 1.14. *Let $\mathcal{H}$ and t be as in Theorem 1.13, and let $d\in\mathbb{N}^{\mathcal{H}}$. Assume there is given, for each $L\in\mathcal{L}$, an element $u_L\in S(V_L^{\perp})$ such that*

(a) *$\{L\in\mathcal{L}\mid u_L\neq 0\}$ is finite,*
(b) *for all $\varphi\in\mathcal{P}(V,\mathcal{H},d)$ we have*

$$0=\sum_{L\in\mathcal{L}}\;\sum_{C'\in\mathrm{comp}(L,\mathcal{H}_L)}t(C')\int_{\mathrm{pt}(C')+iV_L}u_L(q_{L,d}\varphi)\,d\mu_L.$$

Then $u_L=0$ for all L.

Proof. We shall proceed by downward induction on the dimension of L. Thus let $l\in\mathbb{N}$ and assume that it has been established already that $u_L=0$ for all $L\in\mathcal{L}$ whose dimension is strictly greater than l (if $l=\dim V$ this is certainly all right, as there are no such subspaces L). Let $L_0\in\mathcal{L}$ be of dimension l. We claim that $u_{L_0}=0$. Let $\mathcal{L}_*=\{L\in\mathcal{L}\mid \dim L\le l, L\neq L_0\}$. We have

$$0=\sum_{L\in\mathcal{L}_*\cup\{L_0\}}\;\sum_{C'\in\mathrm{comp}(L,\mathcal{H}_L)}t(C')\int_{\mathrm{pt}(C')+iV_L}u_L(q_{L,d}\varphi)\,d\mu_L \tag{1.19}$$

for all $\varphi\in\mathcal{P}(V,\mathcal{H},d)$.

Notice that for each $L\in\mathcal{L}_*$ we have $L_0\not\subset L$ and hence $\mathcal{H}(L)\not\subset\mathcal{H}(L_0)$. Choose $H_L\in\mathcal{H}(L)\setminus\mathcal{H}(L_0)$, then for $N_L\in\mathbb{N}$ sufficiently large we have $u_L(\ell_{H_L}^{N_L}f)=0$ on $\mathrm{reg}(L_{\mathbb{C}},\mathcal{H}_L)$ for all functions f holomorphic on a neighborhood in $V_{\mathbb{C}}$ of this set. Let

$$q=\prod_{L\in\mathcal{L}_*,\,u_L\neq 0}\ell_{H_L}^{N_L}$$

(where, as usual, an empty product is 1), then $u_L(qf) = 0$ on $\mathrm{reg}(L_{\mathbb{C}}, \mathcal{H}_L)$ for f as before. Moreover, q is not identically zero on L_0. We now have (insert $q\varphi$ in place of φ in (1.19))

$$0 = \sum_{C' \in \mathrm{comp}(L_0, \mathcal{H}_{L_0})} t(C') \int_{\mathrm{pt}(C') + iV_{L_0}} u_{L_0}(q_{L_0,d}\, q\varphi)\, d\mu_{L_0}$$

for all $\varphi \in \mathcal{P}(V, \mathcal{H}, d)$. In particular this holds if φ has the form $q_{L_0,d}^{-1}\psi$ with $\psi \in \mathrm{PW}(V)$ (see above Lemma 1.9), and we thus obtain

$$0 = \sum_{C' \in \mathrm{comp}(L_0, \mathcal{H}_{L_0})} t(C') \int_{\mathrm{pt}(C') + iV_{L_0}} u_{L_0}(q\psi)\, d\mu_{L_0}$$

for all $\psi \in \mathrm{PW}(V)$. The space $\mathrm{PW}(V)$ is invariant under multiplication by a polynomial as well as under the application of a constant coefficient differential operator, and functions in $\mathrm{PW}(V)$ restrict to functions in $\mathrm{PW}(L)$ for any $L \in \mathcal{L}$. Hence the integrand in the expression above belongs to $\mathrm{PW}(L_0)$. By Cauchy's theorem we can then replace each point $\mathrm{pt}(C')$ by any other point of L_0, in particular, by the central point, and we obtain (using property (b) in the definition of a residue weight)

$$0 = \int_{c(L_0) + iV_{L_0}} u_{L_0}(q\psi)\, d\mu_{L_0}$$

for $\psi \in \mathrm{PW}(V)$. The space $\mathrm{PW}(V)$ is also invariant under translations by elements of $V_{\mathbb{C}}$, and hence

$$0 = \int_{iV_{L_0}} u_{L_0}(q'\psi)\, d\mu_{L_0} \tag{1.20}$$

for all $\psi \in \mathrm{PW}(V)$, where $q'(\lambda) = q(\lambda + c(L_0))$. Notice that the polynomial q' is not identically zero on $V_{L_0\mathbb{C}}$. The space $\{f|_{iV} \mid f \in \mathrm{PW}(V)\}$ is dense in the Schwartz space $\mathcal{S}(iV)$ (where iV is considered as a real Euclidean space), and the right-hand side of (1.20) is continuous on this space. Hence this identity holds for all $\psi \in \mathcal{S}(iV)$.

Let $\Omega_1 \subset iV_{L_0}^{\perp}$ and $\Omega_2 \subset iV_{L_0}$ be open non-empty sets such that $0 \in \Omega_1$ and such that q' is nowhere zero on $\Omega = \Omega_1 \times \Omega_2 \subset iV$. Then it follows from (1.20) that we have

$$0 = \int_{iV_{L_0}} u_{L_0}\psi\, d\mu_{L_0} \tag{1.21}$$

for $\psi \in C_c^\infty(\Omega)$. If $u_{L_0} \neq 0$ there exists a function $f_1 \in C_c^\infty(\Omega_1)$ such that $u_{L_0} f_1(0) = 1$. Moreover there exists a function $f_2 \in C_c^\infty(\Omega_2)$ such that $\int_{iV_{L_0}} f_2\, d\mu_{L_0} = 1$. Let $\psi = f_1 \otimes f_2 \in C_c^\infty(\Omega)$, then we have

$$\int_{iV_{L_0}} u_{L_0}\psi\, d\mu_{L_0} = 1,$$

contradicting (1.21). Hence $u_{L_0} = 0$ as claimed. This completes the proof of Proposition 1.14, and thus also that of Theorem 1.13. $\square$

1.10 Subconfigurations. In the remainder of Section 1 we give some properties of the residue operators that will be used in the following sections. The properties are easily established by means of the uniqueness in Theorem 1.13.

Suppose $\mathcal{H}_0$ and $\mathcal{H}$ are affine hyperplane configurations in V with $\mathcal{H}_0 \subset \mathcal{H}$. We call $\mathcal{H}_0$ a subconfiguration of $\mathcal{H}$. We have $\mathcal{M}(V,\mathcal{H}) \supset \mathcal{M}(V,\mathcal{H}_0)$ and $\mathcal{P}(V,\mathcal{H}) \supset \mathcal{P}(V,\mathcal{H}_0)$. In general a chamber $C_0 \in \mathrm{comp}(V,\mathcal{H}_0)$ contains several chambers from $\mathrm{comp}(V,\mathcal{H})$; we denote by $\mathrm{comp}(C_0,\mathcal{H})$ the set of these chambers.

Let $L \in \mathcal{A}$, then $\mathcal{H}_{0,L} := (\mathcal{H}_0)_L$ is a subconfiguration of $\mathcal{H}_L$. Moreover, if R is a Laurent operator $\mathcal{M}(V,\mathcal{H}) \to \mathcal{M}(L,\mathcal{H}_L)$, then it follows from (1.9) that the restriction of R to $\mathcal{M}(V,\mathcal{H}_0)$ maps into $\mathcal{M}(L,\mathcal{H}_{0,L})$ and is a Laurent operator.

The set $\mathcal{L}_0 := \mathcal{L}_{\mathcal{H}_0}$ is a subset of $\mathcal{L} = \mathcal{L}_{\mathcal{H}}$. The inclusion map $i\colon \mathcal{H}_0 \hookrightarrow \mathcal{H}$ induces a map i^* from $\mathrm{WT}(\mathcal{H})$ to $\mathrm{WT}(\mathcal{H}_0)$ as follows. Let $t \in \mathrm{WT}(\mathcal{H})$ and define for $L \in \mathcal{L}_0$ and $C_0 \in \mathrm{comp}(L,\mathcal{H}_{0,L})$:

$$i^*(t)(C_0) = \sum_{C\in\mathrm{comp}(C_0,\mathcal{H}_L)} t(C). \tag{1.22}$$

It is easily seen that $i^*(t)$ is a residue weight for $(V,\mathcal{H}_0)$. It is called the induced weight.

Proposition 1.15. *Let $C_0 \in \mathrm{comp}(V,\mathcal{H}_0)$, $C \in \mathrm{comp}(C_0,\mathcal{H})$, $t \in \mathrm{WT}(\mathcal{H})$ and $L \in \mathcal{L}$. Then*

$$\mathrm{Res}_L^{C,\,t}|_{\mathcal{M}(V,\mathcal{H}_0)} = \begin{cases} \mathrm{Res}_L^{C_0,\,i^*(t)} & \text{if } L \in \mathcal{L}_0; \\ 0 & \text{otherwise.} \end{cases} \tag{1.23}$$

Proof. By Theorem 1.13 (for the configuration $\mathcal{H}_0$ and the weight $i^*(t)$) and (1.22) we have, for all $\varphi \in \mathcal{P}(V,\mathcal{H}_0)$:

$$\begin{aligned}\int_{\mathrm{pt}(C_0)+iV} &\varphi\, d\mu_V \\ &= \sum_{L\in\mathcal{L}_0} \sum_{C_0'\in\mathrm{comp}(L,\mathcal{H}_{0,L})} \Big[\sum_{C'\in\mathrm{comp}(C_0',\mathcal{H}_L)} t(C')\Big] \int_{\mathrm{pt}(C_0')+iV_L} \mathrm{Res}_L^{C_0,\,i^*(t)}\varphi\, d\mu_L.\end{aligned}$$

Since $\mathrm{pt}(C') \in C_0'$ for all $C' \in \mathrm{comp}(C_0',\mathcal{H}_L)$ we can insert these points for $\mathrm{pt}(C_0')$ on the right-hand side, and we obtain

$$\int_{\mathrm{pt}(C_0)+iV} \varphi\, d\mu_V = \sum_{L\in\mathcal{L}_0} \sum_{C'\in\mathrm{comp}(L,\mathcal{H}_L)} t(C') \int_{\mathrm{pt}(C')+iV_L} \mathrm{Res}_L^{C_0,\,i^*(t)}\varphi\, d\mu_L \tag{1.24}$$

for all $\varphi \in \mathcal{P}(V,\mathcal{H}_0)$.

Let $d_0 \in \mathbb{N}^{\mathcal{H}_0}$, and define $d \in \mathbb{N}^{\mathcal{H}}$ by $d(H) = d_0(H)$ for $H \in \mathcal{H}_0$, and $d(H) = 0$ otherwise. Then $\mathcal{P}(V,\mathcal{H},d) = \mathcal{P}(V,\mathcal{H}_0,d_0)$, and we have the equation (1.18) for all φ in this space. Since $\mathrm{pt}(C) \in C_0$ the left-hand sides of (1.18) and (1.24) coincide, hence so do the right-hand sides, and the identities in (1.23) follow by means of Proposition 1.14. $\square$

In particular, we have the following immediate consequence of Proposition 1.15.

Corollary 1.16. *Let $\mathcal{H}$, t, and C be as in Theorem 1.13, and let $\varphi \in \mathcal{M}(V, \mathcal{H})$. Let $\mathcal{H}_0 \subset \mathcal{H}$ denote the set of hyperplanes along which φ is singular, and let $\mathcal{L}_0 = \mathcal{L}_{\mathcal{H}_0}$. If $L \in \mathcal{L} \setminus \mathcal{L}_0$ (in particular, if φ is holomorphic in a neighborhood of L), then $\operatorname{Res}_L^{C,\,t} \varphi = 0$.*

For the next result we recall (see below Lemma 1.5) that we have identified the spaces $\operatorname{Laur}(V, L, \mathcal{H})$ and $\operatorname{Laur}(V, L, \mathcal{H}(L))$.

Corollary 1.17. *Let $\mathcal{H}$, t, and C be as above, and let $L \in \mathcal{L}$ be fixed. Furthermore, let $C_0 \in \operatorname{comp}(V, \mathcal{H}(L))$ be determined by $C \subset C_0$, and let $i\colon \mathcal{H}(L) \hookrightarrow \mathcal{H}$ be the inclusion map. Then $\operatorname{Res}_L^{C,\,t} = \operatorname{Res}_L^{C_0,\,i^*(t)}$.*

(In other words, when computing the residue operator $\operatorname{Res}_L^{C,\,t}$, or more precisely, its representative in $S_{\leftarrow}(V_L^{\perp}, X(L))$, we can ignore all the hyperplanes from $\mathcal{H}$ that do not contain L.)

Proof. This follows immediately from Proposition 1.15, since $\mathcal{H}(L)$ is a subconfiguration of $\mathcal{H}$ and since in this case we have $L \in \mathcal{L}_0 = \mathcal{L}_{\mathcal{H}(L)}$. $\square$

1.11 Invariance under isometries. Let $T\colon V \to V$ be an isometry. Then T maps hyperplanes to hyperplanes, hence it maps $\mathcal{H}$ to the affine hyperplane configuration $T\mathcal{H} := \{TH \mid H \in \mathcal{H}\}$. It is easily seen that T maps $\operatorname{comp}(\mathcal{H})$ bijectively to $\operatorname{comp}(T\mathcal{H})$, and that $\varphi \mapsto \varphi \circ T^{-1}$ is a bijective linear map from $\mathcal{M}(V, \mathcal{H})$ to $\mathcal{M}(V, T\mathcal{H})$, as well as from $\mathcal{P}(V, \mathcal{H})$ to $\mathcal{P}(V, T\mathcal{H})$.

