The Flexibility of Mathematics

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Abstract

Mathematics is quite unlike physics: it does not possess empirical content and lives in an independent realm of its own. It seems surprising that the partnership of these dissimilar companions, mathematics and physics, is so extremely successful. But I argue that on further reflection this success is not 'unreasonable': the very difference between the nature of mathematics and that of physics makes it possible for mathematics to be highly flexible and adaptable to the most diverse needs. By means of a number of examples, drawn from fundamental physics, I illustrate how mathematics, through its flexibility and versatility, achieves its great effectiveness.

1 Introduction

"Mathematics may be defined as the subject where we never know what we are talking about, nor whether what we are saying is true", Bertrand Russell famously remarked [8]. What Russell wanted to express is that mathematics is very different from empirical science: it is not about the physical world in which we live and which we can see, touch and smell. One does not have to subscribe to the details of Russell's philosophy of mathematics to agree with this point. For example, Platonists think that mathematics *does* describe something and *can* be true (in the correspondence sense): it describes an Ideal mathematical world. Still, this mathematical heaven is completely separate from the physical world around us. Formalists, on the other hand, believe that mathematics does not describe anything at all but is a mere play with symbols, according to man-made rules. Here too, mathematics is without physical content.

In view of this great distance between physics and mathematics it may come as a surprise that mathematics plays such an important role in modern physics. In some instances mathematical considerations are even the dominant force in physical research. Is this proven effectiveness of mathematics in physics not hard to understand, very 'unreasonable', as Wigner put it in a famous essay [9]?

Now, it seems plausible that it is not *a priori* necessary that a scientific account, in the usual sense, of the physical world is possible. The world might have been dramatically irregular, with no structural permanence at all. The concept of a 'law of nature' would not be usable in such a situation and the question of whether natural laws can be couched in mathematical language would not even have arisen. So, perhaps, it may be considered astounding that there is regularity in our world at all, that the concept of a law of nature actually makes sense. Perhaps one could argue that *a priori* it is more probable that there is no order than that there is—that we therefore find ourselves in an improbable situation, which justifies surprise. I am not sure about arguments of this sort—the status of the *a priori* probabilities used in them, and the justification of the values assigned to these probabilities, seem very much open to question. Moreover, Kantian or anthropic counter-arguments may be defensible, about the physical conditions that have to be satisfied in order to make our own existence possible. But in this essay I will not embark on speculations about whether there might be reasons why regularities in nature exist: I am going to take the existence of such regularities as something given.

The Wignerian question then becomes: even if we accept as a fact that there is order and structure in nature, isn't it unreasonable to expect that mathematics is highly effective? Isn't it strange that mathematics not infrequently plays an inspiring role in physical research, and points the way to new results?

The answer that I want to suggest is that the very observation that mathematics has no physical content can take away most of the surprise. Indeed, exactly because mathematics is a 'freely floating construction', not tightly bound to sense experience, it is extremely flexible and versatile—and therefore useful. I will illustrate some aspects of this flexibility and versatility below, by examples from fundamental physics.

One thing I want to make clear by these examples is that the same physical situation can usually be described in a variety of mathematical ways. The mathematical toolbox is so well-stocked that researchers of different approaches and persuasions can find a way of dealing with a subject that suits their tastes and enables them to pursue their own programs. Conversely, since mathematics itself is empirically empty, the same mathematical techniques and results can often be applied to a diversity of physical situations; new insights can thus be gained at small costs by transporting old results to new contexts. The effectiveness of mathematics thus appears as a built-in feature: because of its flexible applicability anywhere where some type of order reigns, and because of its adaptability to research preferences, mathematics is likely to be effective.

