Stability Limit of the Cryogenic Hydrogen Maser

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It is pointed out that the usual oscillation condition of the H maser is only a necessary condition for steady operation. Reducing the coupled field-matter dynamics to the complex Lorenz equations we derive a second requirement which together with the first forms a set of necessary and sufficient conditions for the steady operation to be stable. The instability of the steady state predicted by the equations should be easily accessible experimentally for the cryogenic H maser. It will be characterized by a pulsed output power which, depending on the detuning, is either periodic or chaotic.

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Since its first realization by Goldenberg, Kleppner, and Ramsey[1] the hydrogen maser has been the most stable of all atomic frequency standards for short and intermediate measuring times. The relative stability of the hydrogen maser is observed to be better than one part in 10^{15}, which makes it a very useful instrument for long-baseline interferometry, tests of general relativity, precision interplanetary navigation (Voyager 2 mission), and various other applications both inside and outside physics.

A hydrogen maser operating at liquid-helium temperatures should have an even better frequency stability.[2,3] Although early estimations predicted an improvement of more than 2 orders of magnitude,[4] present indications show that it has an increased frequency stability of close to 1 order of magnitude.[5–8]

The operation of the hydrogen maser has been described in Refs. 1 and 9. A central result (see Fig. 1) is the so-called oscillation condition,

$$\frac{P}{P_c} = -2q^2\left(\frac{I}{I_{\rm th}}\right)^2 + (1-Cq)\frac{I}{I_{\rm th}} - 1 > 0\,, \qquad (1)$$

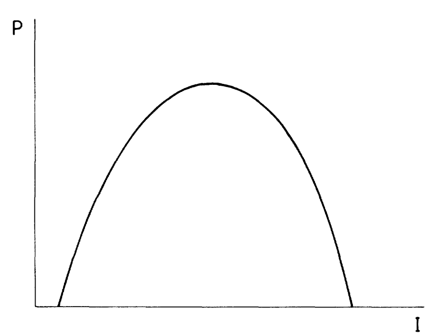

FIG. 1. Power of H maser vs surplus flux of upper level.

in which P is the total power radiated by the atoms, I is the surplus flux of atoms entering the storage bulb in the upper level of the maser transition relative to the lower level, $I_{\rm th}$ is the threshold flux if we neglect density-dependent relaxation, P_c is the critical power, q is the maser quality factor, and $C=(T_2^0/T_1^0)^{1/2}+2(T_1^0/T_2^0)^{1/2}$ with T_1^0 (T_2^0) the density-independent longitudinal (transverse) relaxation time.[9]

It has not been recognized until now that the condition (1) is only necessary but by no means sufficient. More precisely, condition (1) expresses the existence of the steady-state solution for the number of photons in the cavity and therefore for the output microwave power. It remains to determine the domain in which this is also a stable solution.

The purpose of this Letter is to show that within the flux limits determined by condition (1) two regimes can exist, one in which the steady-state solution is stable and another in which spontaneous modulation of the amplitude and phase of the electromagnetic field takes place. This spontaneous modulation can be either periodic or chaotic, depending on the values adopted for the various parameters, and will affect in a similar way the output microwave power. We show that this time-dependent regime can be rather easily reached for the subkelvin hydrogen maser.

The maser dynamics can be described by essentially the same Maxwell-Bloch equations as a single-mode laser with homogeneous broadening. However, interesting differences exist between these two systems. A first aspect is that the two classes of systems display periodic and chaotic behavior in a very different range of parameters, because the maser decay rates are very different from the usual values found in the visible or ir domains (see Table I). As a consequence, the analysis of the dynamical equations has to be specialized for this new domain of parameters. A second aspect which is worth stressing is that the derivation of the Maxwell-Bloch

TABLE I. Time constants (in $\sec^{-1}$) for cryogenic H maser compared to typical laser.

	$\gamma_\parallel$	$\gamma_\perp$	κ	g
Maser	1	1	10^5	10^{-2}
Laser	10^8	10^8	10^7–10^{10}	10^4

equations [see Eqs. (2)] for the H maser requires far less simplifying assumptions than in the case of the laser. For instance, the H maser is naturally homogeneously broadened. Furthermore, the cavity dimension is of the order of the maser wavelength so that effects related to the space dependence of the coupling constant are negligible.

Apart from the intrinsic interest associated with the availability of a system displaying deterministic chaos with a very low noise level, the observation of instability may have important applications for obtaining information on the maser which would be very difficult to obtain otherwise. It is *a priori* to be expected that the nonsteady regime will offer much more information than the frequency and the amplitude, obtainable from stationary operation. This is especially welcome in view of the overwhelming number of experimental parameters, such as hyperfine populations, which determine the maser operation and are notoriously difficult to diagnose.

