Homology of classical groups and K-theory

Homologie van de klassieke groepen en K-theorie
(met een samenvatting in het Nederlands)

Proefschrift

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Contents

Introduction

The study of the homology groups $H_i(G_n, A)$, where G_n is a classical group and A is a commutative ring with trivial G_n-action, seems important, notably because of their close relation to algebraic and Hermitian K-theory and their appearance in the study of scissors congruence of polyhedra. Unfortunately, these groups are much too big and complicated to be computed explicitly. Therefore all results allowing to compare the groups $H_i(G_n, A)$ for different values of n become quite important. This idea goes back to Quillen who proved certain homology stability theorems in order to study the K-groups of rings of integers [**32**].

We say that a sequence of groups $G_0 \subseteq G_1 \subseteq \ldots \subseteq G_n \subseteq \ldots$ has the homology stability property if $H_i(G_n, R) \to H_i(G_{n+1}, R)$ is an isomorphism when n is big enough with respect to i. After Quillen's work, there has been considerable interest in homological stability for general linear groups. The most general results in this direction are due to Van der Kallen [**46**] and Suslin [**40**].

Parallel to this, similar questions for other classical groups such as orthogonal and symplectic groups were conjectured back in early 1980's. For work in this direction, see [**49**], [**50**],[**2**], [**3**], [**8**], [**29**], [**30**]. The most general result is due to Charney [**8**]. She proved the homology stability for orthogonal and symplectic groups over a Dedekind domain. Panin in [**30**] proved a similar result but with a different method and with better range of stability.

It is a very general and well known technique that one can reduce the homology stability of unitary groups to the higher connectivity of the 'poset of isotropic sequences'. For a commutative ring R with 1, the symplectic group $Sp_{2n}(R)$ can be considered as the group of automorphisms of the free module R^{2n} which preserve the symplectic bilinear form.

Let $w \in R^{2n}$ be a unimodular vector, that is the submodule generated by w is a summand of R^{2n}. Let R^{2n-2} be the submodule of R^{2n} generated by the $(2n-2)$-elements $e_1, \ldots, e_{2n-2}$ of the standard basis of R^{2n}. The simple fact that R^{2n-2} may not contain any unimodular element that is orthogonal to w (for example consider $R = \mathbb{R}[[x_1, \ldots, x_{2n-2}]]$ and $w = (x_1, \ldots, x_{2n-2}, 1, 0)$) made it impossible to use techniques analogous to [**46**] to study the poset of isotropic sequences. (The example given above is important because R is a local ring and thus has the smallest unitary stable rank, that is 1.) To study this poset, in Chapters 1 and 2, we develop a quantitative analogue for posets of the nerve theorem. The new nerve theorem allows us to exploit the higher connectivity of the poset of unimodular sequences due to Maazen and Van der Kallen [**46**]. The higher connectivity of the posets of isotropic unimodular sequences follows inductively.

In the first main theorem of this thesis we prove that homology stabilizes for the hyperbolic unitary groups over rings with finite unitary stable rank, for example over rings with finite Krull dimension (see chapter 3, 1.1). We should mention that orthogonal, symplectic and classical unitary groups are special cases of hyperbolic unitary groups.

General stability theorem. *Let R be a ring with finite unitary stable rank* $\mathrm{usr}(R)$ *and let* $G_n = U_{2n}^{\epsilon}(R,\Lambda)$ *or* $EU_{2n}^{\epsilon}(R,\Lambda)$. *Let A be an abelian group with trivial G_n-action. Then* $H_i(\mathrm{inc}) : H_i(G_n, A) \to H_i(G_{n+1}, A)$ *is surjective for* $n \geq 2i + \mathrm{usr}(R) + 2$ *and is injective for* $n \geq 2i + \mathrm{usr}(R) + 3$.

It was believed that some additional assumptions on the ring R provides a better range of homology stability. In [**43**], Suslin proved that for an infinite field F the map $H_i(GL_n(F), \mathbb{Z}) \to H_i(GL_{n+1}(F), \mathbb{Z})$ is an isomorphism if $n \geq i$. The existence of the infinite center in $GL(F)$ was essential in his proof. Absence of such a center in $Sp_{2n}(F)$ or in $O_{2n}(F)$ made the homology of these groups more challenging. In chapter 3, we overcome this difficulty. As a result, for the hyperbolic unitary groups we get much better homology stability rank.

Theorem. *Let k be a field and let R be a commutative local ring with an infinite residue field. Then*

(i) $H_i(\mathrm{inc}) : H_i(U_{2n}^{\epsilon}(R,\Lambda), k) \to H_i(U_{2n+2}^{\epsilon}(R,\Lambda), k)$ *is surjective for* $n \geq i$ *and is injective for* $n \geq i+1$.

(ii) $H_i(\mathrm{inc}) : H_i(U_{2n}^{\epsilon}(R,\Lambda), \mathbb{Z}) \to H_i(U_{2n+2}^{\epsilon}(R,\Lambda), \mathbb{Z})$ *is surjective for* $n \geq i+1$ *and is injective for* $n \geq i+2$.

Homology stability type theorems have many important and interesting applications. An interesting application is in its relation with the Friedlander-Milnor conjecture, which we formulate in the following. (An application of Suslin's stability theorem is discussed in chapter 4.)

Let G be a topological group and let BG^{top} be its classifying space with the underlying topology. Let BG be the classifying space of G as the topological group with the discrete topology. By the functorial property of B we have a natural map $\psi : BG \to BG^{\mathrm{top}}$.

Friedlander-milnor conjecture. *Let G be a Lie group. The canonical map* $\psi : BG \to BG^{\mathrm{top}}$ *induces isomorphism of homology and cohomology with any finite abelian coefficient group.*

The results of Suslin in [**41**],[**42**], confirms the validity of the Friedlander-Milnor conjecture for the stable groups $GL(F)$, $SL(F)$, where F is the field of real or complex numbers. Suslin's proof uses some homology stability theorem due to Van der Kallen [**46**]. The bridge between algebraic K-theory and Hermitian K-theory allowed Karoubi to apply Suslin's machinery in Hermitian K-theory in order to prove similar results for the stable groups $Sp(F)$, $O(F)$ and $U(F)$ [**16**], [**17**]. Now one can use homology stability theorems to prove the conjecture for most of the unstable groups.

For example if $G_n = Sp_{2n}(F), O_{2n}(F), U_{2n}(F)$, then $H_i(BG_n, A) \simeq H_i(BG_n^{\mathrm{top}}, A)$ for $n \geq i+1$, where A is a finite abelian group.

Another aspect of these results obtained in chapter 3 is their application to the homology of general linear groups, which leads us to some new results in algebraic K-theory (see chapter 4).

In the beginning of 1970's two type of K-groups in algebra appeared: Quillen's K-groups and Milnor's K-groups. For a field F, Quillen defined the K-group $K_n(F)$ as the n-th homotopy group of the space $BGL(F)^+$ and Milnor defined the K-group $K_n^M(F)$ as the n-th degree part of $T(F^*)/\langle a \otimes (1-a) : a \in F^* - \{1\}\rangle$, where $T(F^*) := \mathbb{Z} \oplus F^* \oplus F^* \otimes F^* \oplus \cdots$ is the tensor algebra of F^*. There is a canonical ring homomorphism $K_*^M(F) \to K_*(F)$, so a canonical homomorphism $K_n^M(F) \to K_n(F)$. The Hurewicz theorem, in algebraic topology, relates homotopy groups to homology groups, which are much easier to calculate. This in turn provides a homomorphism from $K_n(F)$ to the n-th integral homology of the stable group $GL(F)$.

One of the important approaches to investigate Quillen's K-groups is by means of their relation with integral homology groups of $GL(F)$ and Milnor K-groups. This strategy already has shown its strength, for example Quillen's proof that K-groups of the ring of integers of number fields are finitely generated, his calculation of the K-groups of finite fields, Suslin's proof of the Quillen-Lichtenbaum conjecture on the torsion of the K-groups of algebraically closed fields, etc.

In [**43**], Suslin constructs a map from $H_n(GL_n(F), \mathbb{Z})$ to $K_n^M(F)$, we denote it by κ_n, such that the sequence

$$H_n(GL_{n-1}(F), \mathbb{Z}) \overset{H_n(\mathrm{inc})}{\longrightarrow} H_n(GL_n(F), \mathbb{Z}) \overset{\kappa_n}{\longrightarrow} K_n^M(F) \to 0,$$

is exact. This exact sequence together with his homology stability theorem, that is mentioned above, allowed him to construct a map from $K_n(F)$ to $K_n^M(F)$ and proved that the composite homomorphism

$$K_n^M(F) \to K_n(F) \to K_n^M(F)$$

coincides with the multiplication by $(-1)^{n-1}(n-1)!$.

Now one might ask about the kernel of $H_n(\mathrm{inc})$ in the above exact sequence. Suslin leaves it as a conjecture [**35**], [**10**] that for any infinite field F the natural homomorphism

$$H_n(\mathrm{inc}) : H_n(GL_{n-1}(F), \mathbb{Q}) \to H_n(GL_n(F), \mathbb{Q})$$

is injective. This conjecture is easy if $i = 1, 2$. For $i = 3$ the conjecture was proved positively by Elbaz-Vincent [**13**].

One even suspects that this injectivity would be true not only "rationally" but in a stronger form.

Conjecture. *Let $A = \mathbb{Z}[\frac{1}{(n-1)!}]$. Then for any infinite field F*

$$H_n(GL_n(F), A) = H_n(GL_{n-1}(F), A) \oplus K_n^M(F) \otimes A.$$

This conjecture is trivial for $n = 1$. For $n = 2$ it is due to Dennis, who proved the decomposition $H_2(GL_2(F), \mathbb{Z}) = H_2(GL_1(F), \mathbb{Z}) \oplus K_2^M(F)$ a long time ago [**11**]. The proof of the above conjecture for $n = 3$ is the main purpose of chapter 4.

Establishing the above conjecture for $n = 3$ enabled us to study the third K-group of F. In fact we show that

$$K_3(F)_{\mathrm{ind}} \otimes \mathbb{Z}[\frac{1}{2}] \simeq H_0(F^*, H_3(SL_2(F), \mathbb{Z}[\frac{1}{2}])),$$

where $K_3(F)_{\mathrm{ind}} = \mathrm{coker}(K_3^M(F) \to K_3(F))$ is the indecomposable part of $K_3(F)$.

As we mentioned above, there is a homomorphism $K_n^M(F) \to H_n(GL_n(F), \mathbb{Z})$ that factors through $K_n(F)$, denoted by h_n. For $n \geq 3$ no explicit expression of this homomorphism is known. (This was only known for $n = 2$. Note that this is trivial for $n = 0, 1$.) Surprisingly an explicit map from $K_n^M(F)$ to $H_n(GL_n(F), \mathbb{Z})$ is constructed in chapter 4, which fits in our previous theory, for example the composite homomorphism

$$K_n^M(F) \to H_n(GL_n(F), \mathbb{Z}) \stackrel{\kappa_n}{\to} K_n^M(F)$$

coincides with the multiplication by $(-1)^{n-1}(n-1)!$. The question whether this new map coincides with h_n remains open.

As an application we give a possible approach towards the solution of the above conjecture. For technical reasons we replace A by a field k such that $(n-1)!$ is invertible in k. Then for $n \geq 3$ we reduced the conjecture, for $A = k$, to the exactness of the complex

$$H_n(F^{*2} \times GL_{n-2}(F), k) \xrightarrow{\beta_2^{(n)}} H_n(F^* \times GL_{n-1}(F), k) \xrightarrow{\beta_1^{(n)}} H_n(GL_n(F), k) \to 0$$

where $\beta_1^{(n)} = H_n(\mathrm{inc})$ and $\beta_2^{(n)} = H_n(\alpha) - H_n(\mathrm{inc})$, $\alpha : F^{*2} \times GL_{n-2}(F) \to F^* \times GL_{n-1}(F)$ given by $(a, b, A) \to (b, \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix})$.

So far we could prove the exactness of the above complex for $n = 3$. It should be mentioned that the surjectivity of $\beta_1^{(n)}$ already was well known for all n.

The results of Chapters 1 and 2 and the General Stability Theorem, mentioned in this introduction, is joint work with W. van der Kallen and was already published in [**27**]. The results of chapters 3 and 4 is appeared in preprints [**25**] and [**26**].

CHAPTER 1

Posets

To every partially ordered set or briefly *poset* one can attach a topological space called the geometric realization of the poset. In this chapter we will study the homology and homotopy of these spaces.

We first extend a theorem of Quillen [**34**, Thm 9.1] which was his main tool to prove that certain posets are highly connected. We use it to develop a quantitative analogue for posets of the nerve theorem, which expresses the homotopy type of a space in terms of the the nerve of a suitable cover. In our situation both the elements of the cover and the nerve are replaced with posets. We work with posets of ordered sequences 'satisfying the chain condition', as this is a good replacement for simplicial complexes in the presence of group actions. (Alternatively one might try to work with barycentric subdivisions of a simplicial complex.)

1. Preliminaries

Recall that a topological space X is (-1)-*connected* if it is non-empty, 0-*connected* if it is non-empty and path connected, 1-*connected* if it is non-empty and simply connected. In general for $n \geq 1$, X is called n-*connected* if X is nonempty, X is 0-connected and $\pi_i(X, x) = 0$ for every base point $x \in X$ and $1 \leq i \leq n$. For $n \geq -1$ a space X is called n-*acyclic* if it is nonempty and $\tilde{H}_i(X, \mathbb{Z}) = 0$ for $0 \leq i \leq n$. For $n < -1$ the conditions of n-connectedness and n-acyclicness are vacuous.

THEOREM 1.1 (Hurewicz). *For $n \geq 0$, a topological space X is n-connected if and only if the reduced homology groups $\tilde{H}_i(X, \mathbb{Z})$ are trivial for $0 \leq i \leq n$ and X is 1-connected if $n \geq 1$.*

PROOF. See [**52**], Chap. IV, Corollaries 7.7 and 7.8. □

Let X be a poset. Consider the simplicial complex $S(X)$ associated to X, that is the simplicial complex where vertices or 0-simplices are the elements of X and the k-simplices are the $(k+1)$-tuples $(x_0, \ldots, x_k)$ of elements of X with $x_0 < \cdots < x_k$. We denote the geometric realization of $S(X)$ by $|X|$ and we consider it with the weak topology. We call $|X|$ the *geometric realization* of X. It is well known that $|X|$ is a CW-complex [**23**]. By a *morphism* or *map* of posets $f : X \to Y$ we mean an order-preserving map i. e. if $x \leq x'$ then $f(x) \leq f(x')$. Such a map induces a continuous map $|f| : |X| \to |Y|$.

REMARK 1. If K is a simplicial complex and X the partially ordered set of simplices of K, then the space $|X|$ is the barycentric subdivision of K. Thus every simplicial complex, with weak topology, is homeomorphic to the geometric realization of some, and in fact many, posets. Furthermore since it is well known that any CW-complex is homotopy equivalent to a simplicial complex, it follows that any interesting homotopy type is realized as the geometric realization of a poset.

PROPOSITION 1.2. *Let X and Y be posets.*
(i) (Segal[**36**]) *If $f, g : X \to Y$ are maps of posets such that $f(x) \leq g(x)$ for all $x \in X$, then $|f|$ and $|g|$ are homotopic.*
(ii) *If the poset X has a minimal or maximal element then $|X|$ is contractible.*
(iii) *If X^{op} denotes the opposite poset of X, i. e. with opposite ordering, then $|X^{op}| \simeq |X|$.*

PROOF. (i) Consider the poset $I = \{0, 1 : 0 < 1\}$ and define the poset map $h : I \times X \to Y$ as $h(0, x) = f(x)$, $h(1, x) = g(x)$. Since $|I| \simeq [0, 1]$, we have $|h| : [0, 1] \times |X| \to |Y|$ with $|h|(0, x) = |f|(x)$ and $|h|(1, x) = |g|(x)$. This shows that $|f|$ and $|g|$ are homotopic.
(ii) Suppose X has a maximal element z. Consider the map $f : X \to X$ with $f(x) = z$ for every $x \in X$. Clearly for every $x \in X$, $\mathrm{id}_X(x) \leq f(x)$. By part (i), $|\mathrm{id}_X|$ and the constant map $|f|$ are homotopic. Therefore $|X|$ is contractible. If X has a minimal element the proof is similar.
(iii) This is natural and easy. □

The construction $X \mapsto |X|$ allows us to assign topological concepts to posets. For example we define the homology groups of a poset X to be those of $|X|$, we call X *n-connected* or *contractible* if $|X|$ is n-connected or contractible etc. Note that X is connected if and only if X is connected as a poset. By the *dimension* of a poset X, we mean the dimension of the space $|X|$, or equivalently the supremum of the integers n such that there is a chain $x_0 < \cdots < x_n$ in X. By convention the empty set has dimension -1.

Let X be a poset and $x \in X$. Define $Link_X^+(x) := \{u \in X : u > x\}$ and $Link_X^-(x) := \{u \in X : u < x\}$. Given a map $f : X \to Y$ of posets and an element $y \in Y$, define subposets f/y and $y\backslash f$ of X as follows

$$f/y := \{x \in X : f(x) \leq y\}, \qquad y\backslash f := \{x \in X : f(x) \geq y\}.$$

In fact $f/y = f^{-1}(Y_{\leq y})$ and $y\backslash f = f^{-1}(Y_{\geq y})$, where $Y_{\leq y} = \{z \in Y : z \leq y\}$ and $Y_{\geq y} = \{z \in Y : z \geq y\}$. Note that by 1.2(ii), $Y_{\leq y}$ and $Y_{\geq y}$ are contractible. If $\mathrm{id}_Y : Y \to Y$ is the identity map, then $\mathrm{id}_Y/y = Y_{\leq y}$ and $y\backslash \mathrm{id}_Y = Y_{\geq y}$.

Let $\mathcal{F} : X \to \underline{\mathrm{Ab}}$ be a functor from a poset X, regarded as a category in the usual way, to the category of abelian groups. We define the homology groups $H_i(X, \mathcal{F})$ of X with coefficients $\mathcal{F}$ to be the homology of the complex $C_*(X, \mathcal{F})$ given by

$$C_n(X, \mathcal{F}) = \bigoplus_{x_0 < \cdots < x_n} \mathcal{F}(x_0),$$

where the direct sum is taken over all n-simplices in X, with differential $\partial_n = \sum_{i=0}^{n}(-1)^i d_i^n$, where $d_i^n : C_n(X,\mathcal{F}) \to C_{n-1}(X,\mathcal{F})$ takes the $(x_0 < \cdots < x_n)$-component of $C_n(X,\mathcal{F})$ to the $(x_0 < \cdots < \widehat{x_i} < \cdots < x_n)$-component of $C_{n-1}(X,\mathcal{F})$ via $d_i^n = id_{\mathcal{F}(x_0)}$ if $i > 0$ and $d_0^n : \mathcal{F}(x_0) \to \mathcal{F}(x_1)$. In particular, for $i \geq 0$ we have $H_i(\varnothing,\mathcal{F}) = 0$.

Let $\mathcal{F}$ be the constant functor $\mathbb{Z}$. Then the homology groups with this coefficients coincide with the integral homology of $|X|$, that is $H_k(X,\mathbb{Z}) = H_k(|X|,\mathbb{Z})$ for all $k \in \mathbb{Z}_{\geq -1}$, [**14**, App. II]. Let $\tilde{H}_i(X,\mathbb{Z})$ denote the *reduced integral homology* of the poset X, that is $\tilde{H}_i(X,\mathbb{Z}) = \ker\{H_i(X,\mathbb{Z}) \to H_i(pt,\mathbb{Z})\}$ if $X \neq \varnothing$ and $\tilde{H}_i(\varnothing,\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = -1 \\ 0 & \text{if } i \neq -1 \end{cases}$. So $\tilde{H}_i(X,\mathbb{Z}) = H_i(X,\mathbb{Z})$ for $i \geq 1$ and for $i = 0$ we have the exact sequence

$$0 \to \tilde{H}_0(X,\mathbb{Z}) \to H_0(X,\mathbb{Z}) \to \mathbb{Z} \to \tilde{H}_{-1}(X,\mathbb{Z}) \to 0,$$

where $\mathbb{Z}$ is identified with the group $H_0(pt,\mathbb{Z})$. Notice that $H_0(X,\mathbb{Z})$ is identified with the free abelian group generated by the connected components of X.

A *local system* of abelian groups on a space (resp. poset) X is a functor $\mathcal{F}$ from the groupoid of X (resp. X viewed as a category), to the category of abelian groups which is morphism-inverting, i. e. such that the map $\mathcal{F}(x) \to \mathcal{F}(x')$ associated to a path from x to x' (resp. $x \leq x'$) is an isomorphism. Clearly, a local system $\mathcal{F}$ on a path connected space (resp. 0-connected poset) is determined, up to canonical isomorphism, by the following data: if $x \in X$ is a base point, it suffices to be given the group $\mathcal{F}(x)$ and an action of $\pi_1(X,x)$ on $\mathcal{F}(x)$.

The homology groups $H_k(X,\mathcal{F})$ of a space X with a local system $\mathcal{F}$ are a generalization of the ordinary homology groups. In fact if X is a 0-connected space and if $\mathcal{F}$ is a constant local system on X, then $H_k(X,\mathcal{F}) \simeq H_k(X,\mathcal{F}(x_0))$ for every $x_0 \in X$ [**52**, Chap. VI, 2.1].

Let X be a poset and $\mathcal{F}$ a local system on $|X|$. Then the restriction of $\mathcal{F}$ to X is a local system on X, so we can define $H_k(X,\mathcal{F})$ as in the above. Conversely if $\mathcal{F}$ is a local system on the poset X, then there is a local system, unique up to isomorphism, on $|X|$ such that its restriction to X is $\mathcal{F}$ [**52**, Chap. VI, Thm 1.12], [**31**, I, Prop. 1]. We denote both local systems by $\mathcal{F}$.

Theorem 1.3. *Let X be a poset and $\mathcal{F}$ a local system on X. Then the homology groups $H_k(|X|,\mathcal{F})$ are isomorphic with the homology groups $H_k(X,\mathcal{F})$.*

Proof. See [**52**, Chap. VI, Thm. 4.8] or [**31**, I, p. 91]. □

Theorem 1.4. *Let X be a path connected space with a base point x and let $\mathcal{F}$ be a local system on X. Then the inclusion $\{x\} \hookrightarrow X$ induces an isomorphism $\mathcal{F}(x)/N \xrightarrow{\simeq} H_0(X,\mathcal{F})$, where N is the subgroup of $\mathcal{F}(x)$ generated by all the elements of the form $a - \beta a$ with $a \in \mathcal{F}(x)$, $\beta \in \pi_1(X,x)$.*

Proof. See [**52**], Chap. VI, Thm. 2.8* and Thm. 3.2. □

We need the following interesting and well known lemma about the covering spaces of the space $|X|$, where X is a poset (or more generally a simplicial set). For a definition of a covering space, useful for our purpose, and some more information, see [**37**, Chap. 2].

LEMMA 1.5. *For a poset X the category of the covering spaces of the space $|X|$ is equivalent to the category $\mathcal{L}_S(X)$, the category of functors $\mathcal{F} : X \to \underline{\mathrm{Set}}$, where $\underline{\mathrm{Set}}$ is the category of sets, such that $\mathcal{F}(x) \to \mathcal{F}(x')$ is a bijection for every relation $x \leq x'$.*

PROOF. See [**31**, I, p. 90]. For the same proof with more details see [**39**, lem. 6.1]. $\square$

2. Homology and homotopy of posets

THEOREM 2.1. *Let $f : X \to Y$ be a map of posets. Then there is a first quadrant spectral sequence*

$$E^2_{p,q} = H_p(Y, y \mapsto H_q(f/y, \mathbb{Z})) \Rightarrow H_{p+q}(X, \mathbb{Z}).$$

The spectral sequence is functorial, in the sense that if there is a commutative diagram of posets

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g_X} & & \downarrow{\scriptstyle g_Y} \\ X & \xrightarrow{f} & Y, \end{array}$$

then there is a natural map from the spectral sequence arising from f' to the spectral sequence arising from f. Moreover the map $g_{X*} : H_i(X', \mathbb{Z}) \to H_i(X, \mathbb{Z})$ is compatible with this natural map.

PROOF. Let $C_{*,*}(f)$ be the double complex such that $C_{p,q}(f)$ is the free abelian group generated by the set $\{(x_0 < \cdots < x_q, y_0 < \cdots < y_p) : x_i \in X,\ y_i \in Y, f(x_q) < y_0\}$. The first spectral sequence of this double complex has as E^1-terms $E^1_{p,q}(\mathrm{I}) = H_q(C_{p,*}(f)) = \bigoplus_{y_0 < \cdots < y_p} H_q(f/y_0, \mathbb{Z})$. By the general theory of double complexes, see for example [**6**], we know that $E^2_{p,q}(\mathrm{I})$ is the homology of the chain complex $E^1_{*,q}(\mathrm{I}) = C_*(Y, \mathcal{G}_q)$, where $\mathcal{G}_q : Y \to \underline{\mathrm{Ab}}$ with $\mathcal{G}_q(y) = H_q(f/y, \mathbb{Z})$. Hence $E^2_{p,q}(\mathrm{I}) = H_p(Y, \mathcal{G}_q) = H_p(Y, y \mapsto H_q(f/y, \mathbb{Z}))$. The second spectral sequence has as E^1-terms $E^1_{p,q}(\mathrm{II}) = H_q(C_{*,p}(f)) = \bigoplus_{f(x_p) < y_0 < \cdots < y_q} H_q(f(x_p)\backslash \mathrm{id}_Y, \mathbb{Z})$. By 1.2 (ii), $f(x_p)\backslash \mathrm{id}_Y = Y_{\geq f(x_p)}$ is contractible, so $E^1_{*,0}(\mathrm{II}) = C_*(X^{op}, \mathbb{Z})$ and $E^1_{*,q}(\mathrm{II}) = 0$ for $q > 0$. Hence $H_i(\mathrm{Tot}(C_{*,*}(f))) \simeq H_i(X^{op}, \mathbb{Z}) \simeq H_i(X, \mathbb{Z})$. This completes the proof of existence and convergence of the spectral sequence. The functorial behavior of the spectral sequence follows from the functorial behavior of the spectral sequence of a filtration [**51**, 5.5.1] and the fact that the first and the second spectral sequences of the double complex arise from some filtration. $\square$

REMARK 2. Theorem 2.1 is a special case of a more general theorem [**14**, App. II]. The above proof is taken from [**20**, Chap. I], where the functorial behavior of the spectral sequence is more visible. For more details see [**20**].

DEFINITION 2.2. Let X be a poset. A map $\mathrm{ht}_X : X \to \mathbb{Z}_{\geq 0}$ is called *height function* if it is a strictly increasing map.

EXAMPLE 1. The height function $\mathrm{ht}_X(x) = 1 + \dim(Link_X^-(x))$ is the usual one considered in [**34**], [**20**] and [**8**].

LEMMA 2.3. *Let X be a poset such that $Link_X^+(x)$ is $(n - \mathrm{ht}_X(x) - 2)$-acyclic for every $x \in X$, where ht_X is a height function on X. Let $\mathcal{F} : X \to \underline{\mathrm{Ab}}$ be a functor such that $\mathcal{F}(x) = 0$ for all $x \in X$ with $\mathrm{ht}_X(x) \geq m$, where $m \geq 1$. Then $H_k(X, \mathcal{F}) = 0$ for $k \leq n - m$.*

PROOF. First consider the case of a functor $\mathcal{F}$ such that $\mathcal{F}(x) = 0$ if $\mathrm{ht}_X(x) \neq m - 1$. Then $C_k(X, \mathcal{F}) = \bigoplus_{\substack{x_0 < \cdots < x_k \\ \mathrm{ht}_X(x_0) = m-1}} \mathcal{F}(x_0)$. Clearly $0 = d_0^k = \mathcal{F}(x_0 < x_1) = \mathcal{F}(x_0) \to \mathcal{F}(x_1)$. Thus $\partial_k = \sum_{i=1}^k (-1)^i d_i^k$. Define $C_{-1}(Link_X^+(x_0), \mathcal{F}(x_0)) = \mathcal{F}(x_0)$ and complete the singular complex of $Link_X^+(x_0)$ with coefficient in $\mathcal{F}(x_0)$ to

$$\cdots \to C_0(Link_X^+(x_0), \mathcal{F}(x_0)) \xrightarrow{\varepsilon} C_{-1}(Link_X^+(x_0), \mathcal{F}(x_0)) \to 0,$$

where $\varepsilon((g_i)) = \sum_i g_i$. Then

$$\begin{aligned} C_k(X, \mathcal{F}) &= \bigoplus_{\mathrm{ht}_X(x_0) = m-1} \Big(\bigoplus_{\substack{x_1 < \cdots < x_k \\ x_0 < x_1}} \mathcal{F}(x_0) \Big) \\ &= \bigoplus_{\mathrm{ht}_X(x_0) = m-1} C_{k-1}(Link_X^+(x_0), \mathcal{F}(x_0)). \end{aligned}$$

The complex $C_{k-1}(Link_X^+(x_0), \mathcal{F}(x_0))$ is the standard complex for computing the reduced homology of $Link_X^+(x_0)$ with constant coefficient $\mathcal{F}(x_0)$. So

$$H_k(X, \mathcal{F}) = \bigoplus_{\mathrm{ht}_X(x) = m-1} \tilde{H}_{k-1}(Link_X^+(x), \mathcal{F}(x)).$$

If $\mathrm{ht}_X(x_0) = m - 1$, then $Link_X^+(x_0)$ is $(n - (m-1) - 2)$-acyclic, and by the universal coefficient theorem [**37**, Chap. 5, Thm. 8], $\tilde{H}_{k-1}(Link_X^+(x_0), \mathcal{F}(x_0)) = 0$ for $-1 \leq k - 1 \leq n - (m-1) - 2$. This shows that $H_k(X, \mathcal{F}) = 0$ for $0 \leq k \leq n - m$. To prove the lemma in general, we argue by induction on m. If $m = 1$, then for $\mathrm{ht}_X(x) \geq 1$, $\mathcal{F}(x) = 0$. So the lemma follows from the special case above. Suppose $m \geq 2$. Define $\mathcal{F}_0$ and $\mathcal{F}_1$ to be the functors

$$\mathcal{F}_0(x) = \begin{cases} \mathcal{F}(x) & \text{if } \mathrm{ht}_X(x) < m - 1 \\ 0 & \text{if } \mathrm{ht}_X(x) \geq m - 1 \end{cases}, \quad \mathcal{F}_1(x) = \begin{cases} \mathcal{F}(x) & \text{if } \mathrm{ht}_X(x) = m - 1 \\ 0 & \text{if } \mathrm{ht}_X(x) \neq m - 1 \end{cases},$$

respectively. Then there is a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_0 \to 0$. By the above discussion, $H_k(X, \mathcal{F}_1) = 0$ for $0 \leq k \leq n - m$ and by induction for $m - 1$, we have $H_k(X, \mathcal{F}_0) = 0$ for $k \leq n - (m-1)$. By the long exact sequence for the above short exact sequence of functors it is easy to see that $H_k(X, \mathcal{F}) = 0$ for $0 \leq k \leq n - m$. $\square$

Theorem 2.4. *Let $f : X \to Y$ be a map of posets and ht_Y a height function on Y. Assume for every $y \in Y$, that $Link_Y^+(y)$ is $(n - \mathrm{ht}_Y(y) - 2)$-acyclic and f/y is $(\mathrm{ht}_Y(y) - 1)$-acyclic. Then $f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$ is an isomorphism for $0 \leq k \leq n - 1$.*

Proof. By theorem 2.1, we have the first quadrant spectral sequence

$$E^2_{p,q} = H_p(Y, y \mapsto H_q(f/y, \mathbb{Z})) \Rightarrow H_{p+q}(X, \mathbb{Z}).$$

Since $H_q(f/y, \mathbb{Z}) = 0$ for $0 < q \leq \mathrm{ht}_Y(y) - 1$, the functor $\mathcal{G}_q : Y \to \underline{\mathrm{Ab}}$ with $\mathcal{G}_q(y) = H_q(f/y, \mathbb{Z})$, is trivial for $\mathrm{ht}_Y(y) \geq q+1$, $q > 0$. By lemma 2.3, $H_p(Y, \mathcal{G}_q) = 0$ for $p \leq n-(q+1)$. Hence $E^2_{p,q} = 0$ for $p+q \leq n-1$, $q > 0$. If $q = 0$, by writing the long exact sequence for the short exact sequence $0 \to \tilde{H}_0(f/y, \mathbb{Z}) \to H_0(f/y, \mathbb{Z}) \to \mathbb{Z} \to 0$ (valid because f/y is nonempty) we have

$$\begin{aligned} \cdots \to H_n(Y, \mathbb{Z}) \to H_{n-1}(Y, y \mapsto \tilde{H}_0(f/y, \mathbb{Z})) \to E^2_{n-1,0} \to \\ \cdots \to H_1(Y, \mathbb{Z}) \to H_0(Y, y \mapsto \tilde{H}_0(f/y, \mathbb{Z})) \to E^2_{0,0} \to H_0(Y, \mathbb{Z}) \to 0. \end{aligned}$$

If $\mathrm{ht}_Y(y) \geq 1$, then $\tilde{H}_0(f/y, \mathbb{Z}) = 0$. By lemma 2.3, $H_k(Y, y \mapsto \tilde{H}_0(f/y, \mathbb{Z})) = 0$ for $0 \leq k \leq n - 1$. Thus

$$E^2_{p,q} = \begin{cases} H_p(Y, \mathbb{Z}) & \text{if } q = 0,\ 0 \leq p \leq n - 1 \\ 0 & \text{if } p + q \leq n - 1,\ q > 0 \end{cases}.$$

This shows that $E^2_{p,q} \simeq \cdots \simeq E^\infty_{p,q}$ for $0 \leq p + q \leq n - 1$. Therefore $H_k(X, \mathbb{Z}) \simeq H_k(Y, \mathbb{Z})$ for $0 \leq k \leq n - 1$. Now consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \mathrm{id}_Y} \\ Y & \xrightarrow{\mathrm{id}_Y} & Y. \end{array}$$

By functoriality of the spectral sequence, Theorem 2.1, and the above calculation we get the following commutative diagram

$$\begin{array}{ccc} H_k(Y, y \mapsto H_0(f/y, \mathbb{Z})) & \xrightarrow{\simeq} & H_k(X, \mathbb{Z}) \\ \Big\downarrow{\scriptstyle \mathrm{id}_{Y*}} & & \Big\downarrow{\scriptstyle f_*} \\ H_k(Y, y \mapsto H_0(\mathrm{id}_Y/y, \mathbb{Z})) & \xrightarrow{\simeq} & H_k(Y, \mathbb{Z}). \end{array}$$

Since $\mathrm{id}_Y/y = Y_{\leq y}$ is contractible, $H_k(Y, y \mapsto H_0(\mathrm{id}_Y/y, \mathbb{Z})) = H_k(Y, \mathbb{Z})$. From the above long exact sequence the map id_{Y*} is an isomorphism for $0 \leq k \leq n - 1$. This shows that f_* is an isomorphism for $0 \leq k \leq n - 1$. $\square$

Lemma 2.5. *Let X be a 0-connected poset. Then X is 1-connected if and only if for every local system $\mathcal{F}$ on X and every $x \in X$ the map $\mathcal{F}(x) \to H_0(X, \mathcal{F})$, induced from the inclusion $\{x\} \hookrightarrow X$, is an isomorphism (or equivalently, every local system on X is isomorphic with a constant local system).*

Proof. If X is 1-connected, then by Theorem 1.4 and the connectedness of X, one has $\mathcal{F}(x) \xrightarrow{\simeq} H_0(X, \mathcal{F})$ for every $x \in X$. Now let every local system on X be isomorphic with a constant local system. Let $\mathcal{F} : X \to \underline{\mathrm{Set}}$ be in $\mathcal{L}_S(X)$.

Define the functor $\mathcal{G} : X \to \underline{\text{Ab}}$, where $\mathcal{G}(x)$ is the free abelian group generated by $\mathcal{F}(x)$. Clearly $\mathcal{G}$ is a local system and so it is constant system. It follows that $\mathcal{F}$ is isomorphic to a constant functor. So by lemma 1.5, any connected covering space of $|X|$ is isomorphic to $|X|$. This shows that the universal covering of $|X|$ is $|X|$. Note that the universal covering of a connected simplicial simplex exists and is simply connected [**37**, Chap. 2, Cor. 14 and 15]. Therefore X is 1-connected. □

THEOREM 2.6. *Let $f : X \to Y$ be a map of posets and* ht_Y *a height function on Y. Assume for every $y \in Y$, that $Link_Y^+(y)$ is $(n - \text{ht}_Y(y) - 2)$-connected and f/y is $(\text{ht}_Y(y) - 1)$-connected. Then X is $(n-1)$-connected if and only if Y is $(n-1)$-connected.*

PROOF. By 1.1 and 2.4 we may assume $n \geq 2$. So it is sufficient to prove that X is 1-connected if and only if Y is 1-connected. Let $\mathcal{F} : X \to \underline{\text{Ab}}$ be a local system. Define the functor $\mathcal{G} : Y \to \underline{\text{Ab}}$ with

$$\mathcal{G}(y) = \begin{cases} H_0(f/y, \mathcal{F}) & \text{if } \text{ht}_Y(y) \neq 0 \\ H_0(Link_Y^+(y), y' \mapsto H_0(f/y', \mathcal{F})) & \text{if } \text{ht}_Y(y) = 0 \end{cases}.$$

We prove that $\mathcal{G}$ is a local system. If $\text{ht}_Y(y) \geq 2$, then f/y is 1-connected and by 2.5, $\mathcal{F}|_{f/y}$ is a constant system, and $H_0(f/y, \mathcal{F}) \simeq \mathcal{F}(x)$ for every $x \in f/y$. If $\text{ht}_Y(y) = 1$, then f/y is 0-connected and $Link_Y^+(y)$ is nonempty. Choose $y' \in Y$ such that $y < y'$. Now f/y' is 1-connected, so $\mathcal{F}|_{f/y'}$ is a constant system on f/y'. The relation $f/y \subset f/y'$ implies that $\mathcal{F}|_{f/y}$ is a constant system. Since f/y is 0-connected, by 1.4 and the fact that we mentioned before theorem 1.3, for every $x \in f/y$, $H_0(f/y, \mathcal{F}) \simeq \mathcal{F}(x)$. Now let $\text{ht}_Y(y) = 0$. Then $Link_Y^+(y)$ is 0-connected, f/y is nonempty and for every $y' \in Link_Y^+(y)$, $H_0(f/y', \mathcal{F}) \simeq H_0((f/y)^\circ, \mathcal{F})$, where $(f/y)^\circ$ is a component of f/y, which we fix. This shows that the local system $\mathcal{F}' : Link_Y^+(y) \to \underline{\text{Ab}}$ with $y' \mapsto H_0(f/y', \mathcal{F})$ is isomorphic to a constant system, so $H_0(Link_Y^+(y), y' \mapsto H_0(f/y', \mathcal{F})) = H_0(Link_Y^+(y), \mathcal{F}') \simeq \mathcal{F}'(y') \simeq \mathcal{F}(x)$ for every $x \in f/y'$. Therefore $\mathcal{G}$ is a local system.