2 Non-uniqueness of mathematical models

In the years during which the genesis of modern quantum theory took place, mathematical techniques from different directions were employed. Heisenberg's matrix mechanics and Schrödinger's wave mechanics, respectively, had a radically different mathematical form and fitted in with very different methodological programmes. However, both formalisms were able to handle the discreteness of spectral lines, and therefore succeeded in explaining the most crucial experimental fact that classical theory could not handle. This already furnishes a first example of the flexibility of mathematics. Schrödinger, repelled by the abstract character of Heisenberg's theory, was able to find an alternative mathematical treatment that satisfied his own philosophical and aesthetic demands but made the same observable physical predictions as the abhorred rival theory. This new mathematical scheme enabled him to pursue his favorite idea, according to which quantum objects are inherently wave-like.

However, there is a limit to this kind of adaptability. One cannot impose any philosophical preference whatsoever on nature. Although mathematics is very flexible and will go a long way in meeting a researcher's wishes, it cannot guarantee that all desiderata will be implementable. Nature itself, experimental results, obviously limit the possibilities: Schrödinger was in fact unable to carry through his pet notion that particles are local spots of high density in an omnipresent continuous field. The mathematical reason for this is that the wave field is defined in configuration space rather than in ordinary three-dimensional space, which becomes important as soon as systems consisting of more than one particle are considered; an additional problem is that local regions of high field intensity will not be stable because of dispersion. These features of the wave theory prove inevitable if justice is to be done to the observed phenomena.

Nevertheless, the flexibility of the mathematical treatment permitted Schrödinger to make the most of his research programme. The discrete nature of Heisenberg's calculus clearly turned out to be avoidable; a continuum treatment could be put in its place. Mathematics afforded the maximum of flexibility compatible with empirical results.

Not long after the discussions about these issues in the twenties, von Neumann showed that both Heisenberg's and Schrödinger's theories could be seen as versions of one encompassing mathematical scheme—quantum theory as formulated in Hilbert space [7]. In spite of the fact that matrix mechanics and wave mechanics are so very different—the former a calculus of discrete quantities, the latter a continuum theory—mathematics was able to provide a unifying framework. By going up one level of abstractness, it proved possible to transcend the seemingly unbridgeable differences and to turn the two theories into one. This demonstrates the power of mathematics in bringing out hidden similarities and common structures.

But it should be noted that this unifying power of mathematics is not directly related to effectiveness in dealing with natural phenomena. The two theories under discussion—matrix and wave mechanics—can be regarded as purely mathematical schemes. They are unified by von Neumann's Hilbert space formulation, which itself can also be seen as purely mathematical. Mathematics is able to do its unifying work here because it is designed to describe structure (in this case the hidden common structure of wave and matrix mechanics), quite independently of whether this structure represents something in physical reality. *If* some kind of structure is realized in physical reality, mathematics can be counted on to give a fitting description. This statement cannot be reversed: if mathematics defines a certain structure, we cannot count on its importance in physical theory. The unifying power of mathematics does therefore not testify to an *a priori* rapport between mathematics and physical reality.

Von Neumann's Hilbert space formulation, with its non-commuting observables, has become standard. Still, it has not remained unchallenged. The *Bohm formulation of quantum mechanics* does not work with Hilbert space, but with configuration space as the fundamental arena of physical processes.

It operates with the classical particle concept, according to which particles possess definite positions and momenta at all times. By contrast, in the standard scheme physical systems cannot have both a definite momentum and position, because the corresponding operators do not commute.

There is no need to repeat the mathematical details of the Bohm approach, which are well-known. The point of mentioning this alternative to the standard formulation is that we have here another example of two completely different mathematical schemes that agree about the results of empirical observation. As in our previous example, it again is true that we cannot impose *everything* we might wish. In order to achieve empirical adequacy we have to accept non-locality of interactions in the Bohm theory, for instance. But the example provides another illustration of how mathematics allows us the maximum possible latitude in accommodating our methodological, interpretational and philosophical preferences. It thereby facilitates the formulation and execution of diverse research programmes, and consequently enhances the chances of progress. In the case at hand, it gives us the means to investigate to what extent the classical particle concept is still viable within the quantum context. Again, given that there is something out there to be discovered (this is what we assumed to begin with), it is no miracle that the richness of mathematical tools and the corresponding variety of possible research paths help us to actually do the discovering.