Since T is an isometry there is a unique linear orthogonal transformation of V, which we denote by T', such that

$$(T'u)(\varphi) = u(\varphi \circ T) \circ T^{-1} \tag{1.25}$$

for $u \in V$ and $\varphi \in C^{\infty}(V)$. Thus if T itself is linear then $T' = T$, and if T is a translation then $T' = \mathrm{I}$. Let T' denote as well the natural extension to $S(V)$ of this map, such that (1.25) holds for $u \in S(V)$. Let $L \in \mathcal{A}$. Then T' maps $S(V_L^{\perp})$ to $S(V_{TL}^{\perp})$ and $S_{\leftarrow}(V_L^{\perp}, X(L))$ to $S_{\leftarrow}(V_{TL}^{\perp}, TX(L))$. It follows (cf. Lemma 1.5) that T' induces a linear isomorphism, also denoted T', of $\operatorname{Laur}(V, L, \mathcal{H})$ onto $\operatorname{Laur}(V, TL, T\mathcal{H})$, and by (1.9) and (1.25) we have

$$T'R(\varphi) = R(\varphi \circ T) \circ T^{-1} \tag{1.26}$$

for $R \in \operatorname{Laur}(V, L, \mathcal{H})$, $\varphi \in \mathcal{M}(V, \mathcal{H})$.

Lemma 1.18. *Let $T\colon V \to V$ be as above, and let $t \in \operatorname{WT}(\mathcal{H})$. Then $Tt := t \circ T^{-1} \in \operatorname{WT}(T\mathcal{H})$. Moreover, let $C \in \operatorname{comp}(\mathcal{H})$ and $L \in \mathcal{L}$. Then*

$$T' \operatorname{Res}_L^{C,\,t} = \operatorname{Res}_{TL}^{TC,\,Tt}. \tag{1.27}$$

Proof. The first statement is clear from the definition of WT. Let $\varphi \in \mathcal{P}(V, T\mathcal{H})$, then the claim in (1.27) amounts to

$$\operatorname{Res}_L^{C,\,t}(\varphi \circ T) \circ T^{-1} = \operatorname{Res}_{TL}^{TC,\,Tt} \varphi. \tag{1.28}$$

Since T preserves Lebesgue measure we have

$$\int_{\operatorname{pt}(C)+iV} (\varphi \circ T)\, d\mu_V = \int_{\operatorname{pt}(TC)+iV} \varphi\, d\mu_V. \tag{1.29}$$

The identity (1.28) follows easily, if we apply (1.18) to the left hand side of the expression (1.29) and use the definition (Theorem 1.13) of the residue operators $\operatorname{Res}_{TL}^{TC,\,Tt}$. $\square$

1.12 Extensions. Let $A \subset V$ be an affine subspace, and let $\mathcal{H}_A$ be an affine hyperplane configuration in A. Then by

$$\mathcal{H} = \{H' + V_A^\perp \mid H' \in \mathcal{H}_A\}$$

we define an affine hyperplane configuration in V, which we call the extension of $\mathcal{H}_A$. It satisfies

$$V_A \supset V_H^\perp, \quad \text{for all } H \in \mathcal{H}. \tag{1.30}$$

Conversely, if a given affine hyperplane configuration $\mathcal{H}$ in V satisfies (1.30), then $H = (H \cap A) + V_A^\perp$ for all $H \in \mathcal{H}$, and hence $\mathcal{H}$ is the extension of the hyperplane configuration

$$\mathcal{H}_A := \{H \cap A \mid H \in \mathcal{H}\} \tag{1.31}$$

in A.

Lemma 1.19. *Let $\mathcal{H}$ be the extension of $\mathcal{H}_A$, and let $L \in \mathcal{A}$. Assume $V_A^\perp \subset V_L$. Then*

$$L = L \cap A + V_A^\perp.$$

(i) *Define, for $\varphi \in \mathcal{M}(V, \mathcal{H})$ and $\nu \in V_{A\mathbb{C}}^\perp$, a function φ_ν on $A_\mathbb{C}$ by*

$$\varphi_\nu(\lambda) = \varphi(\lambda + \nu) \qquad (\lambda \in A_\mathbb{C}). \tag{1.32}$$

Then $\varphi_\nu \in \mathcal{M}(A, \mathcal{H}_A)$. Moreover, if $\varphi \in \mathcal{P}(V, \mathcal{H})$ then $\varphi_\nu \in \mathcal{P}(A, \mathcal{H}_A)$.

(ii) *Let a Laurent operator $R_A \colon \mathcal{M}(A, \mathcal{H}_A) \to \mathcal{M}(L \cap A, \mathcal{H}_{L\cap A})$ be given, and define, for $\varphi \in \mathcal{M}(V, \mathcal{H})$, a function $R\varphi$ on $L_\mathbb{C}$ by*

$$R\varphi(\lambda + \nu) = R_A(\varphi_\nu)(\lambda) \tag{1.33}$$

for $\lambda \in (L \cap A)_\mathbb{C}$, $\nu \in V_{A\mathbb{C}}^\perp$. Then $R\varphi \in \mathcal{M}(L, \mathcal{H}_L)$ and $R \colon \mathcal{M}(V, \mathcal{H}) \to \mathcal{M}(L, \mathcal{H}_L)$ is a Laurent operator.

(iii) *The map $R_A \mapsto R$ defined in (ii) is an isomorphism of* $\mathrm{Laur}(A, L \cap A, \mathcal{H}_A)$ *onto* $\mathrm{Laur}(V, L, \mathcal{H})$.

Proof. That $L = L \cap A + V_A^\perp$ is obvious. Let a normal vector $\alpha_H \in V_H^\perp$ be chosen for each $H \in \mathcal{H}$, then α_H is also a normal vector for $H \cap A$ in V_A. With these choices fixed, it follows that the associated first degree polynomials $\ell_H \colon V_\mathbb{C} \to \mathbb{C}$ and $\ell_{H\cap A} \colon A_\mathbb{C} \to \mathbb{C}$ in (1.1) are related by the equation

$$\ell_H(\lambda + \nu) = \ell_{H\cap A}(\lambda), \qquad (\lambda \in A_\mathbb{C}, \nu \in V_{A\mathbb{C}}^\perp). \tag{1.34}$$

The bijection $H \mapsto H \cap A$ from $\mathcal{H}$ to $\mathcal{H}_A$ induces a bijection $\mathbb{N}^\mathcal{H} \simeq \mathbb{N}^{\mathcal{H}_A}$. Let $K \subset A$ be a compact subset. It follows from (1.34) that for every $d \in \mathbb{N}^\mathcal{H} \simeq \mathbb{N}^{\mathcal{H}_A}$ we have

$$\pi_{K,d}(\lambda + \nu) = \pi_{K,d}(\lambda), \qquad (\lambda \in A_\mathbb{C}, \nu \in V_{A\mathbb{C}}^\perp).$$

Now (i) easily follows.

Notice that $\mathcal{H}(L)$ is the extension of $\mathcal{H}_A(L\cap A) = \{H' \in \mathcal{H}_A \mid H' \supset L \cap A\}$. It follows from this observation and from the identity (1.34) that for a given $d \in \mathbb{N}^{\mathcal{H}(L)} \simeq \mathbb{N}^{\mathcal{H}_A(L\cap A)}$ the polynomials $q_L\colon V_{\mathbb{C}} \to \mathbb{C}$ and $q_{L\cap A}\colon A_{\mathbb{C}} \to \mathbb{C}$ in (1.5) are related by

$$q_L(\lambda+\nu) = q_{L\cap A}(\lambda), \qquad (\lambda \in A_{\mathbb{C}}, \nu \in V_{A\mathbb{C}}^{\perp}). \tag{1.35}$$

Let R_A be given, as in (ii), and let $u \in S_{\leftarrow}(V_{L\cap A}^{\perp} \cap V_A, X(L\cap A))$ be its image by the isomorphism in Lemma 1.5. Here the set $X(L\cap A)$ consists of the normal vectors in V_A to the hyperplanes in $\mathcal{H}_A(L\cap A)$. With the choice of normal vectors mentioned earlier in the proof we have $X(L\cap A) = X(L)$. Since $V_{L\cap A}^{\perp} \cap V_A = V_L^{\perp}$ we conclude that

$$S_{\leftarrow}(V_L^{\perp}, X(L)) = S_{\leftarrow}(V_{L\cap A}^{\perp} \cap V_A, X(L\cap A)). \tag{1.36}$$

Hence $u \in S_{\leftarrow}(V_L^{\perp}, X(L))$. As in Lemma 1.5 let R_u be the corresponding Laurent operator $\mathcal{M}(V,\mathcal{H}) \to \mathcal{M}(L,\mathcal{H}_L)$, then $R_u\varphi$ is given by (1.9) for $\varphi \in \mathcal{M}(V,\mathcal{H})$. It is now easily seen from (1.35) that the function $R\varphi$ defined by (1.33) is equal to $R_u\varphi$. Hence $R = R_u$ and (ii) is proved. Moreover, (iii) is an immediate consequence of (1.36) and Lemma 1.5. □

Remark 1.20. Given a Laurent operator $R \in \mathrm{Laur}(V,L,\mathcal{H})$ we denote by R_A its preimage in $\mathrm{Laur}(A, L\cap A, \mathcal{H}_A)$ by the isomorphism of (iii). Notice that if we identify the spaces of Laurent operators with their projective limit models, as mentioned below Lemma 1.5, then it follows from the proof above that the map $R \mapsto R_A$ is just the identity map on the space (1.36).

Let $\mathcal{L} = \mathcal{L}_{\mathcal{H}}$, $\mathcal{L}_A = \mathcal{L}_{\mathcal{H}_A}$. The map $L \mapsto L\cap A$ is a bijection from $\mathcal{L}$ to $\mathcal{L}_A$. The map $C \mapsto C\cap A$ is a bijection from $\mathrm{comp}(\mathcal{H})$ to $\mathrm{comp}(\mathcal{H}_A)$. Hence if $t \in \mathrm{WT}(\mathcal{H})$ we obtain a residue weight $t_A \in \mathrm{WT}(\mathcal{H}_A)$ by defining

$$t_A(C\cap A) = t(C), \qquad (C \in \mathrm{comp}(\mathcal{H})). \tag{1.37}$$

The map $t \mapsto t_A$ is then a bijection from $\mathrm{WT}(\mathcal{H})$ to $\mathrm{WT}(\mathcal{H}_A)$.

Lemma 1.21. *Let $\mathcal{H}$ be the extension of $\mathcal{H}_A$ as above, and let $t \in \mathrm{WT}(\mathcal{H})$, $C \in \mathrm{comp}(\mathcal{H})$. Then*

$$(\mathrm{Res}_L^{C,\,t})_A = \mathrm{Res}_{L\cap A}^{C\cap A,\,t_A} \tag{1.38}$$

for every $L \in \mathcal{L}$.

Proof. We define for each $L \in \mathcal{L}$ the Laurent operator $R_L\colon \mathcal{M}(V,\mathcal{H}) \to \mathcal{M}(L,\mathcal{H}_L)$ by $(R_L)_A = \mathrm{Res}_{L\cap A}^{C\cap A,\,t_A}$. The lemma follows if we establish the identity $R_L = \mathrm{Res}_L^{C,\,t}$ for every $L \in \mathcal{L}$. By the uniqueness in Theorem 1.13 it suffices to prove that

$$\int_{\mathrm{pt}(C)+iV} \varphi\, d\mu_V = \sum_{L\in\mathcal{L}} \sum_{C'\in\mathrm{comp}(L,\mathcal{H}_L)} t(C') \int_{\mathrm{pt}(C')+iV_L} R_L\varphi\, d\mu_L \tag{1.39}$$

for $\varphi \in \mathcal{P}(V,\mathcal{H})$.

For each fixed $\nu \in V_{A\mathbb{C}}^{\perp}$ we have

$$R_L\varphi(\lambda+\nu) = \mathrm{Res}_{L\cap A}^{C\cap A,\, t_A}(\varphi_\nu)(\lambda), \qquad (\lambda \in (L\cap A)_{\mathbb{C}}) \tag{1.40}$$

(cf. (1.33)), and

$$\int_{\mathrm{pt}(C\cap A)+iV_A} \varphi_\nu \, d\mu_A = \sum_{L'\in\mathcal{L}_A} \sum_{C''\in\mathrm{comp}(L',\mathcal{H}_{L'})} t_A(C'') \int_{\mathrm{pt}(C'')+iV_{L'}} \mathrm{Res}_{L'}^{C\cap A,\, t_A}(\varphi_\nu)\, d\mu_{L'},$$

by the definition of the residue operators for $\mathcal{H}_A$. Substituting $L' = L\cap A$ and $C'' = C'\cap A$ ($L \in \mathcal{L}, C' \in \mathrm{comp}(L,\mathcal{H}_L)$), and applying (1.37) and (1.40), we obtain

$$\int_{\mathrm{pt}(C\cap A)+iV_A} \varphi_\nu \, d\mu_A = \sum_{L\in\mathcal{L}} \sum_{C'\in\mathrm{comp}(L,\mathcal{H}_L)} t(C') \int_{\mathrm{pt}(C'\cap A)+iV_{L\cap A}} (R_L\varphi)_\nu \, d\mu_{L\cap A}. \tag{1.41}$$

Now $\varphi \in \mathcal{P}(V,\mathcal{H})$, and for every $L \in \mathcal{L}$ we have $R_L\varphi \in \mathcal{P}(L,\mathcal{H}_L)$, by Lemma 1.10. Hence the expressions on both sides of (1.41) are integrable over $\nu \in iV_A^{\perp}$ with respect to the measure $d\mu_{V_A^{\perp}}(\nu)$. Moreover, the desired equation (1.39) follows by application of the Fubini theorem. $\square$

2 Support conditions

As mentioned in the introduction we would ideally like to replace the η in an integral of the form (0.3) by the origin of V, at the cost of residual terms. This means that for the terms in (1.18) corresponding to $L = V$ we want to have $t(C') = 0$ unless $0 \in C'$. Likewise, in the contributions to (1.18) from $L \neq V$ (the residual terms) we would like to have $t(C') = 0$ unless $c(L) \in C'$. In the application, in [5], to the Plancherel decomposition, the tempered part of the spectrum is to be found on (real) affine subspaces in $V_{\mathbb{C}}$ of the form $c(L) + iV_L$. Therefore, we call this affine subspace of $L_{\mathbb{C}}$ the tempered real form of $L_{\mathbb{C}}$. What we want is that only integrals over tempered real forms contribute in (1.18). However, in general we cannot quite obtain this, since $c(L)$ may belong to the singular set $\mathrm{sing}(L,\mathcal{H}_L)$ for some $L \in \mathcal{L}$. What we can obtain is that an integral over $\mathrm{pt}(C') + iV_L$ only contributes if $c(L)$ is in the closure of C'. For this purpose we introduce in this section the notion of a central residue weight; this is a weight that is supported on chambers C' with closure containing $c(L)$ (where $C' \in \mathrm{comp}(L,\mathcal{H}_L)$). Our main result here is Theorem 2.6, which gives necessary conditions for an element $L \in \mathcal{L}$ to produce a non-vanishing residue operator, relative to a central weight.