If Y is 1-connected, by 2.5, $\mathcal{G}$ is a constant system. But it is easy to see that $\mathcal{F} \simeq \mathcal{G} \circ f$. Therefore $\mathcal{F}$ is a constant system. Since X is connected, by our homology calculation, we conclude that X is 1-connected 2.5. Now let X be 1-connected. If $\mathcal{E}$ is a local system on Y, then $f^*\mathcal{E} := \mathcal{E} \circ f$ is a local system on X. So it is a constant local system. As above we can construct a local system $\mathcal{G}'$ on Y from $\mathcal{F}' := \mathcal{E} \circ f$. This gives a natural transformation from $\mathcal{G}'$ to $\mathcal{E}$ which is an isomorphism. Since $\mathcal{E} \circ f$ is constant, by 1.4 and 2.5 and an argument as above one sees that $\mathcal{G}'$ is constant. Therefore $\mathcal{E}$ is isomorphic to a constant local system and 2.5 shows that Y is 1-connected. □

In the proof of Theorem 2.6 in fact we proved the following proposition.

PROPOSITION 2.7. *Let $f : X \to Y$ be a map of posets and* ht_Y *a height function on Y. Assume that for every $y \in Y$, $Link_Y^+(y)$ is $(-\text{ht}_Y(y))$-connected and f/y is $(\text{ht}_Y(y) - 1)$-connected. Then $f^* : \mathcal{L}_S(Y) \to \mathcal{L}_S(X)$, $\mathcal{E} \mapsto \mathcal{E} \circ f$, is an equivalence of categories.*

REMARK 3. Theorem 2.6 is a generalization of a theorem of Quillen [**34**, Thm. 9.1]. We proved that the converse of that theorem is also valid. Our proof is similar in outline to the proof by Quillen. Furthermore, lemma 2.3 is a generalized version of lemma 1.3 from [**8**].

3. Posets of sequences

Let V be a nonempty set. We denote by $\mathcal{O}(V)$ the poset of finite ordered sequences of distinct elements of V, the length of each sequence being at least one. The partial ordering on $\mathcal{O}(V)$ is defined by refinement: $(v_1,\dots,v_m)\le(w_1,\dots,w_n)$ if and only if there is a strictly increasing map $\phi:\{1,\dots,m\}\to\{1,\dots,n\}$ such that $v_i=w_{\phi(i)}$, in other words if $(v_1,\dots,v_m)$ is an order preserving subsequence of $(w_1,\dots,w_n)$. We call $\mathcal{O}(V)$ the *simplex poset* of V. If $v=(v_1,\dots,v_m)$ and $w=(w_1,\dots,w_n)$, then by vw we mean $(v_1,\dots,v_m,w_1,\dots,w_n)$ and by $|v|$ we mean the length of v, that is $|v|=m$. Let F be a subposet of $\mathcal{O}(V)$. For $v\in F$, and only for such v, we define F_v to be the set of $w\in F$ such that $wv\in F$. Note that $(F_v)_w=F_{wv}$. A subset F of $\mathcal{O}(V)$ is said to satisfy the *chain condition* if $v\in F$ whenever $w\in F$, $v\in\mathcal{O}(V)$ and $v\le w$. The subposets of $\mathcal{O}(V)$ which satisfy the chain condition are extensively studied in [**20**], [**46**] and [**7**]. In this section we will study them some more.

Let $F\subseteq\mathcal{O}(V)$. For a nonempty set S we define the poset $F\langle S\rangle$ as

$$F\langle S\rangle:=\{((v_1,s_1),\dots,(v_r,s_r))\in\mathcal{O}(V\times S):(v_1,\dots,v_r)\in F\}.$$

Assume $s_0\in S$ and consider the injective poset map $l_{s_0}:F\to F\langle S\rangle$ with $(v_1,\dots,v_r)\mapsto((v_1,s_0),\dots,(v_r,s_0))$. We have clearly a projection $p:F\langle S\rangle\to F$ with $((v_1,s_1),\dots,(v_r,s_r))\mapsto(v_1,\dots,v_r)$ such that $p\circ l_{s_0}=\mathrm{id}_F$.

LEMMA 3.1. *Let $n\in\mathbb{Z}_{\ge1}$. Suppose $F\subseteq\mathcal{O}(V)$ satisfies the chain condition and S is a nonempty set. Assume that for every $v\in F$, F_v is $(n-|v|)$-connected.*

(i) *If $s_0\in S$, then $(l_{s_0})_*:H_k(F,\mathbb{Z})\to H_k(F\langle S\rangle,\mathbb{Z})$ is an isomorphism for $0\le k\le n$.*

(ii) *If F is* $\min\{1,n-1\}$*-connected, then $(l_{s_0})_*:\pi_k(F,v)\to\pi_k(F\langle S\rangle,l_{s_0}(v))$ is an isomorphism for $0\le k\le n$.*

PROOF. This follows by [**7**, Prop. 1.6] from the fact that $p\circ l_{s_0}=\mathrm{id}_F$. □

LEMMA 3.2. *Let $F\subseteq\mathcal{O}(V)$ satisfies the chain condition. Then $|Link_F^-(v)|\simeq S^{|v|-2}$ for every $v\in F$.*

PROOF. Let $v=(v_1,\dots,v_n)$. By definition $Link_F^-(v)=\{w\in F:w<v\}=\{(v_{i_1},\dots,v_{i_k}):k<n,i_1<\dots<i_k\}$. Hence $|Link_F^-(v)|$ is isomorphic to the barycentric subdivision of the boundary of the standard simplex Δ_{n-1}. It is well known that $\partial\Delta_{n-1}\simeq S^{n-2}$, hence $|Link_F^-(v)|\simeq S^{|v|-2}$. □

THEOREM 3.3 (Nerve Theorem for Posets). *Let V and T be two nonempty sets, $F\subseteq\mathcal{O}(V)$ and $X\subseteq\mathcal{O}(T)$. Assume $X=\bigcup_{v\in F}X_v$ such that if $v\le w$ in F, then $X_w\subseteq X_v$. Let F, X and X_v, for every $v\in F$, satisfy the chain condition. Also assume*

(i) *for every* $v \in F$, X_v *is* $(l-|v|+1)$*-acyclic (resp.* $(l-|v|+1)$*-connected),*

(ii) *for every* $x \in X$, $\mathcal{A}_x := \{v \in F : x \in X_v\}$ *is* $(l-|x|+1)$*-acyclic (resp.* $(l-|x|+1)$*-connected).*

Then $H_k(F,\mathbb{Z}) \simeq H_k(X,\mathbb{Z})$ *for* $0 \le k \le l$ *(resp.* F *is* l*-connected if and only if* X *is* l*-connected).*

Proof. Let $F_{\le l+2} = \{v \in F : |v| \le l+2\}$. If we consider $|F|$ as a cell complex whose k-cells are $|F_{\le v}|$ with $|v| = k+1$, then $|F_{\le l+2}|$ is the $(l+1)$-skeleton of $|F|$. It is well known that $i_* : H_k(F_{\le l+2},\mathbb{Z}) \to H_k(F,\mathbb{Z})$ and $i_* : \pi_k(F_{\le l+2}, v) \to \pi_k(F,v)$ are isomorphisms for $0 \le k \le l$, where i is the inclusion map (see [**52**], Chap. II, corollary 2.14, and [**52**], Chap. II, Corollary 3.10 and Chap. IV lemma 7.12). So it is sufficient to prove the theorem for $F_{\le l+2}$ and $X_{\le l+2}$, thus assume $F = F_{\le l+2}$ and $X = X_{\le l+2}$. Define $Z \subseteq X \times F$ as $Z = \{(x,v) : x \in X_v\}$ and consider the projections

$$f : Z \to F,\ (x,v) \mapsto v \qquad , \qquad g : Z \to X,\ (x,v) \mapsto x.$$

First we prove that $f^{-1}(v) \sim v\backslash f$ and $g^{-1}(x) \sim x\backslash g$, where $\sim$ means homotopy equivalence. By definition $v\backslash f = \{(x,w) : w \ge v,\ x \in X_w\}$. Define $\phi : v\backslash f \to f^{-1}(v)$, $(x,w) \mapsto (x,v)$. and consider the inclusion $j : f^{-1}(v) \to v\backslash f$. Clearly $\phi \circ j(x,v) = \phi(x,v) = (x,v)$ and $j \circ \phi(x,w) = j(x,v) = (x,v) \le (x,w)$. So by 1.2(i), $v\backslash f$ and $f^{-1}(v)$ are homotopy equivalent. Similarly $x\backslash g \sim g^{-1}(x)$.

We prove that the maps $f^{op} : Z^{op} \to Y^{op}$ and $g^{op} : Z^{op} \to X^{op}$ satisfy the conditions of 2.4. First $f^{op} : Z^{op} \to Y^{op}$; define the height function $\mathrm{ht}_{F^{op}}$ on F^{op} as $\mathrm{ht}_{F^{op}}(v) = l+2-|v|$. It is easy to see that $f^{op}/v \simeq v\backslash f \sim f^{-1}(v) \simeq X_v$. Hence f^{op}/v is $(l-|v|+1)$-acyclic (resp. $(l-|v|+1)$-connected). But $l-|v|+1 = (l+2-|v|)-1 = \mathrm{ht}_{F^{op}}(v)-1$, so f^{op}/v is $(\mathrm{ht}_{F^{op}}(v)-1)$-acyclic (resp. $(\mathrm{ht}_{F^{op}}(v)-1)$-connected). Let $n := l+1$. Clearly $Link^+_{F^{op}}(v) = Link^-_F(v)$. By lemma 3.2, $|Link^-_F(v)|$ is $(|v|-3)$-connected. But $|v|-3 = l+1-(l+2-|v|)-2 = n-\mathrm{ht}_{F^{op}}(v)-2$. Thus $Link^+_{F^{op}}(v)$ is $(n-\mathrm{ht}_{F^{op}}(v)-2)$-acyclic (resp. $(n-\mathrm{ht}_{F^{op}}(v)-2)$-connected). Therefore by theorem 2.4, $f_* : H_i(Z,\mathbb{Z}) \to H_i(F,\mathbb{Z})$ is an isomorphism for $0 \le i \le l$ (resp. by 2.6, F is l-connected if and only if Z is l-connected). Now consider $g^{op} : Z^{op} \to X^{op}$. We saw in the above that $g^{op}/x \simeq x\backslash g \sim g^{-1}(x)$ and $g^{-1}(x) = \{(x,v) : x \in X_v\} \simeq \{v \in F : x \in X_v\}$. It is similar to the case of f^{op} to see that g^{op} satisfies the conditions of theorem 2.4, hence $g_* : H_i(Z,\mathbb{Z}) \to H_i(X,\mathbb{Z})$ is an isomorphism for $0 \le i \le l$ (resp. by 2.6, X is l-connected if and only if Z is l-connected). This completes the proof. $\square$

Let K be a simplicial complex and $\{K_i\}_{i\in I}$ a family of subcomplexes such that $K = \bigcup_{i\in I} K_i$. The *nerve* of this family of subcomplexes of K is the simplicial complex $\mathcal{N}(K)$ on the vertex set I so that a finite subset $\sigma \subseteq I$ is in $\mathcal{N}(K)$ if and only if $\bigcap_{i\in\sigma} K_i \neq \varnothing$. The nerve $\mathcal{N}(K)$ of K, with the inclusion relation, is a poset. As we already said we can consider a simplicial complex as a poset of its simplices.

Corollary 3.4 (Nerve Theorem). *Let* K *be a simplicial complex and* $\{K_i\}_{i\in I}$ *a family of subcomplexes such that* $K = \bigcup_{i\in I} K_i$*. Suppose every nonempty finite intersection* $\bigcap_{j=1}^{t} K_{i_j}$ *is* $(l-t+1)$*-acyclic (resp.* $(l-t+1)$*connected). Then*

$H_k(K,\mathbb{Z}) \simeq H_k(\mathcal{N}(K),\mathbb{Z})$ for $0 \le k \le l$ (resp. K is l-connected if and only if $\mathcal{N}(K)$ is l-connected).

PROOF. Let V be the set of vertices of K. We give a total ordering to V and I. Put $F = \{(i_1,\dots,i_r) : i_1 < \cdots < i_r \text{ and } \bigcap_{j=1}^{r} K_{i_j} \neq \varnothing\} \subseteq \mathcal{O}(I)$, $X = \{(x_1,\dots,x_t) : x_1 < \cdots < x_t \text{ and } \{x_1,\dots,x_t\} \text{ is a simplex in } K\} \subseteq \mathcal{O}(V)$ and for every $(i_1,\dots,i_r) \in F$, put $X_{(i_1,\dots,i_r)} = \{(x_1,\dots,x_t) \in X : \{x_1,\dots,x_t\} \in \bigcap_{j=1}^{r} K_{i_j}\}$. It is not difficult to see that $F \simeq \mathcal{N}(K)$ and $X \simeq K$. Also one should notice that $\mathcal{A}_x := \{v \in F : x \in X_v\} \simeq \mathcal{O}(I'_x)$ is contractible for $x \in X$, where $I'_x = \{i \in I : x \in K_i\}$. We leave the details to the interested readers. $\square$

REMARK 4. The nerve theorem for a simplicial complex, Cor. 3.4, in the stated generality, is proved for the first time in [**4**].

LEMMA 3.5. *Let $F \subseteq \mathcal{O}(V)$ satisfy the chain condition and let $\mathcal{G} : F^{op} \to \underline{\mathrm{Ab}}$ be a functor. Then the natural map $\psi : \bigoplus_{v\in F,\ |v|=1} \mathcal{G}(v) \to H_0(F^{op},\mathcal{G})$ is surjective.*

PROOF. By definition

$$C_0(F^{op},\mathcal{G}) = \bigoplus_{v\in F^{op}} \mathcal{G}(v), \quad C_1(F^{op},\mathcal{G}) = \bigoplus_{v<v'\in F^{op}} \mathcal{G}(v)$$

and we have the chain complex

$$\cdots \to C_1(F^{op},\mathcal{G}) \xrightarrow{\partial_1} C_0(F^{op},\mathcal{G}) \to 0,$$

where $\partial_1 = d_0^1 - d_1^1$. Again by definition $H_0(F^{op},\mathcal{G}) = C_0(F^{op},\mathcal{G})/\mathrm{im}(\partial_1)$. If $w \in F$ and $|w| \ge 2$, then there is a $v \in F$, $v < w$, with $|v| = 1$. By definition $\partial_1|_{\mathcal{G}(w)} : \mathcal{G}(w) \to \mathcal{G}(w) \oplus \mathcal{G}(v)$, $x \mapsto d_0^1(x) - d_1^1(x) = d_0^1(x) - x$. This shows that $\mathcal{G}(w) \subseteq \mathrm{im}\partial_1 + \mathrm{im}\psi$. Therefore $H_0(F^{op},\mathcal{G})$ is generated by the groups $\mathcal{G}(v)$ with $|v| = 1$. $\square$

THEOREM 3.6. *Let V and T be two nonempty sets, $F \subseteq \mathcal{O}(V)$ and $X \subseteq \mathcal{O}(T)$. Assume $X = \bigcup_{v\in F} X_v$ such that if $v \le w$ in F, then $X_w \subseteq X_v$ and let F, X and X_v, for every $v \in F$, satisfy the chain condition. Also assume*
(i) *for every $v \in F$, X_v is $\min\{l-1, l-|v|+1\}$-connected,*
(ii) *for every $x \in X$, $\mathcal{A}_x := \{v \in F : x \in X_v\}$ is $(l-|x|+1)$-connected,*
(iii) *F is l-connected.*
Then X is $(l-1)$-connected and the natural map

$$\bigoplus_{v\in F,\ |v|=1} (i_v)_* : \bigoplus_{v\in F,\ |v|=1} H_l(X_v,\mathbb{Z}) \to H_l(X,\mathbb{Z})$$

is surjective, where $i_v : X_v \to X$ is the inclusion. Moreover, if for every v with $|v| = 1$ there is an l-connected Y_v with $X_v \subseteq Y_v \subseteq X$, then X is also l-connected.

PROOF. If $l = -1$, then everything is easy. If $l = 0$, then for v of length one X_v is nonempty, so X is nonempty. This shows that X is (-1)-connected. Also, every connected component of X intersects at least one X_v and therefore also contains a connected component of an X_v with $|v| = 1$. This gives the surjectivity of the homomorphism

$$\bigoplus_{v\in F,\ |v|=1} (i_v)_* : \bigoplus_{v\in F,\ |v|=1} H_0(X_v, \mathbb{Z}) \to H_0(X, \mathbb{Z}).$$

Now assume that for every v of length one $X_v \subseteq Y_v$, where Y_v is connected. We prove, in a combinatorial way, that X is connected. Let $x, y \in X$, $x \in X_{(v_1)}$ and $y \in X_{(v_2)}$, where $(v_1), (v_2) \in F$. Since F is connected, there is a sequence $(w_1), \dots, (w_r) \in F$ such that they give a path, in F, from (v_1) to (v_2), that is

$$\begin{array}{ccccccccc} (v_1) & & & & (w_1) & & (w_r) & & & & (v_2) \\ & \diagdown & & \diagup & & \cdots & & \diagdown & & \diagup & \\ & & (v_1, w_1) & & & & & & (w_r, v_2) & & . \end{array}$$

Since $Y_{(v_1)}$ is connected, $x \in X_{(v_1)} \subseteq Y_{(v_1)}$ and $X_{(v_1,w_1)} \neq \varnothing$, there is an element $x_1 \in X_{(v_1,w_1)}$ such that there is a path, in $Y_{(v_1)}$, from x to x_1. Now $x_1 \in Y_{(w_1)}$. In a similar way, we can find $x_2 \in X_{(w_1,w_2)}$ such that there is a path, in $Y_{(w_1)}$, from x_1 to x_2. Now $x_2 \in Y_{(w_2)}$. Repeating this process finitely many times, we will find a path from x to y. So X is connected.

Now let $l \geq 1$. As we said in the proof of theorem 3.3, we can assume that $F = F_{\leq l+2}$ and $X = X_{\leq l+2}$ and we define Z, f and g as it is defined there. Define the height function $\mathrm{ht}_{F^{op}}$ on F^{op} as $\mathrm{ht}_{F^{op}}(v) = l + 2 - |v|$. As it is proved in the proof of theorem 3.3, $f^{op}/v \simeq v\backslash f \sim f^{-1}(v) \simeq X_v$. Thus f^{op}/v is $(\mathrm{ht}_{F^{op}}(v) - 1)$-connected if $|v| > 1$ and it is $(\mathrm{ht}_{F^{op}}(v)-2)$-connected if $|v| = 1$ and also $|Link^+_{F^{op}}(v)|$ is $(l+1-\mathrm{ht}_{F^{op}}(v)-2)$-connected. By theorem 2.1, we have the first quadrant spectral sequence

$$E^2_{p,q} = H_p(F^{op}, v \mapsto H_q(f^{op}/v, \mathbb{Z})) \Rightarrow H_{p+q}(Z^{op}, \mathbb{Z}).$$

For $0 < q \leq \mathrm{ht}_{F^{op}}(v)-2$, $H_q(f^{op}/v, \mathbb{Z}) = 0$. If $\mathcal{G}_q : F^{op} \to \underline{\mathrm{Ab}}$, $\mathcal{G}_q(v) = H_q(f^{op}/v, \mathbb{Z})$, then $\mathcal{G}_q(v) = 0$ for $\mathrm{ht}_{F^{op}}(v) \geq q + 2$, $q > 0$. By lemma 2.3, $H_p(F^{op}, \mathcal{G}_q) = 0$ for $p \leq l+1-(q+2)$. Therefore $E^2_{p,q} = 0$ for $p+q \leq l-1$, $q > 0$. If $q = 0$, arguing similarly to the proof of theorem 2.4, we get $E^2_{p,0} = 0$ if $0 < p \leq l - 1$ and $E^2_{0,0} = \mathbb{Z}$. Also by the fact that F^{op} is l-connected we get the surjective homomorphism $H_l(F^{op}, v \mapsto \tilde{H}_0(f^{op}/v, \mathbb{Z})) \twoheadrightarrow E^2_{l,0}$. Since $l \geq 1$, $\tilde{H}_0(f^{op}/v, \mathbb{Z}) = 0$ for all $v \in F^{op}$ with $\mathrm{ht}_{F^{op}}(v) \geq 1$ and so by lemma 2.3, $H_l(F^{op}, v \mapsto \tilde{H}_0(f^{op}/v, \mathbb{Z})) = 0$. This implies that $E^2_{l,0} = 0$. Let $\mathcal{G}'_q : F^{op} \to \underline{\mathrm{Ab}}$, $\mathcal{G}'_q(v) = \begin{cases} 0 & \text{if } \mathrm{ht}_{F^{op}}(v) < l+1 \\ H_q(f^{op}/v, \mathbb{Z}) & \text{if } \mathrm{ht}_{F^{op}}(v) = l+1 \end{cases}$ and $\mathcal{G}''_q : F^{op} \to \underline{\mathrm{Ab}}$, $\mathcal{G}''_q(v) = \begin{cases} H_q(f^{op}/v, \mathbb{Z}) & \text{if } \mathrm{ht}_{F^{op}}(v) < l+1 \\ 0 & \text{if } \mathrm{ht}_{F^{op}}(v) = l+1 \end{cases}$. Then we have the short exact sequence $0 \to \mathcal{G}'_q \to \mathcal{G}_q \to \mathcal{G}''_q \to 0$ and the associated long exact sequence

$$\cdots \to H_{l-q}(F^{op}, \mathcal{G}'_q) \to H_{l-q}(F^{op}, \mathcal{G}_q) \to H_{l-q}(F^{op}, \mathcal{G}''_q) \to H_{l-q-1}(F^{op}, \mathcal{G}'_q) \to \cdots .$$

If $q > 0$, then $\mathcal{G}''_q(v) = 0$ for $0 < q \leq \mathrm{ht}_{F^{op}}(v) - 1$ and so by lemma 2.3, $H_p(F^{op}, \mathcal{G}''_q) = 0$ for $p + q \leq l$, $q > 0$. Also if $|v| = 1$, then $H_0(f^{op}/v, \mathbb{Z}) = 0$

for $0 < q \leq \mathrm{ht}_{F^{op}}(v) - 2 = l - 1$. This shows $\mathcal{G}'_q = 0$ for $0 < q \leq l-1$. From the long exact sequence and the above calculation we get, $E^2_{p,q} = \begin{cases} \mathbb{Z} & \text{if } p = q = 0 \\ 0 & \text{if } 0 < p+q \leq l,\ q \neq l \end{cases}$.
Thus for $0 \leq p+q \leq l$, $q \neq l$, $E^2_{p,q} \simeq \cdots \simeq E^\infty_{p,q}$ and there exist an integer r such that $E^2_{0,l} \twoheadrightarrow \cdots \twoheadrightarrow E^r_{0,l} \simeq E^{r+1}_{0,l} \simeq \cdots \simeq E^\infty_{0,l}$. Hence we get a surjective map $H_0(F^{op}, v \mapsto H_l(f^{op}/v, \mathbb{Z})) \twoheadrightarrow H_l(Z^{op}, \mathbb{Z})$. By lemma 3.5 we have a surjective map $\bigoplus_{v \in F,\ |v|=1} H_l(f^{op}/v, \mathbb{Z}) \twoheadrightarrow H_l(Z^{op}, \mathbb{Z})$.

Now consider the map $g^{op} : Z^{op} \to X^{op}$ and define the height function $\mathrm{ht}_{X^{op}}(x) = l + 2 - |x|$ on X^{op}. Arguing similarly to the proof of theorem 3.3 one sees that $g_* : H_k(Z, \mathbb{Z}) \to H_k(X, \mathbb{Z})$ is an isomorphism for $0 \leq k \leq l$. Therefore we get a surjective map $\bigoplus_{v \in F,\ |v|=1} H_l(X_v, \mathbb{Z}) \twoheadrightarrow H_l(X, \mathbb{Z})$, we call it ψ. We prove that this map is the same map that we claimed. Let $v \in F$ be of length one and consider the commutative diagram of posets

$$\begin{tikzcd}
 & \{v\}^{op} \arrow[rr] & & F^{op} \\
f^{-1}(\{v\})^{op} \arrow[ur] \arrow[rr] \arrow[dr] & & Z^{op} \arrow[ur] \arrow[dr] & \\
 & X_v^{op} \arrow[rr] & & X^{op}.
\end{tikzcd}$$

By functoriality of the spectral sequence for the above diagram and lemma 3.5 we get the commutative diagram

$$\begin{tikzcd}
H_l(f_v^{op}/v, \mathbb{Z}) \arrow[r, "(j_v)_*"] \arrow[d] & \bigoplus_{v \in F, |v|=1} H_l(f^{op}/v, \mathbb{Z}) \arrow[d] \\
H_0(\{v\}^{op}, v \mapsto H_l(f_v^{op}/v, \mathbb{Z})) \arrow[r] \arrow[d] & H_0(F^{op}, v \mapsto H_l(f^{op}/v, \mathbb{Z})) \arrow[d] \\
H_l(f^{-1}(v)^{op}, \mathbb{Z}) \arrow[r] \arrow[d] & H_l(Z^{op}, \mathbb{Z}) \arrow[d] \\
H_l(X_v^{op}, \mathbb{Z}) \arrow[r, "(i_v^{op})_*"] & H_l(X^{op}, \mathbb{Z}),
\end{tikzcd}$$

where $j_v : f_v^{op}/v \to f^{op}/v$ is the inclusion which is a homotopy equivalence as we already mentioned. It is not difficult to see that the composition of homomorphisms in the left column of the above diagram induces the identity map from $H_l(X_v, \mathbb{Z})$, the composition of homomorphisms in the right column of above diagram induces the surjective map ψ and the last row induces the homomorphism $(i_v)_*$. This show that $(i_v)_* = \psi|_{H_l(X_v, \mathbb{Z})}$. This completes the proof of surjectiveness.

Now let for v of length one $X_v \subseteq Y_v$, where Y_v is l-connected. Then we have the commutative diagram

$$\begin{tikzcd}
H_l(X_v, \mathbb{Z}) \arrow[rr, "(i_v)_*"] \arrow[dr] & & H_l(X, \mathbb{Z}) \\
 & H_l(Y_v, \mathbb{Z}). \arrow[ur] &
\end{tikzcd}$$

By the assumption $H_l(Y_v, \mathbb{Z})$ is trivial and this shows that $(i_v)_*$ is the zero map. Hence by the surjectivity, $H_l(X, \mathbb{Z})$ is trivial. If $l \geq 2$, the nerve theorem 3.3 says that X is simply connected and by the Hurewicz theorem 1.1, X is l-connected. So the only case that is left is when $l = 1$. By theorem 2.6, X is 1-connected if and only if Z is 1-connected. So it is sufficient to prove that Z^{op} is 1-connected. Note that as we said, we can assume that $F = F_{\leq 3}$ and $X = X_{\leq 3}$. Suppose $\mathcal{F}$ is a local system on Z^{op}. Define the functor $\mathcal{G} : F^{op} \to \underline{\mathrm{Ab}}$, as

$$\mathcal{G}(y) = \begin{cases} H_0(f^{op}/v, \mathcal{F}) & \text{if } |v| = 1, 2 \\ H_0(Link^{+}_{F^{op}}(v), v' \mapsto H_0(f^{op}/v', \mathcal{F})) & \text{if } |v| = 3 \end{cases}.$$

We prove that $\mathcal{G}$ is a local system on F^{op}. Put $Z_w := g^{-1}(Y_w)$ for $|w| = 1$. If $|v| = 1, 2$, then f^{op}/v is 0-connected and $f^{op}/v \subseteq Z_w^{op}$, where $w \leq v$, $|w| = 1$. By Proposition 2.7 we can assume that $\mathcal{F} = \mathcal{E} \circ g^{op}$, where $\mathcal{E}$ is a local system on X^{op}. Then $\mathcal{F}|_{Z_w^{op}} = \mathcal{E}|_{Y_w^{op}} \circ g^{op}|_{Z_w^{op}}$. Since Y_w^{op} is 1-connected, $\mathcal{E}|_{Y_w^{op}}$ is a constant local system. This shows that $\mathcal{F}|_{Z_w^{op}}$ is a constant local system. So $\mathcal{F}|_{f^{op}/v}$ is a constant local system and since f^{op}/v is 0-connected we have $H_0(f^{op}/v, \mathbb{Z}) \simeq \mathcal{F}(x)$, for every $x \in f^{op}/v$. If $|v| = 3$, with an argument similar to the proof of theorem 2.6 and the above discussion we have $\mathcal{G}(v) \simeq \mathcal{F}(x)$ for every $x \in f^{op}/v$. This shows that $\mathcal{G}$ is a local system on F^{op}. Hence it is a constant local system, because F^{op} is 1-connected. It is easy to see that $\mathcal{F} \simeq \mathcal{G} \circ f$. Therefore $\mathcal{F}$ is a constant system. Since X is connected by our homology calculation, by 2.5 we conclude that X is 1-connected. This completes the proof. $\square$

CHAPTER 2

Poset of isotropic unimodular sequences

In this chapter we will define the poset of isotropic unimodular sequences and will prove that if the unitary stable rank of our ring is finite then this poset is highly connected, which is the difficult part of the proof of the homology stability theorem for unitary groups (over these rings). The new nerve theorem, proved in the previous chapter, allows us to exploit the higher connectivity of the poset of unimodular sequences due to W. van der Kallen. The higher connectivity of the poset of isotropic unimodular sequences follows inductively.

In the last section we will concentrate over the local rings with infinite residue field and will prove a sharper result concerning the higher acyclicity of the poset and consequently a better range of homology stability, which is proved in chapter 3.

1. Posets of unimodular sequences

Let R be an associative ring with unit. A vector $(r_1, \dots, r_n) \in R^n$ is called *unimodular* if there exist $s_1, \dots, s_n \in R$ such that $\sum_{i=1}^{n} s_i r_i = 1$, or equivalently if the submodule generated by this vector is a free summand of the right R-module R^n. We denote the standard basis of R^n by $e_1, \dots, e_n$. If $n \le m$, we assume that R^n is the submodule of R^m generated by $e_1, \dots, e_n \in R^m$.

We say that a ring R satisfies the *stable range condition* (S_m), if $m \ge 1$ is an integer so that for every unimodular vector $(r_0, r_1, \dots, r_m) \in R^{m+1}$, there exist $t_1, \dots, t_m$ in R such that $(r_1 + t_1 r_0, \dots, r_m + t_m r_0) \in R^m$ is unimodular. We say that R has *stable rank* m, we denote it with $\mathrm{sr}(R) = m$, if m is the least number such that (S_m) holds. If such a number does not exist we say that $\mathrm{sr}(R) = \infty$.

Let $\mathcal{U}(R^n)$ denote the subposet of $\mathcal{O}(R^n)$ consisting of unimodular sequences. Recall that a sequence of vectors $v_1, \dots, v_k$ in R^n is called unimodular when $v_1, \dots, v_k$ is basis of a free direct summand of R^n. Note that if $(v_1, \dots, v_k) \in \mathcal{O}(R^n)$ and if $n \le m$, it is the same to say that $(v_1, \dots, v_k)$ is unimodular as a sequence of vectors in R^n or as a sequence of vectors in R^m. We call an element $(v_1, \dots, v_k)$ of $\mathcal{U}(R^n)$ a *k-frame.*

THEOREM 1.1 (Van der Kallen). *Let R be a ring with $\mathrm{sr}(R) < \infty$ and $n \le m+1$. Let δ be 0 or 1. Then*

(i) *$\mathcal{O}(R^n + e_{n+1}\delta) \cap \mathcal{U}(R^m)$ is $(n - \mathrm{sr}(R) - 1)$-connected.*

(ii) *$\mathcal{O}(R^n + e_{n+1}\delta) \cap \mathcal{U}(R^m)_v$ is $(n - \mathrm{sr}(R) - |v| - 1)$-connected for all $v \in \mathcal{U}(R^m)$.*

PROOF. See [**46**, Thm. 2.6]. □

EXAMPLE 2. Let R be a ring with $\mathrm{sr}(R) < \infty$. Let $n \geq \mathrm{sr}(R)+k+1$ and assume $(v_1, \dots, v_k) \in \mathcal{U}(R^{2n})$. Set $W = e_2 + \sum_{i=2}^{n} e_{2i}R$. Renumbering the basis one gets, by Theorem 1.1, that the poset $F := \mathcal{O}(W) \cap \mathcal{U}(R^{2n})_{(v_1,\dots,v_k)}$ is $((n-1)-\mathrm{sr}(R)-k-1)$-connected. Since $n \geq \mathrm{sr}(R)+k+1$, it follows that F is not empty. This shows that there is $v \in W$ such that $(v, v_1, \dots, v_k) \in \mathcal{U}(R^{2n})$. We will need such result in the next section but with a different method we can prove a sharper result. Compare this with lemma 1.3.

An $k \times n$-matrix B with $n < k$ is called *unimodular* if B has a left inverse. If B is an $k \times n$-matrix and $C \in GL_k(R)$, then B is unimodular if and only if CB is unimodular. A matrix of the form $\begin{pmatrix} 1 & u \\ 0 & B \end{pmatrix}$, where u is a row vector, is unimodular if and only if the matrix B is unimodular.

We say that the ring R satisfies the *stable range condition* (S_k^n) if for every $(n+k) \times n$-matrix B, there exists a vector $r = (r_1, \dots, r_{n+k-1})$ such that $\begin{pmatrix} 1 & 0 \\ r & I_{n+k-1} \end{pmatrix} B = \begin{pmatrix} u \\ B' \end{pmatrix}$, where the $(n+k-1) \times n$-matrix B' is unimodular and u is the first row of the matrix B. Note that (S_k^1) is the same as (S_k).

THEOREM 1.2 (Vaserstein). *For every $k \geq 1$ and $n \geq 1$, a ring R satisfies (S_k) if and only if it satisfies (S_k^n).*

PROOF. See [**48**, Thm. 3′] of Vaserstein. □

LEMMA 1.3. *Let R be a ring with $\mathrm{sr}(R) < \infty$ and let $n \geq \mathrm{sr}(R)+k$. Then for every $(v_1, \dots, v_k) \in \mathcal{U}(R^{2n})$ there is a $v \in e_2 + \sum_{i=2}^{n} e_{2i}R$ such that $(v, v_1, \dots, v_k) \in \mathcal{U}(R^{2n})$.*

PROOF. There is a permutation matrix $A \in GL_{2n}(R)$ such that

$$A(e_2 + \sum_{i=2}^{n} e_{2i}R) = e_1 + \sum_{j=n+2}^{2n} e_j R.$$

Let $w_i = Av_i$ for $i = 1, \dots, k$. So $(w_1, \dots, w_k) \in \mathcal{U}(R^{2n})$. Consider the $2n \times k$-matrix B whose i-th column is the vector w_i. By Theorem 1.2 there exists a vector $r = (r_2, \dots, r_{2n})$ such that $\begin{pmatrix} 1 & 0 \\ r & I_{2n-1} \end{pmatrix} B = \begin{pmatrix} u_1 \\ B_1 \end{pmatrix}$, where the $(2n-1) \times k$-matrix B_1 is unimodular and u_1 is the first row of the matrix B. Now let $s = (s_3, \dots, s_{2n})$ such that $\begin{pmatrix} 1 & 0 \\ s & I_{2n-2} \end{pmatrix} B_1 = \begin{pmatrix} u_2 \\ B_2 \end{pmatrix}$, where the $(2n-2) \times k$-matrix B_2 is unimodular and u_2 is the first row of the matrix B_1. Now clearly

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & s & I_{2n-2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r & I_{2n-1} \end{pmatrix} B = \begin{pmatrix} u_1 \\ u_2 \\ B_2 \end{pmatrix}.$$

By continuing this process, n times, we find a $2n \times 2n$ matrix C of the form

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ * & \ddots & 0 & \cdots & 0 \\ * & * & 1 & 0 & 0 \\ * & * & * & 1 & 0 \\ \multicolumn{2}{c}{\mathrm{N}} & * & * & I_{n-1} \end{pmatrix},$$

where N is an $(n-1) \times (n-1)$ matrix and $CB = \left(\frac{L}{M}\right)$, where L is a $(n+1) \times k$ matrix and M is a unimodular $(n-1) \times k$ matrix. Now let $(t_{n+2}, \dots, t_{2n}) = -(\text{first column of } N)$. Then

$$C \begin{pmatrix} 1 & \\ 0 & \\ \vdots & \\ 0 & B \\ t_{n+2} & \\ \vdots & \\ t_{2n} & \end{pmatrix} = \begin{pmatrix} 1 & * \\ * & * \\ * & * \\ 0 & \\ \vdots & M \\ 0 & \end{pmatrix}$$

Since M is unimodular, the right hand side of the above equality is unimodular. This shows that the matrix

$$\begin{pmatrix} 1 & \\ 0 & \\ \vdots & \\ 0 & B \\ t_{n+2} & \\ \vdots & \\ t_{2n} & \end{pmatrix}$$

is unimodular. Put $w = (1, 0, \dots, 0, t_{n+2}, \dots, t_{2n})$. Then $(w, w_1, \dots, w_k) \in \mathcal{U}(R^{2n})$. Now $v = A^{-1}w$ is the one that we are looking for. □

2. Hyperbolic spaces and some posets

Let there be an *involution* on R, that is an automorphism of the additive group of R, $R \to R$ with $r \mapsto \overline{r}$, such that $\overline{\overline{r}} = r$ and $\overline{rs} = \overline{s}\,\overline{r}$. Let ϵ be an element in the center of R such that $\epsilon\overline{\epsilon} = 1$. Set $R_\epsilon := \{r - \epsilon\overline{r} : r \in R\}$ and $R^\epsilon := \{r \in R : \epsilon\overline{r} = -r\}$ and observe that $R_\epsilon \subseteq R^\epsilon$. A *form parameter* relative to the involution and ϵ is a subgroup Λ of $(R, +)$ such that $R_\epsilon \subseteq \Lambda \subseteq R^\epsilon$ and $\overline{r}\Lambda r \subseteq \Lambda$, for all $r \in R$. Notice that R_ϵ and R^ϵ are form parameters. We denote them by $\Lambda_{\min}$ and $\Lambda_{\max}$, respectively. If there is an s in the center of R such that $s + \overline{s} \in R^*$, in particular if $2 \in R^*$, then $\Lambda_{\min} = \Lambda_{\max}$.