3 Holism

The Bohm theory differs from standard quantum theory in that it operates with quantities that are defined on space-time points—like we are used to in Newtonian physics. In this sense the Bohm theory is associated with a 'local' world picture. By contrast, standard quantum mechanics is 'holistic', because properties of composite systems often are not built up from properties of the component systems. Think, for example, of the two-electron singlet spin state, in which the total spin is definite but cannot be considered the sum of definite spin values of the individual particles. The empirical results are compatible both with Bohm's theory and with standard quantum mechanics, so they are compatible both with a local and a holistic treatment. More generally, discussions about 'locality' and 'holism' in physics usually cannot be decided by empirical data alone. The empirical findings have to be evaluated within a theoretical scheme—and mathematics is often able to supply schemes of different kinds.

An interesting further case is furnished by electrodynamics. Classical electrodynamics has a purely local form, in the sense that the central quantities $\overrightarrow{E}$ and $\overrightarrow{B}$ are fields, defined 'per point'. That is, an electric and a magnetic field strength are assigned to each spatial position, and these field strengths determine the force on a charged particle there. In addition, the theory works with electromagnetic potentials, ϕ and $\overrightarrow{A}$. These are also defined locally; moreover, they determine $\overrightarrow{E}$ and $\overrightarrow{B}$ via local relations. In the relativistic treatment these electromagnetic quantities are represented by the anti-symmetric electromagnetic field tensor $F_{\mu\nu}$ and the four-potential A_{μ}; again, both are defined locally.

The potentials are not uniquely determined by the observable phenomena: gauge transformations $A_{\mu} \longrightarrow A_{\mu} - \nabla_{\mu}\Lambda$, with Λ an arbitrary scalar field, change the local values of the potentials but leave the field strengths, and the forces exerted on charges, the same. In classical electrodynamics the underdetermination caused by this gauge freedom is usually considered as insignificant, because the electromagnetic potentials are regarded as purely mathematical expediencies—only $F_{\mu\nu}$ is accepted as physically real. In quantum mechanics, however, the situation becomes different: the wave function couples directly to A_{μ}. Even if no electromagnetic fields are present in a region (i.e., $F_{\mu\nu} = 0$), the wave function does not evolve freely if $A_{\mu} \neq 0$. The notorious example is the Aharonov-Bohm effect, in which an electron can move along two paths around a solenoid. Inside the solenoid there is a magnetic field, but $\overrightarrow{E}$ and $\overrightarrow{B}$ vanish outside of it. The electron moves outside the solenoid and therefore cannot experience the fields. Still, the electron's wave function is changed because of the presence of $\overrightarrow{A}$ in the region where the electric and magnetic fields disappear. The wave function incurs a phase $\int_C \overrightarrow{A}.d\overrightarrow{r}$ along a path C. This phase is empirically significant: the phase difference between the two paths around the solenoid, which is given by $\oint \overrightarrow{A}.d\overrightarrow{r}$ (with the integral taken over a closed contour surrounding the solenoid), is responsible for interference effects which can be measured. The presence of A_{μ} thus has observable effects, and the potential therefore cannot simply be dismissed as physically unreal.

Nevertheless, the gauge freedom $A_{\mu} \longrightarrow A_{\mu} - \nabla_{\mu}\Lambda$ is still there, because the integral $\oint \overrightarrow{A}.d\overrightarrow{r}$ is invariant under such gauge transformations. So the value of $\overrightarrow{A}$ *at a point* remains unobservable; it is only the integral taken over a closed path that is measurable.