2.1 Central residue weights. Let $\mathcal{H}$ be an affine hyperplane configuration in V, and let $L \in \mathcal{L}$. A chamber $C \in \mathrm{comp}(L,\mathcal{H}_L)$ is called central (in L), if its closure contains the central point $c(L)$. The set of central chambers in L is denoted $\mathrm{comp}_c(L,\mathcal{H}_L)$; this is a finite set since $\mathcal{H}$ is locally finite. Let $t\colon \mathrm{comp}(\mathcal{H}) \to [0;1]$ be a residue weight. We call t central if it has central support, that is if for every $L \in \mathcal{L}$ and $C \in \mathrm{comp}(L,\mathcal{H}_L)$ we have $t(C) \neq 0$ only if $C \in \mathrm{comp}_c(L, \mathcal{H}_L)$. The set of central residue weights is denoted $\mathrm{WT}_c(\mathcal{H})$.

Example 2.1. A particularly simple case appears if $c(L) \in \mathrm{reg}(L, \mathcal{H}_L)$ for all $L \in \mathcal{L}$. In this case there is only one central residue weight t_c, namely that which associates the weight 1 to the unique central chamber (which contains $c(L)$) for each L, and 0 to all other chambers. For this weight, (1.18) reads

$$\int_{\mathrm{pt}(C)+iV} \varphi \, d\mu_V = \sum_{L \in \mathcal{L}} \int_{c(L)+iV_L} \mathrm{Res}_L^{C,\, t_c} \varphi \, d\mu_L \tag{2.1}$$

for $C \in \mathrm{comp}(V, \mathcal{H})$ and $\varphi \in \mathcal{P}(V, \mathcal{H})$.

As mentioned, we shall give a necessary condition for an element $L \in \mathcal{L}$ to be in the residual support $\mathrm{ressupp}(C, t)$ of a chamber $C \in \mathrm{comp}(V, \mathcal{H})$ relative to a central weight t. If C is also central, the criterion is simple:

Lemma 2.2. *Let $C \in \mathrm{comp}_c(V, \mathcal{H})$ be a central chamber, and let $t \in \mathrm{WT}_c(\mathcal{H})$ be a central weight. Then for every $L \in \mathrm{ressupp}(C, t)$ we have $0 \in L$.*

Proof. Observe first that if C' is another central chamber in V then there exists a sequence $C'_1, \dots, C'_n$ of central chambers in V such that $C'_1 = C$, $C'_n = C'$, and C'_i, C'_{i+1} are adjacent for all i.

Let $L \in \mathcal{L}$. Since t is central it follows from the preceding observation and the proof in Section 1.8 that $\mathrm{Res}_L^{C,\, t}$ is a linear combination of operators of the form

$$\mathrm{Res}_L^{C_1 \wedge C_2,\, t_H} \circ \mathrm{Res}_H^V$$

with adjacent chambers C_1, C_2, both central in V. The hyperplane $H \in \mathcal{H}(L)$ that separates C_1 and C_2 contains 0 since C_1, C_2 are both central. Moreover $C_1 \wedge C_2$ is a central chamber in H. The restriction t_H of t to $\mathrm{comp}(\mathcal{H}_H)$ is also central.

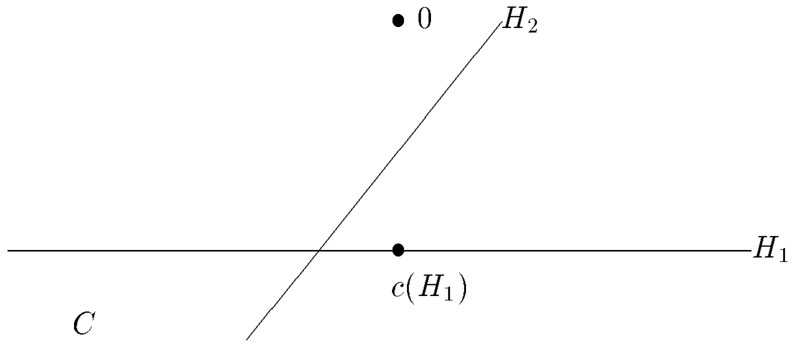
The proof is completed by a straightforward induction on $\dim V$. $\square$

For non-central chambers C our criterion for an element $L \in \mathcal{L}$ possibly to be in $\mathrm{ressupp}(C, t)$ (with $t \in \mathrm{WT}_c(\mathcal{H})$) is more intricate. Let us describe the idea for the simple case of Example 2.1. Using that $\mathrm{Res}_V^{C,\, t_c} = \mathrm{I}$ we rewrite (2.1) as follows:

$$\int_{\mathrm{pt}(C)+iV} \varphi \, d\mu_V - \int_{0+iV} \varphi \, d\mu_V = \sum_{L \in \mathcal{L},\, L \neq V} \int_{c(L)+iV_L} \mathrm{Res}_L^{C,\, t_c} \varphi \, d\mu_L .$$

It follows from the proof of this formula (see Section 1.8) that a hyperplane $H \in \mathcal{H}$ that belongs to $\mathrm{ressupp}(C, t_c)$ must separate C and 0. In other words, the line segment $[\mathrm{pt}(C); 0]$ from $\mathrm{pt}(C)$ to 0 must intersect H, say in a point q. This exactly is our condition if $L = H$ is a hyperplane. The limitation on the lower dimensional spaces in $\mathrm{ressupp}(C, t)$ is inductive: If $L \in \mathrm{ressupp}(C, t_c)$ has codimension 2 in V, it must be contained in one of the above mentioned hyperplanes H, and it must separate q from $c(L)$. Here q is the point mentioned above - notice however that we must take into account that it depends on the choice of the point $\mathrm{pt}(C)$ in C.

An example is given in the following figure, where $\mathcal{H}$ consists of the two lines H_1 and H_2, and C is the lower left chamber.



When we move the two-dimensional integral $\int_{p+iV} \varphi \, d\mu_V$ from $p = \mathrm{pt}(C)$ to $p = 0$, a residue occurs at a point, say q, on H_1, to the left of its intersection with H_2. This residue is itself a one-dimensional integral along $q + iV_{H_1}$, and has to be shifted to an integral along the tempered real form $c(H_1) + iV_{H_1}$ of H_1. In the latter shift another residue occurs at the point of intersection, $H_1 \cap H_2$; this residue is a scalar. Thus we see that $\mathrm{ressupp}(C, t)$ (at most) consists of V, H_1, and $H_1 \cap H_2$.

For the general case when $c(L)$ is allowed to be singular in L, the result is of a similar nature. Besides the complications arising from considering the general case, another difficulty arises from the problem that the point $q \in [\mathrm{pt}(C); 0] \cap H$ (see above) may be a singular point of H. This occurs already in the simple case described above, for example if in the figure we add a third line, H_3, that intersects H_1, resp. H_2, to the left of, resp. below, $H_1 \cap H_2$. If C is again the lower left chamber, the point q where $[\mathrm{pt}(C); 0]$ intersects H_1 could happen to be the point $H_1 \cap H_3$. However, this is not the case if $\mathrm{pt}(C)$ is chosen outside a certain singular subset of C (viz., outside the line through 0 and $H_1 \cap H_3$). This is precisely our aim in the following subsection: We shall define (for finite hyperplane configurations) an open dense subset $\mathrm{reg}^{\sim}(V, \mathcal{H})$ of $\mathrm{reg}(V, \mathcal{H})$ such that the mentioned problem is avoided (on all levels) if $\mathrm{pt}(C)$ is chosen from this subset.

2.2 Weakly singular hyperplanes. For the rest of this section we assume that $\mathcal{H}$ is finite. We shall define $\mathrm{reg}^{\sim}(V, \mathcal{H})$ by means of a larger (but still finite) hyperplane configuration $\mathcal{H}^{\sim}$. The definition of this configuration is inductive.

If $c \in V$ and $A \subset V$ is an affine subspace we denote by $\mathrm{aff}(c, A)$ the affine span of $\{c\} \cup A$, that is the set of all affine combinations $(1-t)c + t\lambda$, $t \in \mathbb{R}$, of c and all points $\lambda \in A$. The set $\mathrm{aff}(c, A)$ is obviously an affine subspace, and its dimension is $\dim A + 1$ unless $c \in A$ in which case $\mathrm{aff}(c, A) = A$.

We define for each $L \in \mathcal{L}$ a finite set $\mathcal{H}_L^{\sim}$ of hyperplanes in L, by induction on $\dim L$, as follows:

$$\mathcal{H}_L^{\sim} = \mathcal{H}_L \cup \{\mathrm{aff}(c(L), H') \mid H \in \mathcal{H}_L, \, c(L) \notin H, \, H' \in \mathcal{H}_H^{\sim}\}. \tag{2.2}$$

If $\dim L = 0$ then $\mathcal{H}_L = \emptyset$, and (2.2) gives $\mathcal{H}_L^{\sim} = \emptyset$. If $\dim L = 1$ then (2.2) gives $\mathcal{H}_L^{\sim} = \mathcal{H}_L$. Let $\mathcal{H}^{\sim} = \mathcal{H}_V^{\sim}$; this is a finite hyperplane configuration in V, and it has $\mathcal{H}$ as a subconfiguration. We call the hyperplanes in $\mathcal{H}^{\sim}$ weakly singular with respect to $\mathcal{H}$.

Notice that by the inductive construction it is obvious that $\mathcal{H}_L^\sim$ is the set of hyperplanes in L that are weakly singular with respect to $\mathcal{H}_L$.

Let $\mathrm{sing}^\sim(V,\mathcal{H}) = \mathrm{sing}(V,\mathcal{H}^\sim) = \cup\mathcal{H}^\sim$ and $\mathrm{reg}^\sim(V,\mathcal{H}) = \mathrm{reg}(V,\mathcal{H}^\sim) = V \setminus \cup\mathcal{H}^\sim$. The crucial property of the refined configuration $\mathcal{H}^\sim$ is expressed in the following lemma:

Lemma 2.3. *Let $\lambda \in \mathrm{reg}^\sim(V,\mathcal{H})$ and let $q \in \mathbb{R}\lambda \cap \mathrm{sing}^\sim(V,\mathcal{H})$, $q \neq 0$. Then $q \in H$ for a unique hyperplane $H \in \mathcal{H}$, and $\mathbb{R}\lambda \cap H = \{q\} \subset \mathrm{reg}^\sim(H,\mathcal{H}_H)$.*

Proof. Let $H \in \mathcal{H}^\sim$ be such that $q \in H$. The set $\mathbb{R}\lambda \cap H$ is affine, hence either it is a point or it equals $\mathbb{R}\lambda$. The latter is excluded since λ is $\sim$-regular and hence is not in H. Thus $\mathbb{R}\lambda \cap H = \{q\}$. In particular, $0 \notin H$. It follows from (2.2) (with $L = V$) that the hyperplanes from $\mathcal{H}^\sim \setminus \mathcal{H}$ contain 0. Hence $H \in \mathcal{H}$. Assume $q \in \mathrm{sing}^\sim(H,\mathcal{H}_H)$. Then $q \in H'$ for some $H' \in \mathcal{H}_H^\sim$, and since (2.2) (again with $L = V$) implies that $\mathrm{aff}(0, H') \subset \mathrm{sing}^\sim(V,\mathcal{H})$, we conclude that $\mathbb{R}q \subset \mathrm{sing}^\sim(V,\mathcal{H})$. Again, this contradicts the assumption on λ. Hence $q \in \mathrm{reg}^\sim(H,\mathcal{H}_H)$. In particular, this implies the stated uniqueness of H. All statements in the lemma have now been proved. $\square$

More generally, let $L_0 \in \mathcal{L}$ and $\lambda \in \mathrm{reg}^\sim(L_0,\mathcal{H}_{L_0})$, and let

$$\ell = \{(1-t)c(L_0) + t\lambda \mid t \in \mathbb{R}\}.$$

If $q \in \ell \cap \mathrm{sing}^\sim(L_0,\mathcal{H}_{L_0})$, $q \neq c(L_0)$, then $q \in L$ for a unique $L \in \mathcal{H}_{L_0}$, and we have $\ell \cap L = \{q\} \subset \mathrm{reg}^\sim(L,\mathcal{H}_L)$. This follows immediately from the preceding lemma, applied to L_0, $\mathcal{H}_{L_0}$.

2.3 The chambers of the refined configuration. We call a connected component of $\mathrm{reg}^\sim(V,\mathcal{H})$ a $\sim$-chamber and denote by $\mathrm{comp}^\sim(V,\mathcal{H})$ the (finite) set ($= \mathrm{comp}(V,\mathcal{H}^\sim)$) of these $\sim$-chambers. Since $\mathrm{reg}^\sim(V,\mathcal{H}) \subset \mathrm{reg}(V,\mathcal{H})$ there is a natural surjective map $\iota_V \colon \mathrm{comp}^\sim(V,\mathcal{H}) \to \mathrm{comp}(V,\mathcal{H})$ defined by $\iota_V(C) \supset C$ for $C \in \mathrm{comp}^\sim(V,\mathcal{H})$.

Put $\mathrm{comp}^\sim(\mathcal{H}) = \cup_{L\in\mathcal{L}}\mathrm{comp}^\sim(L,\mathcal{H}_L)$. Notice that here the set $\mathcal{L}$ is defined relative to the original configuration $\mathcal{H}$; in general not all intersections of elements from $\mathcal{H}^\sim$ belong to $\mathcal{L}$. This has the effect that in general $\mathrm{comp}^\sim(\mathcal{H})$ does not cover all of V (whereas $\mathrm{comp}(\mathcal{H})$ does cover V). Notice also that by the inductive construction of $\mathcal{H}^\sim$ we immediately have for all $L_0 \in \mathcal{L}$ that $\mathrm{comp}^\sim(\mathcal{H}_{L_0})$ is the subset of $\mathrm{comp}^\sim(\mathcal{H})$ consisting of those $\sim$-chambers C for which $C \cap L_0 \neq \emptyset$ (and hence $C \subset L_0$).