Let $e_{i,j}(r)$ be the $2n \times 2n$-matrix with $r \in R$ in the (i, j) place and zero elsewhere. Consider $Q_n = \sum_{i=1}^n e_{2i-1,2i}(1) \in M_{2n}(R)$ and $F_n^{(\epsilon)} = Q_n + \epsilon\ {}^tQ_n = \sum_{i=1}^n (e_{2i-1,2i}(1) + e_{2i,2i-1}(\epsilon)) \in GL_{2n}(R)$. Define the bilinear map $h : R^{2n} \times$

$R^{2n} \to R$ by $h(x,y) = \sum_{i=1}^{n}(\overline{x_{2i-1}}y_{2i} + \epsilon\overline{x_{2i}}y_{2i-1})$ and $q : R^{2n} \to R/\Lambda$ by $q(x) = \Sigma_{i=1}^{n}\overline{x_{2i-1}}x_{2i}$ mod Λ, where $x = (x_1, \dots, x_{2n})$, $y = (y_1, \dots, y_{2n})$. The triple (R^{2n}, h, q) is called a *hyperbolic space*. By definition the *unitary group* relative to Λ is the group

$$U_{2n}^{\epsilon}(R,\Lambda) := \{A \in GL_{2n}(R) : h(Ax, Ay) = h(x,y), q(Ax) = q(x), x, y \in R\}.$$

For more general definitions and the properties of these spaces and groups see [**18**].

Example 3. (i) Let $\Lambda = \Lambda_{\max} = R$. Then $U_{2n}^{\epsilon}(R,\Lambda) = \{A \in GL_{2n}(R) : h(Ax, Ay) = h(x,y)$ for all $x, y \in R^{2n}\} = \{A \in GL_{2n}(R) : {}^t\overline{A}F_n^{(\epsilon)}A = F_n^{(\epsilon)}\}$. In particular if $\epsilon = -1$ and if the involution is the identity map id_R, then $\Lambda_{\max} = R$. In This case $U_{2n}^{\epsilon}(R, \Lambda_{\max}) := Sp_{2n}(R)$ is the usual *symplectic group*. Note that R is commutative in this case.

(ii) Let $\Lambda = \Lambda_{\min} = 0$. Then $U_{2n}^{\epsilon}(R,\Lambda) = \{A \in GL_{2n}(R) : q(Ax) = q(x)$ for all $x \in R^{2n}\} = \{A \in GL_{2n}(R) : {}^t\overline{A}Q_nA = Q_n\}$. In particular if $\epsilon = 1$ and if the involution is the identity map id_R, then $\Lambda_{\min} = 0$. In this case $U_{2n}^{\epsilon}(R, \Lambda_{\min}) := O_{2n}(R)$ is the usual *orthogonal group*. As in the symplectic case, R is necessarily commutative.

(iii) Let $\epsilon = -1$ and the involution is not the identity map id_R. If $\Lambda = \Lambda_{\max}$ then $U_{2n}^{\epsilon}(R,\Lambda) := U_{2n}(R)$ is the *classical unitary group* corresponding to the involution.

Let σ be the permutation of the set of natural numbers given by $\sigma(2i) = 2i - 1$ and $\sigma(2i-1) = 2i$. For $1 \le i, j \le 2n$, $i \ne j$, and every $r \in R$ define

$$E_{i,j}(r) = \begin{cases} I_{2n} + e_{i,j}(r) & \text{if } i = 2k-1, j = \sigma(i), r \in \Lambda \\ I_{2n} + e_{i,j}(r) & \text{if } i = 2k, j = \sigma(i), \overline{r} \in \Lambda \\ I_{2n} + e_{i,j}(r) + e_{\sigma(j),\sigma(i)}(-\overline{r}) & \text{if } i + j = 2k,\ i \ne j \\ I_{2n} + e_{i,j}(r) + e_{\sigma(j),\sigma(i)}(-\epsilon^{-1}\overline{r}) & \text{if } i \ne \sigma(j),\ i = 2k-1,\ j = 2l \\ I_{2n} + e_{i,j}(r) + e_{\sigma(j),\sigma(i)}(\epsilon\overline{r}) & \text{if } i \ne \sigma(j),\ i = 2k,\ j = 2l-1, \end{cases}$$

where I_{2n} is the identity element of $GL_{2n}(R)$. It is easy to see that $E_{i,j}(r) \in U_{2n}^{\epsilon}(R,\Lambda)$. Let $EU_{2n}^{\epsilon}(R,\Lambda)$ be the group generated by the matrices $E_{i,j}(r)$, $r \in R$. We call it *elementary unitary group*.

A nonzero vector $x \in R^{2n}$ is called *isotropic* if $q(x) = 0$. This shows automatically that if x is isotropic then $h(x,x) = 0$. We say that a subset S of R^{2n} is isotropic if for every $x \in S$, $q(x) = 0$ and for every $x, y \in S$, $h(x,y) = 0$. If $h(x,y) = 0$, then we say that x is perpendicular to y. We denote by $\langle S \rangle$ the submodule of R^{2n} generated by S, and by $\langle S \rangle^{\perp}$ the submodule consisting of all the elements of R^{2n} which are perpendicular to all the elements of S.

From now, we fix an involution, an ϵ, a form parameter Λ and we consider the triple (R^{2n}, h, q) as defined above.

Definition 2.1 (Transitivity condition). Let $r \in R$ and define $C_r^{\epsilon}(R^{2n},\Lambda) = \{x \in \mathrm{Um}(R^{2n}) : q(x) = r \text{ mod } \Lambda\}$, where $\mathrm{Um}(R^{2n})$ is the set of all unimodular vectors of R^{2n}. We say that R satisfies the transitivity condition (T_n), if $EU_{2n}^{\epsilon}(R,\Lambda)$ acts transitively on $C_r^{\epsilon}(R^{2n},\Lambda)$, for every $r \in R$. It is easy to see that $e_1 + e_2r \in C_r^{\epsilon}(R^{2n},\Lambda)$.

Definition 2.2 (Unitary stable range). We say that a ring R satisfies the *unitary stable range condition* (US_m) if R satisfies the conditions (S_m) and (T_{m+1}). We say that R has *unitary stable rank* m, we denote it with $\mathrm{usr}(R)$, if m is the least number such that (US_m) is satisfied. If such a number does not exist we say that $\mathrm{usr}(R) = \infty$. Clearly $\mathrm{sr}(R) \leq \mathrm{usr}(R)$.

Remark 5. Our definition of unitary stable range is a little different than the one in [**18**]. In fact if (USR_{m+1}) is satisfied then, by [**18**, Chap. VI, Thm. 4.7.1], (US_m) is satisfied where (USR_{m+1}) is the unitary stable range as defined in [**18**, Chap. VI, 4.6]. In comparison with the absolute stable rank $\mathrm{asr}(R)$ from [**22**], we have that if $m \geq \mathrm{asr}(R) + 1$ or if the involution is the identity map (so R is commutative) and $m \geq \mathrm{asr}(R)$, then (US_m) is satisfied [**22**, 8.1].

Example 4. Let R be a commutative Noetherian ring, where the dimension d of the maximal spectrum $\mathrm{Mspec}(R)$ is finite. If A is a finite R-algebra, then $\mathrm{usr}(A) \leq d + 1$ (see [**47**, Thm. 2.8], [**18**, Thm. 6.1.4]). In particular if R is local ring or more generally a semilocal ring, then $\mathrm{usr}(R) = 1$ [**18**, 6.1.3].

Lemma 2.3. *Let R be a ring with $\mathrm{usr}(R) < \infty$. Assume $n \geq \mathrm{usr}(R) + k$ and $(v_1, \dots, v_k) \in \mathcal{U}(R^{2n})$. Then there is a hyperbolic basis $\{x_1, y_1, \dots, x_n, y_n\}$ of R^{2n} such that $v_1, \dots, v_k \in \langle x_1, y_1, \dots, x_k, y_k\rangle$.*

Proof. The proof is by induction on k. If $k = 1$, then by definition of unitary stable range there is an $E \in EU_{2n}^{\epsilon}(R, \Lambda)$ such that $Ev_1 = e_1 + e_2r$. So the base of the induction is true. Let $k \geq 2$ and assume the induction hypothesis. Arguing as in the base of the induction we can assume that $v_1 = (1, r, 0, \dots, 0)$, $r \in R$. Let $W = e_2 + \sum_{i=2}^{n} e_{2i}R$. By lemma 1.3, choose $w \in W$ so that $(w, v_1, \dots, v_k) \in \mathcal{U}(R^{2n})$. Then $(w, v_1 - wr, v_2, \dots, v_k) \in \mathcal{U}(R^{2n})$. But $(w, v_1 - wr)$ is a hyperbolic pair, so there is an $E \in EU_{2n}^{\epsilon}(R, \Lambda)$ such that $Ew = e_{2n-1}, E(v_1 - wr) = e_{2n}$ [**18**, Chap. VI, Thm. 4.7.1]. Let $(Ew, E(v_1 - wr), Ev_2, \dots, Ev_k) =: (w_0, w_1, \dots, w_k)$, where $w_i = (r_{i,1}, \dots, r_{i,2n})$. If $u_i := w_i - e_{2n-1}r_{i,2n-1} - e_{2n}r_{i,2n}$ for $2 \leq i \leq k$, then $(u_2, \dots, u_k) \in \mathcal{U}(R^{2n-2})$. Now by induction there is a hyperbolic basis $\{a_2, b_2, \dots, a_n, b_n\}$ of R^{2n-2} such that $u_i \in \langle a_2, b_2, \dots, a_k, b_k\rangle$. Let $a_1 = e_{2n-1}$ and $b_1 = e_{2n}$, then $w_i \in \langle a_1, b_1, \dots, a_k, b_k\rangle$. But $Ev_1 = w_1 + Ewr = e_{2n} + e_{2n-1}r$, $Ev_i = w_i$ for $2 \leq i \leq k$ and considering $x_i = E^{-1}a_i$, $y_i = E^{-1}b_i$, one sees that $v_1, \dots, v_k \in \langle x_1, y_1, \dots, x_k, y_k\rangle$. □

Definition 2.4. Let $Z_n = \{x \in R^{2n} : q(x) = 0\}$. We define the poset $\mathcal{U}'(R^{2n})$ as $\mathcal{U}'(R^{2n}) := \mathcal{O}(Z_n) \cap \mathcal{U}(R^{2n})$.

Lemma 2.5. *Let R be a ring with $\mathrm{sr}(R) < \infty$ and $n \leq m$. Then*
(i) *$\mathcal{O}(R^{2n}) \cap \mathcal{U}'(R^{2m})$ is $(n - \mathrm{sr}(R) - 1)$-connected,*
(ii) *$\mathcal{O}(R^{2n}) \cap \mathcal{U}'(R^{2m})_v$ is $(n - \mathrm{sr}(R) - |v| - 1)$-connected for every $v \in \mathcal{U}'(R^{2m})$,*
(iii) *$\mathcal{O}(R^{2n}) \cap \mathcal{U}'(R^{2m}) \cap \mathcal{U}(R^{2m})_v$ is $(n - \mathrm{sr}(R) - |v| - 1)$-connected for every $v \in \mathcal{U}(R^{2m})$.*

Proof. Let $W = \langle e_2, e_4, \dots, e_{2n}\rangle$ and $F := \mathcal{O}(R^{2n}) \cap \mathcal{U}'(R^{2m})$. It is easy to see that $\mathcal{O}(W) \cap F = \mathcal{O}(W) \cap \mathcal{U}(R^{2m})$ and $\mathcal{O}(W) \cap F_u = \mathcal{O}(W) \cap \mathcal{U}(R^{2m})_u$ for every $u \in \mathcal{U}'(R^{2m})$. By theorem 1.1, the poset $\mathcal{O}(W) \cap F$ is $(n - \mathrm{sr}(R) - 1)$-connected and the poset $\mathcal{O}(W)\cap F_u$ is $(n-\mathrm{sr}(R)-|u|-1)$-connected for every $u\in F$. It follows from lemma [**46**, 2.13 (i)] that F is $(n-\mathrm{sr}(R)-1)$-connected. The proof of (ii) and (iii) is similar to the proof of (i). □

LEMMA 2.6. *Let R be a ring with $\mathrm{usr}(R)<\infty$ and let $(v_1,\dots,v_k)\in\mathcal{U}'(R^{2n})$. If $n\geq\mathrm{usr}(R)+k$, then $\mathcal{O}(\langle v_1,\dots,v_k\rangle^{\perp})\cap\mathcal{U}'(R^{2n})_{(v_1,\dots,v_k)}$ is $(n-\mathrm{usr}(R)-k-1)$-connected.*

PROOF. By lemma 2.3 there is a hyperbolic basis $\{x_1,y_1,\dots,x_n,y_n\}$ of R^{2n} such that $v_1,\dots,v_k\in\langle x_1,y_1,\dots,x_k,y_k\rangle$. Let $W=\langle x_{k+1},y_{k+1},\dots,x_n,y_n\rangle\simeq R^{2(n-k)}$ and $F:=\mathcal{O}(\langle v_1,\dots,v_k\rangle^{\perp})\cap\mathcal{U}'(R^{2n})_{(v_1,\dots,v_k)}$. It is easy to see that $\mathcal{O}(W)\cap F=\mathcal{O}(W)\cap\mathcal{U}'(R^{2n})$. Let $V=\langle v_1,\dots,v_k\rangle$, then $\langle x_1,y_1,\dots,x_k,y_k\rangle=V\oplus P$, where P is a (finitely generated) projective module. Consider the element $(u_1,\dots,u_l)\in F\backslash\mathcal{O}(W)$ and let $u_i=x_i+y_i$, where $x_i\in V$ and $y_i\in P\oplus W$. One should notice that $(u_1-x_1,\dots,u_l-x_l)\in\mathcal{U}(R^{2n})$ and not necessarily in $\mathcal{U}'(R^{2n})$. It is not difficult to see that $\mathcal{O}(W)\cap F_{(u_1,\dots,u_l)}=\mathcal{O}(W)\cap\mathcal{U}'(R^{2n})\cap\mathcal{U}(R^{2n})_{(u_1-x_1,\dots,u_l-x_l)}$. By lemma 2.5, $\mathcal{O}(W)\cap F$ is $(n-k-\mathrm{usr}(R)-1)$-connected and $\mathcal{O}(W)\cap F_u$ is $(n-k-\mathrm{usr}(R)-|u|-1)$-connected for every $u\in F\backslash\mathcal{O}(W)$. It follows from lemma [**46**, 2.13 (i)] that F is $(n-\mathrm{usr}(R)-k-1)$-connected. □

3. Posets of isotropic and hyperbolic unimodular sequences

Let $\mathcal{IU}(R^{2n})$ be the set of sequences $(x_1,\dots,x_k)$, $x_i\in R^{2n}$, such that $x_1,\dots,x_k$ form a basis for an isotropic direct summand of R^{2n}. Let $\mathcal{HU}(R^{2n})$ be the set of sequences $((x_1,y_1),\dots,(x_k,y_k))$ such that $(x_1,\dots,x_k)$, $(y_1,\dots,y_k)\in\mathcal{IU}(R^{2n})$, $h(x_i,y_j)=\delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta. We call $\mathcal{IU}(R^{2n})$ and $\mathcal{HU}(R^{2n})$ the *poset of isotropic unimodular sequences* and the *poset of hyperbolic unimodular sequences*, respectively. For $1\leq k\leq n$, let $\mathcal{IU}(R^{2n},k)$ and $\mathcal{HU}(R^{2n},k)$ be the set of all elements of length k of $\mathcal{IU}(R^{2n})$ and $\mathcal{HU}(R^{2n})$ respectively. We call the elements of $\mathcal{IU}(R^{2n},k)$ and $\mathcal{HU}(R^{2n},k)$ the *isotropic k-frames* and the *hyperbolic k-frames*, respectively. Define the poset $\mathcal{MU}(R^{2n})$ as the set of $((x_1,y_1),\dots,(x_k,y_k))\in\mathcal{O}(R^{2n}\times R^{2n})$ such that,

(i) $(x_1,\dots,x_k)\in\mathcal{IU}(R^{2n})$,

(ii) for each i, either $y_i=0$ or $h(x_j,y_i)=\delta_{ji}$ for all $1\leq j\leq k$,

(iii) $\langle y_1,\dots,y_k\rangle$ is isotropic.

We identify $\mathcal{IU}(R^{2n})$ with $\mathcal{MU}(R^{2n})\cap\mathcal{O}(R^{2n}\times\{0\})$ and $\mathcal{HU}(R^{2n})$ with $\mathcal{MU}(R^{2n})\cap\mathcal{O}(R^{2n}\times(R^{2n}\backslash\{0\}))$.

LEMMA 3.1. *Let R be a ring with $\mathrm{usr}(R)<\infty$. If $n\geq\mathrm{usr}(R)+k$, then $EU_{2n}^{\epsilon}(R,\Lambda)$ acts transitively on $\mathcal{IU}(R^{2n},k)$ and $\mathcal{HU}(R^{2n},k)$.*

PROOF. The proof is by induction on k. If $k=1$, then by definition $EU_{2n}^{\epsilon}(R,\Lambda)$ acts transitively on $\mathcal{IU}(R^{2n},1)$ and by [**18**, Chap. VI, Thm. 4.7.1] the group $EU_{2n}^{\epsilon}(R,\Lambda)$ acts transitively on $\mathcal{HU}(R^{2n},1)$. The rest is an easy induction and the fact that for every isotropic k-frame $(x_1,\dots,x_k)$ there is an isotropic k-frame $(y_1,\dots,y_k)$ such that $((x_1,y_1),\dots,(x_k,y_k))$ is a hyperbolic k-frame [**18**, Chap. I, Cor. 3.7.4]. □

LEMMA 3.2. *Let R be a ring with* $\mathrm{usr}(R) < \infty$ *and let* $n \geq \mathrm{usr}(R) + k$. *Let* $((x_1, y_1), \dots, (x_k, y_k)) \in \mathcal{HU}(R^{2n})$, $(x_1, \dots, x_k) \in \mathcal{IU}(R^{2n})$ *and* $V = \langle x_1, \dots, x_k \rangle$. *Then*

(i) $\mathcal{IU}(R^{2n})_{(x_1,\dots,x_k)} \simeq \mathcal{IU}(R^{2(n-k)})\langle V \rangle$,

(ii) $\mathcal{HU}(R^{2n}) \cap \mathcal{MU}(R^{2n})_{((x_1,0),\dots,(x_k,0))} \simeq \mathcal{HU}(R^{2n})_{((x_1,y_1),\dots,(x_k,y_k))}\langle V \times V \rangle$,

(iii) $\mathcal{HU}(R^{2n})_{((x_1,y_1),\dots,(x_k,y_k))} \simeq \mathcal{HU}(R^{2(n-k)})$.

PROOF. See the proof of lemma 3.4 and the proof of Theorem 3.2 in [**8**]. $\square$

For a real number l, by $\lfloor l \rfloor$ we mean the largest integer n with $n \leq l$.

THEOREM 3.3. *The poset* $\mathcal{IU}(R^{2n})$ *is* $\lfloor \frac{n-\mathrm{usr}(R)-2}{2} \rfloor$*-connected and* $\mathcal{IU}(R^{2n})_x$ *is* $\lfloor \frac{n-\mathrm{usr}(R)-|x|-2}{2} \rfloor$*-connected for every* $x \in \mathcal{IU}(R^{2n})$.

PROOF. If $n \leq \mathrm{usr}(R)$, the result is clear, so let $n > \mathrm{usr}(R)$. Let $X_v = \mathcal{IU}(R^{2n}) \cap \mathcal{U}'(R^{2n})_v \cap \mathcal{O}(\langle v \rangle^{\perp})$ for every $v \in \mathcal{U}'(R^{2n})$ and put $X := \bigcup_{v \in F} X_v$, where $F = \mathcal{U}'(R^{2n})$. It follows from lemma 3.1 that $\mathcal{IU}(R^{2n})_{\leq n-\mathrm{usr}(R)} \subseteq X$. So to treat $\mathcal{IU}(R^{2n})$, it is enough to prove that X is $\lfloor \frac{n-\mathrm{usr}(R)-2}{2} \rfloor$-connected. First we prove that X_v is $\lfloor \frac{n-\mathrm{usr}(R)-|v|-2}{2} \rfloor$-connected for every $v \in F$. The proof is by descending induction on $|v|$. If $|v| > n - \mathrm{usr}(R)$, then $\lfloor \frac{n-\mathrm{usr}(R)-|v|-2}{2} \rfloor < -1$. In this case there is nothing to prove. If $n - \mathrm{usr}(R) - 1 \leq |v| \leq n - \mathrm{usr}(R)$, then $\lfloor \frac{n-\mathrm{usr}(R)-|v|-2}{2} \rfloor = -1$, so we must prove that X_v is nonempty. This follows from lemma 2.3. Now assume $|v| \leq n - \mathrm{usr}(R) - 2$ and assume by induction that X_w is $\lfloor \frac{n-\mathrm{usr}(R)-|w|-2}{2} \rfloor$-connected for every w, with $|w| > |v|$. Let $l = \lfloor \frac{n-\mathrm{usr}(R)-|v|-2}{2} \rfloor$ and observe that $n-|v|-\mathrm{usr}(R) \geq l+2$. Set $T_w = \mathcal{IU}(R^{2n}) \cap \mathcal{U}'(R^{2n})_{wv} \cap \mathcal{O}(\langle wv \rangle^{\perp})$, where $w \in G_v = \mathcal{U}'(R^{2n})_v \cap \mathcal{O}(\langle v \rangle^{\perp})$ and set $T := \bigcup_{w \in G_v} T_w$. It follows, by lemma 2.3, that $(X_v)_{\leq n-|v|-\mathrm{usr}(R)} \subseteq T$. So it is sufficient to prove that T is l-connected. The poset G_v is l-connected by lemma 2.6. By induction T_w is $\lfloor \frac{n-\mathrm{usr}(R)-|v|-|w|-2}{2} \rfloor$-connected. But $\min\{l-1, l-|w|+1\} \leq \lfloor \frac{n-\mathrm{usr}(R)-|v|-|w|-2}{2} \rfloor$, so T_w is $\min\{l-1, l-|w|+1\}$-connected. For every $y \in T$, $\mathcal{A}_y = \{w \in G_v : y \in T_w\}$ is isomorphic to $\mathcal{U}'(R^{2n})_{vy} \cap \mathcal{O}(\langle vy \rangle^{\perp})$ so, by lemma 2.6, it is $(l-|y|+1)$-connected. Let $w \in G_v$ with $|w| = 1$. For every $z \in T_w$ we have $wz \in X_v$, so T_w is contained in a cone, call it C_w, inside X_v. Put $C(T_w) = T_w \cup (C_w)_{\leq n-|v|-\mathrm{usr}(R)}$. Thus $C(T_w) \subseteq T$. The poset $C(T_w)$ is l-connected because $C(T_w)_{\leq n-|v|-\mathrm{usr}(R)} = (C_w)_{\leq n-|v|-\mathrm{usr}(R)}$. Now by theorems 1.1 and 3.6 in chapter 1, T is l-connected. In other words, we have now shown that X_v is $\lfloor \frac{n-\mathrm{usr}(R)-|v|-2}{2} \rfloor$-connected. By knowing this one can prove, in a similar way, that X is $\lfloor \frac{n-\mathrm{usr}(R)-2}{2} \rfloor$-connected. (Just pretend that $|v| = 0$.)

Now consider the poset $\mathcal{IU}(R^{2n})_x$ for an $x = (x_1, \dots, x_k) \in \mathcal{IU}(R^{2n})$. The proof is by induction on n. If $n = 1$, everything is easy. Similarly, we may assume $n - \mathrm{usr}(R) - |x| \geq 0$. Let $l = \lfloor \frac{n-\mathrm{usr}(R)-|x|-2}{2} \rfloor$. By lemma 3.2, $\mathcal{IU}(R^{2n})_x \simeq \mathcal{IU}(R^{2(n-|x|)})\langle V \rangle$, where $V = \langle x_1, \dots, x_k \rangle$. In the above we proved that $\mathcal{IU}(R^{2(n-|x|)})$ is l-connected and by induction the poset $\mathcal{IU}(R^{2(n-|x|)})_y$ is $\lfloor \frac{n-|x|-\mathrm{usr}(R)-|y|-2}{2} \rfloor$-connected for every $y \in \mathcal{IU}(R^{2(n-|x|)})$. But $l - |y| \leq \lfloor \frac{n-|x|-\mathrm{usr}(R)-|y|-2}{2} \rfloor$. So by lemma 3.1 in chapter 1, $\mathcal{IU}(R^{2(n-|x|)})\langle V \rangle$ is l-connected. Therefore $\mathcal{IU}(R^{2n})_x$ is l-connected. $\square$

THEOREM 3.4. *The poset* $\mathcal{HU}(R^{2n})$ *is* $\lfloor\frac{n-\mathrm{usr}(R)-3}{2}\rfloor$*-connected and* $\mathcal{HU}(R^{2n})_x$ *is* $\lfloor\frac{n-\mathrm{usr}(R)-|x|-3}{2}\rfloor$*-connected for every* $x \in \mathcal{HU}(R^{2n})$.

PROOF. The proof is by induction on n. If $n = 1$, then everything is trivial. Let $F = \mathcal{IU}(R^{2n})$ and $X_v = \mathcal{HU}(R^{2n}) \cap \mathcal{MU}(R^{2n})_v$, for every $v \in F$. Put $X := \bigcup_{v\in F} X_v$. It follows from lemma 3.1 that $\mathcal{HU}(R^{2n})_{\leq n-\mathrm{usr}(R)} \subseteq X$. Thus to treat $\mathcal{HU}(R^{2n})$, it is sufficient to prove that X is $\lfloor\frac{n-\mathrm{usr}(R)-3}{2}\rfloor$-connected and we may assume $n \geq \mathrm{usr}(R) + 1$. Take $l = \lfloor\frac{n-\mathrm{usr}(R)-3}{2}\rfloor$ and $V = \langle v_1, \dots, v_k\rangle$, where $v = (v_1, \dots, v_k)$. By lemma 3.2, there is an isomorphism $X_v \simeq \mathcal{HU}(R^{2(n-|v|)})\langle V \times V\rangle$, if $n \geq \mathrm{usr}(R) + |v|$. By induction $\mathcal{HU}(R^{2(n-|v|)})$ is $\lfloor\frac{n-|v|-\mathrm{usr}(R)-3}{2}\rfloor$-connected and again by induction $\mathcal{HU}(R^{2(n-|v|)})_y$ is $\lfloor\frac{n-|v|-\mathrm{usr}(R)-|y|-3}{2}\rfloor$-connected for every $y \in \mathcal{HU}(R^{2(n-|v|)})$. So by lemma 3.1 in chapter 1, X_v is $\lfloor\frac{n-|v|-\mathrm{usr}(R)-3}{2}\rfloor$-connected. Thus the poset X_v is $\min\{l-1, l-|v|+1\}$-connected. Let $x = ((x_1, y_1), \dots, (x_k, y_k))$. It is easy to see that $\mathcal{A}_x = \{v \in F : x \in X_v\} \simeq \mathcal{IU}(R^{2n})_{(x_1,\dots,x_k)}$. By theorem 3.3, $\mathcal{A}_x$ is $\lfloor\frac{n-\mathrm{usr}(R)-k-2}{2}\rfloor$-connected. But $l - |x| + 1 \leq \lfloor\frac{n-\mathrm{usr}(R)-k-2}{2}\rfloor$, so $\mathcal{A}_x$ is $(l - |x| + 1)$-connected. Let $v = (v_1) \in F$, $|v| = 1$, and let $D_v := \mathcal{HU}(R^{2n})_{(v_1,w_1)} \simeq \mathcal{HU}(R^{2(n-1)})$, where $w_1 \in R^{2n}$ is a hyperbolic dual of $v_1 \in R^{2n}$. Then $D_v \subseteq X_v$ and D_v is contained in a cone, call it C_v, inside $\mathcal{HU}(R^{2n})$. Take $C(D_v) := D_v \cup (C_v)_{\leq n-\mathrm{usr}(R)}$. By induction D_v is $\lfloor\frac{n-1-\mathrm{usr}(R)-3}{2}\rfloor$-connected and so $(l-1)$-connected. Let $Y_v = X_v \cup C(D_v)$. By the Mayer-Vietoris theorem and the fact that $C(D_v)$ is l-connected, we get the exact sequence

$$\tilde{H}_l(D_v, \mathbb{Z}) \overset{(i_v)_*}{\to} \tilde{H}_l(X_v, \mathbb{Z}) \to \tilde{H}_l(Y_v, \mathbb{Z}) \to 0,$$

where $i_v : D_v \to X_v$ is the inclusion. By induction $(D_v)_w$ is $\lfloor\frac{n-1-\mathrm{usr}(R)-|w|-3}{2}\rfloor$-connected and so $(l - |w|)$-connected, for $w \in D_v$. By lemma 3.1(i) in chap. 1 and lemma 3.2, $(i_v)_*$ is an isomorphism and by exactness of the above sequence we get $\tilde{H}_l(Y_v, \mathbb{Z}) = 0$. If $l \geq 1$, then by the Van Kampen theorem $\pi_1(Y_v, x) \simeq \pi_1(X_v, x)/N$, where $x \in D_v$ and N is the normal subgroup generated by the image of the map $(i_v)_* : \pi_1(D_v, x) \to \pi_1(X_v, x)$. By lemma 3.1(ii) in Chap. 1, $\pi_1(Y_v, x)$ is trivial. Thus by the Hurewicz theorem, 1.1 in Chap. 1, Y_v is l-connected. By having all this we can apply theorem 3.6 in Chap. 1 and so X is l-connected. The fact that $\mathcal{HU}(R^{2n})_x$ is $\lfloor\frac{n-\mathrm{usr}(R)-|x|-3}{2}\rfloor$-connected follows from the above and lemma 3.2. $\square$

REMARK 6. One can define a more generalized version of hyperbolic space $H(P) = P \oplus P^*$, where P is a finitely generated projective module. Charney in [**8**, 2.10] introduced the posets $\mathcal{IU}(P)$, $\mathcal{HU}(P)$ and conjectured that if P contains a free summand of rank n then $\mathcal{IU}(P)$ and $\mathcal{HU}(P)$ are in fact highly connected. We leave it as exercise to the interested reader to prove this conjecture using theorems 3.3 and 3.4 as in the proof of lemma 2.5. In fact one can prove that if P contains a free summand of rank n, then $\mathcal{IU}(P)$ is $\lfloor\frac{n-\mathrm{usr}(R)-2}{2}\rfloor$-connected and $\mathcal{HU}(P)$ is $\lfloor\frac{n-\mathrm{usr}(R)-3}{2}\rfloor$-connected. Also, by assuming the high connectivity of the poset $\mathcal{IU}(R^{2n})$, Charney proved that $\mathcal{HU}(R^{2n})$ is highly connected. Our proof is different and relies on our theory, but we use ideas from her paper, such as the lemma 3.2 and her lemma 3.1, which is a modified version of the work of Maazen.

4. The case of a local ring with infinite residue field

In this section R is a local ring with infinite residue field. The main statement of this section, Theorem 4.5, is rather well known (see [**29**]). We give the details of the proof to make sure that everything is working for our case, Theorem 4.7. For an alternative proof in the case of a field different from $\mathbb{F}_2$ see Remark 7 and Theorem 3.1 in Chap 3.

DEFINITION 4.1. Let $S = \{v_1, \dots, v_k\}$ and $T = \{w_1, \dots, w_{k'}\}$ be basis of two isotropic free summands of R^{2n}. We say that T is in *general position* with S, if $k \leq k'$ and the $k' \times k$-matrix $(h(w_i, v_j))$ has a left inverse.

PROPOSITION 4.2. *Let $n \geq 2$ and assume T_i, $1 \leq i \leq l$, are finitely many finite subsets of R^{2n} such that each T_i is a basis of a free isotropic summand of R^{2n} with k elements, where $k \leq n-1$. Then there is a basis, $T = \{w_1, \dots, w_n\}$, of a free isotropic summand of R^{2n} such that T is in general position with all T_i. Moreover $\dim(W \cap V_i^{\perp}) = n - k$, where $W = \langle T \rangle$ and $V_i = \langle T_i \rangle$.*

PROOF. The proof of the first part is by induction on l. Let $T_i = \{v_{i,1}, \dots, v_{i,k}\}$. For $l = 1$, take a basis of a free isotropic direct summand of R^{2n}, for example $\{w_1, \dots, w_k\}$, such that $h(w_j, v_{1,m}) = \delta_{j,m}$, where $\delta_{j,m}$ is the Kronecker delta and choose T an extension of this basis to a basis of a maximal isotropic free subspace. Assume that the claim is true for $1 \leq i \leq l-1$. This means that there is a basis $\{u_1, \dots, u_n\}$ of a free isotropic summand of R^{2n}, in general position with T_i, $1 \leq i \leq l-1$. Let $\{x_1, \dots, x_k\}$ be a basis of a free isotropic summand of R^{2n} such that $h(x_j, v_{l,m}) = \delta_{j,m}$ and take $A = \prod E_{r,s}(a) \in E_n$ such that $Au_j = x_j$, $1 \leq j \leq k$ [**27**, 6.5, 7.1]. If $x_j := Au_j$ for $k+1 \leq j \leq n$, then $\{x_1, \dots, x_n\}$ is in general position with T_l. Set $B_i = (h(u_j, v_{i,m}))$, $1 \leq i \leq l-1$ and $B_l = (h(x_j, v_{l,m}))$. Let $B_i^{(k)}$ be the matrix obtained from B_i by deleting $(j_{1,i}, \dots, j_{n-k,i})$-th rows such that $f_i^{(k)} := \det(B_i^{(k)}) \in R^*$ for all i. Set $A(t) = \prod E_{r,s}(ta)$, $B_i(t) = (h(A(t)u_j, v_{i,m}))$, for $1 \leq i \leq l$ and let $B_i^{(k)}(t)$ be the matrix obtained by deleting $(j_{1,i}, \dots, j_{n-k,i})$-th rows of $B_i(t)$ and set $f_i^{(k)}(t) := \det(B_i^{(k)}(t)) \in R[t]$. Clearly $f_i^{(k)}(0) = f_i^{(k)}$ for $1 \leq i \leq l-1$ and $f_l^{(k)}(1) = f_l^{(k)}$. It is not difficult to see that there is a $t_1 \in R$ such that $f_i^{(k)}(t_1) \in R^*$ for $1 \leq i \leq l$ [**45**, 1.4, 1.5]. Take $W = \{A(t_1)u_1, \dots, A(t_1)u_n\}$. The second part of the proposition follows from the exact sequence

$$0 \to W \cap V_1^{\perp} \to W \xrightarrow{\psi} R^k \to 0$$

with $\psi(w) := (h(w, v_{1,1}), \dots, h(w, v_{1,k}))$ and the fact that projective modules over local rings are free. □

LEMMA 4.3. *Let n, m be two natural numbers and $n \leq m$. If $n \geq k+1$ then for every $(v_1, \dots, v_k) \in \mathcal{U}(R^m)$ there is a $v \in R^n = \langle e_1, \dots, e_n \rangle$ such that $(v, v_1, \dots, v_k) \in \mathcal{U}(R^m)$.*

PROOF. The proof is similar to the proof of lemma [**27**, 5.4], using the fact that $\mathrm{sr}(R) = 1$. □

THEOREM 4.4. *Let n, m be two natural numbers and $n \leq m$. Then the poset $\mathcal{O}(R^n) \cap \mathcal{U}(R^m)$ is $(n-2)$-acyclic and $\mathcal{O}(R^n) \cap \mathcal{U}(R^m)_w$ is $(n-|w|-2)$-acyclic for every $w = (w_1, \dots, w_r) \in \mathcal{U}(R^m)$.*

PROOF. Let $X = \mathcal{O}(R^n) \cap \mathcal{U}(R^m) = \mathcal{U}(R^n)$ and $\sigma = \sum_{i=1}^{l} n_i(v_0^i, \dots, v_k^i)$ be a cycle in $C_k(X)$, $k \leq n-2$. It is not difficult to see that there is a unimodular vector $v \in R^n$ such that $\{v, v_0^i, \dots, v_k^i\}$ is linearly independent, $1 \leq i \leq l$. If $\beta := \sum_{i=1}^{l} n_i(v, v_0^i, \dots, v_k^i) \in C_{k+1}(X)$, then $\partial_{k+1}(\beta) = \sigma$, so X is $(n-2)$-acyclic.