One can now choose between two positions. One is that we are dealing with a completely local theory, characterized by the real physical fields $F_{\mu\nu}$ and A_μ. It is true that the local values of A_μ cannot be observed. But according to the position under discussion this does not automatically entail that there is nothing real corresponding to A_μ. Indeed, there are many things in physics which are not directly accessible, and about which information can only be obtained in a roundabout way. It is often taken for granted nevertheless that the entities in question exist—think of atoms or elementary particles. It is natural, however, if one is convinced of their reality, to look for better or more ways of observing them. In our case, it seems plausible to think of ways by which A_μ *could* be observed directly; to accommodate this theoretically, the theory should be modified. Now, suppose that such an attempt succeeds and results in a better theory, one that is able to predict more. Perhaps one is then inclined to say: "Mathematics has miraculously led the way; even before we could measure A_μ, mathematics already indicated its existence! Mathematics is unreasonably effective."

But we may also take the position that A_μ does *not* represent a real physical field. In that case it is plausible to look for formulations of the theory in which A_μ does not occur; one would like to be parsimonious and only represent quantities that do possess physical significance. Mathematics is an obedient servant: a formulation of electrodynamics in which the phases over closed contours (the gauge-invariant quantities that can be observed, as we saw above) are central can readily be found [10]. In this formulation one starts with the 'anholonomy' (the mentioned phase) associated with closed curves, and there is no need to introduce local potentials. Now, suppose no evidence for the reality of potentials is ever found. One is then perhaps inclined to say: "Mathematics has miraculously led the way: even before we learnt from experiments that A_μ has no physical existence, mathematics already indicated the holistic nature of electrodynamics! Mathematics is unreasonably effective."

The moral is that mathematics is so versatile that it can be effective regardless of the details of the situation and the actual development of physics.

4 Relativity

General relativity is sometimes adduced as an example of a situation in which a mathematical framework that was developed completely independently of physics proved unreasonably efficient. Differential geometry was first developed as a branch of geometry by Gauß—as a metrical theory of curved two-dimensional surfaces—and then generalized to an arbitrary number of dimensions by Riemann. In the second half of the 19th century the subject underwent further evolution, through the work of mathematicians like Levi-Civita. After Einstein got acquainted with differential geometry, this branch of mathematics proved to be of decisive importance in achieving a break-through in his struggle for a relativistic theory of gravitation.

I do not think that the great effectiveness of mathematics in this episode qualifies as unreasonable, in spite of the magnificent character of the achievement in question (the general theory of relativity). First, the considerable development of differential geometry in the 19th century shows no signs of a pre-established harmony between mathematics and physical needs. Rather, this development matches what we have stressed before: the freedom of mathematics from physical content and the concomitant possibility of evolution free from external influences. Indeed, Gauß's theory fits in perfectly with the historical tradition of work in geometry. The abstract character of mathematics made it subsequently possible and natural to construct a geometrical theory of spaces of an arbitrary number of dimensions, *in spite of* the fact that this notion seemed completely superfluous in physics.

Second, differential geometry did not inspire relativity in its initial stages. It is true, as demonstrated by Minkowski, that already special relativity can be regarded as a geometrical theory of a four-dimensional space-time manifold. But mathematics did not really anticipate this application of its concepts to physics (a point regretfully noted by Minkowski in his essay). The geometrical approach did not play a role in the genesis of special relativity, and it took Einstein considerable time to recognize the value of the geometrical viewpoint. Indeed, one can very well defend the viewpoint that Einstein's original three-dimensional treatment is closer to physical experience than the abstract four-dimensional approach. The situation is similar to the ones discussed above: there are more ways than one to formulate special relativity mathematically, and it cannot be decided beforehand which way will proffer the best chances of fruitful generalization. But one *can* see beforehand that once the geometrical formulation is taken seriously, going from flat Minkowski space-time to curved Riemannian space-time constitutes a way of generalizing special relativity; this generalization is in its mathematical essence identical to what Gauß did in going from flat to curved surfaces. So differential geometry is evidently a suitable instrument to achieve one type of generalization of special relativity.