If $C \in \mathrm{comp}^\sim(\mathcal{H})$ we denote by $L(C)$ the (unique) element $L \in \mathcal{L}$ for which $C \in \mathrm{comp}^\sim(L,\mathcal{H}_L)$, and we put $\dim C = \dim L(C)$. Let $\iota\colon \mathrm{comp}^\sim(\mathcal{H}) \to \mathrm{comp}(\mathcal{H})$ be given by $\iota(C) = \iota_{L(C)}(C)$. Furthermore, let $\mathrm{ressupp}(C,t) = \mathrm{ressupp}(\iota(C),t)$ for $t \in \mathrm{WT}(\mathcal{H})$.

If $p, q \in V$ we write $[p;q]$ for the line segment $\{(1-t)p + tq \mid t \in [0;1]\}$ from p to q, and $[p;q[:= [p;q] \setminus \{q\}$.

Lemma 2.4. *Let $C_0 \in \mathrm{comp}^\sim(\mathcal{H})$ be given and put $L_0 = L(C_0)$. Let $p \in C_0$. The set*

$$\beta(p) = \{C \in \mathrm{comp}^\sim(\mathcal{H}) \mid \dim C < \dim C_0,\ [p;c(L_0)[\cap C \neq \emptyset\}$$

is independent of p. Moreover, $\dim C = \dim C_0 - 1$ for each $C \in \beta(p)$, and $[p;c(L_0)] \cap C$ has exactly one element. Denote this element by $q(p,C)$, then

$$[p;c(L_0)[\cap \mathrm{sing}^\sim(L_0,\mathcal{H}_{L_0}) = \{q(p,C) \mid C \in \beta(p)\}. \tag{2.3}$$

Proof. Fix $C \in \beta(p)$ and let $q \in [p; c(L_0)[\cap C$. It follows from the observation below Lemma 2.3 that $L := L(C)$ is a hyperplane in L_0, and that $[p; c(L_0)] \cap L = \{q\}$. The hyperplane L separates C_0 from $c(L_0)$, hence $[\lambda; c(L_0)] \cap L$ consists of a single point $q(\lambda)$ for all $\lambda \in C_0$. Again by Lemma 2.3 we have $q(\lambda) \in \mathrm{reg}^\sim(L, \mathcal{H}_L)$ for all $\lambda \in C_0$. The map $\lambda \mapsto q(\lambda)$ is affine. Hence its image $q(C_0)$ is a convex subset of $\mathrm{reg}^\sim(L, \mathcal{H}_L)$, and as it contains $q = q(p)$ we conclude that $q(C_0) \subset C$. This shows that $C \in \beta(\lambda)$ for all $\lambda \in C_0$. Hence $\beta(p) \subset \beta(\lambda)$. The converse statement holds by symmetry of the argument. Thus $\beta(p)$ is independent of p.

It remains only to prove (2.3). That $q(p, C)$ belongs to $\mathrm{sing}^\sim(L_0, \mathcal{H}_{L_0})$ for each $C \in \beta(p)$ is clear. Conversely, if $q \in [p; c(L_0)[\, \cap \, \mathrm{sing}^\sim(L_0, \mathcal{H}_{L_0})$ then the observation below Lemma 2.3 shows that $q \in \mathrm{reg}^\sim(L, \mathcal{H}_L)$ for some $L \in \mathcal{H}_{L_0}$. Hence $q \in C$ for some $C \in \mathrm{comp}^\sim(L, \mathcal{H}_L)$. Hence $C \in \beta(p)$ and $q = q(p, C)$. $\square$

We write $\beta(C_0)$ for the set $\beta(p) \subset \mathrm{comp}^\sim(\mathcal{H})$ of the preceding lemma. We now define the partial order relation $\preceq_{\mathcal{H}}$ on $\mathrm{comp}^\sim(\mathcal{H})$ by $C' \preceq_{\mathcal{H}} C$ if and only if there exists an integer $k \geq 0$ and a sequence $C_0, \dots, C_k \in \mathrm{comp}^\sim(\mathcal{H})$ such that $C_0 = C$, $C_k = C'$, and

$$C_j \in \beta(C_{j-1})$$

for $0 < j \leq k$.

Notice that a $\sim$-chamber $C_0 \in \mathrm{comp}^\sim(\mathcal{H})$ is central (i.e., its closure contains $c(L(C_0))$) if and only if $\beta(C_0)$ is empty. Thus the central $\sim$-chambers are the minimal elements in $\mathrm{comp}^\sim(\mathcal{H})$ with respect to $\preceq_{\mathcal{H}}$.

It is easily seen that if $L_0 \in \mathcal{L}$ and $C_0 \in \mathrm{comp}^\sim(L_0, \mathcal{H}_{L_0})$, then a $\sim$-chamber $C' \in \mathrm{comp}^\sim(\mathcal{H})$ satisfies $C' \preceq_{\mathcal{H}} C_0$ if and only if it lies in L_0 and satisfies $C' \preceq_{\mathcal{H}_{L_0}} C_0$. In particular, $\preceq_{\mathcal{H}_{L_0}}$ equals the restriction of $\preceq_{\mathcal{H}}$ to $\mathrm{comp}^\sim(\mathcal{H}_{L_0})$.

2.4 Bounds on the residual support.

Proposition 2.5. *Let $\mathcal{H}$ be finite and $t \in \mathrm{WT}_c(\mathcal{H})$ a central weight. Then for every $C_0 \in \mathrm{comp}^\sim(V, \mathcal{H})$ and for every $L \in \mathrm{ressupp}(C_0, t)$ there exists a $\sim$-chamber $C \preceq_{\mathcal{H}} C_0$ such that*

$$c(L(C)) \in L \subset L(C). \tag{2.4}$$

Proof. For any $C_0 \in \mathrm{comp}^\sim(\mathcal{H})$ we denote by $\mathcal{L}[\preceq_{\mathcal{H}} C_0]$ the set of those $L \in \mathcal{L}$ for which there exist a $\sim$-chamber $C \preceq_{\mathcal{H}} C_0$ such that (2.4) holds. We must show that $\mathrm{ressupp}(C_0, t) \subset \mathcal{L}[\preceq_{\mathcal{H}} C_0]$ for $C_0 \in \mathrm{comp}^\sim(V, \mathcal{H})$. By the uniqueness of the residue operators (cf. Theorem 1.13) it suffices to prove that for every $C_0 \in \mathrm{comp}^\sim(V, \mathcal{H})$, $L \in \mathcal{L}[\preceq_{\mathcal{H}} C_0]$, there exists a Laurent operator $R_L \colon \mathcal{M}(V, \mathcal{H}) \to \mathcal{M}(L, \mathcal{H}_L)$ such that we have

$$\int_{\mathrm{pt}(C_0)+iV} \varphi \, d\mu_V = \sum_{L \in \mathcal{L}[\preceq_{\mathcal{H}} C_0]} \; \sum_{C' \in \mathrm{comp}(L, \mathcal{H}_L)} t(C') \int_{\mathrm{pt}(C')+iV_L} R_L \varphi \, d\mu_L \tag{2.5}$$

for all $\varphi \in \mathcal{P}(V, \mathcal{H})$. We shall achieve this by induction on $\dim V$.

Let $m \in \mathbb{N}$ and assume the existence of operators R_L such that (2.5) holds has been established for all pairs $(V, \mathcal{H})$ with $\dim V < m$ and all central residue weights $t \in \mathrm{WT}_c(\mathcal{H})$

(if $m = 0$ this is certainly all right, as there are no such pairs). Let a pair $(V, \mathcal{H})$ be given with $\dim V = m$, and let $t \in \mathrm{WT}_c(\mathcal{H})$ and $C_0 \in \mathrm{comp}^{\sim}(V, \mathcal{H})$.

Fix $p \in C_0$ and let $C'_1, \dots, C'_r$ be an enumeration (possibly empty) of the elements from $\beta(C_0)$. Then (cf. (2.3)) the hyperplanes $H_j := L(C'_j) \in \mathcal{H}$ cut the line segment $[p; 0[$ into $r+1$ disjoint, non-empty pieces:

$$[p; 0[= \cup_{j=0}^{r} [q_j; q_{j+1}[,$$

where $q_0 := p$, $q_j := q(p, C'_j)$ for $j = 1, \dots, r$, and $q_{r+1} := 0$ (we have assumed that the C'_j are numbered in suitable order). For each $j = 1, \dots, r$ there is a unique chamber $C_j \in \mathrm{comp}(V, \mathcal{H})$ such that $]q_j; q_{j+1}[\subset C_j$. Moreover, C_j is adjacent to C_{j-1}, and we have $C_{j-1} \wedge C_j = \iota(C'_j)$. The chamber C_r is central. It now follows from Proposition 1.12 that for all $\varphi \in \mathcal{P}(V, \mathcal{H})$ we have

$$\int_{p+iV} \varphi \, d\mu_V = \int_{\mathrm{pt}(C_r)+iV} \varphi \, d\mu_V + \sum_{j=1}^{r} \epsilon_j \int_{q_j + iV_{H_j}} \mathrm{Res}^V_{H_j} \varphi \, d\mu_{H_j} \tag{2.6}$$

with $\epsilon_j = \pm 1$.

By Theorem 1.13 and Lemma 2.2 we have

$$\int_{\mathrm{pt}(C_r)+iV} \varphi \, d\mu_V = \sum_{L \in \mathcal{L}, 0 \in L} \; \sum_{C' \in \mathrm{comp}(L, \mathcal{H}_L)} t(C') \int_{\mathrm{pt}(C')+iV_L} \mathrm{Res}_L^{C_r, t} \varphi \, d\mu_L.$$

If $0 \in L$, then $L \in \mathcal{L}[\preceq_{\mathcal{H}} C_0]$ because (2.4) holds with $C = C_0$. Hence the first term in (2.6) has the form desired for (2.5).

It remains to be seen that each of the terms

$$\int_{q_j + iV_{H_j}} \mathrm{Res}^V_{H_j} \varphi \, d\mu_{H_j}$$

in (2.6) also has the desired form. This follows easily from our induction hypothesis and Lemma 1.8 (use that $C \preceq_{\mathcal{H}_{H_j}} C'_j \Rightarrow C \preceq_{\mathcal{H}} C_0$). $\square$

Theorem 2.6. *Let $\mathcal{H}$ be a hyperplane configuration in V and $t \in \mathrm{WT}_c(\mathcal{H})$ a central weight. Let $C_0 \in \mathrm{comp}(V, \mathcal{H})$. Then*

$$|c(L)| \leq \inf_{\lambda \in C_0} |\lambda| \tag{2.7}$$

and

$$\langle c(L), \lambda \rangle \geq 0, \qquad (\lambda \in C_0) \tag{2.8}$$

for all $L \in \mathrm{ressupp}(C_0, t)$.

Proof. Fix $L \in \mathcal{L}$. It follows from Corollary 1.17 that we may assume that $\mathcal{H} = \mathcal{H}(L)$. In particular, then $\mathcal{H}$ is finite. In order to prove the inequalities (2.7) and (2.8) for a chamber $C_0 \in \mathrm{comp}(V, \mathcal{H})$ it suffices, by density, to establish them for each $\sim$-chamber inside C_0. We may therefore assume that $C_0 \in \text{comp}^{\sim}(V, \mathcal{H})$ and $L \in \text{ressupp}(C_0, t)$. According to Proposition 2.5 there exists a $\sim$-chamber $C \preceq_{\mathcal{H}} C_0$ such that (2.4) holds. Then $c(L) = c(L(C))$. Let $C_1, \ldots, C_k \in \text{comp}^{\sim}(\mathcal{H})$ with $C_k = C$ and $C_j \in \beta(C_{j-1})$ for $j = 1, \ldots, k$. Let $\lambda_0 \in C_0$ be arbitrary and determine $\lambda_j \in C_j$ for $j = 1, \ldots, k$ recursively such that $\lambda_j \in [\lambda_{j-1}; c(L(C_{j-1}))[\cap C_j$. Then $|\lambda_j| \leq |\lambda_{j-1}|$ for $j = 1, \ldots, k$, and since $\lambda_k \in C$ we also have $|c(L(C))| \leq |\lambda_k|$. We conclude that $|c(L)| \leq |\lambda_0|$.

Put $L_j = L(C_j)$ and $c_j = c(L_j)$ for $j = 0, \ldots, k$, then $c_0 = 0$, $c_k = c(L)$ and

$$V = L_0 \supset L_1 \supset \ldots \supset L_k = L(C) \supset L.$$

Let $j = 1, \ldots, k$. Then $\langle \nu, c_{j-1} \rangle = \langle c_{j-1}, c_{j-1} \rangle \leq \langle c_j, c_j \rangle = \langle \nu, c_j \rangle$ for all $\nu \in L_j$. Hence

$$\langle \lambda_j, c(L) - c_{j-1} \rangle \geq \langle \lambda_j, c(L) - c_j \rangle.$$

Since $\lambda_j \in [\lambda_{j-1}, c_{j-1}[$ and $c_{j-1} \perp c(L) - c_{j-1}$ we have

$$\langle \lambda_{j-1}, c(L) - c_{j-1} \rangle = t_j \langle \lambda_j, c(L) - c_{j-1} \rangle$$

for some $t_j \geq 1$. Hence $\langle \lambda_{j-1}, c(L) - c_{j-1} \rangle \geq t_j \langle \lambda_j, c(L) - c_j \rangle$ for $j = 1, \ldots, k$, and since $\langle \lambda_k, c(L) - c_k \rangle = \langle \lambda_k, 0 \rangle = 0$ we conclude that $\langle \lambda_0, c(L) \rangle = \langle \lambda_0, c(L) - c_0 \rangle \geq 0$. $\square$

3 The residue scheme for root systems

In this final section we assume Σ to be a (possibly non-reduced) root system in the finite dimensional real inner product space V. Let ${}^\circ V$ denote the span of Σ, and V_0 its orthocomplement in V; we do not require that ${}^\circ V = V$. We shall apply the theory developed so far to meromorphic functions with singular hyperplanes of the form $c + \alpha^\perp$, with $c \in V$, $\alpha \in \Sigma$.