Let $Y = \mathcal{O}(R^n) \cap \mathcal{U}(R^m)_w$ and assume that $n - |w| - 2 \geq -1$. Let σ be a k-cycle in $C_k(Y)$ with $k \leq n - |w| - 2$. To prove the second part of the theorem it is sufficient to find a unimodular vector $v \in R^n$ such that $\{v, v_0^i, \dots, v_k^i, w_1, \dots, w_r\}$ is linearly independent, $1 \leq i \leq l$. The proof is by induction on l. The case $l = 1$ follows from 4.3. By induction assume that there are $u_1, u_2 \in R^n$ such that $(u_1, v_0^i, \dots, v_k^i, w_1, \dots, w_r) \in \mathcal{U}(R^m)$ for $1 \leq i \leq l-1$ and $(u_2, v_0^l, \dots, v_k^l, w_1, \dots, w_r) \in \mathcal{U}(R^m)$. Let $A = \prod E_{r,s}(a)$ be an element of the elementary group $E_n(R) \subseteq GL_n(R)$ such that $Au_1 = u_2$ and set $A(t) = \prod E_{r,s}(ta)$. Let B_i be the matrix whose columns are the vectors $u_1, v_0^i, \dots, v_k^i, w_1, \dots, w_r$ for $1 \leq i \leq l-1$, B_l the matrix whose columns are $u_2, v_0^l, \dots, v_k^l, w_1, \dots, w_r$ and $B_i(t)$ is the matrix whose columns are $A(t)u_1, v_0^i, \dots, v_k^i, w_1, \dots, w_r$, $1 \leq i \leq l$. The rest of the proof is similar to the proof of proposition 4.2. □

THEOREM 4.5. *The poset $\mathcal{IU}(R^{2n})$ is $(n-2)$-acyclic.*

PROOF. If $n = 1$, then everything is trivial, so we assume that $n \geq 2$. Let $\sigma = \sum_{i=1}^{r} n_i v_i$ be a k-cycle. Thus v_i, $1 \leq i \leq r$, are isotropic $(k+1)$-frames with $k \leq n-2$. By 4.2, there is an isotropic n-frame w in general position with v_i, $1 \leq i \leq r$. Set $W = \langle w \rangle$ and let E_σ be the set of all $(u_1, \dots, u_m, t_1, \dots, t_l) \in \mathcal{IU}(R^{2n})$ such that $m, l \geq 0$, $(u_1, \dots, u_m) \in \mathcal{U}(W)$, if $m \geq 1$, and for every $l \geq 1$ there exist an i such that $(t_1, \dots, t_l) \leq v_i$. The poset E_σ satisfies the chain condition and $v_i \in E_\sigma$. It is sufficient to prove that E_σ is $(n-2)$-acyclic, because then $\sigma \in \partial_{k+1}(E_\sigma) \subseteq \partial_{k+1}(C_{k+1}(X))$. Let $F := E_\sigma$. Since $\mathcal{O}(W) \cap F = \mathcal{U}(W)$, by 4.4 the poset $\mathcal{O}(W) \cap F$ is $(n-2)$-acyclic. If $u \in F \backslash \mathcal{O}(W)$, then u is of the form $(u_1, \dots, u_m, t_1, \dots, t_l)$, $l \geq 1$. By 4.2, $\dim(V) = n - l$, where $V = W \cap \langle t_1 \dots t_l \rangle^\perp$. With all this we have

$$\mathcal{O}(W) \cap F_u = \mathcal{O}(V) \cap \mathcal{IU}(R^{2n})_{(u_1, \dots, u_m)} = \mathcal{O}(V) \cap \mathcal{U}(W)_{(u_1, \dots, u_m)}.$$

Again by 4.4, $\mathcal{O}(V) \cap \mathcal{U}(W)_{(u_1, \dots, u_m)}$ is $((n-l) - m - 2)$-acyclic, so $\mathcal{O}(W) \cap F_u$ is $(n - |u| - 2)$-acyclic. Therefore F is $(n-2)$-acyclic [**46**, 2.13 (i)]. □

REMARK 7. (i) The concept of *being in general position* and the idea of the proof of 4.5 is taken from [**29**]. Because the details of the proof in [**29**] never appeared we wrote it down.

(ii) In fact Theorem 4.5 is true for every field $R \neq \mathbb{Z}/2\mathbb{Z}$. Let

$$\mathcal{IV}(R^{2n}) = \{V \subseteq R^{2n} : V \neq 0 \text{ and isotropic subspace}\}.$$

Define the map of the posets $f : \mathcal{IU}(R^{2n}) \to \mathcal{IV}(R^{2n})$, $v \mapsto \langle v \rangle$. As Vogtmann proved, [**50**, Thm. 1.6], $\mathcal{IV}(R^{2n})$ is $(n-2)$-connected (Vogtmann proved this for $G_n = O_{2n}(R)$, but her proof works without modification in our more general setting [**8**, p. 115]). On the one hand it is easy to see that $Link^{+}_{\mathcal{IV}(R^{2n})}(V) \simeq \mathcal{IV}(R^{2(n-\dim(V))})$, so it is $(n-\dim(V)-2)$-connected and on the other hand $f/V = \mathcal{U}(V)$ which is $(\dim(V)-2)$-connected [**46**, 2.6], hence defining the height function $\mathrm{ht}_{\mathcal{IV}(R^{2n})}(V) = \dim(V)-1$ [**27**, section 2], one sees that $\mathcal{IU}(R^{2n})$ is $(n-2)$-connected [**27**, Thm. 3.8].

(iii) We expect that over a ring with no finite ring as a homomorphic image and finite unitary stable rank the poset $\mathcal{IU}(R^{2n})$ is $(n-\mathrm{usr}(R)-1)$-connected. For this it is sufficient to prove 4.2 over such ring. For example Theorem 4.5, without any change in its proof, is true over a semi-local ring with infinite residue fields. Therefore the results of this note are also valid for these rings.

(iii) Using 4.5, (iii) and the same argument as in (ii) one can prove that over a semi-local ring with infinite residue fields, $\mathcal{IV}(R^{2n})$ is $(n-2)$-acyclic. Over an infinite field this gives much easier proof of Vogtmann's theorem mentioned in (ii).

(iv) Using a theorem of Van der Kallen [**46**, Thm. 2.6] and a similar argument as (iii) we can generalize the Tits-Solomon theorem over a ring with stable range one (for example any Artinian ring). Let R be a ring with stable range one and consider the following poset, which we call it the *Tits poset*,

$$\mathcal{T}(R^n) = \{V \subseteq R^n : V \text{ free summand of } R^n, V \neq 0, R^n\}.$$

Let $X = \mathcal{U}(R^n)_{\leq n-1}$ and consider the poset map $g : X \to \mathcal{T}(R^n)$, $v \mapsto \langle v \rangle$. By induction and a similar argument as in (ii), using the fact that X is $(n-3)$connected, one can prove that $\mathcal{T}(R^n)$ is $(n-3)$-connected (note that any stably free projective module of rank ≥ 1 is free). We leave the details of the proof to the interested readers.

Definition 4.6. Define $\underline{\mathcal{U}}(R^n) = \{(\langle v_1 \rangle, \dots, \langle v_k \rangle) : (v_1, \dots, v_k) \in \mathcal{U}(R^n)\}$ and $\underline{\mathcal{IU}}(R^{2n}) = \{(\langle v_1 \rangle, \dots, \langle v_k \rangle) : (v_1, \dots, v_k) \in \mathcal{IU}(R^{2n})\}$.

Theorem 4.7. *Let n, m be two natural numbers and $n \leq m$. Then the poset $\mathcal{O}(\mathbb{P}^{n-1}) \cap \underline{\mathcal{U}}(R^m)$ is $(n-2)$-acyclic, the poset $\mathcal{O}(\mathbb{P}^{n-1}) \cap \underline{\mathcal{U}}(R^m)_w$ is $(n-|w|-2)$-acyclic for every $w \in \underline{\mathcal{U}}(R^m)$ and the poset $\underline{\mathcal{IU}}(R^{2n})$ is $(n-2)$-acyclic.*

Proof. The proof is similar to the proof of 4.4 and 4.5. □

CHAPTER 3

Homology stability for unitary groups

Let $G_1 \subseteq G_2 \subseteq \ldots \subseteq G_n \subseteq \ldots$ be a chain of groups and set $G := \bigcup_i G_i$. Let M be an abelian group with trivial G-action. Then for every $i \geq 0$ we have a chain of maps $H_i(G_1, M) \to H_i(G_2, M) \to \cdots \to H_i(G_i, M) \to \cdots \to H_i(G, M)$.

Question. Is there $n \gg i$ such that the homomorphisms $H_i(G_n, M) \to H_i(G_{n+1}, M) \to \cdots \to H_i(G, M)$ are isomorphisms?

If this is the case, then we say that G_i has the homology stability property. It is easy to see that this is not true and even not interesting in general. But there are some interesting cases that the answer is yes, but not easy to prove.

Usually there is a general strategy to prove the homology stability problem. First prove that a certain poset such that our group G_n acts on it nicely, is highly acyclic. Using this property one can come up with spectral sequence which relates the homology of G_i, $i < n$, to the homology of G_n. The stability theorem is a result of analyzing this spectral sequence, inductively.

In section 1 of this chapter we will prove a general stability theorem for unitary groups over a ring with finite stable rank. In this case driving the spectral sequence and analyzing that is not difficult. In section 2, we will concentrate on the case of a local ring with infinite residue field and we will sharpen the range of homology stability obtained in the first section. Contrary to the first section, obtaining the spectral sequence and analyzing that is difficult. Note that in this case the proof of the higher connectivity of the corresponding poset is not difficult, which is done in Sec. 4 of previous chapter.

1. General homology stability theorem

From theorem 3.4 in Chap. 2, one can get the homology stability of unitary groups and elementary unitary groups. Let $G_n = U_{2n}^{\epsilon}(R, \Lambda)$ and $E_n := EU_{2n}^{\epsilon}(R, \Lambda)$. The embedding $G_n \to G_{n+1}$ and $E_n \to E_{n+1}$ are given by $A \mapsto \mathrm{diag}(I_2, A) = \begin{pmatrix} I_2 & 0 \\ 0 & A \end{pmatrix}$.

THEOREM 1.1. *Let R be a ring with $\mathrm{usr}(R) < \infty$ and let the action of the unitary group on the abelian group A be trivial. Then the homomorphisms* $\mathrm{inc}_* : H_i(G_n, A) \to H_i(G_{n+1}, A)$ *and* $\mathrm{inc}_* : H_i(E_n, A) \to H_i(E_{n+1}, A)$ *are surjective for* $n \geq 2i + \mathrm{usr}(R) + 2$ *and injective for* $n \geq 2i + \mathrm{usr}(R) + 3$.

Proof. Knowing the high acyclicity of $\mathcal{HU}(R^{2n})$, the proof of homology stability is classical, so we refer to [**8**, Section 4] and [**46**, sections 3, 4] (see also the proof of Thm. 2.5 in the next section). $\square$

Remark 8. To prove homology stability of this type one only needs high acyclicity of the corresponding poset, not high connectivity. But usually this type of posets are also highly connected. In the previous chapter we also proved the high connectivity. In particular we wished to confirm the conjecture of Charney [**8**, 2.10], with some reasonable bounds (see remark 6 in chap. 2).

Remark 9. One also can prove homology stability of the unitary groups with twisted coefficients. For more information in this direction see [**46**, §5] and [**8**, 4.2].

Definition 1.2. Let R be a commutative ring with unit and let $G := \bigcup_n G_n$, where $G_n = U_{2n}^{\epsilon}(R, \Lambda)$. For $n \geq 3$, $E_n := EU_{2n}^{\epsilon}(R, \Lambda)$ is a perfect normal subgroup of G_n [**1**, Thm. p.2]. Define $K_i^{(\epsilon,\mathrm{inv})}(R, \Lambda)$ and $K_i^{(\epsilon,\mathrm{inv})}(R, n, \Lambda)$ to be the groups $\pi_i(BG^+)$ and $\pi_i(BG_n^+)$ for $i \geq 1$, $n \geq 3$, respectively. The plus constructions are with respect to the perfect normal subgroups $E = \bigcup_{n\geq 1} E_n$ and E_n, respectively. (For BG^+ see the definitions of plus construction and classifying space.) The group $K_i^{(\epsilon,\mathrm{inv})}(R, \Lambda)$ is called the i-th *Hermitian K-group* of the hyperbolic space (R^{2n}, h, q). The groups $K_iSp(R) := K_i^{(-1,\mathrm{id}_R)}(R, R)$ and $K_iO(R) := K_i^{(1,\mathrm{id}_R)}(R, 0)$ are called the i-th *symplectic K-group* and the i-th *orthogonal K-group* of R, respectively. $K_iSp_{2n}(R)$ and $K_iO_{2n}(R)$ can be defined in a similar way.

Theorem 1.3. *Let X and Y be path-connected pointed spaces and let $f : (X, x) \to (Y, y)$ be a continuous map. If there is an $n \geq 1$ such that $\pi_q(f) : \pi_q(X, x) \to \pi_q(Y, y)$ is an isomorphism for all $q < n$ and epimorphism for $q = n$, then $H_q(f) : H_q(X, \mathbb{Z}) \to H_q(Y, \mathbb{Z})$ is an isomorphism for all $q < n$ and epimorphism for $q = n$. Conversely if X and Y are simply connected and $H_q(f)$ is an isomorphism for all $q < n$ and epimorphism for $q = n$, then $\pi_q(f)$ is an isomorphism for all $q < n$ and epimorphism for $q = n$.*

Proof. See [**37**, Chap. 7, Sec. 5, Thm. 9]. $\square$

Theorem 1.4. *Let $i \geq 1$ be an integer. The map of non-stable Hermitian K-groups $K_i^{(\epsilon,\mathrm{inv})}(R, n, \Lambda) \to K_i^{(\epsilon,\mathrm{inv})}(R, n+1, \Lambda)$ is surjective for $n \geq 2i + \mathrm{usr}(R) + 2$ and is injective for $n \geq 2i + \mathrm{usr}(R) + 3$.*

Proof. This follows from theorems 1.1, 1.3 and the fact that BE_n^+ and BE_{n+1}^+ are simply connected. $\square$

A topological space X is called an H-*space* if there is a continuous map $m : X \times X \to X$ and an element $e \in X$, a base point, such that $m_l(x) := m(x, e)$ and $m_r(x) := m(e, x)$ are homotopic to id_X.

Example 5. (i) Any topological group is an H-space.
(ii) If X is a space, then the loop space ΩX is an H-space.
(iii) If G is an abelian group, then $K(G, n)$ is an H-space for every $n \in \mathbb{N}$.

(iv) Let G be a group such that $[G, G]$ is perfect and assume there is a homomorphism $\oplus : G \times G \to G$. Let we have the following conditions

(a) for any $g_1, \dots, g_n \in [G, G]$ and $g \in G$, there is an $h \in [G, G]$ with $gg_ig^{-1} = hg_ih^{-1}$ for $1 \le i \le n$,

(b) for any finite set $g_1, \dots, g_n \in G$ there are elements c and d in G with $c(g_i \oplus 1_G)c^{-1} = d(1_G \oplus g_i)d^{-1} = g_i$.

Then BG^+ is an H-space, where the plus construction is with respect to $[G, G]$ [**19**, p. 323].

(v) Let $G = \bigcup_{n=1}^{\infty} U_{2n}^{\epsilon}(R, \Lambda)$. Then, using (iv), one can prove that BG^+ is an H-space [**19**, p. 324].

THEOREM 1.5. *Let X be an* H*-space and let $\pi_1(X)$ be finitely generated. Then $H_i(X, \mathbb{Z})$ is finitely generated for all $i \ge 1$ if and only if $\pi_i(X)$ is finitely generated for all $i \ge 1$.*

PROOF. See [**37**, Chap. 9, Sec. 6, Example 18, Thm. 20]. □

For the definition of arithmetic groups and some of their properties see [**38**].

THEOREM 1.6. *Let G be an arithmetic group. Then $H_i(G, \mathbb{Z})$ is finitely generated abelian group for all $i \ge 1$.*

PROOF. It is well known that G has a torsion free subgroup G' of finite index [**38**, 1.3 (4)]. By a theorem of Borel-Serre [**5**, §11], G' is of type FL and so of type FP (see [**6**, Chap. VIII] for definition of FL and FP properties). By [**6**, Chap. VIII, Prop. 5.1], G is of type FP and so $H_i(G, \mathbb{Z})$ is finitely generated abelian group for all $i \ge 1$ [**6**, Chap. VIII, Section 5, Exercise 1]. □

THEOREM 1.7. *Let R be the ring of algebraic integers of a number field. Choose $(\epsilon, \mathrm{inv}, \Lambda)$ such that $U_{2n}^{\epsilon}(R, \Lambda)$ is an arithmetic group. Then $K_i^{(\epsilon,\mathrm{inv})}(R, \Lambda)$ is finitely generated for all $i \ge 1$.*

PROOF. By stability theorem 1.1 there is an $n \ge 2i + 6$ such that $H_i(G, \mathbb{Z}) \simeq H_i(G_n, \mathbb{Z})$, where $G_n = U_{2n}^{\epsilon}(R, \Lambda)$. By 1.6, $H_i(G_n, \mathbb{Z})$ is finitely generated and so $H_i(G, \mathbb{Z})$ is finitely generated. By the properties of the plus construction $K_1^{(\epsilon,\mathrm{inv})}(R, \Lambda) = G/[G, G] = H_1(G, \mathbb{Z})$ and so $K_1^{(\epsilon,\mathrm{inv})}(R, \Lambda)$ is finitely generated. Now by Example 5(v), Thm. 1.6 and Thm. 1.5, $K_i^{(\epsilon,\mathrm{inv})}(R, \Lambda)$ is finitely generated. □

2. The case of local ring with infinite residue field

Here we will establish some notations that we will use in this section. By a ring R, we will always mean a commutative local ring with an infinite residue field unless it is mentioned otherwise. The ring R has an involution (which may be the identity) and we set $R_1 := \{r \in R : \bar{r} = r\}$. This is also a local ring with an infinite residue field. By convention G_0 will be the trivial group. For a group G, by $H_i(G)$ we mean $H_i(G, k)$, where k is a field with trivial G-action. In this section, k will be a field, S_i a k-algebra, $i \in \mathbb{N}$, $S_i^{\otimes n} := S_i \otimes_k \cdots \otimes_k S_i$ (n-times) and $V_n(S_i) := (S_i^{\otimes n})^{\Sigma_n}$, where Σ_n is the symmetric group of degree n.

LEMMA 2.1. *Let $\varphi_i : R \to S_i$ be a ring homomorphism, $i = 1, \dots, d$. Consider the action of R^* on $\bigotimes_{i=1}^d S_i^{\otimes n_i}$ and $\bigotimes_{i=1}^d V_{n_i}(S_i)$ as*

$$r \bigotimes_{i=1}^{d} (a_{1,i} \otimes \cdots \otimes a_{n_i,i}) = \bigotimes_{i=1}^{d} (\varphi_i(r)^{t_i} a_{1,i} \otimes \cdots \otimes \varphi_i(r)^{t_i} a_{n_i,i}),$$

where $t_i \geq 1$. Then $H_0(R^*, \bigotimes_{i=1}^d S_i^{\otimes n_i}) = H_0(R^*, \bigotimes_{i=1}^d V_{n_i}(S_i)) = 0$.

PROOF. The proof is similar to the proof of [**28**, 1.5, 1.6] with minor generalization. If $B := \bigotimes_{i=1}^d S_i^{\otimes n_i}$, then $H_0(R^*, B) = B/I$, where I is the ideal of B generated by the elements $\bigotimes_{i=1}^d (\varphi_i(r)^{t_i} \otimes \cdots \otimes \varphi_i(r)^{t_i}) - 1$. Consider the collection $\{\psi_1^{(j_i)}, \dots, \psi_{t_i}^{(j_i)}\}$, $i = 1, \dots, d$, $1 \leq j_1 \leq n_1$ and $\sum_{i=1}^{m-1} n_i < j_m \leq \sum_{i=1}^{m} n_i$ for $m \geq 2$, of homomorphisms $R \to B/I$ given by $\psi_l^{(j_i)}(r) = 1 \otimes \cdots \otimes \varphi_i(r) \otimes \cdots \otimes 1 \mod I$, with $\varphi_i(r)$ in the j_i-th position and $1 \leq l \leq t_i$. For simplicity we denote this collection by $\{\psi_{l'} : 1 \leq l' \leq \sum_{i=1}^d t_i n_i\}$. If I is a proper ideal, we obtain a collection of ring homomorphisms $\psi_{l'}$ such that $\prod_{l'} \psi_{l'}(r) = 1$ for every $r \in R^*$, but this is impossible [**28**, Cor. 1.3, Lem. 1.4]. Thus $I = B$ and therefore $H_0(R^*, \bigotimes_{i=1}^d S_i^{\otimes n_i}) = 0$. For the proof of the second part let $l_1^{(i)}, \dots, l_{s_i}^{(i)}$, $i = 1, \dots, d$, be the natural numbers such that $\sum_{j=1}^{s_i} l_j^{(i)} = n_i$, and denote by $V_{n_i}^{l_1^{(i)}, \dots, l_{s_i}^{(i)}}$ the subspace of $V_{n_i}(S_i)$ generated by the elements of the form

$$y_{c,l^{(i)},i} := \sum_{\delta \in \Sigma_{n_i}/\Sigma_{l_1^{(i)}} \times \cdots \times \Sigma_{l_{s_i}^{(i)}}} (\underbrace{c_1^{(i)} \otimes \cdots \otimes c_1^{(i)}}_{l_1^{(i)}} \otimes \cdots \otimes \underbrace{c_{s_i}^{(i)} \otimes \cdots \otimes c_{s_i}^{(i)}}_{l_{s_i}^{(i)}})^{\delta}.$$

Clearly $V_{n_i}^{l_1^{(i)}, \dots, l_{s_i}^{(i)}}$ is an R^*-invariant subspace of the space $V_{n_i}(S_i)$ and $V_{n_i}(S_i) = \sum_{l_1^{(i)} + \cdots + l_{s_i}^{(i)} = n_i} V_{n_i}^{l_1^{(i)}, \dots, l_{s_i}^{(i)}}$. Let $V_{n_i}^{(j)}(S_i) = \sum_{s_i \geq n_i - j} V_{n_i}^{l_1^{(i)}, \dots, l_{s_i}^{(i)}}$ and set

$$T_h := \sum_{h_1 + \cdots + h_d = h} V_{n_1}^{(h_1)}(S_1) \otimes \cdots \otimes V_{n_d}^{(h_d)}(S_d).$$

It is not difficult to see that if $\sum_{i=1}^d n_i - s_i = h$ and $l_1^{(i)} + \cdots + l_{s_i}^{(i)} = n_i$, then

$$\bigotimes_{i=1}^{d} S_i^{\otimes s_i} \to T_h/T_{h-1},$$

$$\bigotimes_{i=1}^{d} c_1^{(i)} \otimes \cdots \otimes c_{s_i}^{(i)} \mapsto y_{c,l^{(1)},1} \otimes \cdots \otimes y_{c,l^{(d)},d} \mod T_{h-1}$$

is multilinear, so it gives an R^*-equivariant homomorphism. In this way we obtain an R^*-equivariant epimorphism

$$\coprod_{n_1 - s_1 + \cdots + n_d - s_d = h} \bigotimes_{i=1}^{d} S_i^{\otimes s_i} \to T_h/T_{h-1}.$$

Since the functor H_0 is right exact, by applying the first part of the lemma we get $H_0(R^*, T_h/T_{h-1}) = 0$. By induction on h we prove that $H_0(R^*, T_h) = 0$. If $h = 0$, then $T_0 = \bigotimes_{i=1}^d V_{n_i}^{(0)}(S_i)$ and $\bigotimes_{i=1}^d S_i^{\otimes n_i} \to T_0$ is surjective, so $H_0(R^*, T_0) = 0$. By induction and applying the functor H_0 to the short exact sequence $0 \to T_{h-1} \to T_h \to T_h/T_{h-1} \to 0$, we see that $H_0(R^*, T_h) = 0$. □

LEMMA 2.2. *Let P_i and Q_i be two S_i-modules for $i = 1, \ldots, d$. Then $\bigotimes_{i=1}^d \bigwedge^{n_{i,1}} P_i \otimes_k V_{n_{i,2}}(Q_i)$ has a natural structure of $\bigotimes_{i=1}^d V_{n_i}(S_i)$-module, where $n_i = n_{i,1} + n_{i,2}$. Moreover for all $l \geq 0$*

$$H_l(R^*, \bigotimes_{i=1}^d \bigwedge^{n_{i,1}} P_i \otimes_k V_{n_{i,2}}(Q_i)) = 0.$$

PROOF. The first part follows immediately from [**28**, Lem. 1.7] and the second part follows from 2.1 and [**28**, Lem. 1.8]. □

Let B be a k-vector space and let $\Gamma(B)$ be the *algebra of divided powers* of B, which is a graded commutative algebra concentrated in even degrees and endowed with a system of divided powers with $\Gamma_{2n}(B) = V_n(B)$ (see ([**6**, Chap. V, No. 6] and [**28**, §1] for more details). The homology of an abelian group A with rational coefficients coincides with exterior powers: $H_p(A, \mathbb{Q}) = \bigwedge^p(A \otimes \mathbb{Q})$. The homology with coefficients in the prime field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is more complicated. The ring $H_*(A, \mathbb{F}_p)$ has a canonical structure of divided powers [**6**, Chap. V, Example 6.5.4]. Moreover, $H_1(A, \mathbb{F}_p) = A/pA$ and there is an exact sequence

$$0 \to \bigwedge^2(A/pA) \to H_2(A, \mathbb{F}_p) \xrightarrow{\beta} {}_pA \to 0.$$

Any choice of a section for β gives a homomorphism $\varphi : {}_pA \to H_2(A, \mathbb{F}_p)$, which by the property of the algebra Γ, uniquely extends to an $\mathbb{F}_p$-algebra homomorphism $\bigwedge^*(A/pA) \otimes_k \Gamma({}_pA) \to H_*(A, \mathbb{F}_p)$, thus giving rise to an isomorphism of graded $\mathbb{F}_p$-algebras [**6**, Chap. V, Thm. 6.6], [**33**, §8, Prop. 3]. We identify $H_j(A, \mathbb{F}_p)$ with $\coprod_i \bigwedge^{j-2i}(A/pA) \otimes_k \Gamma_{2i}({}_pA)$ and introduce a filtration on $H_j(A, \mathbb{F}_p)$ setting

$$H_j^{(r)} = \coprod_{i \leq r} \bigwedge^{j-2i}(A/pA) \otimes_k \Gamma_{2i}({}_pA).$$

This filtration does not depend on our choice of section φ and successive factors $H_j^{(r)}/H_j^{(r-1)}$ are canonically isomorphic to $\bigwedge^{j-2r}(A/pA) \otimes_k \Gamma_{2r}({}_pA)$.

THEOREM 2.3. *Let M_i be a T_i-module and let $R \to T_i$ be a ring homomorphism. Consider the action of R^* on M_i given by $r \cdot m = \varphi_i(r)^{t_i} m$, where $t_i \geq 1$. If k is a prime field then, $H_l(R^*, \bigotimes_{i=1}^d H_{l_i}(M_i)) = 0$, for $l \geq 0$, where $l_i > 0$ for some i.*

PROOF. Let $P_i = M_i \otimes_{\mathbb{Z}} k$ and $S_i = T_i \otimes_{\mathbb{Z}} k$. If $k = \mathbb{Q}$, then $H_{l_i}(M_i) = \bigwedge^{l_i} P_i$ and if $k = \mathbb{F}_p$ for some prime number p, then we can find an R^*-invariant filtration of $\bigotimes_{i=1}^d H_{l_i}(M_i)$ whose successive factors are isomorphic to $\bigotimes_{i=1}^d \bigwedge^{j_i - 2m_i} P_i \otimes_k \Gamma_{2m_i}(Q_i)$ for some j_i and m_i, where $Q_i = {}_p(P_i \otimes_{\mathbb{Z}} k)$. Note that P_i and Q_i are S_i-modules. Then both cases follow from 2.2. □

Let $\overline{\sigma_2} = (\langle e_1\rangle, \langle e_3\rangle) \in \underline{\mathcal{IU}}(R^{2n})$. The elements of $\mathrm{Stab}_{G_n}(\overline{\sigma_2}) = \{B \in G_n : B\overline{\sigma_2} = \overline{\sigma_2}\}$ are of the form

$$\begin{pmatrix} a_1 & * & 0 & * & * & * \\ 0 & \overline{a_1}^{-1} & 0 & 0 & 0 & 0 \\ 0 & * & a_2 & * & * & * \\ 0 & 0 & 0 & \overline{a_2}^{-1} & 0 & 0 \\ 0 & * & 0 & * & & \\ 0 & * & 0 & * & & A \end{pmatrix},$$

where $a_i \in R^*$ and $A \in G_{n-2}$. Let $N_{n,2}$ and $L_{n,2}$ be the subgroups of $\mathrm{Stab}_{G_n}(\overline{\sigma_2})$ of elements of the form

$$\begin{pmatrix} 1 & * & 0 & * & * & * \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & * & 1 & * & * & * \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & * & 0 & * & & \\ 0 & * & 0 & * & & I_{2(n-2)} \end{pmatrix}, \begin{pmatrix} a_1 & * & 0 & * & * & * \\ 0 & \overline{a_1}^{-1} & 0 & 0 & 0 & 0 \\ 0 & * & a_2 & * & * & * \\ 0 & 0 & 0 & \overline{a_2}^{-1} & 0 & 0 \\ 0 & * & 0 & * & & \\ 0 & * & 0 & * & & I_{2(n-2)} \end{pmatrix}$$

respectively. It is a matter of an easy calculation to see that the elements of the group $N'_{n,2} = [N_{n,2}, N_{n,2}]$ are of the form

$$\begin{pmatrix} 1 & r & 0 & t & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\epsilon^{-1}\overline{t} & 1 & s & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & I_{2(n-2)} \end{pmatrix},$$

where $r,\ s \in \Lambda = \{r \in R : \epsilon^{-1}\overline{r} = -r\}$ and $t \in R$. In general one can define $N_{n,p}$, $L_{n,p}$ and $N'_{n,p}$ for all p, $1 \le p \le n$, in a similar way. Embed $R^{*p} \times G_{n-p}$ in $\mathrm{Stab}_{G_n}(\overline{\sigma_p})$ as $(a_1, \dots, a_p, A) \mapsto \mathrm{diag}(\begin{pmatrix} a_1 & 0 \\ 0 & \overline{a_1}^{-1} \end{pmatrix}, \dots, \begin{pmatrix} a_p & 0 \\ 0 & \overline{a_p}^{-1} \end{pmatrix}, A)$.

THEOREM 2.4. *Let $\overline{\sigma_p} = (\langle e_1\rangle, \langle e_3\rangle, \dots, \langle e_{2p-1}\rangle) \in \underline{\mathcal{IU}}(R^{2n})$. Then the inclusion $R^{*p} \times G_{n-p} \to \mathrm{Stab}_{G_n}(\overline{\sigma_p})$ induces isomorphism $H_i(R^{*p} \times G_{n-p}) \xrightarrow{\simeq} H_i(\mathrm{Stab}_{G_n}(\overline{\sigma_p}))$, for all i.*

PROOF. It is sufficient to prove the theorem when k is a prime field. Fix a natural number p, $1 \le p \le n$, and set $N = N_{n,p}$, $L = L_{n,p}$, $N' = N'_{n,p}$ and $T = \mathrm{Stab}_{G_n}(\overline{\sigma_p})$. The extensions $1 \to N' \to L \to L/N' \to 1$ and $1 \to N/N' \to L/N' \to L/N \to 1$ give the Lyndon-Hochschild-Serre spectral sequences

$$E^2_{p,q} = H_p(L/N', H_q(N')) \Rightarrow H_{p+q}(L),$$
$$E^2_{p',q'} = H_{p'}(L/N, H_{q'}(N/N', H_q(N'))) \Rightarrow H_{p'+q'}(L/N', H_q(N')),$$

respectively. Since $L/N \simeq R^{*p}$ and N/N' acts trivially on N', $E^2_{p',q'} = H_{p'}(R^{*p}, H_{q'}(N/N') \otimes_k H_q(N'))$. It is not difficult to see that $N/N' \simeq R^h$ and $N' \simeq R^l \times \Lambda^m$ for some h, l, m and the action of R^*_1 on N/N' and N' is linear-diagonal and quadratic-diagonal respectively. Again the extension $1 \to R_1^* \to R^{*p} \to R^{*p}/R_1^* \to 1$ (R_1^* embeds in R^{*p} diagonally) gives

$$E^2_{r,s} = H_r(R^{*p}/R_1^*, H_s(R_1^*, M)) \Rightarrow H_{r+s}(R^{*p}, M),$$

where $M = H_{q'}(N/N') \otimes_k H_q(N')$. Since the homology functor commutes with the direct sum functor,

$$H_s(R_1^*, M) \simeq \bigoplus_{i=0}^{q} H_s(R_1^*, H_{q'}(R^h) \otimes_k H_i(R^l) \otimes_k H_{q-i}(\Lambda^m)),$$

where the action of R_1^* on R^h, R^l and Λ^m is linear-diagonal, quadratic-diagonal and quadratic-diagonal respectively. By theorem 2.3, $H_s(R_1^*, M) = 0$ for $s \geq 0$ and $q > 0$ or $q' > 0$. This shows that $E^2_{p',q'} = 0$ for $p' \geq 0$ and $q > 0$ or $q' > 0$. Therefore $H_{p'}(L/N', H_q(N')) = 0$ for $p' \geq 0$ and $q > 0$. Hence $E^2_{p,q} = 0$ for $p \geq 1$ and $q > 0$. By the convergence of the spectral sequence we get

$$H_p(L) \overset{\simeq}{\to} H_p(L/N'). \tag{1}$$

The extension $1 \to N/N' \to L/N' \to L/N \to 1$ gives

$$E^2_{i,j} = H_i(L/N, H_j(N/N')) \Rightarrow H_{i+j}(L/N').$$

and by a similar approach to (1),

$$H_i(R^{*p}) \overset{\simeq}{\to} H_i(L/N'). \tag{2}$$

From the embedding $R^{*p} \to L$, (1) and (2) we get the isomorphism $H_i(R^{*p}) \overset{\simeq}{\to} H_i(L)$, $i \geq 0$. The commutative diagram

$$\begin{array}{ccccccccc}
1 & \to & R^{*p} & \to & R^{*p} \times G_{n-p} & \to & G_{n-p} & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & L & \to & T & \to & G_{n-p} & \to & 1
\end{array}$$

gives the map of the spectral sequences

$$\begin{array}{ccc}
E^2_{p,q} = H_p(G_{n-p}, H_q(R^{*p})) & \Rightarrow & H_{p+q}(R^{*p} \times G_{n-p}) \\
\Big\downarrow & & \Big\downarrow \\
{E'}^2_{p,q} = H_p(G_{n-p}, H_q(L)) & \Rightarrow & H_{p+q}(T).
\end{array}$$

By what we proved in the above we have the isomorphism $E^2_{p,q} \simeq {E'}^2_{p,q}$. This gives an isomorphism on the abutments and so $H_i(R^{*p} \times G_{n-p}) \simeq H_i(T)$. $\square$

THEOREM 2.5. *There is a first quadrant spectral sequence converging to zero with*

$$E^1_{p,q}(n) = \begin{cases} H_q(R^{*p} \times G_{n-p}) & \text{if } 0 \leq p \leq n \\ H_q(G_n, H_{n-1}(X_n)) & \text{if } p = n+1 \\ 0 & \text{if } p \geq n+2 \end{cases},$$

where $X_n = \underline{\mathcal{IU}}(R^{2n})$.
For $1 \leq p \leq n$ *the differential* $d^1_{p,q}(n)$ *equals* $\sum_{i=1}^{p}(-1)^{i+1}H_q(\alpha_{i,p})$, *where* $\alpha_{i,p}$: $R^{*p} \times G_{n-p} \to R^{*p-1} \times G_{n-p+1}$ *with*

$$(a_1, \dots, a_p, A) \mapsto (a_1, \dots, \widehat{a_i}, \dots, a_p, \begin{pmatrix} a_i & 0 & 0 \\ 0 & \overline{a_i}^{-1} & 0 \\ 0 & 0 & A \end{pmatrix}).$$

In particular for $0 \leq p \leq n$, $d^1_{p,0}(n) = \begin{cases} \mathrm{id}_k & \textit{if } p \textit{ is odd} \\ 0 & \textit{if } p \textit{ is even} \end{cases}$, *so* $E^2_{p,0} = 0$ *for* $0 \leq p \leq n-1$.