Reformulating the dynamics of the maser,[9] the operation of the cryogenic H maser can conveniently be described by the Maxwell-Bloch equations

$$\begin{aligned} \dot{B} &= -i\omega_c B - \kappa B + gM , \\ \dot{M} &= -i\omega_{\rm at} M - \gamma_\perp M + gB\Delta , \\ \dot{\Delta} &= -\gamma_\parallel(\Delta - \Delta_0) - 2g(BM^* + B^*M) , \end{aligned} \tag{2}$$

for the complex magnetic field B, the complex magnetization M, and the inversion Δ. Explicitly B is defined as the expectation value of the photon annihilation operator. In terms of the one-atom spin-density matrix we have, furthermore, $M = N\rho_{ca}$, $\Delta = N(\rho_{cc} - \rho_{aa})$, where N is the number of atoms. The cavity resonance frequency is denoted by ω_c, the atomic frequency by $\omega_{\rm at}$, the cavity damping rate by $\kappa = \omega_c/2Q_c = 1/T_c$, and the damping rates for M and Δ by $\gamma_\perp = 1/T_2$ and $\gamma_\parallel = 1/T_1$. The equilibrium value of Δ in the absence of atom-field interaction is denoted by Δ_0. Finally g, the one-photon Rabi frequency, is given by

$$g^2 = \mu_0(\mu_e + \mu_p)^2 \eta\omega_c/2\hbar V_c , \tag{3}$$

with V_c being the cavity volume, μ_e (μ_p) the electron (proton) magnetic moment, and η the filling factor.

In this Letter we will only discuss the stability properties of the steady state, following the analysis made by Mandel and Zeghlache[10] for a detuned laser. We transform Eqs. (2) by introducing the new parameters

$$\sigma = \kappa/\gamma_\perp, \quad b = \gamma_\parallel/\gamma_\perp, \quad R = g^2\Delta_0/\kappa\gamma_\perp , \tag{4}$$

and new variables

$$\begin{aligned} B &= (\gamma_\perp/2g)(x_1 + ix_2)\exp(-i\omega_m t) , \\ M &= (\Delta_0/2R)(y_1 + iy_2)\exp(-i\omega_m t) , \\ \Delta &= \Delta_0(1 - z/R) , \end{aligned} \tag{5}$$

where ω_m is the yet unknown operating frequency of the maser. After rescaling the time according to $t = T_2\tau$, Eqs. (2) take the form

$$\begin{aligned} x_1' &= -\sigma(x_1 + \delta_c x_2 - y_1) , \\ x_2' &= -\sigma(x_2 - \delta_c x_1 - y_2) , \\ y_1' &= -y_1 + Rx_1 + \delta_{\rm at} y_2 - x_1 z , \\ y_2' &= -y_2 + Rx_2 - \delta_{\rm at} y_1 - x_2 z , \\ z' &= -bz + x_1 y_1 + x_2 y_2 , \end{aligned} \tag{6}$$

in which the prime stands for $d/d\tau$ and

$$\delta_{\rm at} = \frac{\omega_{\rm at} - \omega_m}{\gamma_\perp}, \quad \delta_c = \frac{\omega_m - \omega_c}{\kappa} . \tag{7}$$

In the case of perfect tuning ($\omega_{\rm at} = \omega_m = \omega_c$), Eqs. (6) have a class of solutions for which $x_2 = y_2 = 0$ for all times. The remaining variables $x = x_1$, $y = y_1$, and z obey the usual Lorenz equations.[11]

To study the steady-state solutions of Eqs. (6), we set the derivatives in these equations equal to zero. Since ω_m is the operating frequency in the steady state, x_2 can be chosen to be zero. This leads to the dispersion equation or cavity-pulling relation $\delta_{\rm at} = \delta_c \equiv \delta$ and to three fixed points: $x_1 = y_1 = y_2 = z = 0$ and $x_1 = y_1 = \pm(bz)^{1/2}$, $y_2 = \mp\delta(bz)^{1/2}$, $z = R - 1 - \delta^2$, respectively. The last two solutions are physically identical, since they differ in phase only. For $R \le 1 + \delta^2$ only the trivial zero-field solution exists and is stable. A necessary condition for the finite-field solution to exist is $R > 1 + \delta^2$, which generalizes Eq. (1) for a detuned cavity.