3.1 Admissible hyperplane configurations. By definition an affine root hyperplane in V (with respect to Σ) is an affine hyperplane H for which there exists a root $\alpha \in \Sigma$ such that $V_H = \alpha^\perp$. Thus $H = c(H) + \alpha^\perp = H_{\alpha,s}$, where

$$H_{\alpha,s} := \{\lambda \in V \mid \langle \lambda, \alpha \rangle = s\} \tag{3.1}$$

and $s = \langle c(H), \alpha \rangle \in \mathbb{R}$. Let $\mathcal{H}_\Sigma$ denote the set of all affine root hyperplanes in V and $\mathcal{H}_\Sigma(0) = \{\alpha^\perp \mid \alpha \in \Sigma\}$ the (finite) subset of the hyperplanes that contain 0. An affine hyperplane configuration $\mathcal{H}$ in V is called Σ-admissible if $\mathcal{H} \subset \mathcal{H}_\Sigma$, that is if it consists of affine root hyperplanes. Notice that $\mathcal{H}_\Sigma$ itself is not an affine hyperplane configuration, since it is not locally finite (unless $\dim {}^\circ V = 0$).

A root space in V (with respect to Σ) is defined to be a linear subspace $\mathfrak{b}$ in V of the form $\mathfrak{b} = \alpha_1^\perp \cap \ldots \cap \alpha_l^\perp$ for some roots $\alpha_1, \ldots, \alpha_l \in \Sigma$; we agree that V itself is a root space. Let $\mathcal{R} = \mathcal{R}_\Sigma$ denote the set of root spaces, and let $\mathcal{L}_\Sigma$ be the set of all affine subspaces L of V for which $V_L \in \mathcal{R}$. The elements of $\mathcal{L}_\Sigma$ and $\mathcal{R}$ are the non-empty intersections of hyperplanes from $\mathcal{H}_\Sigma$ and $\mathcal{H}_\Sigma(0)$, respectively. Given $L \in \mathcal{L}_\Sigma$ we put

$$\mathcal{H}_\Sigma(L) = \{H \in \mathcal{H}_\Sigma \mid H \supset L\} = \{c(L) + \alpha^\perp \mid \alpha \in \Sigma,\ \alpha \perp V_L\}; \tag{3.2}$$

this is a finite set, hence a Σ-admissible hyperplane configuration. The set of intersections associated with this configuration is

$$\mathcal{L}_\Sigma(L) := \{L' \in \mathcal{L}_\Sigma \mid L' \supset L\} = \{c(L) + \mathfrak{b} \mid \mathfrak{b} \in \mathcal{R},\ \mathfrak{b} \supset V_L\}.$$

In particular we have $\mathcal{H}_\Sigma(V_0) = \mathcal{H}_\Sigma(0)$ and $\mathcal{L}_\Sigma(V_0) = \mathcal{R}$.

Given $\mathfrak{b} \in \mathcal{R}$ we write $\mathrm{sing}(\mathfrak{b}, \Sigma)$ and $\mathrm{reg}(\mathfrak{b}, \Sigma)$ for the sets of singular, resp. regular, elements in $\mathfrak{b}$, associated with the hyperplane configuration $\mathcal{H}_\Sigma(0)$. This means that

$$\mathrm{sing}(\mathfrak{b}, \Sigma) = \bigcup_{\alpha \in \Sigma \setminus \mathfrak{b}^\perp} \alpha^\perp \cap \mathfrak{b}, \qquad \mathrm{reg}(\mathfrak{b}, \Sigma) = \mathfrak{b} \setminus \mathrm{sing}(\mathfrak{b}, \Sigma).$$

As usual the connected components of the latter set are called the chambers of $\mathfrak{b}$; we write $\mathcal{P}(\mathfrak{b})$ for the set ($= \mathrm{comp}(\mathfrak{b}, \mathcal{H}_\Sigma(0)_\mathfrak{b})$) of these, and $\mathcal{P}$ for the set ($= \mathrm{comp}(\mathcal{H}_\Sigma(0))$) of all chambers of all $\mathfrak{b} \in \mathcal{R}$:

$$\mathcal{P} = \cup_{\mathfrak{b} \in \mathcal{R}} \mathcal{P}(\mathfrak{b}).$$

This union is disjoint; if $P \in \mathcal{P}$ there is a unique root space $\mathfrak{b}_P \in \mathcal{R}$ such that $P \in \mathcal{P}(\mathfrak{b})$. Notice that if $P \in \mathcal{P}(\mathfrak{b})$ then the subset $-P$ of $\mathfrak{b}$ also belongs to $\mathcal{P}(\mathfrak{b})$; it is called the chamber opposite to P. The set $\mathcal{P}$ is called the Coxeter complex.

Notice that if $\mathfrak{b}$ is a root space, then the set $\Sigma_{\mathfrak{b}^\perp} := \Sigma \cap \mathfrak{b}^\perp$ is a root system in the subspace $\mathfrak{b}^\perp$ of V. Notice also that W, the Weyl group of Σ, acts on $\mathcal{R}$: If $\mathfrak{b} = \alpha_1^\perp \cap \ldots \cap \alpha_l^\perp$ and $w \in W$ then $w\mathfrak{b} = \{w\lambda \mid \lambda \in \mathfrak{b}\} = (w\alpha_1)^\perp \cap \ldots \cap (w\alpha_l)^\perp$. Moreover, $w(\mathrm{reg}(\mathfrak{b}, \Sigma)) = \mathrm{reg}(w\mathfrak{b}, \Sigma)$. Hence there is also a natural action of W on $\mathcal{P}$.

The set $\mathcal{P}(V)$ is in one-to-one correspondence with the set of positive systems for Σ; the correspondence is given by

$$P \leftrightarrow \Sigma(P) := \{\alpha \in \Sigma \mid \alpha > 0 \text{ on } P\}.$$

Let $P \in \mathcal{P}(V)$ be given. Each affine root hyperplane $H \in \mathcal{H}_\Sigma$ has the form (3.1) with $\alpha \in \Sigma(P)$ and $s \in \mathbb{R}$. Let $V^+(P, H)$ denote the component of $V \setminus H$ pointed at by α, and $V^-(P, H)$ the other component. Then

$$V^\pm(P, H) = \{\lambda \in V \mid \langle \lambda, \alpha \rangle \gtrless s\}.$$

Furthermore, if $\mathcal{H}$ is a Σ-admissible hyperplane configuration we put

$$V^\pm(P, \mathcal{H}) = \cap_{H \in \mathcal{H}} V^\pm(P, H). \tag{3.3}$$

Clearly if $V^+(P, \mathcal{H})$ or $V^-(P, \mathcal{H})$ is not empty, it belongs to $\mathrm{comp}(V, \mathcal{H})$. We say that $\mathcal{H}$ is P-bounded if there exists $s_0 \in \mathbb{R}$ such that if $H_{\alpha,s} \in \mathcal{H}$ for some $\alpha \in \Sigma(P)$, $s \in \mathbb{R}$, then $s \geq s_0$.

Lemma 3.1. *Let $P \in \mathcal{P}(V)$ and let $\mathcal{H}$ be a Σ-admissible hyperplane configuration. The following properties of $\mathcal{H}$ are equivalent:*

(i) *$\mathcal{H}$ is P-bounded,*

(ii) *$V^-(P, \mathcal{H}) \neq \emptyset$,*

(iii) $\exists \lambda_0 \in V : \lambda_0 - P \subset \mathrm{reg}(V, \mathcal{H})$.

Proof. (i) $\Rightarrow$ (ii). Let s_0 be as above, and choose $\lambda_0 \in V$ such that $\langle \lambda_0, \alpha \rangle < s_0$ for all $\alpha \in \Sigma(P)$. Then $\lambda_0 \in V^-(P, H_{\alpha,s})$ for all $\alpha \in \Sigma(P)$ and $s \geq s_0$, and hence $\lambda_0 \in V^-(P, \mathcal{H})$.

(ii) $\Rightarrow$ (iii). Take $\lambda_0 \in V^-(P, \mathcal{H})$. Then $\lambda_0 - P \subset V^-(P, \mathcal{H}) \subset \mathrm{reg}(V, \mathcal{H})$.

(iii) $\Rightarrow$ (i). Suppose $\lambda_0 - P \subset \mathrm{reg}(V, \mathcal{H})$, and let $s_0 = \min_{\alpha \in \Sigma(P)} \langle \lambda_0, \alpha \rangle$. If $H = H_{\alpha,s} \in \mathcal{H}$, where $\alpha \in \Sigma(P)$, then $(\lambda_0 - P) \cap H = \emptyset$. Hence $\langle \lambda_0 - \lambda, \alpha \rangle \neq s$ for all $\lambda \in P$, from which it easily follows that $\langle \lambda_0, \alpha \rangle \leq s$. Thus $s_0 \leq s$. $\square$

3.2 Residue weights.

Definition 3.2. The elements of $\mathrm{WT}(\Sigma) := \mathrm{WT}(\mathcal{H}_\Sigma(0))$ are called residue weights associated with Σ. Thus, by definition, these are the functions $t\colon \mathcal{P} \to [0;1]$ such that $\sum_{P \in \mathcal{P}(\mathfrak{b})} t(P) = 1$ for all $\mathfrak{b} \in \mathcal{R}$.

For $t \in \mathrm{WT}(\Sigma)$ and $w \in W$ we define $wt \in \mathrm{WT}(\Sigma)$ by $wt(P) = t(w^{-1}P)$ for $P \in \mathcal{P}$. Likewise, we define $t^\vee \in \mathrm{WT}(\Sigma)$ by $t^\vee(P) = t(-P)$. If $wt = t$ for all $w \in W$, resp. if $t^\vee = t$, we call t Weyl invariant, resp. even.

Example 3.3. The map $P \mapsto 1/|\mathcal{P}(\mathfrak{b}_P)|^{-1}$ is a residue weight. We call it the standard weight. It is both Weyl invariant and even.

Our goal in this subsection is to define a suitable map from $\mathrm{WT}(\Sigma)$ to $\mathrm{WT}_c(\mathcal{H})$, for each Σ-admissible hyperplane configuration $\mathcal{H}$. For this we need the following lemma:

Lemma 3.4. *Let a chamber $Q \in \mathcal{P}(V)$ be given, and let $\mathcal{H}$ be a Σ-admissible hyperplane configuration. Then there exists a unique central chamber $C \in \mathrm{comp}_c(V, \mathcal{H})$ for which $Q \cap C \neq \emptyset$.*

Proof. Since $\mathcal{H}$ is locally finite there exists a positive number ϵ such that $0 \in H$ for all hyperplanes $H \in \mathcal{H}$ that meet the open ball $B_\epsilon := B(0, \epsilon)$ in V. Moreover, since $\mathcal{H}$ is Σ-admissible such a hyperplane is contained in $\mathrm{sing}(V, \Sigma)$. It follows that

$$\emptyset \neq B_\epsilon \cap Q \subset B_\epsilon \cap \mathrm{reg}(V, \Sigma) \subset B_\epsilon \cap \mathrm{reg}(V, \mathcal{H}) \subset \cup_{C \in \mathrm{comp}_c(V, \mathcal{H})} C.$$

Moreover, for $C \in \mathrm{comp}_c(V, \mathcal{H})$ we have $Q \cap C \neq \emptyset$ if and only if $B_\epsilon \cap Q \cap C \neq \emptyset$, since Q and C are both central and stable under contraction. However, since $B_\epsilon \cap Q$ is convex, it follows from the above inclusions that $B_\epsilon \cap Q \cap C \neq \emptyset$ for one and only one chamber $C \in \mathrm{comp}_c(V, \mathcal{H})$. $\square$

Let $\mathcal{H}$ be as above and let $L \in \mathcal{L} = \mathcal{L}_\mathcal{H}$. Then it follows from Lemma 3.4 that for each chamber $Q \in \mathcal{P}(V_L)$ there is a unique central chamber $C_Q = C_{Q,L,\mathcal{H}} \in \mathrm{comp}_c(L, \mathcal{H}_L)$ intersecting non-trivially with $c(L) + Q$.

Let now $t \in \mathrm{WT}(\Sigma)$ be given. We define a map $\omega_\mathcal{H}(t)\colon \mathrm{comp}(\mathcal{H}) \to [0;1]$ as follows. Let $L \in \mathcal{L}$ and $C \in \mathrm{comp}(L, \mathcal{H}_L)$. Then

$$\omega_\mathcal{H}(t)(C) := \sum_{Q \in \mathcal{P}(V_L),\, C_Q = C} t(Q) \tag{3.4}$$

if C is central in L, and $\omega_\mathcal{H}(t)(C) := 0$ otherwise. It is straightforward to check that $\omega_\mathcal{H}(t) \in \mathrm{WT}_c(\mathcal{H})$.

3.3 Laurent operators. Let $L \in \mathcal{L}_\Sigma$ and let $\mathcal{H}_\Sigma(L)$ be the finite hyperplane configuration in V given by (3.2). The Laurent operators $R \in \mathrm{Laur}(V, L, \mathcal{H}_\Sigma(L))$ map $\mathcal{M}(V, \mathcal{H}_\Sigma(L))$ into $\mathcal{O}(L_\mathbb{C})$. Fix a chamber $P \in \mathcal{P}(V)$, and let $\bar{\Sigma}(P)$ denote the set of indivisible roots in $\Sigma(P)$. For each $H \in \mathcal{H}_\Sigma(L)$ we require that the chosen normal vector α_H (see Section 1.1) belongs to $\bar{\Sigma}(P)$ (it is then unique). Let $\mathfrak{b} = V_L$. As in Section 1.3 (see (1.10)) we form the projective limit $S_\leftarrow(\mathfrak{b}^\perp, X)$, $X = \bar{\Sigma}(P) \cap \mathfrak{b}^\perp$. The space $S_\leftarrow(\mathfrak{b}^\perp, X)$ is isomorphic to $\mathrm{Laur}(V, L, \mathcal{H}_\Sigma(L))$ (cf. Lemma 1.5); we denote it by $S_\leftarrow(\mathfrak{b}^\perp, P)$. The map $u \mapsto R = R_u$ that takes an element $u \in S_\leftarrow(\mathfrak{b}^\perp, P)$ into $\mathrm{Laur}(V, L, \mathcal{H}_\Sigma(L))$ is given by (1.9), that is, by

$$R\varphi(\lambda) = u_d(\varpi_{X,d}\varphi_\lambda)(0), \qquad (\lambda \in L_\mathbb{C}), \tag{3.5}$$

for $d \in \mathbb{N}^{\mathcal{H}_\Sigma(L)} \simeq \mathbb{N}^{\bar{\Sigma}(P)\cap\mathfrak{b}^\perp}$, $\varphi \in \mathcal{M}(V, \mathcal{H}_\Sigma(L), d)$. Here

$$\varphi_\lambda \colon \mathfrak{b}^\perp \ni \nu \mapsto \varphi(\lambda + \nu),$$

and

$$\varpi_{X,d} \colon \mathfrak{b}^\perp \ni \nu \mapsto \prod_{\alpha \in \bar{\Sigma}(P)\cap\mathfrak{b}^\perp} \langle \alpha, \nu \rangle^{d(\alpha)}.$$

In particular, we emphasize that we have in $S_\leftarrow(\mathfrak{b}^\perp, P)$ a model for $\mathrm{Laur}(V, L, \mathcal{H}_\Sigma(L))$ that depends only on L through its tangent space $\mathfrak{b} = V_L \in \mathcal{R}$.