Proof. Let $C'_l(X_n)$ be the k-vector space with the basis consisting of l-simplices (isotropic $(l+1)$-frames) of X_n. Since X_n is $(n-2)$-acyclic, Chap. 2, Thm. 4.7, we get an exact sequence

$$0 \leftarrow k \leftarrow C'_0(X_n) \leftarrow C'_1(X_n) \leftarrow \cdots \leftarrow C'_{n-1}(X_n) \leftarrow H_{n-1}(X_n, k) \leftarrow 0.$$

Call this exact sequence L_*: $L_0 = k, L_i = C'_{i-1}(X_n)$, $1 \leq i \leq n$, $L_{n+1} = H_{n-1}(X_n, k)$ and $L_i = 0$ for $i \geq n+2$. Let $F_* \to k$ be a resolution of k by free (left) G_n-modules and consider the bicomplex $C_{*,*} = L_* \otimes_{G_n} F_*$. Here we convert the left action of G_n on L_* into a right action with $vg := g^{-1}v$. By the general theory of the spectral sequence for a bicomplex we have $E^1_{p,q}(I) = H_q(C_{p,*}) = H_q(L_p \otimes_{G_n} F_*)$ and $E^1_{p,q}(II) = H_q(C_{*,p}) = H_q(L_* \otimes_{G_n} F_p)$. Since F_p is a free G_n-module, $L_* \otimes_{G_n} F_p$ is exact and this shows that $E^1_{p,q}(II) = 0$. Therefore $E^1_{p,q}(n) := E^1_{p,q}(I)$ converges to zero. If $p = 0$, then $E^1_{0,q}(n) = H_q(k \otimes_{G_n} F_*) = H_q(G_n)$. The group G_n acts transitively on the l-frames of X_n, $1 \leq l \leq n$, so by the Shapiro lemma [**6**, Chap. III, 6.2], $L_p \otimes_{G_n} F_* \simeq k \otimes_{\mathrm{Stab}_{G_n}(\overline{\sigma_p})} F_*$ and thus $E^1_{p,q}(n) = H_q(\mathrm{Stab}_{G_n}(\overline{\sigma_p}))$, $1 \leq p \leq n$. By 2.4, $E^1_{p,q}(n) = H_q(R^{*p} \times G_{n-p})$ for $0 \leq p \leq n$, hence $E^1_{p,q}(n)$ is of the form that we mentioned. Now we look at the differential $d^1_{p,q}(n) : E^1_{p,q}(n) \to E^1_{p-1,q}(n)$, $1 \leq p \leq n$; $d^1_{1,q}(n)$ is induced by $C'_0(X_n) \otimes_{G_n} F_* \to k \otimes_{G_n} F_*$, $\overline{\sigma_1} \otimes x \mapsto 1 \otimes x$. Considering the isomorphism $k \otimes_{\mathrm{Stab}_{G_n}(\overline{\sigma_1})} F_* \to C'_0(X) \otimes_{G_n} F_*$, $1 \otimes x \mapsto \overline{\sigma_1} \otimes x$, one sees that $d^1_{1,q}(n)$ is induced by $k \otimes_{\mathrm{Stab}_{G_n}(\overline{\sigma_1})} F_* \to k \otimes_{G_n} F_*$. This shows that $d^1_{1,q}(n)$ is the map $H_q(\mathrm{Stab}_{G_n}(\overline{\sigma_1})) \to H_q(G_n)$ induced by the inclusion map, therefore $d^1_{1,q}(n) = H_q(\mathrm{inc}) : H_q(\mathrm{inc}) : H_q(R^* \times G_{n-1}) \to H_q(G_n)$. For $2 \leq p \leq n$, $d^1_{p,q}(n)$ is induced by the map $\sum_{i=1}^{p}(-1)^{i+1}d_i : L_p \to L_{p-1}$, where d_i deletes the i-th component of the isotropic p-frames. Let $g_{i,p}$ be the permutation matrix such that $(e_{2h-1}, e_{2h})g^{-1}_{i,p} = (e_{2h-1}, e_{2h})$, $1 \leq h \leq i-1$, $(e_{2i-1}, e_{2i})g^{-1}_{i,p} = (e_{2p-1}, e_{2p})$ and $(e_{2l-1}, e_{2l})g^{-1}_{i,p} = (e_{2l-3}, e_{2l-2})$, $i+1 \leq l \leq p$, where $vg^{-1} := gv$ for $v \in R^{2n}$. It is easy to see that $d_i(\overline{\sigma_p}) = \overline{\sigma_{p-1}}g_{i,p}$, and so $\partial(\overline{\sigma_p}) = \sum_{i=1}^{p}(-1)^{i+1}\overline{\sigma_{p-1}}g_{i,p}$. Consider $d_i \otimes \mathrm{id}_{F_*} : L_p \otimes_{G_n} F_* \to L_{p-1} \otimes_{G_n} F_*$, $\overline{\sigma_p} \otimes x \mapsto d_i(\overline{\sigma_p}) \otimes x$. Let $\mathrm{inn}_{g_{i,p}} : G_n \to G_n$, $g \mapsto g_{i,p}gg^{-1}_{i,p}$ and $l_{g_{i,p}} : F_* \to F_*$, $x \mapsto g_{i,p}x$. The map $l_{g_{i,p}}$ is an $\mathrm{inn}_{g_{i,p}}$-homomorphism, and $d_i \otimes \mathrm{id}_{F_*}$ induces the map $k \otimes_{\mathrm{Stab}_{G_n}(\overline{\sigma_p})} F_* \to k \otimes_{\mathrm{Stab}_{G_n}(\overline{\sigma_{p-1}})} F_*$, $1 \otimes x \mapsto 1 \otimes l_{g_{i,p}}(x)$. This shows that d_i induces $H_q(\mathrm{inn}_{g_{i,p}}) : H_q(\mathrm{Stab}_{G_n}(\overline{\sigma_p})) \to H_q(\mathrm{Stab}_{G_n}(\overline{\sigma_{p-1}}))$ and hence the map $H_q(\mathrm{inn}_{g_{i,p}}) : H_q(R^{*p} \times G_{n-p}) \to H_q(R^{*p-1} \times G_{n-p+1})$. Set $\alpha_{i,p} = \mathrm{inn}_{g_{i,p}}$. Since G_n

acts transitively on the generators of $C'_p(X_n)$, $E^1_{*,0}(n)$ is of the following form

$$0 \leftarrow k \leftarrow k \leftarrow k \leftarrow \cdots \leftarrow k \leftarrow H_0(G_n H_{n-1}(X_n, k)) \leftarrow 0,$$

where $d^1_{p,0}(n) = \begin{cases} \mathrm{id}_k & \text{if } p \text{ is odd} \\ 0 & \text{if } p \text{ is even} \end{cases}$. Clearly $E^2_{p,0}(n) = 0$ if $0 \leq p \leq n-1$. □

REMARK 10. In fact $E^2_{n,0}(n) = 0$. For a proof see the proof of theorem 2.8.

To prove the homology stability result we have to study the spectral sequence that we obtained in theorem 2.5.

LEMMA 2.6. *Let $n \geq 1$, $l \geq 0$ be integers such that $n-1 \geq l$. Let $H_q(\mathrm{inc}) : H_q(G_{n-2}) \to H_q(G_{n-1})$ be surjective if $0 \leq q \leq l-1$. Then the following conditions are equivalent;*
(i) *$H_l(\mathrm{inc}) : H_l(G_{n-1}) \to H_l(G_n)$ is surjective,*
(ii) *$H_l(\mathrm{inc}) : H_l(R^* \times G_{n-1}) \to H_l(G_n)$ is surjective.*

PROOF. For $n = 1$ every thing is easy so let $n \geq 2$. By the Künneth theorem [**21**, Chap. V, §10, Thm. 10.1] we have $H_l(R^* \times G_{n-1}) = S_1 \oplus S_2$, where $S_1 = H_l(G_{n-1})$ and $S_2 = \bigoplus_{i=1}^{l} H_i(R^*) \otimes_k H_{l-i}(G_{n-1})$. The case (i)⇒(ii) is trivial. To prove (ii)⇒(i) it is sufficient to prove that $\tau_1(S_2) \subseteq \tau_1(S_1)$, where $\tau_1 = H_l(\mathrm{inc})$. From $i \geq 1$ and $n-1 \geq l$, we have $n-2 \geq l-1 \geq l-i$, so by hypothesis $H_{l-i}(\mathrm{inc}) : H_{l-i}(G_{n-2}) \to H_{l-i}(G_{n-1})$ is surjective, $1 \leq i \leq l$. Consider the following diagram

$$\begin{array}{ccccc} H_i(R^*) \otimes_k H_{l-i}(G_{n-1}) & \xrightarrow{\beta_1} & H_l(R^* \times G_{n-1}) & \xrightarrow{\tau_1} & H_l(G_n) \\ \uparrow{\scriptstyle \alpha_1} & & & & \uparrow{\scriptstyle \alpha_2} \\ H_i(R^*) \otimes_k H_{l-i}(G_{n-2}) & \xrightarrow{\beta_2} & H_l(R^* \times G_{n-2}) & \xrightarrow{\tau'_1} & H_l(G_{n-1}), \end{array}$$

where β_j is the shuffle product, $j = 1, 2$ [**6**, Chap. V, Sec. 5], $\alpha_1 = \mathrm{id} \otimes H_{l-i}(\mathrm{inc})$ is surjective and $\alpha_2 = H_l(\mathrm{inc})$. By giving an explicit description of the above maps we prove that this diagram is commutative. For this purpose we use the bar resolution of a group [**6**, Chap. I, Sec. 5] (see also appendix A). If $x = \sum [a_1 | \dots | a_i] \otimes [A_1 | \dots | A_{l-i}] \in H_i(R^*) \otimes_k H_{l-i}(G_{n-2})$, then

$$\begin{aligned} x &\overset{\alpha_1}{\longmapsto} \sum [a_1 | \dots | a_i] \otimes [\mathrm{diag}(I_2, A_1) | \dots | \mathrm{diag}(I_2, A_{l-i})] \\ &\overset{\tau_1 \circ \beta_1}{\longmapsto} \sum \sum_{\delta} \mathrm{sign}(\delta) [\dots | \mathrm{diag}(a_{\delta(i')}, \overline{a_{\delta(i')}}^{-1}, I_{2(n-1)}) | \dots | \mathrm{diag}(I_4, A_{\delta(j')}) | \dots] \end{aligned}$$

and

$$\begin{aligned} x &\overset{\tau'_1 \circ \beta_2}{\longmapsto} \sum \sum_{\delta} \mathrm{sign}(\delta) [\dots | \mathrm{diag}(a_{\delta(i')}, \overline{a_{\delta(i')}}^{-1}, I_{2(n-2)}) | \dots | \mathrm{diag}(I_2, A_{\delta(j')}) | \dots] \\ &\overset{\alpha_2}{\longmapsto} \sum \sum_{\delta} \mathrm{sign}(\delta) [\dots | \mathrm{diag}(I_2, a_{\delta(i')}, \overline{a_{\delta(i')}}^{-1}, I_{2(n-2)}) | \dots | \mathrm{diag}(I_4, A_{\delta(j')}) | \dots]. \end{aligned}$$

See [**21**, Chap. VIII, §8] and appendix A for more details about the shuffle product. Let $P \in G_n$ be the permutation matrix that permutes the first and second columns with third and forth columns respectively and let $\mathrm{inn}_P : G_n \to G_n$, $A \mapsto PAP^{-1} = PAP$. It is well known that $H_q(\mathrm{inn}_P) = \mathrm{id}_{H_q(G_n)}$ [**6**, Chap. II, §8], hence

$$\begin{aligned} H_l(\mathrm{inn}_P)([\dots|\mathrm{diag}(I_2, a_{\delta(i')}, \overline{a_{\delta(i')}}^{-1}, I_{2(n-2)})|\dots|\mathrm{diag}(I_2, I_2, A_{\delta(j')})|\dots]) \\ = [\dots|\mathrm{diag}(a_{\delta(i')}, \overline{a_{\delta(i')}}^{-1}, I_2, I_{2(n-2)})|\dots|\mathrm{diag}(I_2, I_2, A_{\delta(j')})|\dots]. \end{aligned}$$

This shows that the above diagram is commutative. Therefore $\tau_1(S_2) \subseteq \tau_1(S_1)$. □

Lemma 2.7. *Let $n \geq 2$, $l \geq 0$ be integers such that $n-1 \geq l+1$. Let $H_q(\mathrm{inc}) : H_q(G_{m-1}) \to H_q(G_m)$ be isomorphism for $m = n-1, n-2$ and $0 \leq q \leq \min\{l-1, m-2\}$. Then the following conditions are equivalent;*

(i) $H_l(\mathrm{inc}) : H_l(G_{n-1}) \to H_l(G_n)$ *is bijective,*

(ii) $H_l({R^*}^2 \times G_{n-2}) \overset{\tau_2}{\to} H_l(R^* \times G_{n-1}) \overset{\tau_1}{\to} H_l(G_n) \to 0$ *is exact, where $\tau_1 = H_l(\mathrm{inc})$ and $\tau_2 = H_l(\alpha_{1,2}) - H_l(\alpha_{2,2})$.*

Proof. Let $H_l(R^* \times G_{n-1}) = S_1 \oplus S_2$, where S_1 and S_2 are as in the proof of lemma 2.6 and $H_l({R^*}^2 \times G_{n-2}) = \bigoplus_{h=1}^4 T_h$, where

$$\begin{aligned} T_1 &= H_l(G_{n-2}), \\ T_2 &= \bigoplus_{i=1}^{l} H_i(R^* \times 1) \otimes_k H_{l-i}(G_{n-2}), \\ T_3 &= \bigoplus_{i=1}^{l} H_i(1 \times R^*) \otimes_k H_{l-i}(G_{n-2}), \\ T_4 &= \bigoplus_{\substack{i,j\geq 1\\ i+j\leq l}} H_i(R^* \times 1) \otimes_k H_j(1 \times R^*) \otimes_k H_{l-i-j}(G_{n-2}). \end{aligned}$$

Set $\sigma_1^{(2)} = H_l(\alpha_{1,2})$ and $\sigma_2^{(2)} = H_l(\alpha_{2,2})$. First (i)⇒(ii). The surjectivity of τ_1 is trivial. Let $(x, v) \in S_1 \oplus S_2$ such that $\tau_1((x, v)) = 0$. The relations $n-1 \geq l+1$ and $i \geq 1$ imply that $n-3 \geq l-1 \geq l-i$ and hence $H_{l-i}(G_{n-2}) \to H_{l-i}(G_{n-1})$ is bijective, so there exists $w \in T_2$ such that $-\sigma_1^{(2)}(w) = v$. If $y = (0, w, 0, 0) \in \bigoplus_{h=1}^4 T_h$, then $\tau_2(y) = (\sigma_2^{(2)}(w), -\sigma_1^{(2)}(w)) = (\sigma_2^{(2)}(w), v)$. Since $\tau_1 \circ \tau_2 = 0$, $\tau_1(\sigma_2^{(2)}(w)) = -\tau_1(v)$. Combining this with $\tau_1(x) = -\tau_1(v)$, we obtain $\tau_1(\sigma_2^{(2)}(w)) = \tau_1(x)$. By injectivity of $H_l(\mathrm{inc})$, $\sigma_2^{(2)}(w) = x$, thus $\tau_2(y) = (x, v)$. This shows that the complex is exact. The proof of (ii)⇒(i) is more difficult. The surjectivity of $\tau_1 = H_l(\mathrm{inc})$ follows from lemma 2.6. Let $x \in \ker(H_l(G_{n-1}) \to H_l(G_n))$, then $(x, 0) \in \ker(\tau_1)$. By exactness of the complex there is a $y = (0, y_2, y_3, y_4) \in \bigoplus_{h=1}^4 T_h$ such that $\tau_2(y) = (x, 0)$ (one should notice that $\tau_2(T_1) = 0$). First we prove that we can assume $y_4 = 0$. If $n = 2$, then $l \leq 1$ and so $T_4 = 0$. Therefore we may assume $n \geq 3$. Consider the summand

$$U = \bigoplus_{\substack{i,j\geq 1\\ i+j\leq l}} H_i(R^* \times 1 \times 1) \otimes_k H_j(1 \times R^* \times 1) \otimes_k H_{l-i-j}(G_{n-3})$$

of $H_l({R^*}^3 \times G_{n-3})$ and set $\tau_3 := d_{3,l}^1(n) = \sigma_1^{(3)} - \sigma_2^{(3)} + \sigma_3^{(3)}$, where $\sigma_i^{(3)} = H_l(\alpha_{i,3})$. It is easy to see that $\sigma_3^{(3)}(U) \subseteq T_4$ and $-\sigma_2^{(3)} + \sigma_1^{(3)}(U) \subseteq T_2 \oplus T_3$. By assumption,

$\sigma_3^{(3)}|_U : U \to T_4$ is an isomorphism. If $\sigma_3^{(3)}(z) = y_4$, then $y - \tau_3(z) = (0, y_2', y_3', 0)$ and $\tau_2(y - \tau_3(z)) = (x, 0)$. So we can assume that $y = (0, y_2, y_3, 0)$. Let

$$y_2 = (\sum [a_1|\dots|a_i] \otimes [A_1|\dots|A_{l-i}])_{1 \le i \le l}$$
$$y_3 = (\sum [b_1|\dots|b_i] \otimes [B_1|\dots|B_{l-i}])_{1 \le i \le l}.$$

By an explicit computation

$$\tau_2(y) = (\sigma_1^{(2)}(y_2) - \sigma_2^{(2)}(y_3), -\sigma_2^{(2)}(y_2) + \sigma_1^{(2)}(y_3)).$$

This shows that $x = \sigma_1^{(2)}(y_2) - \sigma_2^{(2)}(y_3)$ is equal to

$$\sum_{i=1}^{l} \sum \sum_{\delta} \mathrm{sign}(\delta)[\dots|\mathrm{diag}(a_{\delta(i')}, \overline{a_{\delta(i')}}^{-1}, I_{2(n-1)})|\dots|\mathrm{diag}(I_2, A_{\delta(j')})|\dots]$$
$$- \sum_{i=1}^{l} \sum \sum_{\delta} \mathrm{sign}(\delta)[\dots|\mathrm{diag}(b_{\delta(i')}, \overline{b_{\delta(i')}}^{-1}, I_{2(n-1)})|\dots|\mathrm{diag}(I_2, B_{\delta(j')})|\dots]$$

and for $1 \le i \le l$,

$$0 = \sum [a_1|\dots|a_i] \otimes [\mathrm{diag}(I_2, A_1)|\dots|\mathrm{diag}(I_2, A_{l-i})]$$
$$- \sum [b_1|\dots|b_i] \otimes [\mathrm{diag}(I_2, B_1)|\dots|\mathrm{diag}(I_2, B_{l-i})].$$

By the injectivity of $H_{l-i}(G_{n-2}) \to H_{l-i}(G_{n-1})$, we see that $y_2 = y_3$, (note that we view y_2 and y_3 as elements of T_2 or T_3). Now it is easy to see that $x = 0$. □

Consider $R^{2(n-2)}$ as the submodule of R^{2n} generated by $e_5, e_6, \dots, e_{2n}$ (so G_{n-2} embeds in G_n as $\mathrm{diag}(I_2, I_2, G_{n-2})$). Let L'_* be the complex

$$\cdots \leftarrow C'_{n-3}(X_{n-2}) \leftarrow H_{n-3}(X_{n-2}, k) \leftarrow 0$$
$$0 \leftarrow 0 \leftarrow 0 \leftarrow k \leftarrow C'_0(X_{n-2}) \leftarrow C'_1(X_{n-2}) \leftarrow$$

with $X_{n-2} = \underline{\mathcal{IU}}(R^{2(n-2)})$. Define the map of complexes $L'_* \xrightarrow{\alpha_*} L_*$, given by

$$(\langle v_1\rangle, \dots, \langle v_k\rangle) \overset{\alpha_k}{\mapsto} (\langle e_1\rangle, \langle e_3\rangle, \langle v_1\rangle, \dots, \langle v_k\rangle) - (\langle e_1\rangle, \langle e_1 + e_3\rangle, \langle v_1\rangle, \dots, \langle v_k\rangle)$$
$$+ (\langle e_3\rangle, \langle e_1 + e_3\rangle, \langle v_1\rangle, \dots, \langle v_k\rangle)$$

Note that this is similar to one defined in the proof of proposition 2.6 in [**28**]. This gives the maps of bicomplexes

$$L'_* \otimes_{G_{n-2}} F'_* \to L_* \otimes_{G_n} F_* \to L_* \otimes_{G_n} F_*/L'_* \otimes_{G_{n-2}} F'_*,$$

where L_* and F_* are as in the proof of theorem 2.5 and F'_* is F_* as G_{n-2}-module, so it induces the maps of spectral sequences

$$E'^{r}_{p,q}(n) \to E^{r}_{p,q}(n) \to E''^{r}_{p,q}(n),$$

where all the three spectral sequences converge to zero. By a similar argument as in the proof of 2.5, one sees that the spectral sequence ${E'}^1_{p,q}(n)$ is of the form

$$ {E'}^1_{p,q}(n) = \begin{cases} E^1_{p-2,q}(n-2) & \text{if } p \geq 2 \\ 0 & \text{if } p = 0,1 \end{cases}. $$

For $2 \leq p \leq n$, ${E'}^1_{p,q}(n) \to E^1_{p,q}(n)$ is induced by $\mathrm{inc} : R^{*p-2} \times G_{n-p} \to R^{*p} \times G_{n-p}$, $A \mapsto (1,1,A)$, and

$$ {E''}^1_{p,q}(n) = E^1_{p,q}(n)/{E'}^1_{p,q}(n). $$

From the complexes

$$ \begin{aligned} D_*(q): &\quad 0 \to E^1_{n,q}(n) \to E^1_{n-1,q}(n) \to \cdots \to E^1_{0,q}(n) \to 0 \\ {D'}_*(q): &\quad 0 \to {E'}^1_{n,q}(n) \to {E'}^1_{n-1,q}(n) \to \cdots \to {E'}^1_{0,q}(n) \to 0 \\ {D''}_*(q): &\quad 0 \to {E''}^1_{n,q}(n) \to {E''}^1_{n-1,q}(n) \to \cdots \to {E''}^1_{0,q}(n) \to 0 \end{aligned} $$

we obtain a short exact sequence

$$ 0 \to {D'}_*(q) \to D_*(q) \to {D''}_*(q) \to 0 $$

and by applying the homology long exact sequence to this short exact sequence we get the following exact sequence

$$ \begin{aligned} {E'}^2_{n-1,q}(n) \to E^2_{n-1,q}(n) \to {E''}^2_{n-1,q}(n) \to {E'}^2_{n-2,q}(n) \\ \to \cdots \to {E'}^2_{0,q}(n) \to E^2_{0,q}(n) \to {E''}^2_{0,q}(n) \to 0. \end{aligned} $$

THEOREM 2.8. *Let $n \geq 1$, $l \geq 0$ be integers. Then $H_l(\mathrm{inc}) : H_l(G_{n-1}) \to H_l(G_n)$ is surjective for $n-1 \geq l$ and is injective for $n-1 \geq l+1$.*

PROOF. The proof is by induction on l. If $l = 0$, then everything is obvious. Assume the induction hypothesis, that is $H_i(G_{m-1}) \to H_i(G_m)$ is surjective if $m-1 \geq i$ and is bijective if $m-1 \geq i+1$, where $1 \leq i \leq l-1$. Let $n-1 \geq l$ and consider the spectral sequence ${E''}^2_{p,q}(n)$. To prove the surjectivity, it is sufficient to prove that ${E''}^2_{p,q}(n) = 0$ if $n \geq p+q$, $0 \leq q \leq l-1 \leq n-2$ and $2 \leq p \leq n$, because then we obtain ${E''}^2_{0,l}(n) = {E''}^\infty_{0,l}(n) = 0$ and applying lemma 2.6 we have the desired result. Let R^*_i denote the i-th factor of R^{*m}. By the Künneth theorem ${E''}^1_{p,q}(n) = T_1 \oplus T_2 \oplus T_3 \mod {E'}^1_{p,q}$, where

$$ \begin{aligned} T_1 &= \bigoplus_{i_1 \geq 1} H_{i_1}(R^*_1) \otimes H_{i_3}(R^*_3) \otimes \cdots \otimes H_{i_p}(R^*_p) \otimes H_{q-\Sigma i_j}(G_{n-p}), \\ T_2 &= \bigoplus_{i_2 \geq 1} H_{i_2}(R^*_2) \otimes H_{i_3}(R^*_3) \otimes \cdots \otimes H_{i_p}(R^*_p) \otimes H_{q-\Sigma i_j}(G_{n-p}), \\ T_3 &= \bigoplus_{k_1,k_2 \geq 1} H_{k_1}(R^*_1) \otimes H_{k_2}(R^*_2) \otimes \cdots \otimes H_{k_p}(R^*_p) \otimes H_{q-\Sigma k_s}(G_{n-p}). \end{aligned} $$

Consider the following summand of ${E''}^1_{p+1,q}(n)$

$$U_1 = \bigoplus_{j_2, j_3 \geq 1} H_{j_2}(R_2^*) \otimes H_{j_3}(R_3^*) \otimes \cdots \otimes H_{j_{p+1}}(R_{p+1}^*) \otimes H_{q-\Sigma j_t}(G_{n-p-1}),$$

where $j_t = k_{t-1}$, $2 \leq t \leq p+1$. Let $\sigma_i^{(m)} := H_l(\alpha_{i,m})$. It is easy to see that $\sigma_1^{(p+1)}(U_1) \subseteq T_3$ and $\sum_{i=2}^{p+1}(-1)^{i+1}\sigma_i^{(p+1)}(U_1) \subseteq T_2$. Let $x = (x_1, x_2, x_3) \in \ker({d''}^1_{p,q})$. Since $n-p-1 \geq q-1 \geq q - \sum j_t$, by a similar argument as in the proof of lemma 2.7, we can assume that $x_3 = 0$. If

$$U_2 = \bigoplus_{j_2 \geq 1} H_{j_2}(R_2^*) \otimes H_{j_4}(R_4^*) \otimes \cdots \otimes H_{j_{p+1}}(R_{p+1}^*) \otimes H_{q-\Sigma j_t}(G_{n-p-1}),$$

then we have $\sigma_1^{(p+1)}(U_2) \subseteq T_1$, $\sigma_2^{(p+1)}(U_2) = 0 \bmod {E'}^1_{p,q}(n)$ and

$$\sum_{i=3}^{p+1}(-1)^{i+1}\sigma_i^{(p+1)}(U_2) \subseteq T_2.$$

In the same way, using our assumption we can again assume that $x_1 = 0$. So $x = (0, x_2, 0)$. Once again we have $\sigma_1^{(p)}(T_2) \subseteq S_1$ and $\sum_{i=2}^{p}(-1)^{i+1}\sigma_i^{(p)}(T_2) \subseteq S_2$, where

$$S_1 = \bigoplus_{k_1 \geq 1} H_{k_1}(R_1^*) \otimes H_{k_2}(R_2^*) \otimes \cdots \otimes H_{k_{p-1}}(R_{p-1}^*) \otimes H_{q-\Sigma k_t}(G_{n-p+1}),$$

$$S_2 = \bigoplus_{l_2 \geq 1} H_{l_2}(R_2^*) \otimes H_{l_3}(R_3^*) \otimes \cdots \otimes H_{l_{p-1}}(R_{p-1}^*) \otimes H_{q-\Sigma l_t}(G_{n-p+1}).$$

By the induction hypothesis $\sigma_1^{(p)}$ is an isomorphism, so $x_2 = 0$. Therefore ${E''}^2_{p,q}(n) = 0$ if $n \geq p+q$, $2 \leq p \leq n-1$, $1 \leq q \leq l-1$. To prove that ${E''}^2_{p,0}(n) = 0$ for $0 \leq p \leq n$, it is sufficient to prove that $E^2_{p,0}(n) = 0$ for $0 \leq p \leq n$. For $0 \leq p \leq n-1$ this follows from 2.5. If n is odd then $E^2_{n,0}(n) = 0$, because $d^1_{n,0}(n) = \mathrm{id}_k$. So let n be even. We prove by induction on n that $E^2_{n,0}(n) = 0$. If $n = 2$, then

$$\theta := (\langle e_1 \rangle, \langle e_3 \rangle) - (\langle e_1 \rangle, \langle e_1 + e_3 \rangle) + (\langle e_3 \rangle, \langle e_1 + e_3 \rangle) \in H_1(X_2, k)$$

and so $d^1_{3,0}(2)(\theta \mod G_2) = 1 \in \mathbb{Z}$. Assume that this is true for $n-2$, that is $E^2_{n-2,0}(n-2) = 0$. From the map ${E'}^1_{p,q}(n) \to E^1_{p,q}(n)$ we get the commutative diagram

$$\begin{array}{ccccc}
H_0(G_{n-2}, H_{n-3}(X_{n-2}, k)) & \xrightarrow{d^1_{n-1,0}(n-2)} & k & \longrightarrow & 0 \\
\big\downarrow \alpha' & & \big\downarrow \mathrm{id}_k & & \\
H_0(G_n, H_{n-1}(X_n, k)) & \xrightarrow{d^1_{n+1,0}(n)} & k & \longrightarrow & 0,
\end{array}$$

where the map α' is induced by the map α_*. By induction and the commutativity of the above diagram we see that $d^1_{n+1,0}(n)$ is surjective and therefore $E^2_{n,0}(n) = 0$. This shows that ${E''}^2_{p,0}(n) = 0$, $0 \leq p \leq n$ and so the proof of the claim is complete. To complete the proof of the theorem we must prove the injectivity claimed in the theorem. This can be done by a similar argument as in the above with suitable changes and applying lemma 2.7. □

COROLLARY 2.9. *If $n - p \geq q$, then the complex*

$$H_q(R^{*p} \times G_{n-p}) \xrightarrow{\tau_p} H_q(R^{*p-1} \times G_{n-p+1}) \xrightarrow{\tau_{p-1}} \cdots$$
$$\xrightarrow{\tau_2} H_q(R^* \times G_{n-1}) \xrightarrow{\tau_1} H_q(G_n) \longrightarrow 0$$

is exact, where $\tau_i := d^1_{i,q}(n)$.

PROOF. This comes out of the proof of 2.8. □

THEOREM 2.10. *Let $n \geq 1$, $l \geq 0$ be integer numbers. Then $H_l(\mathrm{inc})$: $H_l(G_n, \mathbb{Z}) \to H_l(G_{n+1}, \mathbb{Z})$ is surjective for $n \geq l+1$ and is injective for $n \geq l+2$.*

PROOF. For $n \geq l+1$, theorem 2.8 implies $H_{l+1}(G_{n+1}, G_n) = 0$. Here $H_{l+1}(G_{n+1}, G_n)$ is the homology of the mapping cone of the map of complexes $F_*^{(n)} \to F_*^{(n+1)}$ with coefficients in k where $F_*^{(m)}$ is the G_m-resolution of k. Applying the homology long exact sequence to the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

we have the exact sequence

$$\cdots \to H_{l+1}(G_{n+1}, G_n, \mathbb{Q}/\mathbb{Z}) \to H_l(G_{n+1}, G_n, \mathbb{Z})$$
$$\to H_l(G_{n+1}, G_n, \mathbb{Q}) \to H_l(G_{n+1}, G_n, \mathbb{Q}/\mathbb{Z}) \to \cdots .$$

We must prove that $H_{l+1}(G_{n+1}, G_n, \mathbb{Q}/\mathbb{Z}) = 0$. Since $\mathbb{Q}/\mathbb{Z} = \oplus_p \varinjlim \mathbb{Z}/p^d\mathbb{Z}$ and since the homology functor commutes with the direct limit functor, it is sufficient to prove that $H_{l+1}(G_{n+1}, G_n, \mathbb{Z}/p^d\mathbb{Z}) = 0$. This can be deduced from writing the homology long exact sequence of the short exact sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^d\mathbb{Z} \to \mathbb{Z}/p^{d-1}\mathbb{Z} \to 0$$

and induction on d. Therefore $H_l(G_{n+1}, G_n, \mathbb{Z}) = 0$. The surjectivity, claimed in the theorem, follows from the long exact sequence

$$\cdots \to H_{l+1}(G_{n+1}, G_n, \mathbb{Z}) \to H_l(G_n, \mathbb{Z})$$
$$\to H_l(G_{n+1}, \mathbb{Z}) \to H_l(G_{n+1}, G_n, \mathbb{Z}) \to \cdots .$$

The proof of the other claim follows from a similar argument. □

REMARK 11. Theorem 2.10 gives almost a positive answer to a question asked by Sah in [**35**, 4.9]. Also it gives a better range of homology stability in comparison to other results [**27**], [**50**].

Let G be a topological group and let BG^{top} be the quotient space $\bigcup_n \Delta^n \times G^n/\sim$, where $\sim$ is the relation

$$(t_0, \dots, t_n, g_1, \dots, g_n) \sim$$
$$\begin{cases} (t_0, \dots, \widehat{t_i}, \dots, t_n, g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_n) & \text{if } t_i = 0 \\ (t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_n, g_1, \dots, \widehat{g_i}, \dots, g_n) & \text{if } g_i = e. \end{cases}$$

It is easy to see that B is a functor from the category of topological groups to the category of topological spaces. The topological space BG^{top} is called the *classifying space* of G with the underlying topology. Let BG be the classifying space of G as the topological group with the discrete topology. By the functorial property of B we have a natural map $\psi : BG \to BG^{\mathrm{top}}$.

CONJECTURE 2.11 (Friedlander-Milnor Conjecture). *Let G be a Lie group. The canonical map $\psi : BG \to BG^{\mathrm{top}}$ induces isomorphism of homology and cohomology with any finite abelian coefficient group.*

See [**23**] and [**35**] for more information in this direction.

THEOREM 2.12. *Let $F = \mathbb{R}$ or $\mathbb{C}$. If $G = O(F)$ or $Sp(F)$, then $H_i(BG, A) \simeq H_i(BG^{\mathrm{top}}, A)$ for all i and any finite coefficient group A.*

PROOF. See [**17**, Thm. 1, 2] □

COROLLARY 2.13. *Let $F = \mathbb{R}$ or $\mathbb{C}$. If $G_n = O_{2n}(F)$ or $Sp_{2n}(F)$, then $H_i(BG_n, A) \simeq H_i(BG_n^{\mathrm{top}}, A)$ for $n \geq i + 1$ and any finite coefficient group A.*

PROOF. This follows from 2.8 and 2.12. □

THEOREM 2.14. *Let F be an algebraically closed field with* $\mathrm{char}(F) = p \neq 2$. *Let $G = Sp(F)$ or $O(F)$. Then*

(i) $H_*(G, \mathbb{Z}/p) = \mathbb{Z}/p$.

(ii) *If q is a prime such that $p \neq q$ and $q \neq 2$, then $H_*(G, \mathbb{Z}/q) = \mathbb{Z}/q[c_1, c_2, \dots]$, where $c_i \in H_{4i}(G, \mathbb{Z}/q)$.*

(iii) *$H_*(Sp(F), \mathbb{Z}/2) = \mathbb{Z}/2[c_1, c_2, \dots]$, where $c_i \in H_{4i}(Sp(F), \mathbb{Z}/2)$ and $H_*(O(F), \mathbb{Z}/2) = \mathbb{Z}/2[d_1, d_2, \dots]$, where $d_i \in H_i(O(F), \mathbb{Z}/2)$.*

PROOF. See the corollary on p. 250 of [**16**]. □

COROLLARY 2.15. *Let F be an algebraically closed field with* $\mathrm{char}(F) = p \neq 2$. *Let $G_n = O_{2n}(F)$ or $Sp_{2n}(F)$. Let $n \geq i + 1$.*

(i) $H_i(G_n, \mathbb{Z}/p) = \begin{cases} \mathbb{Z}/p & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$.

(ii) *If $p \neq q$ and $q \neq 2$, then* $H_i(G_n, \mathbb{Z}/q) = \begin{cases} 0 & \text{if } i = 1,2,3 \mod 4 \\ V_i & \text{if } i = 0 \mod 4 \end{cases}$, *where V_i is a finite dimensional $\mathbb{Z}/q$-vector space.*

PROOF. This follows from 2.14 and theorem 2.8. □

3. The case of finite fields

In this section we will explain which part of the above results is true if R is a finite field, so in this section we assume that $R := F$ is a finite field.

LEMMA 3.1. *Let F be a field different from $\mathbb{F}_2$. Then $\underline{\mathcal{U}}(F^n)$ is $(n-2)$-connected, $\underline{\mathcal{U}}(F^n)_w$ is $(n - |w| - 2)$-connected for every $w \in \underline{\mathcal{U}}(F^n)$ and the poset $\underline{\mathcal{IU}}(F^{2n})$ is $(n - 2)$-connected.*

PROOF. The proof of the first two claims is by induction on n. Let $Z := \underline{\mathcal{U}}(F^n)$ and $Y := \mathcal{O}(\mathbb{P}^{n-2})$. For any $v = (\langle v_1\rangle, \dots, \langle v_k\rangle) \in Z\backslash Y$, there is an i, for example $i = 1$, such that $v_i \notin R^{n-1}$. This means that the n-th coordinate of v_1 is not zero. Choose $r_i \in F$ such that $v'_i = v_i - r_i v_1 \in F^{n-1}$, $2 \le i \le k$. It is not difficult to see that

$$\begin{aligned} Y \cap Z_v &\simeq Y \cap \underline{\mathcal{U}}(F^n)_{(\langle v_1\rangle, \langle v'_2\rangle, \dots, \langle v'_k\rangle)} \simeq Y \cap \underline{\mathcal{U}}(F^n)_{(\langle v'_2\rangle, \dots, \langle v'_k\rangle)} \\ &\simeq \underline{\mathcal{U}}(F^{n-1})_{(\langle v'_2\rangle, \dots, \langle v'_k\rangle)}. \end{aligned}$$

By induction $\underline{\mathcal{U}}(F^{n-1})_{(\langle v'_2\rangle, \dots, \langle v'_k\rangle)}$ is $((n-1)-(|v|-1)-2)$-connected, so $Y \cap Z_v$ is $((n-3)-|v|+1)$-connected. Since $Y \cap Z \subseteq Z_{(\langle e_n\rangle)}$, Z is $(n-2)$-connected [**46**, 2.13 (ii)]. To complete the proof we have to prove that $Z' := \underline{\mathcal{U}}(F^n)_w$ is $(n-|w|-2)$-connected. If $w \in Y$, then replacing Z by Z' in the above and using the induction assumption one sees that Z' is $(n-|w|-2)$-connected. If $w \notin Y$, then by induction $Y \cap Z'$ is $(n-|w|-2)$-connected and $Y \cap Z'_u$ is $(n-|w|-|u|-2)$-connected for every $u \in Z'\backslash Y$ as we proved in the above. Now by [**46**, 2.13 (i)] the poset Z' is $(n-|w|-2)$-connected. The proof of the last claim is similar to the proof given in Remark 7(ii)in the previous chapter. □

LEMMA 3.2. *Let* $\mathrm{char}(F) \neq \mathrm{char}(k)$. *Then we have the isomorphism* $H_i(F^{*p} \times G_{n-p}, k) \simeq H_i(\mathrm{Stab}_{G_n}(\overline{\sigma_p}), k)$ *for all* i.