So, the development of differential geometry can be understood from the internal dynamics of mathematics, without reference to its later application in relativity. The mathematics of differential geometry did not play a role in the genesis of special relativity. After the special theory had been developed, it turned out that differential geometry could be used as a tool—but that mathematics is able to give a geometrical description of special relativity cannot be considered remarkable, given its nature of a theory of invariants. It was not obvious beforehand that the use of differential geometry, and the type of generalization of special relativity suggested by it, would lead to a revolutionary new physical theory. Indeed, many physicists made attempts to incorporate gravitation into relativity in a non-geometrical way. That the application of differential geometry to relativity was in fact highly successful is very understandable with hindsight, given that general relativity has uncovered that the space-time of our world is curved. That is almost tautological, and does not point into the direction of a pre-established harmony between the developments of mathematics and physics. If one of the other research programmes that were pursued after 1905 (e.g., Abraham's or Lorentz's) had been successful, the geometrical approach might have been forgotten by now.

One might answer that it is still an unreasonable coincidence that the geometrical tools lay ready just in time, waiting for Einstein to come along. I do not think that this is a convincing manoeuvre, however. We already saw that it is of the essence in mathematics that developments take place freely, in diverse directions. There is a steady rate of addition of new tools to the mathematical repertoire. In line with this, cases in which there are several mathematical approaches to choose from abound in the history of physics; perhaps they occur more frequently than cases in which no suitable mathematical instruments are available at all. But situations of the latter kind do certainly happen too. For example, in present-day elementary particle physics physicists feel obliged to develop their own specialized new mathematics, adapted to the particular needs of string and membrane theories. This underscores the fact that the development of mathematics is not tuned to needs about to arise in physics.

5 Transporting insight

It is a well-known phenomenon that mathematical models and techniques used in the context of one physical problem are often also applicable to completely different areas in physics. This is made possible by the neutrality, in the sense of freedom of physical content, of mathematics: the same mathematical objects and symbols can receive completely different physical interpretations. Results achieved in one context can thus be translated to other contexts. For example, the same equations apply to electrostatics and laminar flow in fluids; these very different phenomena can both be regarded as models (in the sense of model theory) of the equations.

Mathematical correspondences between different fields are often used for illustrative purposes, for instance in physics education. But, importantly, they also play a significant role at the forefront of physical research, in breaking new ground. An interesting example from recent research in the foundations of physics is provided by a translation of the famous Bell theorem to a space-time context.

Bell's theorem demonstrates that the measurement results that are predicted by quantum mechanics cannot be interpreted as simply mirroring system properties that already existed independently of the measurements. This is to be contrasted with the situation in classical physics. According to classical mechanics, for example, particles possess properties like position and momentum quite independently of whether any measurements of these quantities take place. *If* measurements are made, the results should of course reflect the values that were already there. But as just said, this cannot be maintained in quantum theory. Here, the outcome of a measurement is in general not the reflection of an object system property that was already there; and what is more, it cannot even be considered to be independent of measurements performed far away.

The latter point is illustrated by experiments of the Einstein-Podolsky-Rosen type [4]. In a modern version, two electrons whose total spin state is the singlet state, in which the total spin is zero, fly apart until their mutual distance has become very great. Subsequently, spin measurements are made on the individual particles. For each particle, there is the choice of measuring the spin in one of two directions. The experiment can be repeated with different choices of these directions, so that four combinations of directions will be measured in the series of repetitions. Correlations between outcomes in these four pairs of directions are predicted by quantum mechanics, and are verified in actual experiments. As is well known, these correlations violate the Bell inequality. Now, it is a mathematical fact that the Bell inequality is satisfied as soon as the spin values found in the measurements on the individual particles can be regarded as coming from one joint probability distribution of spin values [5]. The latter would be the case if the individual electrons possessed spin values in all directions, as in classical theory, independently of which—or whether—measurements are going to be made. If that were true, there would be well-defined, definite spin values in the four directions under discussion in each run of the experiment; in repetitions of the experiment these values would vary and form an ensemble that defines a joint distribution of the four spin quantities. Only two of them could actually be measured in any single experimental run (one direction for each particle); the measured values would therefore be samples from this joint distribution. The violation of Bell's inequality by the predictions of quantum mechanics, and by the experimental results, shows that we cannot think of the EPR situation in such classical terms—the measurements do not reveal pre-existing jointly defined quantities. The four spin quantities do not have a joint probability distribution and can therefore not be thought of as co-existing independently of the measurements.