Let $\mathcal{H}$ be an arbitrary Σ-admissible hyperplane configuration in V, and let $L \in \mathcal{L}_\Sigma$ and $P \in \mathcal{P}(V)$ be given. Again we require that the normal vector α_H has been taken from $\bar{\Sigma}(P)$ for all $H \in \mathcal{H}$. Let $\mathcal{H}(L)$ be defined by (1.3), then $\mathcal{H}(L) \subset \mathcal{H}_\Sigma(L)$. Let $\mathfrak{b} = V_L$. Given an element $d \in \mathbb{N}^{\mathcal{H}(L)}$ we extend it trivially to an element of $\mathbb{N}^{\mathcal{H}_\Sigma(L)} \simeq \mathbb{N}^{\bar{\Sigma}(P)\cap\mathfrak{b}^\perp}$ (that is, so that it vanishes outside $\mathcal{H}(L)$). Then the polynomial q_L defined in (1.5) is related to the polynomial $\varpi_{X,d}$ defined above by $q_{L,d}(\lambda + \nu) = \varpi_{X,d}(\nu)$ for $\lambda \in L$, $\nu \in V_L^\perp$ (cf. (1.12)). It follows that (3.5) makes sense for $\varphi \in \mathcal{M}(V, \mathcal{H}, d)$, and moreover that in this way we obtain a Laurent operator $R = R_u \colon \mathcal{M}(V, \mathcal{H}) \to \mathcal{M}(L, \mathcal{H}_L)$. In conclusion, there is a natural linear map from $S_\leftarrow(\mathfrak{b}^\perp, P)$ to $\mathrm{Laur}(V, L, \mathcal{H})$, for all Σ-admissible hyperplane configurations $\mathcal{H}$ in V and all $L \in \mathcal{L}_\Sigma$ with $V_L = \mathfrak{b}$, and if $\mathcal{H} = \mathcal{H}_\Sigma(L)$ then this map is an isomorphism.

3.4 The universal residue operator. Let $L \in \mathcal{L}_\Sigma$ and let $\mathcal{H}_\Sigma(L)$ be the finite hyperplane configuration in V given by (3.2). Fix $P \in \mathcal{P}(V)$ and let $V^-(P, L) = V^-(P, \mathcal{H}_\Sigma(L))$ be the chamber in $\mathrm{comp}(V, \mathcal{H}_\Sigma(L))$ defined by (3.3); we have $V^-(P, L) \neq \emptyset$ because $\mathcal{H}_\Sigma(L)$ is finite (use Lemma 3.1). Finally, let $t \in \mathrm{WT}(\Sigma)$ be given and put $\omega_L(t) = \omega_{\mathcal{H}_\Sigma(L)}(t)$. We define the residue operator associated with the data L, P, t by

$$\mathrm{Res}_L^{P,\,t} := \mathrm{Res}_L^{V^-(P,L),\,\omega_L(t)} \colon \mathcal{M}(V, \mathcal{H}_\Sigma(L)) \to \mathcal{O}(L_\mathbb{C}). \tag{3.6}$$

Let $\mathfrak{b} = V_L$. As described in the previous subsection the residue operator (3.6) is given by a unique element in the projective limit space $S_\leftarrow(\mathfrak{b}^\perp, P)$; we denote this element by $\mathrm{Res}_L^{P,\,t}$ as well, and call it the universal residue operator associated with the data L, P, t. It also follows from the previous subsection that it makes sense to apply this element to functions in $\mathcal{M}(V, \mathcal{H})$ for any Σ-admissible hyperplane configuration $\mathcal{H}$; it gives a Laurent operator from $\mathcal{M}(V, \mathcal{H})$ to $\mathcal{M}(L, \mathcal{H}_L)$. In particular, if $L \in \mathcal{L}_\Sigma \setminus \mathcal{L}_\mathcal{H}$ then it follows easily from Corollary 1.16 that $\mathrm{Res}_L^{P,\,t} \varphi = 0$ for $\varphi \in \mathcal{M}(V, \mathcal{H})$.

Example 3.5. Let $V = \mathbb{R}$, $\Sigma = \{\pm\alpha\}$, $P = \{x > 0\}$, and let $t \in \mathrm{WT}(\Sigma)$ be given by $t(P) = t(-P) = 1/2$, $t(\{0\}) = 1$. Fix $\lambda \in \mathbb{R}$ and let $L = \{\lambda\}$. There are exactly two chambers in $\mathrm{comp}(V, \mathcal{H}_\Sigma(L))$, they are the sets $V^-(P, L)$ and $V^+(P, L)$ given by the inequalities $x < \lambda$ and $x > \lambda$, respectively. The induced weight $\omega_L(t)$ takes the following values on these chambers. If $\lambda < 0$ then $\omega_L(t)(V^-(P, L)) = 0$ and $\omega_L(t)(V^+(P, L)) = 1$; if $\lambda > 0$ then $\omega_L(t)(V^-(P, L)) = 1$ and $\omega_L(t)(V^+(P, L)) = 0$; if $\lambda = 0$ then $\omega_L(t)(V^-(P, L)) = \omega_L(t)(V^+(P, L)) = 1/2$. It then follows from the residue theorem that

$$\mathrm{Res}_{\{\lambda\}}^{P,\,t} = \begin{cases} -2\pi u & \text{if } \lambda < 0, \\ -\pi u & \text{if } \lambda = 0, \\ 0 & \text{if } \lambda > 0. \end{cases}$$

where u is the element of $S_{\leftarrow}(\mathfrak{b}^\perp, P)$ that corresponds to the operator $\varphi \mapsto \mathrm{Res}_{z=\lambda}\, \varphi(z)$; it is independent of λ, cf. Example 1.6.

Let $\mathcal{H}$ be a P-bounded Σ-admissible hyperplane configuration in V, then we have $V^-(P, \mathcal{H}) \in \mathrm{comp}(V, \mathcal{H})$ (cf. Lemma 3.1). Let $L \in \mathcal{L}_\mathcal{H}$, and let $\mathcal{H}_\Sigma(L)$ be as in (3.2). Then $\mathcal{H}(L) \subset \mathcal{H}_\Sigma(L)$, where $\mathcal{H}(L)$ is given in (1.3). We shall now identify the residue operator

$$\mathrm{Res}_L^{V^-(P,\mathcal{H}),\,\omega_\mathcal{H}(t)} \colon \mathcal{M}(V, \mathcal{H}) \to \mathcal{M}(L, \mathcal{H}_L)$$

in terms of the element $\mathrm{Res}_L^{P,\,t}$, which was defined independently of $\mathcal{H}$.

Proposition 3.6. *Let $t \in \mathrm{WT}(\Sigma)$, $P \in \mathcal{P}(V)$, let $\mathcal{H}$ be a P-bounded Σ-admissible hyperplane configuration, and let $L \in \mathcal{L}_\mathcal{H}$. Then*

$$\mathrm{Res}_L^{V^-(P,\mathcal{H}),\,\omega_\mathcal{H}(t)} \varphi = \mathrm{Res}_L^{P,\,t} \varphi$$

for all $\varphi \in \mathcal{M}(V, \mathcal{H})$.

For the proof we need the following lemma. Let $\mathcal{H}_1 \subset \mathcal{H}_2$ be Σ-admissible hyperplane configurations, and let the map $i^* \colon \mathrm{WT}(\mathcal{H}_2) \to \mathrm{WT}(\mathcal{H}_1)$ be defined as in (1.22).

Lemma 3.7. *We have $i^*(\omega_{\mathcal{H}_2}(t)) = \omega_{\mathcal{H}_1}(t)$ for all $t \in \mathrm{WT}(\Sigma)$.*

Proof. Let $L \in \mathcal{L}_{\mathcal{H}_1}$ and $C_1 \in \mathrm{comp}(L, (\mathcal{H}_1)_L)$. It is easily seen that it suffices to show the following: Let $Q \in \mathcal{P}(V_L)$. Then

$$C_1 = C_{Q,L,\mathcal{H}_1} \Leftrightarrow C_{Q,L,\mathcal{H}_2} \in \mathrm{comp}(C_1, (\mathcal{H}_2)_L). \tag{3.7}$$

Recall that $C_{Q,L,\mathcal{H}_j} \in \mathrm{comp}(L, (\mathcal{H}_j)_L)$ is the unique central chamber for which

$$C_{Q,L,\mathcal{H}_j} \cap [c(L) + Q] \neq \emptyset.$$

This property immediately implies (3.7). $\square$

Proof of Proposition 3.6. Let $\mathcal{H}_1 = \mathcal{H}(L)$, then $\mathcal{H}_1 \subset \mathcal{H}$ and $V^-(P, \mathcal{H}_1) \supset V^-(P, \mathcal{H})$, as well as $\mathcal{H}_1 \subset \mathcal{H}_\Sigma(L)$ and $V^-(P, \mathcal{H}_1) \supset V^-(P, L)$. By the preceding lemma we have $i^*(\omega_\mathcal{H}(t)) = \omega_{\mathcal{H}_1}(t)$ as well as $i^*(\omega_L(t)) = \omega_{\mathcal{H}_1}(t)$, and by Proposition 1.15 we then have

$$\mathrm{Res}_L^{V^-(P,\mathcal{H}),\, \omega_\mathcal{H}(t)} |_{\mathcal{M}(V,\mathcal{H}_1)} = \mathrm{Res}_L^{V^-(P,\mathcal{H}_1),\, \omega_{\mathcal{H}_1}(t)}$$

as well as

$$\mathrm{Res}_L^{P,\, t} |_{\mathcal{M}(V,\mathcal{H}_1)} = \mathrm{Res}_L^{V^-(P,\mathcal{H}_1),\, \omega_{\mathcal{H}_1}(t)}.$$

The proposition follows immediately, since a Laurent operator $\mathcal{M}(V, \mathcal{H}) \to \mathcal{M}(L, \mathcal{H}_L)$ is uniquely determined by its restriction to $\mathcal{M}(V, \mathcal{H}_1)$ (see Lemma 1.4). □

3.5 The action of the Weyl group. The Weyl group W acts orthogonally on V and it preserves Σ. Hence it also acts on $\mathcal{H}_\Sigma$ and $\mathcal{L}_\Sigma$. We shall now see how this action affects the residue operators.

Lemma 3.8. *Let $\mathcal{H}$ be a Σ-admissible hyperplane configuration in V, and let $w \in W$. Then $w\mathcal{H} = \{wH \mid H \in \mathcal{H}\}$ is also Σ-admissible, and if $L \in \mathcal{L}_\mathcal{H}$ then w maps* $\mathrm{comp}(L, \mathcal{H}_L)$ *bijectively onto* $\mathrm{comp}(wL, (w\mathcal{H})_{wL})$. *Moreover, if $t \in \mathrm{WT}(\Sigma)$ then*

$$\omega_{w\mathcal{H}}(t)(wC) = \omega_\mathcal{H}(w^{-1}t)(C) \tag{3.8}$$

for all $C \in \mathrm{comp}(\mathcal{H})$.

Proof. The first statements are straightforward to verify. The equality in (3.8) follows from (3.4) and Definition 3.2, once it has been observed that if $Q \in \mathcal{P}(V_L)$ then $wQ \in \mathcal{P}(V_{wL})$ and $wC_{Q,L,\mathcal{H}} = C_{wQ,wL,w\mathcal{H}}$. This latter observation is also straightforward (cf. Lemma 3.4). □

We shall now apply Lemma 1.18. Notice that the operator $w' \colon S(V) \to S(V)$ obtained from (1.25) is just the natural action of w. We denote this operator, as well as the corresponding operator in (1.26), by w.

Corollary 3.9. *Let $\mathcal{H}$, w, L, and t be as in Lemma 3.8, and let $C \in \mathrm{comp}(\mathcal{H})$. Then*

$$w\, \mathrm{Res}_L^{C,\, \omega_\mathcal{H}(t)} = \mathrm{Res}_{wL}^{wC,\, \omega_{w\mathcal{H}}(wt)}.$$

If $\mathcal{H}$ is P-bounded for some $P \in \mathcal{P}(V)$, then $w\mathcal{H}$ is wP-bounded and

$$w\, \mathrm{Res}_L^{V^-(P,\mathcal{H}),\, \omega_\mathcal{H}(t)} = \mathrm{Res}_{wL}^{V^-(wP,w\mathcal{H}),\, \omega_{w\mathcal{H}}(wt)}.$$

Proof. The first statement follows immediately from Lemma 3.8 in combination with Lemma 1.18. The other statements then follow from the observation that $wV^-(P, \mathcal{H}) = V^-(wP, w\mathcal{H})$. □

Proposition 3.10. *Let $P \in \mathcal{P}(V)$, $L \in \mathcal{L}_\Sigma$, $t \in \mathrm{WT}(\Sigma)$, and $w \in W$. Then*

$$w\, \mathrm{Res}_L^{P,\, t} = \mathrm{Res}_{wL}^{wP,\, wt}. \tag{3.9}$$

If
$$w(\Sigma(P)\cap V_L^\perp)\subset \Sigma(P), \tag{3.10}$$
then we also have
$$w\,\mathrm{Res}_L^{P,\,t}=\mathrm{Res}_{wL}^{P,\,wt}. \tag{3.11}$$

Proof. Put $\mathcal{H}=\mathcal{H}_\Sigma(L)$. Then $w\mathcal{H}=\mathcal{H}_\Sigma(wL)$, and we obtain (3.9) from (3.6) and Corollary 3.9. Assume (3.10). We claim that then
$$\mathrm{Res}_L^{P,\,t}=\mathrm{Res}_L^{w^{-1}P,\,t}. \tag{3.12}$$
By the definition of $\mathrm{Res}_L^{P,\,t}$ it suffices to show that $V^-(P,\mathcal{H})=V^-(w^{-1}P,\mathcal{H})$, and for this it suffices to show that $V^-(P,H)=V^-(w^{-1}P,H)$ for all hyperplanes $H\in\mathcal{H}$. Such a hyperplane is of the form $c(L)+\alpha^\perp$ with $\alpha\in\Sigma\cap V_L^\perp$ (cf. (3.2)), and we must then show that $\alpha\in\Sigma(P)$ if and only if $\alpha\in\Sigma(w^{-1}P)$. This follows easily from (3.10). Hence (3.12) holds, and by application of w to both sides of it we obtain (3.11) after use of (3.9). $\square$

Notice that we may regard (3.9) as an identity in the space $S_{\leftarrow}(wV_L^\perp,wP)$. When (3.10) holds we have $wV_L^\perp\cap\Sigma(wP)=wV_L^\perp\cap\Sigma(P)$, hence in this case $S_{\leftarrow}(wV_L^\perp,wP)=S_{\leftarrow}(wV_L^\perp,P)$, and we may similarly regard (3.11) as an identity in the latter space.