PROOF. Let M be a finite dimensional F_1-vector space, where $F_1 := \{x \in F : \overline{x} = x\}$. From [**6**, Cor. 10.2, Chap. III] and the fact that for every group G, $H_i(G,k) \simeq \mathrm{Hom}(H^i(G,k),k)$, we deduce that $H_i(M,k) = \begin{cases} k & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$. By a proof similar to the proof of 2.4, one sees that $H_i(F^{*p} \times G_{n-p}, k) \simeq H_i(\mathrm{Stab}_{G_n}(\overline{\sigma_p}), k)$. □

Applying lemmas 3.1 and 3.2 one sees that theorem 2.5 is true if $F \neq \mathbb{F}_2$ and $\mathrm{char}(F) \neq \mathrm{char}(k)$. So we can apply the techniques that we developed in sections 3 and 4 to prove the following theorems.

THEOREM 3.3. *Let* F *be a finite field different from* $\mathbb{F}_2$ *and* $\mathrm{char}(F) \neq \mathrm{char}(k)$. *Then*

(i) *the map* $H_l(\mathrm{inc}) : H_l(G_n, k) \to H_l(G_{n+1}, k)$ *is surjective if* $n \ge l$ *and is injective if* $n \ge l+1$,

(ii) *if* $n - h \ge l$, *then the complex*

$$\begin{aligned} H_l(F^{*h} \times G_{n-h}, k) &\xrightarrow{\tau_h} H_l(F^{*h-1} \times G_{n-h+1}, k) \xrightarrow{\tau_{h-1}} \cdots \\ &\xrightarrow{\tau_2} H_l(F^* \times G_{n-1}, k) \xrightarrow{\tau_1} H_l(G_n, k) \longrightarrow 0 \end{aligned}$$

is exact, where $\tau_i := d^1_{i,l}(n)$.

THEOREM 3.4. *Let* $\mathrm{char}(F) = p$. *Then the map* $H_l(\mathrm{inc}) : H_l(G_n, \mathbb{Z}[\frac{1}{p}]) \to H_l(G_{n+1}, \mathbb{Z}[\frac{1}{p}])$ *is surjective if* $n \ge l+1$ *and is injective if* $n \ge l+2$.

REMARK 12. Let F be a finite field such that $\mathrm{char}(F) = \mathrm{char}(k)$.

(i) We don't know if a similar result as 3.3 is true or not. There is some information from previous results, it is true if $n \geq 2l+3$ [**27**, Thm. 8.2].

(ii) Theorem 2.4 is not true in this case because otherwise it will be true with every prime field k as a coefficient group and so it must be true with integral coefficients (see proof of the theorem 2.10). Hence $R^{*p} \times G_{n-p}$ must be isomorphic to the group $\mathrm{Stab}_{G_n}(\overline{\sigma_p})$ [**9**], which is not true.

CHAPTER 4

Homology of GL_3

The study of the groups $H_i(GL(F), \mathbb{Z})$ seems rather important, in particular, because of their close relation to algebraic K-theory. The main purpose of this chapter is the study of this relation for low degrees. The main theorem of this chapter asserts that $H_3(\mathrm{inc}) : H_3(GL_2(F), \mathbb{Z}[\frac{1}{2}]) \rightarrow H_3(GL_3(F), \mathbb{Z}[\frac{1}{2}])$ is injective for any F infinite field F. Applying this we will get some information about the indecomposable part of $K_3(F)$. In appendix B we give the necessary background for this chapter.

Here we establish some notation that we will use in this chapter. By $H_i(G)$ we mean the i-th integral homology of the group G. We use the bar resolution to define the homology of a group (see appendix A). Define $\mathbf{c}(g_1, g_2, \ldots, g_n) = \sum_{\sigma \in \Sigma_n} \mathrm{sign}(\sigma)[g_{\sigma(1)}|g_{\sigma(2)}|\ldots|g_{\sigma(n)}] \in H_n(G)$, where the elements $g_i \in G$ commute with each other and Σ_n is the symmetric group of degree n. By G_n we mean the general linear group $GL_n(F)$, where F is an infinite field. Note that G_0 is the trivial group and $G_1 = F^*$. The i-th factor of F^{*m} is denoted by F_i^*.

1. Spectral sequences

Let $C_l(F^n)$ and $D_l(F^n)$ be the free abelian groups with a basis consisting of $(\langle v_0 \rangle, \ldots, \langle v_l \rangle)$ and $(\langle w_0 \rangle, \ldots, \langle w_l \rangle)$ respectively, where every $\min\{l+1, n\}$ of $v_i \in F^n$ and every two of $w_i \in F^n$ are linearly independent. By $\langle v_i \rangle$ we mean the line passing through vectors v_i and 0. Let $\partial_0 : C_0(F^n) \rightarrow C_{-1}(F^n) := \mathbb{Z}$, $\sum_i n_i(\langle v_i \rangle) \mapsto \sum_i n_i$ and $\partial_l = \sum_{i=0}^{l} (-1)^i d_i : C_l(F^n) \rightarrow C_{l-1}(F^n)$, $l \geq 1$, where

$$d_i((\langle v_0 \rangle, \ldots, \langle v_l \rangle)) = (\langle v_0 \rangle, \ldots, \widehat{\langle v_i \rangle}, \ldots, \langle v_l \rangle).$$

Define the differential $\tilde{\partial}_l = \sum_{i=0}^{l} (-1)^i \tilde{d}_i : D_l(F^n) \rightarrow D_{l-1}(F^n)$ similar to ∂_l. Set $L_0 = \mathbb{Z}$, $M_0 = \mathbb{Z}$, $L_l = C_{l-1}(F^n)$ and $M_l = D_{l-1}(F^n)$, $l \geq 1$. It is easy to see that the complexes

$$\begin{array}{ll} L_* : & 0 \leftarrow L_0 \leftarrow L_1 \leftarrow \cdots \leftarrow L_l \leftarrow \cdots \\ M_* : & 0 \leftarrow M_0 \leftarrow M_1 \leftarrow \cdots \leftarrow M_l \leftarrow \cdots \end{array}$$

are exact. Take a G_n-resolution $P_* \rightarrow \mathbb{Z}$ of $\mathbb{Z}$ with trivial G_n-action. From the double complexes $L_* \otimes_{G_n} P_*$ and $M_* \otimes_{G_n} P_*$ we obtain two first quadrant spectral sequences converging to zero with

$$E^1_{p,q}(n)=\begin{cases} H_q(F^{*p}\times G_{n-p}) & \text{if } 0\le p\le n\\ H_q(G_n, C_{p-1}(F^n)) & \text{if } p\ge n+1\end{cases},$$

$$\tilde{E}^1_{p,q}(n)=\begin{cases} H_q(F^{*p}\times G_{n-p}) & \text{if } 0\le p\le 2\\ H_q(G_n, D_{p-1}(F^n)) & \text{if } p\ge 3\end{cases}.$$

For $1\le p\le n$, and $q\ge 0$ the differential $d^1_{p,q}(n)$ equals $\sum_{i=1}^{p}(-1)^{i+1}H_q(\alpha_{i,p})$, where $\alpha_{i,p}: F^{*p}\times G_{n-p}\to F^{*p-1}\times G_{n-p+1}$ with

$$(a_1,\dots,a_p,A)\mapsto(a_1,\dots,\widehat{a_i},\dots,a_p,\begin{pmatrix} a_i & 0\\ 0 & A\end{pmatrix}).$$

In particular for $0\le p\le n$, $d^1_{p,0}(n)=\begin{cases}\mathrm{id}_{\mathbb{Z}} & \text{if } p \text{ is odd}\\ 0 & \text{if } p \text{ is even}\end{cases}$, so $E^2_{p,0}(n)=0$ for $p\le n-1$. It is also easy to see that $E^2_{n,0}(n)=E^2_{n+1,0}(n)=0$. See the proof of 2.5 in chapter 3. for more details.

In this note we will use $\tilde{E}^i_{p,q}(n)$ only for $n=3$, so from now on by $\tilde{E}^i_{p,q}$ we mean $\tilde{E}^i_{p,q}(3)$. We describe $\tilde{E}^1_{p,q}$ for $p=3,4$. Let $w_1=(\langle e_1\rangle,\langle e_2\rangle,\langle e_3\rangle)$, $w_2=(\langle e_1\rangle,\langle e_2\rangle,\langle e_1+e_2\rangle)\in D_2(F^3)$ and $u_1,\dots,u_5,u_{6,a}\in D_3(F^3)$, $a\in F^*-\{1\}$, where

$$\begin{aligned} &u_1=(\langle e_1\rangle,\langle e_2\rangle,\langle e_3\rangle,\langle e_1+e_2+e_3\rangle), && u_2=(\langle e_1\rangle,\langle e_2\rangle,\langle e_3\rangle,\langle e_1+e_2\rangle),\\ &u_3=(\langle e_1\rangle,\langle e_2\rangle,\langle e_3\rangle,\langle e_2+e_3\rangle), && u_4=(\langle e_1\rangle,\langle e_2\rangle,\langle e_3\rangle,\langle e_1+e_3\rangle),\\ &u_5=(\langle e_1\rangle,\langle e_2\rangle,\langle e_1+e_2\rangle,\langle e_3\rangle), && u_{6,a}=(\langle e_1\rangle,\langle e_2\rangle,\langle e_1+e_2\rangle,\langle e_1+ae_2\rangle).\end{aligned}$$

By the Shapiro lemma $\tilde{E}^1_{3,q}=H_q(\mathrm{Stab}_{G_3}(w_1))\oplus H_q(\mathrm{Stab}_{G_3}(w_2))$ and $\tilde{E}^1_{4,q}=\bigoplus_{j=1}^5 H_q(\mathrm{Stab}_{G_3}(u_j))\oplus[\bigoplus_{a\in F^*-\{1\}}H_q(\mathrm{Stab}_{G_3}(u_{6,a}))]$. Applying the so called center killer lemma [**43**, Thm. 1.9], one gets

$$\begin{aligned}\tilde{E}^1_{3,q}&=H_q(F^{*3})\oplus H_q(F^*I_2\times F^*),\\ \tilde{E}^1_{4,q}&=H_q(F^*I_3)\oplus H_q(F^*I_2\times F^*)\oplus H_q(F^*\times F^*I_2)\oplus H_q(T)\\ &\quad\oplus H_q(F^*I_2\times F^*)\oplus[\bigoplus_{a\in F^*-\{1\}}H_q(F^*I_2\times F^*)],\end{aligned}$$

where $T=\{(a,b,a)\in F^3: a,b\in F^*\}$. Note that $\tilde{d}^1_{p,q}=d^1_{p,q}(3)$ for $p=1,2$, $\tilde{d}^1_{3,q}|_{H_q(F^{*3})}=d^1_{3,q}(3)$ and $\tilde{d}^1_{3,q}|_{H_q(F^*I_2\times F^*)}=H_q(\mathrm{inc})$, where $\mathrm{inc}: F^*I_2\times F^*\to F^{*3}$.

LEMMA 1.1. *The group $\tilde{E}^2_{p,0}$ is trivial for $0\le p\le 5$.*

PROOF. Triviality of $\tilde{E}^2_{p,0}$ is easy for $0\le p\le 2$. To prove the triviality of $\tilde{E}^2_{3,0}$, note that $\tilde{E}^1_{2,0}=\mathbb{Z}$, $\tilde{E}^1_{3,0}=\mathbb{Z}\oplus\mathbb{Z}$ and $\tilde{d}^1_{3,0}((n_1,n_2))=n_1+n_2$, so if $(n_1,n_2)\in\ker(\tilde{d}^1_{3,0})$, then $n_2=-n_1$. It is easy to see that this sits in $\mathrm{im}(\tilde{d}^1_{4,0})$. We prove the triviality of $\tilde{E}^2_{5,0}$. Triviality of $\tilde{E}^2_{4,0}$ is similar but much easier. This proof is just taken from [**13**, Sec. 1.3.3].

Triviality of $\tilde{E}^2_{5,0}$. The proof will be in four steps;

Step 1. The sequence $0 \to C_*(F^3) \otimes_{G_3} \mathbb{Z} \to D_*(F^3) \otimes_{G_3} \mathbb{Z} \to Q_*(F^3) \otimes_{G_3} \mathbb{Z} \to 0$ is exact, where $Q_*(F^3) := D_*(F^3)/C_*(F^3)$.
Step 2. The group $H_4(Q_*(F^3) \otimes_{G_3} \mathbb{Z})$ is trivial.
Step 3. The map induced in homology by $C_*(F^3) \otimes_{G_3} \mathbb{Z} \to D_*(F^3) \otimes_{G_3} \mathbb{Z}$ is zero in degree 4.
Step 4. The group $\tilde{E}^2_{5,0}$ is trivial.
Proof of step 1. For $i \geq -1$, $D_i(F^3) \simeq C_i(F^3) \oplus Q_i(F^3)$. This decomposition is compatible with the action of G_3, so we get an exact sequence of $\mathbb{Z}[G_3]$-modules

$$0 \to C_i(F^3) \to D_i(F^3) \to Q_i(F^3) \to 0$$

which splits as a sequence of $\mathbb{Z}[G_3]$-modules. One can easily deduce the desired exact sequence from this. Note that this exact sequence does not split as complexes.
Proof of step 2. The complex $Q_*(F^3)$ induces a spectral sequence

$$\hat{E}^1_{p,q} = \begin{cases} 0 & \text{if } 0 \leq p \leq 2 \\ H_q(G_3, Q_{p-1}(F^3)) & \text{if } p \geq 3 \end{cases}$$

which converges to zero. To prove the claim it is sufficient to prove that $\hat{E}^2_{5,0} = 0$ and to prove this it is sufficient to prove that $\hat{E}^2_{3,1} = 0$. One can see that $\hat{E}^1_{3,1} = H_1(F^*I_2 \times F^*)$. If $w = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1 + e_2\rangle) \in Q_3(F^3)$, then $H_1(\mathrm{Stab}_{G_3}(w)) \simeq H_1(F^*I_2 \times F^*)$ is a summand of $\hat{E}^1_{4,1}$ and $\hat{d}^1_{4,1} : H_1(\mathrm{Stab}_{G_3}(w)) \to \hat{E}^1_{3,1}$ is an isomorphism. So $\hat{d}^1_{4,1}$ is surjective and therefore $\hat{E}^2_{3,1} = 0$.
Proof of step 3. Consider the following commutative diagram

$$\begin{array}{ccccc} C_5(F^3) \otimes_{G_3} \mathbb{Z} & \to & C_4(F^3) \otimes_{G_3} \mathbb{Z} & \to & C_3(F^3) \otimes_{G_3} \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ D_5(F^3) \otimes_{G_3} \mathbb{Z} & \to & D_4(F^3) \otimes_{G_3} \mathbb{Z} & \to & D_3(F^3) \otimes_{G_3} \mathbb{Z}. \end{array}$$

The generators of $C_4(F^3) \otimes_{G_3} \mathbb{Z}$ are of the form $x_{a,b} \otimes 1$, where $x_{a,b} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_1 + ae_2 + be_3\rangle, \langle e_3\rangle, \langle e_1 + e_2 + e_3\rangle)$, $a, b \in F^* - \{1\}$ and $a \neq b$. Since $C_3(F^3) \otimes_{G_3} \mathbb{Z} = \mathbb{Z}$, $(x_{a,b} - x_{c,d}) \otimes 1 \in \ker(\partial_4 \otimes 1)$ and the elements of this form generate $\ker(\partial_4 \otimes 1)$. Hence to prove this step it is sufficient to prove that $(x_{a,b} - x_{c,d}) \otimes 1 \in \mathrm{im}(\tilde{\partial}_5 \otimes 1)$. Set $w_{a,b} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_1 + ae_2 + be_3\rangle, \langle e_3\rangle, \langle e_1 + e_2\rangle, \langle e_1 + ae_2\rangle) \in D_5(F^3)$, where $a, b \in F^* - \{1\}$ and $a \neq b$. Let g, g', and g'' be the matrices

$$\begin{pmatrix} 0 & a^{-1} & 0 \\ -1 & 1 + a^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & -b^{-1} \\ 0 & 1 & -ab^{-1} \\ 0 & 0 & b^{-1} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & b^{-1} \end{pmatrix},$$

respectively, then

$$g(\tilde{d}_1(w_{a,b})) = \tilde{d}_0(w_{a,b}), \ g'(\tilde{d}_3(w_{a,b})) = \tilde{d}_2(w_{a,b}), \ g''(\tilde{d}_4(w_{a,b})) = v_{1,1}$$

and so $(\tilde{\partial}_5 \otimes 1)(w_{a,b} \otimes 1) = (v_{1,1} - v_{a,b}) \otimes 1$, where

$$v_{a,b} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_1 + ae_2 + be_3\rangle, \langle e_3\rangle, \langle e_1 + e_2\rangle).$$

Note that the elements of the form $(gw-w)\otimes 1$ are zero in $D_*\otimes_{G_3}\mathbb{Z}$. If

$$\begin{aligned} u_{a,b} &= (\langle e_3\rangle, \langle e_1+ae_2+be_3\rangle, \langle e_1\rangle, \langle e_1+e_2\rangle, \langle e_1+ae_2\rangle),\\ u'_{a,b} &= (\langle e_1+ae_2+be_3\rangle, \langle e_1\rangle, \langle e_2\rangle, \langle e_1+e_2\rangle, \langle e_1+ae_2\rangle), \end{aligned}$$

where $a,b\in F^*-\{1\}$, then

$$\begin{aligned} gu_{a,b} &= (\langle e_3\rangle, \langle e_1+ae_2+be_3\rangle, \langle e_2\rangle, \langle e_1+e_2\rangle, \langle e_1+ae_2\rangle),\\ g'u'_{a,b} &= (\langle e_3\rangle, \langle e_1\rangle, \langle e_2\rangle, \langle e_1+e_2\rangle, \langle e_1+ae_2\rangle). \end{aligned}$$

So if $a,b,c,d\in F^*-\{1\}$, $a\neq b$, $c\neq d$, then $(\tilde{\partial}_5\otimes 1)((z_{a,b}-z_{c,d})\otimes 1)=(t_{c,d}-t_{a,b})\otimes 1$, where

$$\begin{aligned} z_{a,b} &= (\langle e_3\rangle, \langle e_1+ae_2+be_3\rangle, \langle e_1\rangle, \langle e_2\rangle, \langle e_1+e_2\rangle, \langle e_1+ae_2\rangle),\\ t_{a,b} &= (\langle e_3\rangle, \langle e_1+ae_2+be_3\rangle, \langle e_1\rangle, \langle e_2\rangle, \langle e_1+e_2\rangle). \end{aligned}$$

If g_1, g_2 and g_3 are the matrices

$$\begin{pmatrix} -1 & 0 & 1\\ -1 & 0 & 0\\ -1 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -1 & 1\\ 0 & -1 & 0\\ 1 & -1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & -1 \end{pmatrix},$$

respectively, then we have $g_1(\tilde{d}_0(y_{a,b}))=t_{\frac{1}{1-b},\frac{1-a}{1-b}}$, $g_2(\tilde{d}_1(y_{a,b}))=t_{\frac{-a}{b-a},\frac{1-a}{b-a}}$ and $g_3(\tilde{d}_3(y_{a,b}))=v_{\frac{a-b}{1-b},\frac{-b}{1-b}}$, where

$$y_{a,b}=(\langle e_1\rangle, \langle e_2\rangle, \langle e_1+ae_2+be_3\rangle, \langle e_3\rangle, \langle e_1+e_2+e_3\rangle, \langle e_1+e_2\rangle).$$

By an easy computation

$$\begin{aligned} (\tilde{\partial}_5\otimes 1)(y_{a,b}\otimes 1) = {} & t_{\frac{1}{1-b},\frac{1-a}{1-b}}\otimes 1 - t_{\frac{-a}{b-a},\frac{1-a}{b-a}}\otimes 1 + v_{1,1}\otimes 1\\ & - v_{\frac{a-b}{1-b},\frac{-b}{1-b}}\otimes 1 + v_{a,b}\otimes 1 - x_{a,b}\otimes 1. \end{aligned}$$

Now it is easy to see that $(x_{a,b}-x_{c,d})\otimes 1\in(\tilde{\partial}_5\otimes 1)(D_5(F^3)\otimes_{G_3}\mathbb{Z})$. This completes the proof of step 3.

Proof of Step 4. Applying the homology long exact sequence to the short exact sequence obtained in the first step, we get the exact sequence

$$H_4(C_*(F^3)\otimes_{G_3}\mathbb{Z})\to H_4(D_*(F^3)\otimes_{G_3}\mathbb{Z})\to H_4(Q_*(F^3)\otimes_{G_3}\mathbb{Z}).$$

By steps 2 and 3, $H_4(D_*(F^3)\otimes_{G_3}\mathbb{Z})=0$, but $\tilde{E}^2_{5,0}=H_4(D_*(F^3)\otimes_{G_3}\mathbb{Z})$. This completes the proof of the triviality of $\tilde{E}^2_{5,0}$. □

LEMMA 1.2. *The group* $\tilde{E}^2_{p,1}$ *is trivial for* $0\leq p\leq 4$.

PROOF. Triviality of $\tilde{E}^2_{p,1}$, $p=0,1$, is a result of lemma 1.1 and the fact that the spectral sequence converges to zero (one can also prove this directly). If $(a,b,c)\in\ker(\tilde{d}^1_{2,1})$, $a,b,c\in H_1(F^*)$, then $a=b$. It is easy to see that this element sits in $\mathrm{im}(\tilde{d}^1_{3,1})$. Let $x=(x_1,\dots,x_5,(x_{6,a}))\in\tilde{E}^1_{4,1}$, where $x_2=(a_2,a_2,b_2)$,

$x_3 = (a_3, b_3, b_3)$, $x_4 = (a_4, b_4, a_4)$, $x_5 = (a_5, a_5, b_5)$, $a_i, b_i \in H_1(F^*)$. By a direct calculation $\tilde{d}_{4,1}(x) = (z_1, z_2)$, where

$$z_1 = -(a_2, a_2, b_2) - (a_3, b_3, b_3) + (b_4, a_4, a_4) + (a_5, a_5, b_5),$$
$$z_2 = (a_2, a_2, b_2) + (b_3, b_3, a_3) - (a_4, a_4, b_4) - (a_5, a_5, b_5).$$

If $y = ((a, b, c), (d, d, e)) \in \ker(\tilde{d}^1_{3,1})$, $a, b, c, d, e \in H_1(F^*)$, then $b+d = a-b+c+e = 0$. Let $x_2 = (-b, -b, -c)$, $x_3 = (-a+b, 0, 0)$ and set $x' = (0, x_2, x_3, 0, 0, 0) \in \tilde{E}^1_{4,1}$, then $y = \tilde{d}_{4,1}(x')$.

To prove the triviality of the group $\tilde{E}^2_{4,1}$; let $x \in \ker(\tilde{d}_{4,1})$ and set $w_1 = (\langle e_1\rangle, \langle e_2\rangle, \langle e_1+e_2\rangle, \langle e_3\rangle, \langle e_1+ae_2\rangle)$, $w_2 = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1+e_3\rangle, \langle e_1+be_3\rangle)$, $w_3 = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1+e_2+e_3\rangle, \langle e_2+e_3\rangle)$, $w_{4,a} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1+e_2\rangle, \langle e_1+ae_2\rangle)$, $w_5 = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1+e_2+e_3\rangle, \langle e_1+ae_2+be_3\rangle)$, $a, b \in F^* - \{1\}$ and $a \neq b$. The groups $T_i = H_1(\mathrm{Stab}_{G_3}(w_i))$, $i = 1, 2, 3, 5$ and $T_4 = \bigoplus_{a\in F^*-\{1\}} H_1(\mathrm{Stab}_{G_3}(w_{4,a}))$ are summands of $\tilde{E}^1_{5,1}$. Note that $T_1 = H_1(F^*I_2 \times F^*)$, $T_2 = H_1(T)$, $T_3 = T_5 = H_1(F^*I_3)$ and $T_4 = \bigoplus_{a\in F^*-\{1\}} H_1(F^*I_2 \times F^*)$. The restriction of $\tilde{d}^1_{5,1}$ on these summands is as follow;

$$\tilde{d}^1_{5,1}|_{T_1}((c_1, c_1, d_1)) = (0, (c_1, c_1, d_1), 0, 0, (c_1, c_1, d_1), -(c_1, c_1, d_1)),$$
$$\tilde{d}^1_{5,1}|_{T_2}((c_2, d_2, c_2)) = (0, 0, (d_2, c_2, c_2), (c_2, d_2, c_2), 0, -(c_2, c_2, d_2)),$$
$$\tilde{d}^1_{5,1}|_{T_3}((c_3, c_3, c_3) = ((c_3, c_3, c_3), (c_3, c_3, c_3), -(c_3, c_3, c_3), 0, 0, 0),$$
$$\tilde{d}^1_{5,1}|_{T_{4,a}}((c_4, c_4, d_4)) = (0, 0, 0, 0, 0, (c_4, c_4, d_4)),$$
$$\tilde{d}^1_{5,1}|_{T_5} = \mathrm{id}_{H_1(F^*I_3)}.$$

Let $z_1 = (a_5, a_5, b_5) \in T_1$ and $z_2 = (a_4, b_4, a_4) \in T_2$, then $x - \tilde{d}^1_{5,1}(z_1 + z_2) = (x'_1, x'_2, x'_3, 0, 0, (x'_{6,a}))$, so we can assume that $x_4 = x_5 = 0$. An easy calculation shows that $a_2 = b_2 = -a_3 = -b_3$. If $z_3 = (a_2, a_2, a_2) \in T_3$, then $x - \tilde{d}_{5,1}(z_3) = (x'_1, 0, 0, 0, 0, (x'_{6,a}))$. Again we can assume that $x_2 = x_3 = 0$. If $z_4 = (x_{6,a}) \in T_4$, then $x - \tilde{d}^1_{5,1}(z_4) = (x'_1, 0, 0, 0, 0, 0)$. Once more we can assume that $x_{6,a} = 0$. These reduce x to an element of the form $(x_1, 0, 0, 0, 0, 0)$. If $x_1 \in T_5$, then $\tilde{d}^1_{5,1}(x_1) = (x_1, 0, 0, 0, 0, 0)$. This completes the triviality of $\tilde{E}^2_{4,1}$. □

LEMMA 1.3. *The group $\tilde{E}^2_{p,2}$ is trivial for $0 \leq p \leq 3$.*

PROOF. Triviality of $\tilde{E}^2_{0,2}$ and $\tilde{E}^2_{1,2}$ is a result of lemmas 1.1, 1.2 and the fact that the spectral sequence converges to zero. Let $\tilde{E}^1_{1,2} = H_2(F^* \times G_2) = H_2(F^*) \oplus H_2(G_2) \oplus H_1(F^*) \otimes H_1(G_2)$, $\tilde{E}^1_{2,2} = H_2(F^{*3}) = \bigoplus_{i=1}^6 T_i$ and $\tilde{E}^1_{3,2} = H_2(F^{*3}) \oplus H_2(F^*I_2 \times F^*) = \bigoplus_{i=1}^9 T_i$, where $T_i = H_2(F^*_i)$ for $i = 1, 2, 3$, $T_4 = H_1(F^*_1) \otimes H_1(F^*_2)$, $T_5 = H_1(F^*_1) \otimes H_1(F^*_3)$, $T_6 = H_1(F^*_2) \otimes H_1(F^*_3)$, $T_7 = H_2(F^*I_2)$, $T_8 = H_2(I_2 \times F^*)$ and $T_9 = H_1(F^*I_2) \otimes H_1(I_2 \times F^*)$. If $y = (y_1, y_2, y_3, \sum r \otimes s, \sum t \otimes u, \sum v \otimes w) \in \tilde{E}^1_{2,2}$ and $x = (x_1, x_2, x_3, \sum a \otimes b, \sum c \otimes d, \sum e \otimes f, x_7, x_8, \sum g \otimes h) \in \tilde{E}^1_{3,2}$, $a, b, c, \dots, h \in H_1(F^*)$, then $\tilde{d}^1_{2,2}(y) = (h_1, h_2, h_3)$, where $h_1 = -y_1 + y_2$, $h_3 = -\sum s \otimes \mathrm{diag}(1, r) - \sum r \otimes \mathrm{diag}(1, s) - \sum t \otimes \mathrm{diag}(1, u) + \sum v \otimes \mathrm{diag}(1, w)$ and $\tilde{d}^1_{3,2}(x) = (z_i)_{1\leq i\leq 6}$, where

$z_1 = z_2 = x_2 + x_7$, $z_3 = x_1 + x_3 - x_2 + x_8$, $z_4 = \sum a \otimes b - \sum c \otimes d + \sum e \otimes f$, $z_5 = -\sum b\otimes a - \sum a\otimes b + \sum c\otimes d + \sum g\otimes h$, $z_6 = -\sum d\otimes c + \sum f\otimes e + \sum e\otimes f + \sum g\otimes h$.

If $y \in \ker(\tilde{d}^1_{2,2})$, then $y_1 = y_2$ and $h_3 = 0$. By $H_1(F^*) \otimes H_1(G_1) \simeq H_1(F^*) \otimes H_1(G_2)$ and $h_3 = 0$ we have $-\sum s \otimes r - \sum r \otimes s - \sum t \otimes u + \sum v \otimes w = 0$. If $z = (y_1, y_1, y_3, 0, \sum t \otimes u, \sum r \otimes s + \sum t \otimes u, 0, 0, 0) \in \tilde{E}^1_{3,2}$, then $y = \tilde{d}^1_{3,2}(z)$ and therefore $\tilde{E}^2_{2,2} = 0$.

Let $\tilde{d}^1_{3,2}(x) = 0$. Consider the summands $S_2 = H_2(\mathrm{Stab}_{G_3}(u_2)) = H_2(F^* I_2 \times F^*)$ and $S_3 = H_2(\mathrm{Stab}_{G_3}(u_3)) = H_2(F^* \times F^* I_2)$ of $\tilde{E}^1_{4,2}$. Then $S_i \simeq H_2(F^*) \oplus H_2(F^*) \oplus H_1(F^*) \otimes H_1(F^*)$ and by a direct calculation

$$\tilde{d}^1_{4,2}|_{S_2}((y_1, y_2, s \otimes t)) = (-y_1, -y_1, -y_2, 0, -s \otimes t, -s \otimes t, y_1, y_2, s \otimes t),$$
$$\tilde{d}^1_{4,2}|_{S_3}((u_1, u_2, p \otimes q)) = (-u_1, -u_2, -u_2, -p \otimes q, -p \otimes q, 0, u_2, u_1, -q \otimes p).$$

Choose $z'_2 = (-x_2, -x_3, -\sum e \otimes f) \in S_2$ and $z'_3 = (x_3 + x_8, 0, -\sum a \otimes b) \in S_3$. Then $x = \tilde{d}^1_{4,2}(z'_2 + z'_3)$ and therefore $\tilde{E}^2_{3,2} = 0$. $\square$

LEMMA 1.4. *The groups $\tilde{E}^2_{0,3}$, $\tilde{E}^2_{1,3}$ and $\tilde{E}^3_{0,4}$ are trivial.*

PROOF. These follow from 1.1, 1.2 and 1.3 and the fact that the spectral sequence converges to zero. $\square$

PROPOSITION 1.5. (i) *The complex*

$$H_2(F^{*3} \times G_0) \xrightarrow{d^1_{3,2}(3)} H_2(F^{*2} \times G_1) \xrightarrow{d^1_{2,2}(3)} H_2(F^* \times G_2) \xrightarrow{d^1_{1,2}(3)} H_2(G_3) \to 0$$

is exact, where $d^1_{3,2}(3) = H_2(\alpha_{1,3}) - H_2(\alpha_{2,3}) + H_2(\alpha_{3,3})$, $d^1_{2,2}(3) = H_2(\alpha_{1,2}) - H_2(\alpha_{2,2})$ and $d^1_{1,2}(3) = H_2(\mathrm{inc})$.

(ii) *The complex*

$$H_3(F^{*2} \times G_1) \xrightarrow{d^1_{2,3}(3)} H_3(F^* \times G_2) \xrightarrow{d^1_{1,3}(3)} H_3(G_3) \to 0$$

is exact, where $d^1_{2,3}(3) = H_3(\alpha_{1,2}) - H_3(\alpha_{2,2})$ and $d^1_{1,3}(3) = H_3(\mathrm{inc})$.

(iii) *(stability) The map $H_2(\mathrm{inc}) : H_2(G_2) \to H_2(G_3)$ is an isomorphism and the map $H_3(\mathrm{inc}) : H_3(F^* \times G_2) \to H_3(G_3)$ is surjective.*

PROOF. The only case that remains to prove is that $H_2(G_2) \to H_2(G_3)$ is an isomorphism. The proof is similar to the proof of lemma 2.7 in chapter 3 using (i). $\square$

REMARK 13. (i) By a similar approach as the above proposition one can prove that $H_2(G_n) \to H_2(G_{n+1})$ is an isomorphism for $n \geq 3$. For this one should work with $E^1_{p,q}(n)$, $n \geq 3$. This combined with 1.5 will prove the homology stability for the functor H_2: The map $H_2(G_2) \to H_2(G)$ is an isomorphism, where G is the stable group.

(ii) A similar result as 1.5(ii) is not true for $n = 2$, that is the complex

$$H_2(F^{*2} \times G_0) \xrightarrow{d^1_{2,2}(2)} H_2(F^* \times G_1) \xrightarrow{d^1_{1,2}(2)} H_2(G_2) \to 0$$

is not exact. In fact

$$\ker(d^1_{1,2}(2))/\mathrm{im}(d^1_{2,2}(2)) \simeq \langle x \wedge (x-1) - x \otimes (x-1) : x \in F^* \rangle$$

is a subset of $H_2(F^*) \oplus (F^* \otimes F^*)_\sigma$, where $(F^* \otimes F^*)_\sigma = (F^* \otimes F^*)/\langle a \otimes b + b \otimes a : a, b \in F^* \rangle$. To prove this let $Q(F)$ be the free abelian group with the basis $\{[x] : x \in F^* - \{1\}\}$. Denote by $\mathfrak{p}(F)$ the factor group of $Q(F)$ by the subgroup generated by the elements of the form $[x]-[y]+[y/x]-[(1-x^{-1})/(1-y^{-1})]+[(1-x)/(1-y)]$. The homomorphism $\psi : Q(F) \to F^* \otimes F^*$, $[x] \mapsto x \otimes (x-1)$ induces a homomorphism $\mathfrak{p}(F) \to (F^* \otimes F^*)_\sigma$, [**44**, 1.1]. By [**44**, 2.2], $E^2_{4,0}(2) \simeq \mathfrak{p}(F)$ and $E^2_{p,q}(2)$ has the following form

$$\begin{array}{cccccc}
* & * & & & & \\
0 & E^2_{1,2}(2) & * & & & \\
0 & 0 & 0 & 0 & * & \\
0 & 0 & 0 & 0 & \mathfrak{p}(F) & * \quad .
\end{array}$$

An easy calculation shows that $E^2_{1,2}(2) \subseteq H_2(F^*) \oplus (F^* \otimes F^*)_\sigma$. By [**44**, 2.4] $d^3_{4,0}(2) : E^3_{4,0}(2) \to E^3_{1,2}(2) \simeq E^2_{1,2}(2)$ is defined by $d^3_{4,0}(2)([x]) = x \wedge (x-1) - x \otimes (x-1)$. Because the spectral sequence converges to zero we see that $d^3_{4,0}(2)$ is surjective and so $E^2_{1,2}(2)$ is generated by the elements of the form $x \wedge (x-1) - x \otimes (x-1) \in H_2(F^*) \oplus (F^* \otimes F^*)_\sigma$.

Following [**53**, Section 3] we define;

Definition 1.6. We call $\wp^n(F)_{\mathrm{cl}} := H(C_{n+2}(F^n)_{G_n} \to C_{n+1}(F^n)_{G_n} \to C_n(F^n)_{G_n})$ the n-th *classical Bloch group.*

It is well known that $\wp^2(F)_{\mathrm{cl}} \simeq \mathfrak{p}(F)$ [**44**, 1.1], where $\mathfrak{p}(F)$ is defined in remark 13.

Proposition 1.7. *We have an isomorphism $\wp^3(F)_{\mathrm{cl}} \simeq F^*$. In particular if F is algebraically closed, then $\wp^3(F)_{\mathrm{cl}}$ is divisible.*

Proof. Using 1.5 one sees that $E^2_{p,q}(3)$ is of the form

$$\begin{array}{ccccccc}
* & * & & & & & \\
0 & 0 & * & * & 0 & * & \\
0 & 0 & 0 & * & 0 & * & \\
0 & 0 & 0 & F^* & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & \wp^3(F)_{\mathrm{cl}} & * \quad .
\end{array}$$

From this we obtain the exact sequence

$$0 \to E^3_{5,0} \to \wp^3(F)_{\mathrm{cl}} \xrightarrow{d^2_{5,0}} F^* \to 0.$$

Comparing $E^3_{0,4}(3)$ with $\tilde{E}^3_{0,4}(3)$ and applying lemma 1.4 one sees that $E^3_{0,4}(3) = 0$. Now it is easy to see that $E^3_{5,0}(3) = 0$. This proves the first part of the proposition. The second part follows from the fact that for an algebraically closed field F, F^* is divisible. $\square$

REMARK 14. From 1.7 and the existence of a surjective map $\wp^3(F)_{\rm cl} \to \wp^3(F)$ [**53**, Prop. 3.11] we deduce that $\wp^3(F)$ is divisible. See [**53**, 2.7] for the definition of $\wp^3(F)$. This gives a positive answer to conjecture 0.2 in [**53**] for $n = 3$.