The inequality $R > 1 + \delta^2$ gives a lower bound for the domain of existence and stability of the finite-amplitude solution. However, the linear stability analysis of this solution indicates that a second threshold may occur at higher photon numbers, when R reaches a critical value R_H which depends in a rather complicated way on the parameters of the problem. Using the relative magnitudes of the parameter values given in Table I and retaining the dominant contributions in $\epsilon = 1/\sigma$ with b and δ being functions of the order of 1, we find that there is always an upper bound for the stability of the steady state. It is reached when the photon number equals the critical value $|B_H|^2 = (R_H - 1 - \delta^2)/4T_1T_2g^2$. Restricting ourselves to the case of small detuning ($\frac{1}{3} - \delta^2 \gg \epsilon$), the second requirement for stable steady oscillation is

$$\frac{T_1}{T_t}\frac{P - P_H}{P_H} = \left[-2q + \frac{1}{\kappa T_t}\frac{1 - 3\delta^2}{(1+\delta^2)^2}\right]\frac{I}{I_{\rm th}} - \frac{T_2^0}{T_t} < 0 , \tag{8}$$

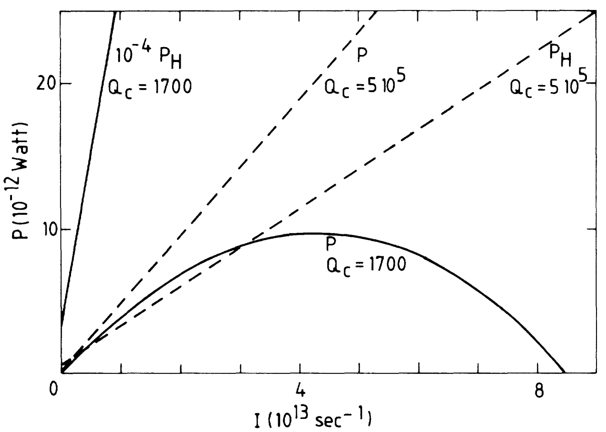

FIG. 2. Power P vs surplus flux, compared to threshold power P_H for onset of nonsteady oscillation. Solid lines: actual parameter values for the University of British Columbia maser. Dashed lines: increased Q_c value.

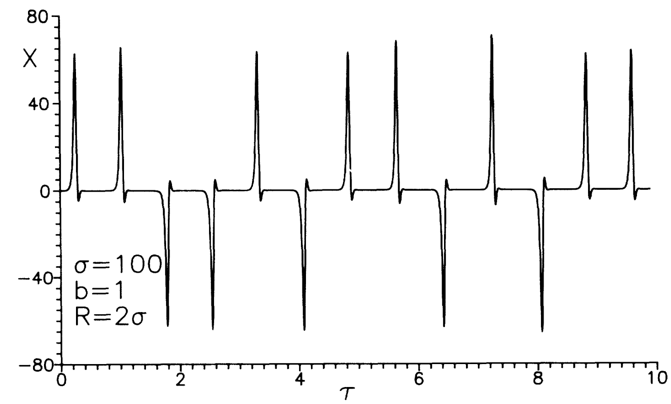

FIG. 3. Field amplitude as a function of time for zero detuning, both scaled as described in text.

in which $T_t = (T_1^0 T_2^0)^{1/2}$.

In Fig. 2 we display graphically the two conditions of stability (1) and (8) for the steady state. We present the radiated power P as well as the quantity P_H. For definiteness we take $\delta = 0$, and for $T_1^0 = T_2^0$ and the product qQ_c we take the University of British Columbia cryogenic-maser[7] values 0.6 and 27 sec, respectively. Clearly, from Fig. 2 we see that the actual Q_c value of order 1700 does not admit nonsteady oscillation. It has been kept low deliberately to reduce fluctuations of the maser frequency due to cavity pulling, but may easily be increased to reach the unstable domain $Q_c > \omega_c / 2g\sqrt{\Delta_0}$, which is equivalent to the condition $R > R_H$ when $\delta = 0$. For instance, Fig. 2 shows that the new oscillation regime can be reached by increasing Q_c to values of order 5×10^5.

Calculations for the Harvard-Smithsonian maser[12] with $qQ_c = 17$ show that in this case an increase of Q_c from the present magnitude of 7×10^4 by a factor of order 20 would be sufficient. From a similar analysis it follows that unstable oscillation is very difficult to achieve for room-temperature H masers. This is mainly due to the much lower maximum densities allowed by the oscillation condition (1) due to the faster collisional relaxation.

Hence, the cryogenic H maser promises to be an experimental realization of the Lorenz equations in the domain $R, \sigma \gg 1$ and $b \approx 1$. This domain, partly investigated by Fowler and McGuinness,[13] is characterized by pulses for x and y, i.e., for the field and magnetization amplitudes. An example of this behavior is displayed in Fig. 3. A detailed analysis of Eqs. (2) in the relevant domain of parameter space will be published in a separate paper.

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