By arguments similar to those leading up to (3.9) we obtain the following identity
$$(\mathrm{Res}_L^{P,\,t})^\vee=\mathrm{Res}_{-L}^{-P,\,t^\vee}\in S_{\leftarrow}(V_L^\perp,-P), \tag{3.13}$$
where the element on the left-hand side has been defined by means of the principal automorphism $u\mapsto u^\vee$ of $S(V)$ determined from $X^\vee:=-X$ $(X\in V)$; it is easily seen that this automorphism induces a map from $S_{\leftarrow}(V_L^\perp,P)$ to $S_{\leftarrow}(V_L^\perp,-P)$.

3.6 Transitivity of residues. Let $\mathfrak{b}\in\mathcal{R}$. If $P\in\mathcal{P}(V)$ then $\Sigma(P)\cap\mathfrak{b}^\perp$ is a positive system for $\Sigma_{\mathfrak{b}^\perp}$. Let ${}^*P$ be the associated chamber of $\mathfrak{b}^\perp$, so that
$$\Sigma(P)\cap\mathfrak{b}^\perp=\Sigma_{\mathfrak{b}^\perp}({}^*P). \tag{3.14}$$
Alternatively, ${}^*P$ may be characterized as the unique chamber of $\mathfrak{b}^\perp$ for which
$$P\subset{}^*P+\mathfrak{b}. \tag{3.15}$$
More generally we have the following result. Let $\mathcal{P}_{\mathfrak{b}^\perp}$ denote the set of all chambers of all root spaces in $\mathfrak{b}^\perp$.

Lemma 3.11. *Let $\mathfrak{b}\in\mathcal{R}$, $P\in\mathcal{P}$, and assume that $\mathfrak{b}\subset\mathfrak{b}_P$. Then there is a unique chamber ${}^*P\in\mathcal{P}_{\mathfrak{b}^\perp}$ for which P is an open subset of ${}^*P+\mathfrak{b}$.*

Proof. Let ${}^*\mathfrak{b}_P=\mathfrak{b}_P\cap\mathfrak{b}^\perp\in\mathcal{R}_{\Sigma_{\mathfrak{b}^\perp}}$, then $\mathfrak{b}_P$ decomposes as the orthogonal direct sum ${}^*\mathfrak{b}_P+\mathfrak{b}$. We now have the following inclusions of open subsets:
$$P\subset\mathrm{reg}(\mathfrak{b}_P,\Sigma)\subset\mathrm{reg}({}^*\mathfrak{b}_P,\Sigma_{\mathfrak{b}^\perp})+\mathfrak{b}\subset\mathfrak{b}_P,$$

from which the result easily follows. $\square$

Let $t \in \mathrm{WT}(\Sigma)$ and define ${}^*t\colon \mathcal{P}_{\mathfrak{b}^\perp} \to [0;1]$ by

$$ {}^*t(Q) = \sum_{P\in\mathcal{P},\, \mathfrak{b}_P \supset \mathfrak{b},\, {}^*P=Q} t(P) \tag{3.16} $$

for $Q \in \mathcal{P}_{\mathfrak{b}^\perp}$. It is easily seen that ${}^*t \in \mathrm{WT}(\Sigma_{\mathfrak{b}^\perp})$. Moreover, if t is Weyl invariant or even, then so is ${}^*t$.

Let $\mathcal{H}$ be a Σ-admissible hyperplane configuration in V, and assume that $\mathfrak{b} \subset V_H$ for all $H \in \mathcal{H}$. Let ${}^*\mathcal{H} = \mathcal{H}_{\mathfrak{b}^\perp} = \{H \cap \mathfrak{b}^\perp \mid H \in \mathcal{H}\}$ (cf. (1.31)), then ${}^*\mathcal{H}$ is a hyperplane configuration in $\mathfrak{b}^\perp$, and $\mathcal{H}$ is its extension. It is easily seen that ${}^*\mathcal{H}$ is $\Sigma_{\mathfrak{b}^\perp}$-admissible. If $s \in \mathrm{WT}(\mathcal{H})$ we define ${}^*s = s_{\mathfrak{b}^\perp} \in \mathrm{WT}({}^*\mathcal{H})$ as in (1.37), that is by ${}^*s(C \cap \mathfrak{b}^\perp) = s(C)$ for $C \in \mathrm{comp}(\mathcal{H})$.

Lemma 3.12. *Let $\mathcal{H}$ be as above, and let $t \in \mathrm{WT}(\Sigma)$. Then $\omega_{{}^*\mathcal{H}}({}^*t) = {}^*[\omega_{\mathcal{H}}(t)]$.*

Proof. Let $L \in \mathcal{L}$ and let $P \in \mathcal{P}(V_L)$. Recall from the text following the proof of Lemma 3.4 that P determines a central chamber $C_P = C_{P,L,\mathcal{H}} \in \mathrm{comp}_c(L, \mathcal{H}_L)$ by the condition

$$ C_P \cap (c(L) + P) \neq \emptyset. \tag{3.17} $$

Let ${}^*L = L \cap \mathfrak{b}^\perp$, then $V_{{}^*L} = V_L \cap \mathfrak{b}^\perp$. Let $\mathcal{P}_{\mathfrak{b}^\perp}(V_{{}^*L})$ denote the set of $\Sigma_{\mathfrak{b}^\perp}$-chambers of $V_{{}^*L}$. Then, similarly, each $Q \in \mathcal{P}_{\mathfrak{b}^\perp}(V_{{}^*L})$ determines a unique central chamber $C_Q = C_{Q,{}^*L,{}^*\mathcal{H}} \in \mathrm{comp}_c({}^*L, {}^*\mathcal{H}_{{}^*L})$ by

$$ C_Q \cap (c(L) + Q) \neq \emptyset. \tag{3.18} $$

It follows from (3.15) and (3.17) that

$$ C_P \cap (c(L) + {}^*P + \mathfrak{b}) \neq \emptyset, $$

and since $C = (C \cap \mathfrak{b}^\perp) + \mathfrak{b}$ for all $C \in \mathrm{comp}(L, \mathcal{H}_L)$ this implies that

$$ (C_P \cap \mathfrak{b}^\perp) \cap (c(L) + {}^*P) \neq \emptyset. $$

Invoking (3.18) with $Q = {}^*P$ we conclude that

$$ C_{{}^*P} = C_P \cap \mathfrak{b}^\perp \tag{3.19} $$

for $P \in \mathcal{P}(V_L)$.

Let $C \in \mathrm{comp}_c(L, \mathcal{H}_L)$. It follows from (3.19) that we have the disjoint union

$$ \{P \in \mathcal{P}(V_L) \mid C_P = C\} = \bigcup_{Q\in\mathcal{P}_{\mathfrak{b}^\perp}(V_{{}^*L}),\, C_Q = C\cap\mathfrak{b}^\perp} \{P \in \mathcal{P}(V_L) \mid {}^*P = Q\}. $$

Hence we obtain from (3.4) and (3.16)

$$ \omega_{\mathcal{H}}(t)(C) = \sum_{Q\in\mathcal{P}_{\mathfrak{b}^\perp}(V_{{}^*L}),\, C_Q = C\cap\mathfrak{b}^\perp} {}^*t(Q) = \omega_{{}^*\mathcal{H}}({}^*t)(C \cap \mathfrak{b}^\perp), $$

and the lemma is proved. □

Lemma 3.13. *Let $\mathcal{H}$ be as in Lemma 3.12, and let $P \in \mathcal{P}(V)$. Then*

$$V^-(P, \mathcal{H}) \cap \mathfrak{b}^\perp = V^-({}^*P, {}^*\mathcal{H}).$$

Proof. It follows immediately from (3.14) that

$$V^-(P, H) \cap \mathfrak{b}^\perp = V^-({}^*P, H \cap \mathfrak{b}^\perp)$$

for all $H \in \mathcal{H}$. □

Let $L \in \mathcal{L}_\Sigma$ and assume that $V_L \supset \mathfrak{b}$. Then we can apply the preceding two lemmas to the configuration $\mathcal{H} = \mathcal{H}_\Sigma(L)$. Let ${}^*L = L \cap \mathfrak{b}^\perp$, then we have:

$$\begin{aligned} {}^*\mathcal{H} = {}^*[\mathcal{H}_\Sigma(L)] &= \{H \cap \mathfrak{b}^\perp \mid H \in \mathcal{H}_\Sigma, H \supset L\} \\ &= \{{}^*H \in \mathcal{H}_{\Sigma_{\mathfrak{b}^\perp}} \mid {}^*H \supset {}^*L\} = \mathcal{H}_{\Sigma_{\mathfrak{b}^\perp}}({}^*L), \end{aligned} \tag{3.20}$$

and $\mathcal{H}_\Sigma(L)$ is the extension to V of this configuration in $\mathfrak{b}^\perp$.

Notice that the projective limit model for the set $\mathrm{Laur}(\mathfrak{b}^\perp, {}^*L, {}^*\mathcal{H})$ of Laurent operators $\mathcal{M}(\mathfrak{b}^\perp, {}^*\mathcal{H}) \to \mathcal{M}({}^*L, {}^*\mathcal{H}_{{}^*L})$ is $S_\leftarrow(V_{{}^*L}^\perp \cap \mathfrak{b}^\perp, {}^*P)$. Since $\mathfrak{b} \subset V_L$ we have $V_{{}^*L}^\perp \cap \mathfrak{b}^\perp = V_L^\perp$, and by (3.14) we have $\Sigma({}^*P) \cap V_{{}^*L}^\perp \cap \mathfrak{b}^\perp = \Sigma(P) \cap V_L^\perp$. Hence this model is identical with $S_\leftarrow(V_L^\perp, P)$, the projective limit model for $\mathrm{Laur}(V, L, \mathcal{H})$ (cf. also (1.36)).

Theorem 3.14. *Let $\mathfrak{b} \in \mathcal{R}$, $L \in \mathcal{L}_\Sigma$, and assume that $\mathfrak{b} \subset V_L$. Then, for $t \in \mathrm{WT}(\Sigma)$, $P \in \mathcal{P}(V)$, we have the following identity in $S_\leftarrow(V_L^\perp, P)$:*

$$\mathrm{Res}_L^{P,\,t} = \mathrm{Res}_{{}^*L}^{{}^*P,\,{}^*t}. \tag{3.21}$$

In particular, if $\mathfrak{b} = V_L$, then ${}^*L$ is a point. Thus by this theorem we can reduce the determination of the residue operators $\mathrm{Res}_L^{P,\,t}$ to the case where L is a point.

Proof. Let $\mathcal{H} = \mathcal{H}_\Sigma(L)$. Recall from the definition in (3.6) that

$$\mathrm{Res}_L^{P,\,t} = \mathrm{Res}_L^{V^-(P,\mathcal{H}),\,\omega_\mathcal{H}(t)}.$$

By Lemma 1.21 and Remark 1.20 we have, in $S_\leftarrow(V_L^\perp, P)$:

$$\mathrm{Res}_L^{V^-(P,\mathcal{H}),\,\omega_\mathcal{H}(t)} = \mathrm{Res}_{{}^*L}^{V^-(P,\mathcal{H})\cap\mathfrak{b}^\perp,\,{}^*[\omega_\mathcal{H}(t)]},$$

and combining these identities with Lemmas 3.12 and 3.13 we then have

$$\mathrm{Res}_L^{P,\,t} = \mathrm{Res}_{{}^*L}^{V^-({}^*P,{}^*\mathcal{H}),\,\omega_{{}^*\mathcal{H}}({}^*t)}.$$

Here ${}^*\mathcal{H} = \mathcal{H}_{\Sigma_{\mathfrak{b}^\perp}}({}^*L)$ as in (3.20). On the other hand, by (3.6) we also have

$$\mathrm{Res}_{{}^*L}^{{}^*P,\,{}^*t} = \mathrm{Res}_{{}^*L}^{V^-({}^*P,{}^*\mathcal{H}),\,\omega_{{}^*\mathcal{H}}({}^*t)}.$$

This proves (3.21). □

3.7 The support theorem. Let L, P, and t be as in the beginning of Section 3.4. We shall now give a necessary condition on L in order that $\operatorname{Res}_L^{P,t}$ does not vanish. The condition is on the central point $c(L)$, and the key to the result is Theorem 2.6.