2. Künneth theorem for third homology group

The Künneth theorem claims that the group $H_n(F^* \times F^*)$, $n \geq 1$, sits in the following exact sequence

$$0 \to \bigoplus_{i+j=n} H_i(F^*) \otimes H_j(F^*) \to H_n(F^* \times F^*) \to \bigoplus_{i+j=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_i(F^*), H_j(F^*)) \to 0$$

which splits. In this section we will see that if $n \leq 3$, then it splits canonically. This is clear for $n = 1$. For $n = 2$ it follows from the fact that $\mathrm{Tor}_1^{\mathbb{Z}}(H_i(F^*), H_j(F^*)) = 0$ if $(i, j) = (1, 0), (0, 1)$.

So let $n = 3$. If μ_F is the group of roots of unity of F, then $\mu_F = \varinjlim \mu_{n,F}$, where $\mu_{n,F}$ is the group of n-th roots of unity. By what we know about the homology of finite cyclic groups we obtain $H_2(\mu_F) = 0$. Hence the Künneth theorem for $H_3(\mu_F \times \mu_F)$ finds the following form

$$0 \to H_3(\mu_F) \oplus H_3(\mu_F) \to H_3(\mu_F \times \mu_F) \to \mathrm{Tor}_1^{\mathbb{Z}}(\mu_F, \mu_F) \to 0.$$

Clearly $H_3(\mu_F) \oplus H_3(\mu_F) \xrightarrow{\alpha} H_3(\mu_F \times \mu_F)$ is defined by $H_3(i_1) + H_3(i_2)$, where $i_k : \mu_F \to \mu_F \times \mu_F$ is the usual injection, $k = 1, 2$. Define the map $\beta : H_3(p_1) \oplus H_3(p_2) : H_3(\mu_F \times \mu_F) \to H_3(\mu_F) \oplus H_3(\mu_F)$, where $p_k : \mu_F \times \mu_F \to \mu_F$ is the usual projection, $k = 1, 2$. From $\alpha \circ \beta = \mathrm{id}$ one deduce that the above exact sequence splits canonically. Thus we have the canonical decomposition

$$H_3(\mu_F \times \mu_F) = H_3(\mu_F) \oplus H_3(\mu_F) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\mu_F, \mu_F).$$

We construct the splitting map $\mathrm{Tor}_1^{\mathbb{Z}}(\mu_F, \mu_F) \to H_3(\mu_F \times \mu_F)$. The elements of $\mathrm{Tor}_1^{\mathbb{Z}}(\mu_F, \mu_F) = \mathrm{Tor}_1^{\mathbb{Z}}(H_1(\mu_F), H_1(\mu_F))$ are of the form $\langle \xi, n, \xi \rangle = \langle [\xi], n, [\xi] \rangle$ for some n, where ξ is an n-th root of unity in F [**21**, Chap. V, Section 6]. It is easy to see that $\partial_2(\sum_{i=1}^n [\xi|\xi^i]) = n[\xi]$ in $B_{1_{\mu_F}}$. See [**6**, Chap. I, section 5] for the definition of ∂_2 and B_*. By [**21**, Chap. V, Prop. 10.6] the map $\phi : \mathrm{Tor}_1^{\mathbb{Z}}(H_1(\mu_F), H_1(\mu_F)) \to H_3(B_{*_{\mu_F}} \otimes B_{*_{\mu_F}})$ can be defined by

$$a := \langle [\xi], n, [\xi] \rangle \mapsto [\xi] \otimes \sum_{i=1}^n [\xi|\xi^i] + \sum_{i=1}^n [\xi|\xi^i] \otimes [\xi].$$

Considering the isomorphism $B_{*_{\mu_F}} \otimes B_{*_{\mu_F}} \simeq B_{*(\mu_F \times \mu_F)}$ we have $\phi(a) = \chi(\xi)) \in H_3(\mu_F \times \mu_F)$, where

$$\chi(\xi) := \sum_{i=1}^n \big([(\xi, 1)|(1, \xi)|(1, \xi^i)] - [(1, \xi)|(\xi, 1)|(1, \xi^i)] + [(1, \xi)|(1, \xi^i)|(\xi, 1)] + [(\xi, 1)|(\xi^i, 1)|(1, \xi)] - [(\xi, 1)|(1, \xi)|(\xi^i, 1)] + [(1, \xi)|(\xi, 1)|(\xi^i, 1)]\big).$$

Consider the following commutative diagram

$$\begin{array}{ccccccc} 0 \to & H_3(\mu_F) \oplus H_3(\mu_F) & \to & H_3(\mu_F \times \mu_F) & \to & \mathrm{Tor}_1^{\mathbb{Z}}(\mu_F, \mu_F) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \bigoplus_{i+j=3} H_i(F^*) \otimes H_j(F^*) & \to & H_3(F^* \times F^*) & \to & \mathrm{Tor}_1^{\mathbb{Z}}(F^*, F^*) & \to 0. \end{array}$$

Since $\mathrm{Tor}_1^{\mathbb{Z}}(\mu_F, \mu_F) \simeq \mathrm{Tor}_1^{\mathbb{Z}}(F^*, F^*)$, we see that the second horizontal exact sequence in the above diagram splits canonically.

Proposition 2.1. *We have the canonical decomposition*

$$H_3(F^* \times F^*) = \bigoplus_{i+j=3} H_i(F^*) \otimes H_j(F^*) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(F^*, F^*),$$

where the splitting map $\phi : \mathrm{Tor}_1^{\mathbb{Z}}(F^*, F^*) = \mathrm{Tor}_1^{\mathbb{Z}}(\mu_F, \mu_F) \to H_3(F^* \times F^*)$ *is defined by* $\langle [\xi], n, [\xi] \rangle \mapsto \chi(\xi)$.

3. Injectivity theorem

Let $A := \mathbb{Z}[\frac{1}{2}]$ and let $P_* \to A$ be a $A[G_3]$-resolution of A with trivial G_3-action. Consider the complex

$$D'_* : \quad 0 \leftarrow D'_0(F^3) \leftarrow D'_1(F^3) \leftarrow \cdots \leftarrow D'_l(F^3) \leftarrow \cdots,$$

where $D'_i(F^3) := D_i(F^3) \otimes A$. The double complex $D'_* \otimes_{G_3} P_*$ induces a first quadrant spectral sequence $\tilde{\mathcal{E}}^1_{p,q} \Rightarrow H_{p+q}(G_3, A)$, where $\tilde{\mathcal{E}}^1_{p,q} = \tilde{E}^1_{p+1,q}(3) \otimes A$ and $\tilde{\mathfrak{d}}^1_{p,q} = \tilde{d}^1_{p+1,q} \otimes \mathrm{id}_A$.

Lemma 3.1. *The groups* $\tilde{\mathcal{E}}^2_{3,0}$, $\tilde{\mathcal{E}}^2_{4,0}$, $\tilde{\mathcal{E}}^2_{2,1}$, $\tilde{\mathcal{E}}^2_{3,1}$, $\tilde{\mathcal{E}}^2_{1,2}$ *and* $\tilde{\mathcal{E}}^2_{2,2}$ *are trivial.*

Proof. This follows from the above spectral sequence and lemmas 1.1, 1.2, 1.3. □

Theorem 3.2. *The map* $H_3(\mathrm{inc}) : H_3(G_2, \mathbb{Z}[\frac{1}{2}]) \to H_3(G_3, \mathbb{Z}[\frac{1}{2}])$ *is injective.*

Proof. By lemma 3.1, $\tilde{\mathcal{E}}^2_{0,3} \simeq \tilde{\mathcal{E}}^\infty_{0,3} \simeq H_3(G_3, A)$, so to prove the theorem it is sufficient to prove that $H_3(G_2, A)$ is a summand of $\tilde{\mathcal{E}}^2_{0,3}$. To prove this it is sufficient to define a map $\varphi : H_3(F^* \times G_2, A) \to H_3(G_2, A)$ such that $\varphi|_{H_3(G_2,A)}$ is the identity map and $\tilde{\mathfrak{d}}^1_{1,3}(H_3(F^{*2} \times G_1, A)) \subseteq \ker(\varphi)$.

By a similar argument as in the previous section we have the canonical decompositions $H_3(F^* \times G_2, A) = \bigoplus_{i=0}^4 S_i$, where $S_i = H_i(F^*, A) \otimes H_{3-i}(G_2, A)$ for $0 \le i \le 3$ and $S_4 = \mathrm{Tor}_1^A(H_1(F^*, A), H_1(G_2, A))$. Note that the splitting map is

$$S_4 \simeq \mathrm{Tor}_1^{\mathbb{Z}}(\mu_F, \mu_F) \otimes A \xrightarrow{\phi} H_3(F^* \times F^*, A) \xrightarrow{q_*} H_3(F^* \times G_2, A),$$

where ϕ is defined in the previous section and $q : F^* \times F^* \to F^* \times G_2$, $(a, b) \mapsto \mathrm{diag}(a, b, 1)$.

Define $\varphi : S_0 \to H_3(G_2, A)$ the identity map, $\varphi : S_2 \simeq H_2(F^*, A) \otimes H_1(G_1, A) \to H_3(F^* \times G_1, A) \to H_3(G_2, A)$ the shuffle product, $\varphi : S_3 \to H_3(G_2, A)$ the map induced by $F^* \to G_2$, $a \mapsto \mathrm{diag}(a, 1)$, and $\varphi : S_4 \to H_3(G_2, A)$ the composite map

$$S_4 \xrightarrow{\phi} H_3(F^* \times F^*, A) \xrightarrow{\mathrm{inc}_*} H_2(G_2, A).$$

Consider the decomposition $H_2(G_2, A) = H_2(G_1, A) \oplus K_2^M(F) \otimes A$ [**13**, Prop. A. 11, p. 67]. Then $S_1 = S_1' \oplus S_1''$, where $S_1' = H_1(F^*, A) \otimes H_2(G_1, A)$ and $S_1'' = H_1(F^*, A) \otimes K_2^M(F) \otimes A$. Define $\varphi : S_1' \to H_3(G_2, A)$ the shuffle product and let $\varphi : S_1'' \to H_3(G_2, A)$ be the composite map

$$\begin{aligned} H_1(F^*, A) \otimes K_2^M(F) \otimes A &\xrightarrow{f} H_1(F^*, A) \otimes H_2(G_2, A) \\ &\xrightarrow{g} H_3(F^* \times G_2, A) \xrightarrow{h} H_3(G_2, A), \end{aligned}$$

where f is induced by

$$\begin{aligned} &K_2^M(F) \otimes A \to H_2(G_2, A) \\ &\{a, b\} \mapsto \frac{1}{2}\mathbf{c}(\mathrm{diag}(a, 1), \mathrm{diag}(b, b^{-1})), \end{aligned}$$

g is the shuffle product and h is induced by the map $F^* \times G_2 \to G_2$, $\mathrm{diag}(a, A) \mapsto aA$. By proposition 2.1 we have the canonical decomposition $H_3({F^*}^2 \times G_1, A) = \bigoplus_{i=0}^{8} T_i$, where

$$\begin{aligned} T_0 &= H_3(G_1, A), \\ T_1 &= \bigoplus_{i=1}^{3} H_i(F_1^*, A) \otimes H_{3-i}(G_1, A), \\ T_2 &= \bigoplus_{i=1}^{3} H_i(F_2^*, A) \otimes H_{3-i}(G_1, A), \\ T_3 &= H_1(F_1^*, A) \otimes H_1(F_2^*, A) \otimes H_1(G_1, A), \\ T_4 &= \mathrm{Tor}_1^A(H_1(F_1^*, A), H_1(F_2^*, A)), \\ T_5 &= \mathrm{Tor}_1^A(H_1(F_1^*, A), H_1(G_1, A)), \\ T_6 &= \mathrm{Tor}_1^A(H_1(F_2^*, A), H_1(G_1, A)), \\ T_7 &= H_1(F_1^*, A) \otimes H_2(F_2^*, A), \\ T_8 &= H_2(F_1^*, A) \otimes H_1(F_2^*, A). \end{aligned}$$

We know that $\tilde{\mathfrak{d}}_{1,3}^1 = \sigma_1 - \sigma_2$, where $\sigma_i = H_3(\alpha_{i,2})$. It is not difficult to see that $\tilde{\mathfrak{d}}_{1,3}^1(T_0 \oplus T_1 \oplus T_2 \oplus T_7 \oplus T_8) \subseteq \ker(\varphi)$. Here one should use the isomorphism $H_1(G_1, A) \simeq H_1(G_2, A)$. Now $(\sigma_1 - \sigma_2)(T_4) \subseteq S_4$, $\sigma_1(T_5) \subseteq S_0$ and $\sigma_2(T_5) \subseteq S_4$, $\sigma_1(T_6) \subseteq S_4$ and $\sigma_2(T_6) \subseteq S_0$. With this description one can see that $\tilde{\mathfrak{d}}_{1,3}^1(T_4 \oplus T_5 \oplus T_6) \subseteq \ker(\varphi)$. To finish the proof of the claim we have to prove that $\tilde{\mathfrak{d}}_{1,3}^1(T_3) \subseteq \ker(\varphi)$. Let $x = a \otimes b \otimes c \in T_3$, then

$$\begin{aligned} \tilde{\mathfrak{d}}_{1,3}^1(x) &= -b \otimes \mathbf{c}(\mathrm{diag}(a, 1), \mathrm{diag}(1, c)) - a \otimes \mathbf{c}(\mathrm{diag}(b, 1), \mathrm{diag}(1, c)) \in S_1 \\ &= (-b \otimes \mathbf{c}(a, c) - a \otimes \mathbf{c}(b, c), b \otimes \{a, c\} + a \otimes \{b, c\}) \in S_1' \oplus S_1'' \end{aligned}$$

So

$$\begin{aligned}\varphi(\tilde{\mathfrak{d}}^1_{1,3}(x)) = &-\mathbf{c}(\mathrm{diag}(b,1),\mathrm{diag}(1,a),\mathrm{diag}(1,c))\\ &-\mathbf{c}(\mathrm{diag}(a,1),\mathrm{diag}(1,b),\mathrm{diag}(1,c))\\ &+\frac{1}{2}\mathbf{c}(\mathrm{diag}(b,b),\mathrm{diag}(a,1),\mathrm{diag}(c,c^{-1}))\\ &+\frac{1}{2}\mathbf{c}(\mathrm{diag}(a,a),\mathrm{diag}(b,1),\mathrm{diag}(c,c^{-1})).\end{aligned}$$

Set $p := \mathrm{diag}(p,1), \overline{q} := \mathrm{diag}(1,q), p\overline{qr} := \mathbf{c}(\mathrm{diag}(p,1),\mathrm{diag}(1,q),\mathrm{diag}(1,r))$, etc. Conjugation by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ induces the equality $p\overline{qr} = \overline{p}qr$ and it is easy to see that $pqr = -qpr$ and $\overline{p^{-1}}qr = -\overline{p}qr$. With these notations and the above relations we have

$$\begin{aligned}\varphi(\tilde{\mathfrak{d}}^1_{1,3}(x)) = &-b\overline{ac} - a\overline{bc} + \frac{1}{2}(bac + ba\overline{c^{-1}} + \overline{b}ac + \overline{b}a\overline{c^{-1}})\\ &+\frac{1}{2}(abc + ab\overline{c^{-1}} + \overline{a}bc + \overline{a}b\overline{c^{-1}}) = 0.\end{aligned}$$

This proves that $H_3(G_2, A)$ is a summand of $\tilde{\mathcal{E}}^2_{0,3}$. □

THEOREM 3.3. *We have a short exact sequence*

$$0 \to H_3(G_2, \mathbb{Z}[\frac{1}{2}]) \to H_3(G_3, \mathbb{Z}[\frac{1}{2}]) \to K_3^M(F) \otimes \mathbb{Z}[\frac{1}{2}] \to 0$$

which splits. The splitting map $K_3^M(F) \otimes \mathbb{Z}[\frac{1}{2}] \to H_3(G_3, \mathbb{Z}[\frac{1}{2}])$ *is defined by*

$$\{a,b,c\} \mapsto [a,b,c] := \frac{1}{2}\mathbf{c}(\mathrm{diag}(a,1,a^{-1}),\mathrm{diag}(b,b^{-1},1),\mathrm{diag}(c,1,c^{-1})).$$

PROOF. The exactness follows from 3.2 and [**28**, 3.25]. For the splitting map see proposition 4.3 and remark 15 in the next section. □

4. Indecomposable part of K_3

LEMMA 4.1. *Let G be a group and let $g_1, g_2, h_1, \dots, h_n \in G$ such that each pair commutes. Let $C_G(\langle h_1, \dots, h_n\rangle)$ be the subgroup of G consisting of all elements of G that commute with all h_i, $i = 1, \dots, n$. If $\mathbf{c}(g_1, g_2) = 0$ in $H_2(C_G(\langle h_1, \dots, h_n\rangle))$, then $\mathbf{c}(g_1, g_2, h_1, \dots, h_n) = 0$ in $H_{n+2}(G)$.*

PROOF. The homomorphism $C_G(\langle h_1, \dots, h_n\rangle) \times \langle h_1, \dots, h_n, \rangle \to G$ defined by $(g,h) \to gh$ induces the map $H_2(C_G(\langle h_1, \dots, h_n\rangle)) \otimes H_n(\langle h_1, \dots, h_n\rangle) \to H_{n+2}(G)$. The claim follows from the fact that $\mathbf{c}(g_1, g_2, h_1, \dots, h_n)$ is the image of $\mathbf{c}(g_1, g_2) \otimes \mathbf{c}(h_1, \dots, h_n)$ under this map. □

DEFINITION 4.2. Let $A_{i,n} := \mathrm{diag}(a_i, \dots, a_i, a_i^{-(i-1)}, I_{n-i}) \in G_n$. We define $[a_1, \dots, a_n] := \mathbf{c}(A_{1,n}, \dots, A_{n,n}) \in H_n(G_n)$.

Proposition 4.3. (i) *The map* $\nu_n : K_n^M(F) \longrightarrow H_n(G_n)$ *defined by* $\{a_1, \dots, a_n\} \to [a_1, \dots, a_n]$ *is a homomorphism of groups.*

(ii) *Let* $\kappa_n : H_n(G_n) \to K_n^M(F)$ *be the map defined by Suslin (see appendix B). Then the composite map* $\kappa_n \circ \nu_n$ *coincides with the multiplication by* $(-1)^{(n-1)}(n-1)!$.

Proof. (i) The map $K_2^M(F) \to H_2(G_2)$ is defined by $\{a, b\} \mapsto [a, b]$ [**13**, A. 11]. So by lemma 4.1

$$[a_1, 1 - a_1, a_3, \dots, a_n] = 0.$$

To complete the proof of (i) it is sufficient to prove that

$$[a_1, \dots, a_{n-2}, a_{n-1}, a_n] = -[a_1, \dots, a_{n-2}, a_n, a_{n-1}].$$

This can be done in the following way;

$$\begin{aligned}
&[a_1, \dots, a_{n-2}, a_{n-1}, a_n] = \mathbf{c}(A_{1,n}, \dots, A_{n-2,n}, A_{n-1,n}, A_{n,n}) \\
&\ = \mathbf{c}(A_{1,n}, \dots, A_{n-2,n}, \operatorname{diag}(a_{n-1}I_{n-2}, a_{n-1}, a_{n-1}^{-(n-1)}), A_{n,n}) \\
&\ +\mathbf{c}(A_{1,n}, \dots, A_{n-2,n}, \operatorname{diag}(I_{n-2}, a_{n-1}^{-(n-1)}, a_{n-1}^{(n-1)}), A_{n,n}) \\
&\ = \mathbf{c}(A_{1,n}, \dots, A_{n-2,n}, \operatorname{diag}(a_{n-1}I_{n-2}, a_{n-1}, a_{n-1}^{-(n-1)}), \\
&\qquad\qquad \operatorname{diag}(a_nI_{n-2}, a_n^{-(n-2)}, 1)) \\
&\ +\mathbf{c}(A_{1,n}, \dots, A_{n-2,n}, \operatorname{diag}(a_{n-1}I_{n-2}, a_{n-1}, a_{n-1}^{-(n-1)}), \\
&\qquad\qquad \operatorname{diag}(I_{n-2}, a_n^{(n-1)}, a_n^{-(n-1)})) \\
&\ +\mathbf{c}(A_{1,n}, \dots, A_{n-2,n}, \operatorname{diag}(I_{n-2}, a_{n-1}^{-(n-1)}, a_{n-1}^{(n-1)}), A_{n,n}) \\
&= -[a_1, \dots, a_{n-2}, a_n, a_{n-1}] \\
&\ +\mathbf{c}(A_{1,n}, \dots, A_{n-2,n}, \operatorname{diag}(I_{n-2}, a_{n-1}, a_{n-1}^{-(n-1)}), \\
&\qquad\qquad \operatorname{diag}(I_{n-2}, a_n^{(n-1)}, a_n^{-(n-1)})) \\
&\ +\mathbf{c}(A_{1,n}, \dots, A_{n-2,n}, \operatorname{diag}(a_{n-1}I_{n-2}, 1, 1), \operatorname{diag}(I_{n-2}, a_n^{(n-1)}, a_n^{-(n-1)})) \\
&\ +\mathbf{c}(A_{1,n}, \dots, A_{n-2,n}, \operatorname{diag}(I_{n-2}, a_{n-1}^{-(n-1)}, a_{n-1}^{(n-1)}), \operatorname{diag}(I_{n-2}, a_n, a_n^{-(n-1)})) \\
&\ +\mathbf{c}(A_{1,n}, \dots, A_{n-2,n}, \operatorname{diag}(I_{n-2}, a_{n-1}^{-(n-1)}, a_{n-1}^{(n-1)}), \operatorname{diag}(a_nI_{n-2}, 1, 1)) \\
&= -[a_1, \dots, a_n, a_{n-1}].
\end{aligned}$$

(ii) Let τ_n be the composite map $K_n^M(F) \to K_n(F) \xrightarrow{h_n} H_n(G_n)$. Then $\kappa_n \circ \tau_n$ coincides with the multiplication by $(-1)^{(n-1)}(n-1)!$ [**43**, section 4]. It is well known that the composite map $K_n^M(F) \xrightarrow{\tau_n} H_n(G_n) \to H_n(G_n)/H_n(G_{n-1})$ is an isomorphism and it is defined by $\{a_1, \dots, a_n\} \mapsto (a_1 \cup \dots \cup a_n) \mod H_n(G_{n-1})$, where

$$a_1 \cup a_2 \cup \dots \cup a_n = \mathbf{c}(\operatorname{diag}(a_1, I_{n-1}), \operatorname{diag}(1, a_2, I_{n-2}), \dots, \operatorname{diag}(I_{n-1}, a_n))$$

(See [**28**, Remark 3.27].) Also we know that κ_n factor as

$$H_n(G_n) \to H_n(G_n)/H_n(G_{n-1}) \to K_n^M(F).$$

Our claim follows from the fact that

$$[a_1, \dots, a_n] \mod H_n(G_{n-1}) = (-1)^{n-1}(n-1)!(a_1 \cup \dots \cup a_n) \mod H_n(G_{n-1}).$$

$\square$

REMARK 15. It is easy to see that in $H_3(G_3)$

$$\mathbf{c}(\mathrm{diag}(a,1,a^{-1}), \mathrm{diag}(b,b^{-1},1), \mathrm{diag}(1,c^{-1},c)) = 0.$$

Using this one can prove that

$$[a,b,c] = \mathbf{c}(\mathrm{diag}(a,1,a^{-1}), \mathrm{diag}(b,b^{-1},1), \mathrm{diag}(c,1,c^{-1})).$$

From this one can deduce that $[a,b,c] = -[c,b,a]$.

LEMMA 4.4. (i) *We have the following isomorphisms*

$$H_i(SL(F)) \simeq H_0(F^*, H_i(SL_n(F))) \quad \textit{for} \quad n \geq i,$$
$$H_3(G_3) \simeq H_0(F^*, H_3(SL_3(F))) \oplus K_2(F) \otimes F^* \oplus H_3(F^*),$$
$$H_3(G_2, \mathbb{Z}[\tfrac{1}{2}]) \simeq H_0(F^*, H_3(SL_2(F), \mathbb{Z}[\tfrac{1}{2}])) \oplus$$
$$K_2(F) \otimes F^* \otimes \mathbb{Z}[\tfrac{1}{2}] \oplus H_3(F^*, \mathbb{Z}[\tfrac{1}{2}]).$$

(ii) *Let* $H_3(\mathrm{inc}) : H_3(G_2, \mathbb{Z}[\frac{1}{2}]) \to H_3(G_3, \mathbb{Z}[\frac{1}{2}])$. *Then on the summands*

$$H_3(\mathrm{inc}) = \begin{pmatrix} \mathrm{inc}_* & \beta & 0 \\ 0 & 2.\mathrm{id} & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix},$$

where $\beta : K_2(F) \otimes F^* \otimes \mathbb{Z}[\frac{1}{2}] \to H_0(F^*, H_3(SL_3(F), \mathbb{Z}[\frac{1}{2}]))$ *is induced by*

$$\{a,b\} \otimes c \mapsto \mathbf{c}(\mathrm{diag}(a,1,a^{-1}), \mathrm{diag}(b,b^{-1},1), \mathrm{diag}(c,1,c^{-1})).$$

PROOF. The part (i) of this lemma is rather well known (see [**35**, 2.7], [**44**, p. 233], [**12**, Rem. 1.2.8]). We will include the proofs to clarify the proof of (ii).

For the first isomorphism see [**35**, 2.7, p. 284]. Each element $M \in G$ can be written as $M = \mathrm{diag}(\det(M)^{-1}, M).\mathrm{diag}(\det(M), 1)$. This induces the homotopy equivalence $BG^+ \cong BSL(F)^+ \times BF^{*+}$ [**44**, Lemma 5.3]. The second isomorphism in (i) follows from applying the Künneth theorem to BG^+, the first isomorphism and the stability theorem $H_3(G_3) \simeq H_3(G)$ [**43**, Thm. 3.4]. The inclusions $H_0(F^*, H_3(SL_3(F))) \to H_3(G_3)$ and $H_3(F^*) \to H_3(G_3)$ are induced by the maps $SL_3(F) \to G_3, M \mapsto M$, and $F^* \to G_3, a \mapsto \mathrm{diag}(a,1,1)$, respectively. The inclusion $K_2(F) \otimes F^* \to H_3(G_3)$ is defined by

$$\{a,b\} \otimes c \mapsto \mathbf{c}(\mathrm{diag}(a,1,1), \mathrm{diag}(b,b^{-1},1), \mathrm{diag}(1,1,c)).$$

Now we prove the last isomorphism in (i). Set $A := \mathbb{Z}[\frac{1}{2}]$. From the map $\gamma : SL_2(F) \times F^* \to G_2$, $(M,a) \mapsto aM$, we obtain two short exact sequences

$$1 \to \mu_{2,F} \to SL_2(F) \times F^* \to \mathrm{im}(\gamma) \to 1,$$
$$1 \to \mathrm{im}(\gamma) \to G_2 \to F^*/F^{*2} \to 1.$$

Applying the Lyndon-Hochschild-Serre spectral sequence to the above exact sequences and carrying out not difficult analysis, one gets

$$H_3(\mathrm{im}(\gamma), A) \simeq H_3(SL_2(F) \times F^*, A), \tag{3}$$

$$H_0(F^*/{F^*}^2, H_3(\mathrm{im}(\gamma), A)) \simeq H_3(G_2, A). \tag{4}$$

The action of ${F^*}^2$ on $H_3(\mathrm{im}(\gamma), A)$ is trivial because for every $M \in \mathrm{im}(\gamma)$,

$$\begin{aligned}\mathrm{diag}(a^2, 1).M.\mathrm{diag}(a^{-2}, 1) = \mathrm{diag}(a, a).\mathrm{diag}(a, a^{-1}).M.\\ \mathrm{diag}(a^{-1}, a).\mathrm{diag}(a^{-1}, a^{-1}),\end{aligned}$$

so from (4) we obtain

$$H_0(F^*, H_3(\mathrm{im}(\gamma), A)) \simeq H_3(G_2, A). \tag{5}$$

Relations (3) and (5) imply

$$H_3(G_2, A) \simeq H_0(F^*, H_3(SL_2(F) \times F^*, A)).$$

Now applying the Künneth theorem we get the isomorphism that we are looking for. The inclusions $H_0(F^*, H_3(SL_2(F), A)) \to H_3(G_2, A)$ and $H_3(F^*, A) \to H_3(G_2, A)$ are defined in natural way. (See the proof of the second isomorphism.) The inclusion $K_2(F) \otimes F^* \otimes A \to H_3(G_2, A)$ is defined by

$$\{a, b\} \otimes c \mapsto \mathbf{c}(\mathrm{diag}(a, 1), \mathrm{diag}(b, b^{-1}), \mathrm{diag}(c, c)).$$

Using remak 15

$$\begin{aligned} H_3(\mathrm{inc})(\{a, b\} \otimes c) &= \mathbf{c}(\mathrm{diag}(a, 1, 1), \mathrm{diag}(b, b^{-1}, 1), \mathrm{diag}(c, c, 1)) \\ &= \mathbf{c}(\mathrm{diag}(a, 1, 1), \mathrm{diag}(b, b^{-1}, 1), \mathrm{diag}(c, c, c^{-2})) \\ &\quad + \mathbf{c}(\mathrm{diag}(a, 1, 1), \mathrm{diag}(b, b^{-1}, 1), \mathrm{diag}(1, 1, c^2)) \\ &= \mathbf{c}(\mathrm{diag}(a, 1, a^{-1}), \mathrm{diag}(b, b^{-1}, 1), \mathrm{diag}(c, 1, c^{-1})) \\ &\quad + \mathbf{c}(\mathrm{diag}(a, 1, 1), \mathrm{diag}(b, b^{-1}, 1), \mathrm{diag}(1, 1, c^2)). \end{aligned}$$

Therefore on the summands

$$H_3(\mathrm{inc})(0, \{a, b\} \otimes c, 0) = ([a, b, c], 2\{a, b\} \otimes c, 0).$$

□

COROLLARY 4.5. *There is an exact sequence*

$$\begin{aligned} 0 \to H_0(F^*, H_3(SL_2(F), \mathbb{Z}[\tfrac{1}{2}])) \to \\ H_0(F^*, H_3(SL_3(F), \mathbb{Z}[\tfrac{1}{2}])) \to K_3^M(F) \otimes \mathbb{Z}[\tfrac{1}{2}] \to 0 \end{aligned}$$

which splits. The splitting map is induced by $\{a, b, c\} \mapsto \frac{1}{2}[a, b, c]$.

PROOF. The proof follows from 4.4, 4.3 and 3.3. □

Let $K_3^M(F) \to K_3(F)$ be the natural map from the Milnor K-group to the Quillen K-group. Define $K_3(F)_{\text{ind}} := \text{coker}(K_3^M(F) \to K_3(F))$. This group is called the indecomposable part of the Quillen K_3-group.

THEOREM 4.6. *We have an isomorphism*

$$K_3(F)_{\text{ind}} \otimes \mathbb{Z}[\frac{1}{2}] \simeq H_0(F^*, H_3(SL_2(F), \mathbb{Z}[\frac{1}{2}])).$$

PROOF. Suslin constructed a map $s_3 : K_3(F) \to K_3^M(F)$ such that $K_3^M(F) \xrightarrow{\psi} K_3(F) \xrightarrow{s_3} K_3^M(F)$ coincides with the multiplication by 2. This implies that

$$0 \to K_3^M(F) \otimes \mathbb{Z}[\frac{1}{2}] \xrightarrow{\psi} K_3(F) \otimes \mathbb{Z}[\frac{1}{2}] \to K_3(F)_{\text{ind}} \otimes \mathbb{Z}[\frac{1}{2}] \to 0,$$

is exact and splits. Set $L_i = H_0(F^*, H_3(SL_i(F), \mathbb{Z}[\frac{1}{2}]))$ for $i = 2, 3$. We have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_3(F)_{\text{ind}} \otimes \mathbb{Z}[\frac{1}{2}] & \longrightarrow & K_3(F) \otimes \mathbb{Z}[\frac{1}{2}] & \xrightarrow{s_3} & K_3^M(F) \otimes \mathbb{Z}[\frac{1}{2}] & \longrightarrow & 0 \\
 & & \Big\downarrow f & & \Big\downarrow g & & \Big\downarrow & & \\
0 & \longrightarrow & L_2 & \longrightarrow & L_3 & \longrightarrow & K_3^M(F) \otimes \mathbb{Z}[\frac{1}{2}] & \longrightarrow & 0,
\end{array}$$

where g is the map

$$K_3(F) \otimes \mathbb{Z}[\frac{1}{2}] \simeq H_3(SL(F), \mathbb{Z}[\frac{1}{2}]) \simeq H_0(F^*, H_3(SL_3(F), \mathbb{Z}[\frac{1}{2}]))$$

(see [**35**, Prop. 2.5]) and f is induced by the commutativity of the right part of the diagram. The results follows from the Five lemma. □

REMARK 16. Theorem 4.6 generalizes theorem [**35**, Thm. 4.1], where three torsion was not treated.

5. Homology of general linear group

Let k be a field and $C_i'(F^n) := C_i(F^n) \otimes k$. Consider the following commutative diagram of two complexes

$$\begin{array}{ccccccccccc}
0 & \leftarrow & k & \leftarrow & C_0'(F^n) & \leftarrow & C_1'(F^n) & \leftarrow & C_2'(F^n) & \leftarrow & \cdots \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\
0 & \leftarrow & 0 & \leftarrow & C_0'(F^n) & \leftarrow & C_1'(F^n) & \leftarrow & C_2'(F^n) & \leftarrow & \cdots,
\end{array}$$

where the first vertical map is zero and the other vertical maps are just identity maps. This gives a map of the first quadrant spectral sequences

$$E_{p,q}^1(n) \otimes k \to \mathcal{E}_{p,q}^1(n),$$

where $\mathcal{E}_{p,q}^1(n) \Rightarrow H_{p+q-1}(G_n, k)$ with $\mathcal{E}^1$-terms

$$\mathcal{E}_{p,q}^1(n) = \begin{cases} E_{p,q}^1(n) \otimes k & \text{if } p \geq 1 \\ 0 & \text{if } p = 0 \end{cases}$$

and differentials $\mathfrak{d}^1_{p,q}(n) = \begin{cases} d^1_{p,q}(n)\otimes \mathrm{id}_k & \text{if } p\geq 2 \\ 0 & \text{if } p=1\end{cases}$. It is not difficult to see that $E^\infty_{p,q}\otimes k = \mathcal{E}^\infty_{p,q}$ if $p\neq 1$, $q\leq n$ and $p+q\leq n+1$. Hence $\mathcal{E}^\infty_{p,q}=0$ if $p\neq 1$, $q\leq n$ and $p+q\leq n+1$.

We look at the second spectral sequence in a different way. The complex

$$0\leftarrow C'_0(F^n)\leftarrow C'_1(F^n)\leftarrow\cdots\leftarrow C'_l(F^n)\leftarrow\cdots,$$

induces a first quadrant spectral sequence ${\mathcal{E}'}^1_{p,q}(n)\Rightarrow H_{p+q}(G_n,k)$, where ${\mathcal{E}'}^1_{p,q}(n)=\mathcal{E}^1_{p+1,q}(n)$ and ${\mathfrak{d}'}^1_{p,q}(n)=\mathfrak{d}^1_{p+1,q}(n)$. Thus ${\mathcal{E}'}^\infty_{p,q}(n)=0$ if $p\geq 1$, $q\leq n-1$ and $p+q\leq n$.

Proposition 5.1. *Let $n\geq 3$ and let k be a field such that $(n-1)!\in k^*$. Assume the complex*

$$H_n(F^{*2}\times G_{n-2},k)\xrightarrow{\beta_2^{(n)}} H_n(F^*\times G_{n-1},k)\xrightarrow{\beta_1^{(n)}} H_n(G_n,k)\to 0 \tag{6}$$

is exact, where $\beta_2^{(n)}=H_n(\alpha_{1,2})-H_n(\alpha_{2,2})$, $\beta_1^{(n)}=H_n(\mathrm{inc})$ and assume $H_m(\mathrm{inc}): H_m(G_{m-1},k)\to H_m(G_m,k)$ is injective for $m=n-1,n-2$. Then $H_n(\mathrm{inc}): H_n(G_{n-1},k)\to H_n(G_n,k)$ is injective.