A consequence of this result is that the actually measured spin values cannot reveal local particle spins, independently of the kind of measurement performed on the other, far-away particle. See Figure 1 for a schematic representation of the situation: either σ_1 or σ_1' is measured on electron 1, and σ_2 or σ_2' on electron 2. The two horizontal and two diagonal lines symbolize the four possible combinations of measurements. The vertical double lines represent the electrons.

This result is shocking for the classical intuition. It undermines the classical concept of locality, and even the very concepts of physical properties and physical systems; its ramifications have not been completely digested yet. But it turns out that there is even more in store. Exactly the same mathematical structure can be recognized in a new situation, so that the argument can be repeated there—with results that appear even farther-reaching [6].

Consider two well-localized systems, S_i, $i = 1, 2$. Let α and β be two hyperplanes of simultaneity for some reference frame Σ. Let E_i be the places where the systems S_i are located on α, and let F_i be the corresponding regions on β (see Figure 2). We assume that the two systems are sufficiently far apart that E_1 is spacelike separated from F_2, and E_2 is spacelike separated from

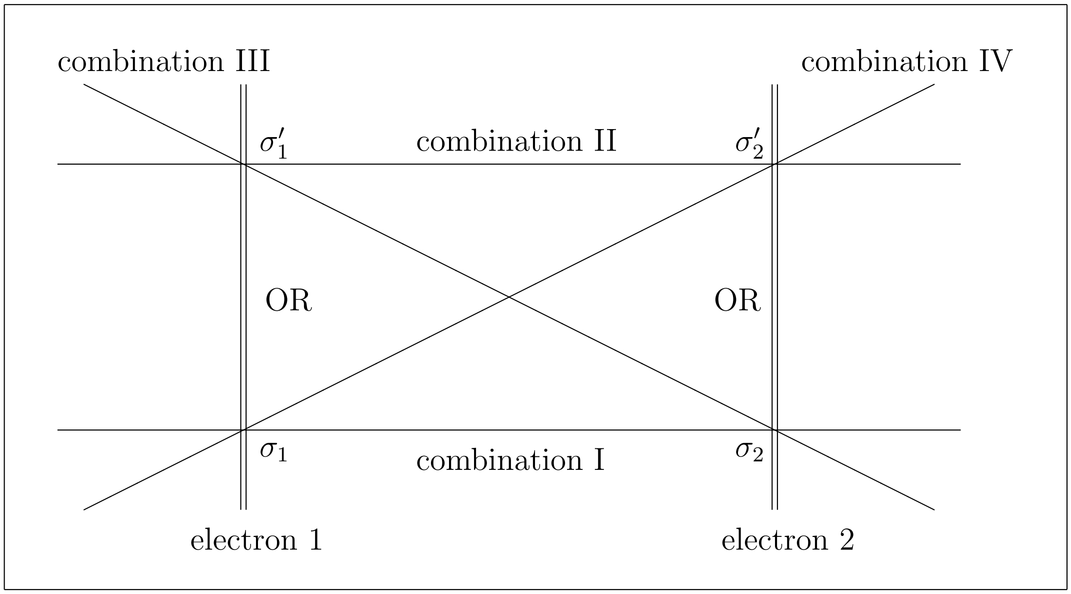

Figure 1. The four possible combinations of spin measurements.

F_1. Let γ be a spacelike hypersurface containing F_1 and E_2, and let δ be a spacelike hypersurface containing E_1 and F_2.