Let $\Gamma^+ = \Gamma^+(V) \subset V$ be the closed cone spanned by the roots of $\Sigma(P)$, that is,

$$\Gamma^+ = \Big\{ \sum_{\alpha \in \Sigma(P)} x_\alpha \alpha \;\Big|\; x_\alpha \in \mathbb{R},\ x_\alpha \geq 0 \Big\},$$

and let $\Gamma^- = -\Gamma^+$. For $\mathfrak{b} \in \mathcal{R}$ we define similarly

$$\Gamma^+(\mathfrak{b}^\perp) = \Big\{ \sum_{\alpha \in \Sigma(P) \cap \mathfrak{b}^\perp} t_\alpha \alpha \;\Big|\; t_\alpha \in \mathbb{R}, t_\alpha \geq 0 \Big\}, \qquad \Gamma^-(\mathfrak{b}^\perp) = -\Gamma^+(\mathfrak{b}^\perp),$$

then $\Gamma^\pm(\mathfrak{b}^\perp) \subset \Gamma^\pm \cap \mathfrak{b}^\perp$. (We agree to set $\Gamma^\pm(\{0\}) = \{0\}$.) We also put

$$(\mathfrak{b}^\perp)^+ = \{\lambda \in \mathfrak{b}^\perp \mid \langle \lambda, \alpha \rangle > 0, \forall \alpha \in \Sigma(P) \cap \mathfrak{b}^\perp\}.$$

Theorem 3.15. *Let $L \in \mathcal{L}_\Sigma$, $P \in \mathcal{P}(V)$, and $t \in \mathrm{WT}(\Sigma)$. If $\operatorname{Res}_L^{P,t} \neq 0$ then*

$$c(L) \in \Gamma^-(V_L^\perp).$$

Proof. Let $\mathcal{H} = \mathcal{H}_\Sigma(L)$ and $C_0 = V^-(P, \mathcal{H})$, then by definition (see (3.6)) we have $\operatorname{Res}_L^{P,t} = \operatorname{Res}_L^{C_0,\, \omega_{\mathcal{H}}(t)}$. Since $\omega_{\mathcal{H}}(t)$ is central we can apply Theorem 2.6. Assume $\operatorname{Res}_L^{P,t} \neq 0$. Then $L \in \operatorname{ressupp}(C_0, \omega_{\mathcal{H}}(t))$ and we obtain that $\langle c(L), \lambda \rangle \geq 0$ for all $\lambda \in C_0$.

Notice that for each $H \in \mathcal{H}_\Sigma(L)$ we have $H = c(L) + \alpha^\perp$ with $\alpha \in V_L^\perp \cap \Sigma(P)$. Hence $c(L) - (V_L^\perp)^+ \subset V^-(P, H)$, and taking the intersection over $H \in \mathcal{H}_\Sigma(L)$ we obtain that $c(L) - (V_L^\perp)^+ \subset C_0$. Hence $\langle c(L), c(L) - \lambda \rangle \geq 0$ for all $\lambda \in (V_L^\perp)^+$. It follows easily that then $\langle c(L), \lambda \rangle \leq 0$, and from this we derive the desired result since $\Gamma^-(V_L^\perp)$ is the dual of the cone $(V_L^\perp)^+$ in $V_L^\perp$. $\square$

3.8 Conclusion. We can now state the main result that will be applied in [5]. Let $P \in \mathcal{P}(V)$, let $\mathcal{H}$ be a P-bounded Σ-admissible hyperplane configuration, and let $\mathcal{L} = \mathcal{L}_{\mathcal{H}}$. Moreover, let

$$R_{P,\mathcal{H}} = \inf\{|\lambda| \mid \lambda \in V^-(P, \mathcal{H})\}$$

and let $\bar{B}(0, R_{P,\mathcal{H}})$ denote the closed ball of radius $R_{P,\mathcal{H}}$, centered at 0. Since $\mathcal{H}$ is locally finite there exists $\epsilon > 0$ such that for all $L \in \mathcal{L}$ with $|c(L)| \leq R_{P,\mathcal{H}}$ and all $H \in \mathcal{H}$ we have $H \cap B(c(L), \epsilon) \neq \emptyset \Rightarrow c(L) \in H$. Choose, for each $Q \in \mathcal{P}$, a point $\varepsilon_Q \in Q \cap B(0, \epsilon)$. Then, for $\lambda \in \bar{B}(0, R_{P,\mathcal{H}})$ and $L \in \mathcal{L}$ with $c(L) = \lambda$ we have $\lambda + \varepsilon_Q \in C_{Q,L,\mathcal{H}}$ (see Section 3.2).

Theorem 3.16. *Let P, $\mathcal{H}$, $\mathcal{L}$, and ε_Q, $Q \in \mathcal{P}$, be as above, and let $t \in \mathrm{WT}(\Sigma)$. Then for each $\mathfrak{b} \in \mathcal{R}$ the set*

$$\{\lambda \in \mathfrak{b}^\perp \mid \operatorname{Res}_{\lambda+\mathfrak{b}}^{P,t} \varphi \neq 0 \textit{ for some } \varphi \in \mathcal{M}(V, \mathcal{H})\} \tag{3.22}$$

is finite and contained in $\Gamma^-(\mathfrak{b}^\perp) \cap \bar{B}(0, R_{P,\mathcal{H}})$. *Moreover, if* $\eta \in V^-(P, \mathcal{H})$ *then*

$$\int_{\eta+iV} \varphi \, d\mu_V = \sum_{\mathfrak{b}\in\mathcal{R}} \sum_{\lambda\in\mathfrak{b}^\perp} \sum_{Q\in\mathcal{P}(\mathfrak{b})} t(Q) \int_{\lambda+\varepsilon_Q+i\mathfrak{b}} \operatorname{Res}^{P,\,t}_{\lambda+\mathfrak{b}} \varphi \, d\mu_{\mathfrak{b}} \tag{3.23}$$

for all $\varphi \in \mathcal{P}(V, \mathcal{H})$.

Notice that in (3.23) the term corresponding to $\mathfrak{b} = V$ reads as follows:

$$\sum_{Q\in\mathcal{P}(V)} t(Q) \int_{\varepsilon_Q+iV} \varphi \, d\mu_V.$$

In particular, if $0 \in \operatorname{reg}(V, \mathcal{H})$ then this equals $\int_{iV} \varphi \, d\mu_V$, and thus (3.23) gives an expression for the difference between the latter integral and $\int_{\eta+iV} \varphi \, d\mu_V$ by means of residues.

Proof. Given $\mathfrak{b} \in \mathcal{R}$ and $\lambda \in \mathfrak{b}^\perp$ we have from Proposition 3.6 that

$$\operatorname{Res}^{P,\,t}_{\lambda+\mathfrak{b}} \varphi = \operatorname{Res}^{V^-(P,\mathcal{H}),\,\omega_{\mathcal{H}}(t)}_{\lambda+\mathfrak{b}} \varphi \tag{3.24}$$

for $\varphi \in \mathcal{M}(V, \mathcal{H})$ if $\lambda + \mathfrak{b} \in \mathcal{L}$. Otherwise $\operatorname{Res}^{P,\,t}_{\lambda+\mathfrak{b}} \varphi = 0$ (see the remarks after (3.6)). The finiteness of the set in (3.22) then follows from Theorem 1.13 (a). That the set is contained in $\Gamma^-(\mathfrak{b}^\perp)$ and $\bar{B}(0, R_{P,\mathcal{H}})$ follows from Theorem 3.15 and Theorem 2.6, respectively.

Combining Theorem 1.13 and (3.24) we have

$$\int_{\eta+iV} \varphi \, d\mu_V = \sum_{L\in\mathcal{L}} \sum_{C'\in\operatorname{comp}(L,\mathcal{H}_L)} \omega_{\mathcal{H}}(t)(C') \int_{\operatorname{pt}(C')+iV_L} \operatorname{Res}^{P,\,t}_L \varphi \, d\mu_L.$$

Hence, by (3.4),

$$\begin{aligned} \int_{\eta+iV} \varphi \, d\mu_V &= \sum_{L\in\mathcal{L}} \sum_{C'\in\operatorname{comp}(L,\mathcal{H}_L)} \sum_{Q\in\mathcal{P}(V_L),C_Q=C'} t(Q) \int_{\operatorname{pt}(C_Q)+iV_L} \operatorname{Res}^{P,\,t}_L \varphi \, d\mu_L \\ &= \sum_{L\in\mathcal{L}} \sum_{Q\in\mathcal{P}(V_L)} t(Q) \int_{\operatorname{pt}(C_Q)+iV_L} \operatorname{Res}^{P,\,t}_L \varphi \, d\mu_L. \end{aligned}$$

In the last expression we choose $c(L) + \varepsilon_Q$ as the point in C_Q. Moreover, we write $\lambda = c(L)$ and $\mathfrak{b} = V_L$. Then $\mathfrak{b} \in \mathcal{R}$ and $\lambda \in \mathfrak{b}^\perp$. Hence (3.23) holds. $\square$

It is convenient to rewrite (3.23) in a somewhat different form. Let $\Delta = \Delta(P)$ denote the set of simple roots for $\Sigma(P)$. Then the Coxeter complex $\mathcal{P} = \cup_{\mathfrak{b}\in\mathcal{R}} \mathcal{P}(\mathfrak{b})$ can be parametrized as follows. For each subset F of Δ we denote by $\mathfrak{b}_F$ the orthocomplement of F in V; then $\mathfrak{b}_F \in \mathcal{R}$. Let $P_F \in \mathcal{P}(\mathfrak{b}_F)$ be the chamber on which the roots of $\Delta \setminus F$ are positive. The chambers P_F, where $F \subset \Delta$, are called the *standard* chambers (relative to P). In particular, we have $P = P_\emptyset$ and $V_0 = P_\Delta$. Given $F \subset \Delta$ we denote by W_F the subgroup of W generated by the reflections in the elements of F, and define the subset W^F of W by

$$W^F = \{v \in W \mid v(F) \subset \Sigma(P)\}.$$

Lemma 3.17. *(i) Let $F \subset \Delta$. Each element $w \in W$ has a unique expression of the form $w = vu$, where $v \in W^F$ and $u \in W_F$. The stabilizer of the standard chamber P_F in W is W_F.*

(ii) Let $Q \in \mathcal{P}$. There exists a unique subset $F \subset \Delta$ such that Q is W-conjugate to P_F. Moreover, there exists a unique $v \in W^F$ for which $Q = vP_F$.

Proof. See [7, Thm. 2.5.8 and Props. 2.6.1, 2.6.3]. $\square$

In the following corollary notation and assumptions are as in Theorem 3.16. We assume in addition that the weight t is Weyl invariant, and that the ε_Q have been chosen so that $\varepsilon_{wQ} = w\varepsilon_Q$ for all $w \in W$, $Q \in \mathcal{P}$ (this is clearly possible). Let $\varepsilon_F = \varepsilon_{P_F}$ for $F \subset \Delta$.

Corollary 3.18. *For each $F \subset \Delta$ the set*

$$\{\lambda \in \mathfrak{b}_F^\perp \mid \mathrm{Res}^{P,\,t}_{\lambda+\mathfrak{b}_F}(\varphi \circ v) \neq 0 \text{ for some } \varphi \in \mathcal{M}(V, \mathcal{H}), v \in W^F\} \tag{3.25}$$

is finite and contained in $\Gamma^-(\mathfrak{b}_F^\perp) \cap \bar{B}(0, R_{P,\mathcal{H}})$. Moreover

$$\int_{\eta+iV} \varphi \, d\mu_V = \sum_{F\subset\Delta} \sum_{\lambda\in\mathfrak{b}_F^\perp} t(P_F) \int_{\lambda+\varepsilon_F+i\mathfrak{b}_F} \mathrm{Res}^{P,\,t}_{\lambda+\mathfrak{b}_F}\Big(\sum_{v\in W^F} \varphi \circ v\Big) \, d\mu_{\mathfrak{b}_F} \tag{3.26}$$

for $\varphi \in \mathcal{P}(V, \mathcal{H})$.

Proof. It follows from (1.26) and Proposition 3.10 that for $v \in W^F$, $\varphi \in \mathcal{M}(V, \mathcal{H})$ we have

$$\mathrm{Res}^{P,\,t}_{\lambda+\mathfrak{b}_F}(\varphi \circ v) = ((v \, \mathrm{Res}^{P,\,t}_{\lambda+\mathfrak{b}_F})\varphi) \circ v = (\mathrm{Res}^{P,\,t}_{v(\lambda+\mathfrak{b}_F)} \varphi) \circ v. \tag{3.27}$$

Hence by the first conclusion of Theorem 3.16, if $\mathrm{Res}^{P,\,t}_{\lambda+\mathfrak{b}_F}(\varphi \circ v) \neq 0$ then $v\lambda$ belongs to a finite subset of $\Gamma^-(v\mathfrak{b}_F^\perp) \cap \bar{B}(0, R_{P,\mathcal{H}})$. It is easily seen that $v\lambda \in \Gamma^-(v\mathfrak{b}_F^\perp)$ implies $\lambda \in \Gamma^-(\mathfrak{b}_F^\perp)$ for $v \in W^F$. The statements about the set (3.25) follow.

It follows from (3.23) and Lemma 3.17 together with our assumptions on t and ε_Q that

$$\int_{\eta+iV} \varphi \, d\mu_V = \sum_{F\subset\Delta} \sum_{v\in W^F} \sum_{\nu\in v\mathfrak{b}_F^\perp} t(P_F) \int_{\nu+v\varepsilon_F+iv\mathfrak{b}_F} \mathrm{Res}^{P,\,t}_{\nu+v\mathfrak{b}_F} \varphi \, d\mu_{v\mathfrak{b}_F}$$

Substitution of $\nu = v\lambda$, $\lambda \in \mathfrak{b}_F^\perp$, in the sum over ν, together with a similar substitution in the integral on the right-hand side, yields

$$\sum_{F\subset\Delta} \sum_{v\in W^F} \sum_{\lambda\in\mathfrak{b}_F^\perp} t(P_F) \int_{\lambda+\varepsilon_F+i\mathfrak{b}_F} (\mathrm{Res}^{P,\,t}_{v(\lambda+\mathfrak{b}_F)} \varphi) \circ v \, d\mu_{\mathfrak{b}_F},$$

and the result follows from (3.27) and a simple rearrangement of the sum. $\square$

Acknowledgment We thank Gerrit Heckman for useful conversations concerning [10]. The results presented here were found in the fall of 1995, when both authors were at the Mittag-Leffler Institute for the 95/96 program, Analysis on Lie Groups. We are grateful to the organizers of the program and the staff of the institute for providing us with this unique opportunity. The first author thanks his home institution and the second author thanks his then home institution, The Royal Veterinary and Agricultural University in Copenhagen, for making the stay in Stockholm possible.

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