Proof. The exactness of (6) shows that the differentials ${\mathfrak{d}'}^r_{r,n-r+1}(n)$: ${\mathcal{E}'}^r_{r,n-r+1}(n)\to{\mathcal{E}'}^r_{0,n}(n)$ are zero for $r\geq 2$. This proves that ${\mathcal{E}'}^2_{0,n}(n)\simeq{\mathcal{E}'}^\infty_{0,n}(n)$. To complete the proof it is sufficient to prove that the group $H_n(G_{n-1},k)$ is a summand of ${\mathcal{E}'}^2_{0,n}(n)$. To prove this it is sufficient to define a map $\varphi: H_n(F^*\times G_{n-1},k)\to H_n(G_{n-1},k)$ such that ${\mathfrak{d}'}^1_{1,n}(H_n(F^{*2}\times G_{n-2},k))\subseteq\ker(\varphi)$. Consider the decompositions $H_n(F^*\times G_n,k)=\bigoplus_{i=0}^n S_i$, where $S_i=H_i(F^*,k)\otimes H_{n-i}(G_{n-1},k)$. For $2\leq i\leq n$, the stability theorem gives the isomorphisms $H_i(F^*,k)\otimes H_{n-i}(G_{n-2},k)\simeq S_i$. Define $\varphi: S_0\to H_n(G_{n-1},k)$ the identity map and for $2\leq i\leq n$, $\varphi: S_i\simeq H_i(F^*,k)\otimes H_{n-i}(G_{n-2},k)\to H_n(G_{n-1},k)$ the shuffle product. To complete the definition of φ we must define it on S_1. By a theorem of Suslin [**43**, 3.4] and the assumption, we have the decomposition $H_{n-1}(G_{n-1},k)\simeq H_{n-1}(G_{n-2},k)\oplus K^M_{n-1}(F)\otimes k$. So $S_1\simeq H_1(F^*,k)\otimes H_{n-1}(G_{n-2},k)\oplus H_1(F^*,k)\otimes K^M_{n-1}(F)\otimes k$. Now define $\varphi: H_1(F^*,k)\otimes H_{n-1}(G_{n-2},k)\to H_n(G_{n-1},k)$ the shuffle product and $\varphi: H_1(F^*,k)\otimes K^M_{n-1}(F)\to H_n(G_{n-1},k)$ the composite map

$$\begin{aligned} H_1(F^*,k)\otimes K^M_{n-1}(F)\otimes k &\xrightarrow{f} H_1(F^*,k)\otimes H_{n-1}(G_{n-1},k)\\ &\xrightarrow{g} H_n(F^*\times G_{n-1},k)\xrightarrow{h} H_n(G_{n-1},k),\end{aligned}$$

where $f=\frac{1}{n-1}(\mathrm{id}\otimes\kappa_{n-1})$, g is the shuffle product and h is induced by the map $F^*\times G_{n-1}\to G_{n-1}$, $\mathrm{diag}(a,A)\mapsto aA$. By the Künnuth theorem we have the decomposition

$$
\begin{aligned}
T_0 &= H_n(G_{n-2}, k),\\
T_1 &= \bigoplus_{i=1}^{n} H_i(F_1^*, k) \otimes H_{n-i}(G_{n-2}, k),\\
T_2 &= \bigoplus_{i=1}^{n} H_i(F_2^*, k) \otimes H_{n-i}(G_{n-2}, k),\\
T_3 &= H_1(F_1^*, k) \otimes H_1(F_2^*, k) \otimes H_{n-2}(G_{n-2}, k),\\
T_4 &= \bigoplus_{\substack{i+j\geq 3\\ i,j\neq 0}} H_i(F_1^*, k) \otimes H_j(F_2^*, k) \otimes H_{n-i-j}(G_{n-2}, k).
\end{aligned}
$$

By lemma 4.3, $T_3 = T_3' \oplus T_3''$, where $T_3' = H_1(F_1^*, k) \otimes H_1(F_2^*, k) \otimes H_{n-2}(G_{n-3}, k)$ and $T_3'' = H_1(F_1^*, k) \otimes H_1(F_2^*, k) \otimes K_{n-2}^M(F) \otimes k$. It is not difficult to see that $\mathfrak{d}'^{1}_{1,n}(T_0 \oplus T_1 \oplus T_2 \oplus T_3' \oplus T_4) \subseteq \ker(\varphi)$. Here one should use the stability theorem. To prove $\mathfrak{d}'^{1}_{1,n}(T_3'') \subseteq \ker(\varphi)$ we apply 4.3;

$$
\begin{aligned}
&\mathfrak{d}'^{1}_{1,n}\Big(a \otimes b \otimes \{c_1, \dots, c_{n-2}\}\Big)\\
&= -\frac{(-1)^{n-3}}{(n-3)!}\Big(b \otimes \mathbf{c}(\mathrm{diag}(a, I_{n-2}), \mathrm{diag}(1, C_{1,n-2}), \dots, \mathrm{diag}(1, C_{n-2,n-2}))\\
&\qquad +a \otimes \mathbf{c}(\mathrm{diag}(b, I_{n-2}), \mathrm{diag}(1, C_{1,n-2}), \dots, \mathrm{diag}(1, C_{n-2,n-2}))\Big)\\
&= \frac{1}{(n-2)!}\Big(b \otimes [c_1, \dots, c_{n-2}, a] + a \otimes [c_1, \dots, c_{n-2}, b],\\
&\qquad -b \otimes \mathbf{c}(\mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(aI_{n-2}, 1))\\
&\qquad -a \otimes \mathbf{c}(\mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(bI_{n-2}, 1))\Big).
\end{aligned}
$$

Therefore $\mathfrak{d}'^{1}_{1,n}\Big(a \otimes b \otimes \{c_1, \dots, c_{n-2}\}\Big) = (x_1, x_2) \in T_3' \oplus T_3''$, where

$$
\begin{aligned}
x_1 &= -\frac{1}{(n-2)!}\Big(b \otimes \mathbf{c}(\mathrm{diag}(C_{1,n-2}), \dots, \mathrm{diag}(C_{n-2,n-2}), \mathrm{diag}(aI_{n-2}))\\
&\qquad +a \otimes \mathbf{c}(\mathrm{diag}(C_{1,n-2}), \dots, \mathrm{diag}(C_{n-2,n-2}), \mathrm{diag}(bI_{n-2}))\Big),\\
x_2 &= (-1)^{n-2}\Big(b \otimes \{c_1, \dots, c_{n-2}, a\} + a \otimes \{c_1, \dots, c_{n-2}, b\}\Big).
\end{aligned}
$$

We have $\phi(x_1) = -\frac{1}{(n-2)!}y$, where

$$
\begin{aligned}
y = &\ \mathbf{c}(\mathrm{diag}(b, I_{n-2}), \mathrm{diag}(1, C_{1,n-2}), \dots, \mathrm{diag}(1, C_{n-2,n-2}), \mathrm{diag}(1, aI_{n-2}))\\
&+\mathbf{c}(\mathrm{diag}(a, I_{n-2}), \mathrm{diag}(1, C_{1,n-2}), \dots, \mathrm{diag}(1, C_{n-2,n-2}), \mathrm{diag}(1, bI_{n-2}))
\end{aligned}
$$

and $\phi(x_2) = \frac{(-1)^{n-2}}{n-1}\frac{(-1)^{n-2}}{(n-2)!}z = \frac{1}{(n-1)!}z$, where

$$
\begin{aligned}
z = &\\
&\mathbf{c}(\mathrm{diag}(bI_{n-1}), \mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(aI_{n-2}, a^{-(n-2)}))\\
&+\mathbf{c}(\mathrm{diag}(aI_{n-1}), \mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(bI_{n-2}, b^{-(n-2)}))\\
&= \mathbf{c}(\mathrm{diag}(bI_{n-1}), \mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(aI_{n-2}, a))\\
&+\mathbf{c}(\mathrm{diag}(bI_{n-1}), \mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(I_{n-2}, a^{-(n-1)}))\\
&+\mathbf{c}(\mathrm{diag}(aI_{n-1}), \mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(bI_{n-2}, b))\\
&+\mathbf{c}(\mathrm{diag}(aI_{n-1}), \mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(I_{n-2}, b^{-(n-1)})).
\end{aligned}
$$

Hence $\phi(x_2) = \frac{-1}{(n-2)!}z'$, where

$$
\begin{aligned}
z' &= \mathbf{c}(\mathrm{diag}(bI_{n-1}), \mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(I_{n-2}, a))\\
&\quad+\mathbf{c}(\mathrm{diag}(aI_{n-1}), \mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(I_{n-2}, b))\\
&= \mathbf{c}(\mathrm{diag}(bI_{n-2}, 1), \mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(I_{n-2}, a))\\
&\quad+\mathbf{c}(\mathrm{diag}(I_{n-2}, b), \mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(I_{n-2}, a))\\
&\quad+\mathbf{c}(\mathrm{diag}(aI_{n-2}, 1), \mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(I_{n-2}, b))\\
&\quad+\mathbf{c}(\mathrm{diag}(I_{n-2}, a), \mathrm{diag}(C_{1,n-2}, 1), \dots, \mathrm{diag}(C_{n-2,n-2}, 1), \mathrm{diag}(I_{n-2}, b))\\
&= -y.
\end{aligned}
$$

Therefore $\varphi(x_2) = \frac{-1}{(n-2)!}z' = -\frac{-1}{(n-2)!}y = -\varphi(x_1)$. This completes the proof of the fact that $\mathfrak{d}'^{1}_{1,n}(H_n(F^{*2} \times G_{n-2}, k)) \subseteq \ker(\varphi)$. □

So it is reasonable to conjecture

CONJECTURE 5.2. *Let $n \geq 3$. Then the following complex is exact*

$$H_n(F^{*2} \times G_{n-2}, k) \xrightarrow{\beta_2^{(n)}} H_n(F^* \times G_{n-1}, k) \xrightarrow{\beta_1^{(n)}} H_n(G_n, k) \to 0.$$

REMARK 17. The surjectivity of $\beta_1^{(n)}$ is already proven by Suslin [**43**].

REMARK 18. All the results of this chapter are true if one replaces the infinite field with a semi-local ring with infinite residue fields.

APPENDIX A

Homology of a group and shuffle product

Let G be a group and let $B_n(G)$, $n \geq 0$, be the free left $\mathbb{Z}[G]$-module generated by the symbols $[g_1|\dots|g_n]$, $g_i \in G-\{1\}$. Note that $B_0(G)$ is generated by the single symbol $[\ ]$. For $n \geq 1$, define $\partial_n = \sum_{i=1}^{n}(-1)^i d_i : B_n(G) \to B_{n-1}(G)$, where

$$\begin{aligned} d_0([g_1|\dots|g_n]) &= g_1[g_2|\dots|g_n], \\ d_i([g_1|\dots|g_n]) &= \begin{cases} [g_1|\dots|g_ig_{i+1}|\dots|g_n] & \text{if } g_ig_{i+1} \neq 1 \\ 0 & \text{if } g_ig_{i+1} = 1 \end{cases}, \\ d_n([g_1|\dots|g_n]) &= [g_1|\dots|g_{n-1}]. \end{aligned}$$

If $\epsilon : B_0(G) \to \mathbb{Z}$ with $[\ ] \mapsto 1$, then the augmented complex

$$\cdots \to B_2(G) \xrightarrow{\partial_2} B_1(G) \xrightarrow{\partial_1} B_0(G) \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

is exact. $B_*(G)$ is called the *bar resolution* of G. (Note that $\mathbb{Z}$ is a G-module with trivial G-action.)

Example A.1. (i) $\partial_1([g]) = g[\ ] - [\ ]$, $\partial_2([g|h]) = g[h] - [gh] + [g]$, $\partial_3([f|g|h]) = f[g|h] - [fg|h] + [f|gh] - [f|g]$.

Convert the left action of G on $B_*(G)$ to the right action by $[g_1|\dots|g_n]g := g^{-1}[g_1|\dots|g_n]$. Let M be a (left) $\mathbb{Z}[G]$-module. The i-th homology group of G with coefficients in M is defined as

$$H_i(G, M) := H_i(B_*(G) \otimes_{\mathbb{Z}[G]} M) = \mathrm{Tor}_i^{\mathbb{Z}[G]}(\mathbb{Z}, M)$$

It is a well known fact that $H_0(G, M) \simeq M_G := M/\langle m - gm : m \in M, g \in G\rangle$ and $H_1(G, \mathbb{Z}) \simeq G/[G, G]$.

A homomorphism $\psi : G \to G'$ induces a map of complexes $\psi_* : B_*(G) \to B_*(G')$ given by $\psi_n([g_1|\dots|g_n]) = [\psi(g_1)|\dots|\psi(g_n)]$ and so a map of i-th homology groups $H_i(\psi) : H_i(G, M) \to H_i(G', M)$.

For groups G and G', we have a map

$$\psi_* : B_*(G) \otimes B_*(G') \to B_*(G \times G')$$

given by

$$g[g_1|\dots|g_p] \otimes g'[g'_1|\dots|g'_q] \mapsto \sum \mathrm{sign}(\sigma)(g, g')[h_{\sigma(1)}|\dots|h_{\sigma(p+q)}],$$

where $(h_1, \dots, h_{p+q}) = ((g, 1), \dots, (g_p, 1), (1, g'_1), \dots, (1, g'_q))$ and σ runs through (p, q)-shuffles of $\{1, \dots, p+q\}$. A (p, q)-shuffle is a permutation of the set $\{1, \dots, p+q\}$

of integers in such a way that $\sigma(1) < \sigma(2) < \cdots < \sigma(p)$ and $\sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q)$.

EXAMPLE A.2. $\psi([g] \otimes [h]) = [(g,1)|(1,h)] - [(1,h)|(g,1)]$ and $\psi([f] \otimes [g|h]) = [(f,1)|(1,g)|(1,h)] - [(1,g)|(f,1)|(1,h)] + [(1,g)|(1,h)|(f,1)]$.

The map ψ_* induces the map

$$H_p(G,\mathbb{Z}) \otimes H_q(G',\mathbb{Z}) \to H_{p+q}(G \times G',\mathbb{Z}).$$

This is called the *shuffle product* of the groups $H_p(G,\mathbb{Z})$ and $H_p(G',\mathbb{Z})$.

Let $g_1,\dots,g_n$ be pairwise commuting element of a group G and set

$$\mathbf{c}(g_1,g_2,\dots,g_n) := \sum_{\sigma\in\Sigma_n} \mathrm{sign}(\sigma)[g_{\sigma(1)}|g_{\sigma(2)}|\dots|g_{\sigma(n)}] \in H_n(G,\mathbb{Z}).$$

LEMMA A.1. Let G be a group.

(i) If g_1' commutes with all the elements $g_1,\dots,g_n \in G$, then

$$\mathbf{c}(g_1g_1',g_2,\dots,g_n) = \mathbf{c}(g_1,g_2,\dots,g_n) + \mathbf{c}(g_1',g_2,\dots,g_n).$$

(ii) For every $\sigma \in \Sigma_n$, $\mathbf{c}(g_{\sigma(1)},\dots,g_{\sigma(n)}) = \mathrm{sign}(\sigma)\mathbf{c}(g_1,g_2,\dots,g_n)$.

(ii) The shuffle product of $\mathbf{c}(g_1,\dots,g_p) \in H_p(G,\mathbb{Z})$ and $\mathbf{c}(g_1',\dots,g_q') \in H_q(G',\mathbb{Z})$ is $\mathbf{c}((g_1,1),\dots,(g_p,1),(1,g_1'),\dots,(1,g_q')) \in H_{p+q}(G \times G')$.

PROOF. The proof is easy, so we leave it to the interested readers. $\square$

APPENDIX B

Quillen and Milnor K-theory

In this appendix we define the Quillen and Milnor K-groups and state some of their properties and relations, which we use in this thesis.

THEOREM B.1. Let (X, x) be a path connected CW-complex and let N be a perfect normal subgroup of $\pi_1(X, x)$, that is $[N, N] = N$. Then there exists a CW-complex (X^+, x^+) and an embedding $f : (X, x) \to (X^+, x^+)$ such that

(i) $\pi_1(X^+, x^+) = \pi_1(X, x)/N$,

(ii) for every local coefficient system L on X^+, $H_i(X, f^*L) \to H_i(X^+, L)$ is an isomorphism for every $i \geq 0$.

(iii) if $g : (X, x) \to (Y, y)$ is a continuous map such that $N \subseteq \ker(\pi_1(g) : \pi_1(X, x) \to \pi_1(Y, y))$, then there exist $h : (X^+, x^+) \to (Y, y)$, unique up to homotopy, such that $h \circ g = f$.

PROOF. See [**39**, Thm. 2.1]. □

The space (X^+, x^+) is called the *plus construction* of (X, x) with respect to N and it is unique up to homotopy equivalent.

THEOREM B.2. Let (X_i^+, x_i^+) be the plus construction of a CW-complex (X_i, x_i) with respect to a perfect normal subgroup $N_i \subseteq \pi_1(X_i, x_i)$, $i = 1, 2$. Then $(X_1^+ \times X_2^+, (x_1^+, x_2^+))$ is homotopy equivalent to $((X_1 \times X_2)^+, (x_1, x_2)^+)$, the result of applying the plus construction to $N_1 \times N_2 \subseteq \pi_1(X_1 \times X_2, (x_1, x_2))$.

PROOF. See [**39**, Prop. 2.3]. □

THEOREM B.3. Let G be a group and let $N = [G, G]$ be perfect. Then BN^+ is homotopy equivalent to the universal covering of BG^+, where both plus construction are with respect to N.

PROOF. See [**19**, Prop. 1.1.7]. □

DEFINITION B.1. Let R be a ring with 1. For $n \geq 1$ the n-th *Quillen K-group* of R, $K_n(R)$, is the group $\pi_n(BGL(R)^+)$, where the plus construction $BGL(R)^+$ is with respect to the elementary group $E(R)$. Not that $E(R) = [E(R), E(R)] = [GL(R), GL(R)]$.

There is an operation $K_n(R) \otimes K_m(R) \to K_{n+m}(R)$ which endows the group $\bigoplus_{n=0}^{\infty} K_n(F)$ with the structure of a skew-commutative graded ring.

Example B.3. (i) For any ring R, $K_1(R) = GL(R)/[GL(R), GL(R)] = H_1(GL(R), \mathbb{Z})$, $K_2(R) = H_2(E(R), \mathbb{Z})$ and $K_3(R) = H_3(St(R), \mathbb{Z})$ [**39**, Cor. 1.2, 2.6].

(ii) (Matsumoto) If F is a field, then $K_2(F) \simeq F^* \otimes F^* / \langle a \otimes (1-a) : a \in F^* - \{1\}\rangle$.

(iii) (Van der Kallen) Let R be a semilocal ring such that its residue fields have more that five elements. Then $K_2(R) \simeq R^* \otimes R^* / \langle a \otimes (1-a) : a \in R^* - \{1\}\rangle$.

Example B.4. (i) (Quillen) $K_{2k}(\mathbb{F}_q) = 0$ and $K_{2k-1}(\mathbb{F}_q) = \mathbb{Z}/(q^k - 1)$.

(ii) (Quillen) Let R be the ring of integers of a number field F. Then $K_n(R)$, $n \geq 1$, and $K_n(F)$,$n \geq 3$, are finitely generated. The group $K_2(F)$ is torsion.

(iii) $K_2(\mathbb{Q}) = \mathbb{Z}/2 \oplus \bigoplus_{p \text{ prime}} (\mathbb{Z}/p)^*$ (Tate) and $K_3(\mathbb{Q}) = \mathbb{Z}/48$ (Lee-Szczarba).

(iv) (Borel) Let R be the ring of integers of a number field F and let $[F : \mathbb{Q}] = r_1 + 2r_2$. Then

$$K_n(R)_{\mathbb{Q}} = \begin{cases} 0 & \text{if } n \text{ is even} \\ \mathbb{Q}^{r_1+r_2} & \text{if } n \equiv 1 \mod 4 \\ \mathbb{Q}^{r_2} & \text{if } n \equiv 3 \mod 4 \end{cases}.$$

Writing the Hurewicz map for $BGL(R)^+$, we obtain

$$h_n : K_n(R) = \pi_n(BGL(R)^+) \to H_n(BGL(R)^+, \mathbb{Z}) = H_n(GL(R), \mathbb{Z}).$$

Example B.5. h_1 are isomorphism and h_2 is injective.

Theorem B.4 (Milnor-Moore). *The map $h_n : K_n(R)_{\mathbb{Q}} \to H_n(GL(R), \mathbb{Q})$ is injective.*

By B.2, $K_n(R) = \pi_n(BE(R)^+)$ for $n \geq 2$. So the Hurewicz map decompose as $K_n(R) \to H_n(E(R), \mathbb{Z}) \to H_n(GL(R), \mathbb{Z})$.

Theorem B.5. *The map $K_3(R) \to H_3(E(R), \mathbb{Z})$ is surjective with 2-torsion kernel. In particular $K_3(R) \otimes \mathbb{Z}[\frac{1}{2}] \to H_3(E(R), \mathbb{Z}[\frac{1}{2}])$ is an isomorphism.*

Proof. see [**35**, Prop. 2.5] □

Definition B.2. Let F be a field and $T(F^*) := \mathbb{Z} \oplus F^* \oplus F^* \otimes F^* \oplus \cdots$ be the tensor algebra of F^*. The n-th *Milnor K-group*, $K_n^M(F)$, is the n-th degree part of $K_*^M(F) := T(F^*)/\langle a \otimes (1-a) : a \in F^* - \{1\}\rangle$.

There is a canonical ring homomorphism from $K_*^M(F) \to K_*(F)$, so a canonical homomorphism $\psi_n : K_n^M(F) \to K_n(F)$.

Example B.6. (i) $K_n^M(F) \simeq K_n(F)$ for $n = 1, 2$. If F is a finite field, then $K_n^M(F) = 0$ for $n \geq 2$. So ψ_n is not surjective in general. One can show that $K_n^M(\mathbb{Q}) \to K_n(\mathbb{Q})$ is not injective for $n \geq 4$.

Theorem B.6 (Suslin). *Let F be an infinite field. Then the map $H_n(\mathrm{inc}) : H_n(GL_n(F), \mathbb{Z}) \to H_n(GL(F), \mathbb{Z})$ is bijective and the map $H_n(\mathrm{inc}) : H_n(F^* \times GL_{n-1}(F), \mathbb{Z}) \to H_n(GL_n(F), \mathbb{Z})$ is surjective.*

Proof. See [**43**, Section 3] □

THEOREM B.7 (Suslin). For any infinite field F, there is map $\kappa_n : H_n(G_n, \mathbb{Z}) \to K_n^M(F)$ such that

$$H_n(GL_{n-1}(F), \mathbb{Z}) \to H_n(GL_n(F), \mathbb{Z}) \xrightarrow{\kappa_n} K_n^M(F) \to 0$$

is exact.

PROOF. See [**43**, Thm. 3.4] □

Using these two theorems, Suslin constructed a map from $K_n(F)$ to $K_n^M(F)$, we call it s_n, in the following way;

$$K_n(F) \to H_n(GL(F), \mathbb{Z}) \xleftarrow{\simeq} H_n(GL_n(F), \mathbb{Z}) \to$$
$$H_n(GL_n(F), \mathbb{Z})/H_n(GL_{n-1}(F), \mathbb{Z}) \xrightarrow{\simeq} K_n^M(F).$$

THEOREM B.8 (Suslin). (i) The composite homomorphism

$$K_n^M(F) \to K_n(F) \xrightarrow{s_n} K_n^M(F)$$

is multiplication by $(-1)^{n-1}(n-1)!$.

(ii) The composite homomorphism

$$K_n(F) \to K_n^M(F) \xrightarrow{s_n} K_n(F)$$

coincides with the characteristic class $c_{n,n} : K_n(F) \to H^0(\mathrm{Spec}(F), \mathcal{K}_n) = K_n(F)$. (For any scheme X we denote by $\mathcal{K}_n$ the sheaf (in the Zariski topology) associated to the presheaf $U \to K_n(\Gamma(U, \mathcal{O}_X))$.

PROOF. See [**43**, Sections 4, 5] □

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Index

Notations

Samenvatting

De belangrijkste doelstelling van deze dissertatie is de bestudering van het stabiliteits-probleem van de homologie van unitaire groepen en hun toepassingen. Vanwege de hechte relatie tussen de homologie van klassieke groepen en K-groepen en vanwege de overeenkomsten tussen de definities van Hermitese en algebraïsche K-groepen, bespreken we het probleem met betrekking tot de algebraïsche K-groepen, met name K_0 en K_1, om genoeg motivatie voor de bestudering van dit type problemen te geven.

Lineaire algebra is zonder enige twijfel een zeer invloedrijk onderwerp binnen het domein van de zuivere wiskunde. Een van de centrale doelstellingen van dit domein is het vinden van de oplossingen van een systeem van lineaire vergelijking over een lichaam, bijvoorbeeld $\mathbb{R}$ of $\mathbb{C}$. Dit leidt ons tot de bestudering van de familie van eindige dimensionale vector-ruimten en lineaire functies tussen hen, bijvoorbeeld matrices.

Vaak is onze basis-structuur een meer algemene ring en net zoals bij lineaire algebra is het nodig de vectorruimten op deze algemene ringen te bestuderen, alsmede de 'lineaire functies' tussen hen.

Lagere K-theorie (K_0, K_1 en later K_2) mag worden beschouwd als een correctie op lineaire algebra. Voor een eindige dimensionale vectorruimte over een lichaam bestaat altijd een basis en iedere basis kan getransformeerd worden in een andere door middel van een reeks van elementaire transformaties. De juiste generalisatie van vectorruimten hier bestaat uit eindig gegenereerde projectieve modulen over een ring. Bases hoeven niet langer te bestaan en wanneer ze dat wel doen, hoeven ze niet equivalent onder elementaire transformaties te zijn. De groep K_0 meet de obstructie voor hun bestaan; K_1 beschrijft hun niet-uniekheid; K_2 meet die relaties tussen elementaire matrixen die afhangen van de ring.

K-theorie werd tot ontstaan gebracht door Grothendiecks Riemann-Roch stelling in 1957. Met betrekking tot zijn groep en de keuze van de letter K zegt Grothendieck in zijn brief van 9 februari 1985 aan Bruce Magurn:

> Eerst visualiseerde ik een K-groep als een groep van 'klassen van objecten' van een abelse (of meer algemeen, additieve) categorie, zoals coherente schoven op een algebraïsche variëteit, of vectorbundels, etc. Ik zou deze groep waarschijnlijk $C(X)$ (X zijnde een variëteit of een andere soort van 'ruimte') hebben genoemd, C de eerste letter van 'class', maar mijn verleden in functionaal-analyse heeft dit wellicht voorkomen, gezien $C(X)$ ook verwijst naar de ruimte van continue functies op X

> (wanneer X een topologische ruimte is). Dus keerde ik terug tot K in plaats van C, aangezien mijn moedertaal Duits is, Class=Klasse (in het Duits), en de klanken die corresponderen met C en K zijn hetzelfde.

Het lijkt erop dat algebraïsche K-theorie begon met Bass' observatie dat twee abelse groepen die vastzitten aan twee objecten in twee verschillende onderwerpen, namelijk Grothendiecks K_0 in algebraïsche geometrie en Whiteheads K_1 in topologie, tot eenheid konden worden gebracht in een mooie en krachtige theorie met wijd verspreide toepassingen in algebra.

In het nu volgende zal ik de definities van K_0 en K_1 geven en de stabiliteitsproblemen uiteenzetten. Het woord 'ring' betekent altijd een commutatieve ring met 1.

De groep $K_0(R)$. Een basis van een R-moduul M is een deelverzameling $\{e_i\}_{i\in I}$ zodanig dat ieder element van M op een unieke manier geuit kan worden als een eindige som $\sum r_i e_i$ met $r_i \in R$. Als M een basis heeft, noemen we het een vrij moduul en kan men bewijzen dat $M \simeq \oplus_{i\in I} R$. Een R-moduul P wordt projectief genoemd als voor iedere surjectieve afbeelding $f : M \to N$ van R-modulen M en N en een R-homomorfisme $g : P \to N$ er een afbeelding is $h : P \to M$ zodanig dat $f \circ h = g$. (Men dient op te merken dat deze eigenschap een van de hoofdeigenschappen van vectorruimten is). We zeggen dat P van constante rang is indien er een geheel getal $n \geq 0$ bestaat zodanig dat $P_{\mathfrak{p}} \simeq R_{\mathfrak{p}}^n$ voor alle priem-idealen $\mathfrak{p}$ van R.

Het is gemakkelijk om te zien dat een vrij moduul projectief is. Maar het tegenovergestelde geldt niet in het algemeen. Om precies te zijn is een moduul projectief dan en slechts dan als het een summand van een vrij moduul is. Maar voor sommige ringen zijn de enige projectieve modulen over hen vrij:

(i) Iedere projectieve moduul over $\mathbb{Z}$ is vrij.
(ii) Iedere projectieve moduul over een locale ring is vrij.
(iii) Iedere eindig gegenereerd projectief moduul over een polynoomring $k[T_1, \ldots, T_n]$, waar k een lichaam is, is vrij (Suslin-Quillen).
(iv) Laat P een eindig gegenereerd projectief R-moduul zijn. Laat $\mathfrak{p}$ een priem-ideaal van R zijn. Dan bestaat er $s \in R - \mathfrak{p}$ zodanig dat $P[\frac{1}{s}] \simeq (R[\frac{1}{s}])^m$ voor een positief geheel getal m. Dit induceert de notie van een lokaal vrije schoof $P^\sim$ op het schema $\mathrm{Spec}(R)$.

Laat $\mathcal{P}(R)$ de categorie van eindig gegenereerde projectieve R-modulen zijn. De groep $K_0(R)$ wordt gedefinieerd als de Grothendieck-groep van $\mathcal{P}(R)$. Het is de abelische groep met generatoren $[P]$, de isomorfie klasse van $P \in \mathcal{P}(R)$, en relaties $[P \oplus Q] = [P] + [Q]$.

In haar vroege stadium van ontwikkeling dankte algebraïsche K-theorie veel aan ideeën uit de topologie. Een groot deel van de inspiratie voor het onderwerp kwam in feite van de theorie van vectorbundels. De sleutelobservatie is dat er een hechte analogie bestaat tussen vectorbundels en projectieve modulen. Laten we om dit expliciet te maken een compacte Hausdorf-ruimte X beschouwen en $Vec_F(X)$, $F = \mathbb{R}$ of $\mathbb{C}$ schrijven, voor de categorie van alle F-vectorbundels over X.

Laat $F(X)$ staan voor de ring van continu F-waardige functies op X. Definieer voor iedere vectorbundel $p : E \to X$ op X dat de F-vectorruimte van globale secties $\Gamma(E) := \{s : X \to E | s \in \mathcal{C}(X, E), p \circ s = \mathrm{id}_X\}$ is. Dan is $\Gamma(E)$ een moduul over $F(X)$ door de actie

$$(f.s)(x) = f(x)s(x), f \in F(X), s \in \Gamma(E), x \in X.$$

Hieruit volgen twee feiten:

(i) $\Gamma(E)$ *is een eindig gegenereerd projectief moduul over* $F(X)$.
(ii) *Als* E *een* n*-bundel is (dit betekent dat de vezels van* E n*-dimensionale* F*-vectorruimten zijn), dan heeft* $\Gamma(E)$ *een constante rang* n *over* $F(X)$.

Het blijkt verder dat iedere module in $\mathcal{P}(F(X))$ noodzakelijk van de vorm $\Gamma(E)$ is, voor passende $E \in Vec_F(X)$. Meer precies:

Stelling van Swan. Γ *definieert een equivalentie van categorieën van* $Vec_F(X)$ *naar* $\mathcal{P}(F(X))$.

Laat X een samenhangend eindig CW-complex zijn van dimensie d. Als E een n-bundel over X is, met $n > d$, is het in de topologie bekend dat we een triviale lijnbundel T van E kunnen afpellen, b.v. $E \simeq E' \oplus T$, voor passende bundel E'. Verder, als $n \geq d+1$, dan is het isomorfisme-type van E' uniek bepaald. Dit worden de *Stabiliteitsstellingen van de topologie* genoemd.

Stabiliteitsstelling voor vectorbundels. *Laat* $Vec_{F,n}(X)$ *de categorie van* n*-bundels over* X *zijn, en laat* $\psi_n : Vec_{F,n}(X) \to Vec_{F,n+1}(X)$ *de afbeelding zijn die* E *naar* $E \oplus T$ *stuurt (een triviale lijnbundel toevoegend). Dan is* ψ_n *surjectief indien* $n \geq d$ *en bijectief indien* $n \geq d+1$.

Men kan een algebraïsche vorm van deze stelling geven. Laat $R = F(X)$ en laat $K_{0,n}(R)$ staan voor de categorie van isomorfie-klassen van eindig gegenereerde projectieve R-modulen van constante rang n. Merk op dat onder de categorie equivalentie Γ, $Vec_{F,n}(X)$ correspondeert met $K_{0,n}(R)$. Verder correspondeert de triviale bundel T met $\Gamma(T) \simeq R$.

K_0-Stabiliteitsstelling. *Laat* $\phi_n : K_{0,n}(R) \to K_{0,n+1}(R)$ *gedefinieerd zijn door* $P \mapsto P \oplus R$. *Dan is* ϕ_n *surjectief indien* $n \geq d$ *en bijectief indien* $n \geq d+1$.

Op grond hiervan zal een algebraïcus onmiddellijk vragen of we een vergelijkbaar resultaat kunnen verkrijgen als we $F(X)$ door een meer algemene ring te vervangen. Hiervoor heeft men een goede vervanging voor d nodig, de dimensie van X. Onderzoek in deze richting toonde dat als $\mathrm{Mspec}(R)$ een Noetherse topologische ruimte is, d vervangen kan worden door de dimensie van $\mathrm{Mspec}(R)$, en $K_{0,n}(R)$ vervangen kan worden door

$$\{[P] : P \in \mathcal{P}(R), \min_{\mathfrak{m} \in \mathrm{Mspec}(R)} (\mathrm{rank}_{R_\mathfrak{m}} P_\mathfrak{m}) = n\}.$$

Een vergelijkbare K_0-stabiliteitsstelling kan nu worden bewezen in een algebraïsche context.

De K_0-stabiliteitsstelling zegt, dat bij de bestudering van het projectieve moduul P over R het voldoende is om te kijken naar die met $\min_{\mathfrak{m}\in \mathrm{Mspec}(R)}(\mathrm{rank}_{R_\mathfrak{m}} P_\mathfrak{m}) \leq d$; verder, indien $\min_{\mathfrak{m}\in \mathrm{Mspec}(R)}(\mathrm{rank}_{R_\mathfrak{m}} P_\mathfrak{m}) \geq d+1$, dan bepaalt de klasse $[P] \in K_0(R)$ het isomorfisme-type van P.

De groep $K_1(R)$. Laat $GL_n(R)$ de groep zijn van inverteerbare $n\times n$-matrices. De identificatie van iedere $n \times n$-matrix A met de grotere matrix $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ geeft een inbedding van $GL_n(R)$ in $GL_{n+1}(R)$. De vereniging van de resulterende rij

$$GL_1(R) \subseteq GL_2(R) \subseteq \ldots \subseteq GL_n(R) \subseteq GL_{n+1}(R) \ldots$$

wordt de stabiele algemene lineaire groep $GL(R)$ genoemd.

In navolging van Bass definieert men $K_1(R)$ als de groep $GL(R)/E(R)$, voorkomend in Whiteheads theorie van het simpele homotopie-type. Hier is $E(R)$ de ondergroep van $GL(R)$, gegenereerd door elementaire matrices. Op grond van Whiteheads lemma geldt $E(R) = [GL(R, GL(R))] = [E(R), E(R)]$, wat toont dat $E(R)$ een normale ondergroep van $GL(R)$ is.

Voor de functor K_1 is het vrij gemakkelijk om analoge stabiliteits-vragen te stellen. De rang van een projectieve moduul moet simpelweg vervangen worden door de grootte van een matrix. Laat $K_{1,n}(R) := GL_n(R)/E_n(R)$. Op grond van een stelling van Suslin is $E_n(R)$ een normale ondergroep van $GL_n(R)$ voor $n \geq 3$. Dus $K_{1,n}(R)$ is een groep voor $n \geq 3$. De inbedding $GL_n(R) \to GL_{n+1}(R)$ induceert de natuurlijke afbeeldingen $K_{1,n}(R) \to K_{1,n+1}(R) \to \ldots \to K_1(R)$.

> K_1-Stabiliteits-stelling. *Laat R een ring zijn zodanig dat* $\mathrm{Mspec}(R)$ *een Noetherse topologische ruimte is en* $d = \dim(\mathrm{Mspec}(R)) < \infty$. *Dan is $K_{1,n}(R) \to K_{1,n+1}(R)$ surjectief indien $n \geq d+1$ en bijectief indien $n \geq d+2$.*

De determinant induceert de afbeelding $\det : K_1(R) \to R^*$ en we schrijven zijn kern als $SK_1(R)$. Het is gemakkelijk om te zien dat det surjectief is en dus $K_1(R) \simeq R^* \times SK_1(R)$. Definieer $SK_{1,n}(R)$ overeenkomstig $K_{1,n}(R)$. Op grond van de stabiliteits-stelling hebben we $SK_{1,n}(R) \simeq SK_{1,n+1}(R) \simeq \cdots \simeq SK_1(R)$ indien $n \geq d+2$.

Laat R de ring zijn van algebraïsch gehele getallen van een getallenlichaam F. Op grond van een diepzinnige stelling van Bass, Milnor en Serre geldt $SK_1(R) = 0$. Gezien $\dim(\mathrm{Mspec}(R)) = 1$ hebben we, op grond van de stabiliteits-stelling voor $SK_1(R)$, $SL_n(R) \simeq E_n(R)$ voor alle $m \geq 3$. Dit betekent dat iedere $n \times n$ matrix over R ($n \geq 3$) van determinant 1 tot I_n gebracht kan worden door middel van elementaire rij- (of kolom-) operaties over R.

Hogere K_i en Hermitese K-theorie. De definitie van hogere K-groepen is in vergelijking met lagere K-groepen gecompliceerder. Ze worden gedefinieerd als de hogere homotopie van zekere topologische ruimten. De Hurewicz stelling in de algebraïsche topologie geeft een afbeelding van Hogere K-groepen naar de homologie van stabiele algemene lineaire groepen met coefficient in $\mathbb{Z}$, $K_n(R) \to H_n(GL(R), \mathbb{Z})$.

Zie Appendixen A en B voor de definities van de homologie van een groep, Hogere K-groepen en sommige van hun eigenschappen.

Nu kan men vergelijkbare stabiliteits-vragen stellen voor hogere K-groepen. De Hurewicz-stelling staat het ons toe het stabiliteits-probleem van de K-groepen te reduceren tot het stabiliteits-probleem van de homologie van stabiele algemene lineaire groepen. De meest algemene resultaten in deze richting zijn te danken aan Van der Kallen en Suslin, die verschillende benaderingen gebruiken.

Men kan vergelijkbare stabiliteits-vragen voor Hogere Hermitese K-groepen stellen en eveneens voor de homologie van unitaire groepen. Het hoofddoel van deze dissertatie is om deze vragen te bestuderen. Als een resultaat daarvan zullen we een algemene stabiliteits-stelling voor de homologie bewijzen, en nog later zullen we voor enige specifieke ringen onze homologische stabiliteits-grens verbeteren.

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Curriculum vitae

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