If S_1 and S_2 are isolated during their evolution between α and β there will be unitary operators U_i such that the state of the combined system $S_1 \oplus S_2$ on β will be related to its state on α by

$$\rho(\beta) = U_1 \otimes U_2 \; \rho(\alpha) \; U_1^\dagger \otimes U_2^\dagger. \tag{1}$$

If the regions E_1, E_2, F_1, F_2 are sufficiently small, they may be treated as points, and we may regard γ and δ as hyperplanes of simultaneity for reference frames Σ', Σ'', respectively. Let $\rho(\gamma)$ be the state on hypersurface γ, and let $\rho(\delta)$ be the state according on δ. On the basis of the assumption of unitary evolution between α and β, the states on the hyperplanes γ and δ can easily be related to $\rho(\alpha)$. We find:

$$\rho(\gamma) = U_1 \otimes I_2 \; \rho(\alpha) \; U_1^\dagger \otimes I_2, \tag{2}$$

and similarly

$$\rho(\delta) = I_1 \otimes U_2 \; \rho(\alpha) \; I_1 \otimes U_2^\dagger. \tag{3}$$

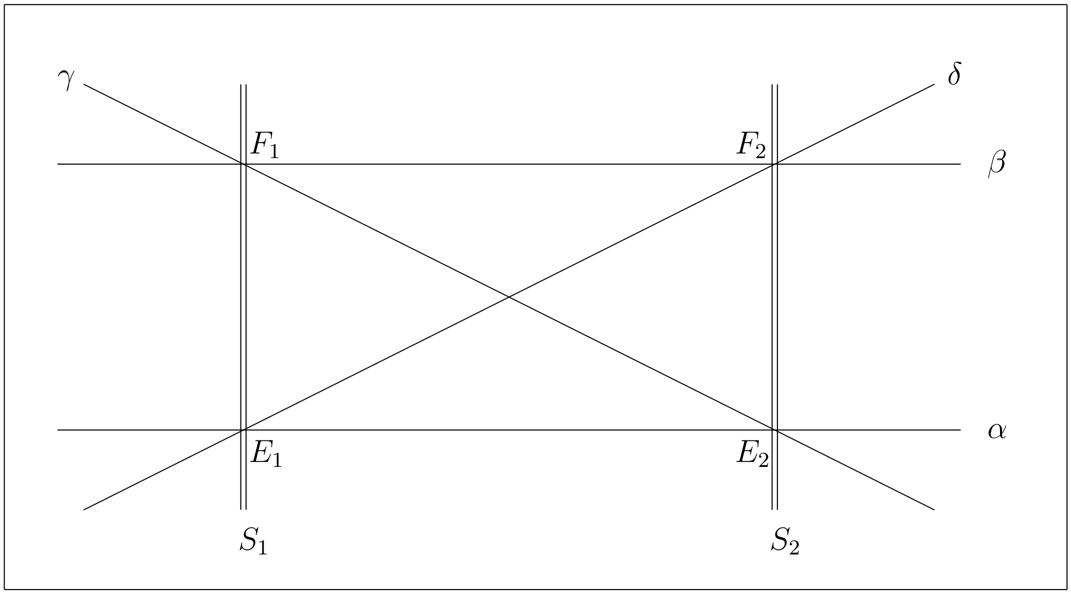

Figure 2. The four simultaneity hyperplanes α, β, γ and δ.

Now suppose that A_1 and A_2 are definite properties of S_1 and S_2, respectively, on α, and B_1 and B_2 are definite properties on β. This supposition fits in with interpretations of quantum mechanics according to which the quantum state assigns probabilities to objectively existing quantities (Bohm's interpretation or modal interpretations, for instance [2, 3]). Suppose further that the value of A_1 possessed by S_1 at E_1 is possessed by it without reference to the hypersurface containing E_1 that is contemplated, and similarly for the other points of intersection E_2, F_1, F_2; this is just the almost self-evident assumption that what happens at these four points are objective events located in space-time. There must then be a joint probability distribution over the values of our four observables, that yields as marginals the quantum mechanical Born probabilities on all four hyperplanes. In this we have assumed the central tenet of special relativity, namely that the different frames of reference are equivalent; in our case that the Born probability rule applies equally on α, β, γ and δ.

But the states on the various hyperplanes are interrelated, as indicated in Eqs. (2, 3). By inspection of these relations we find that the existence of such a joint distribution is equivalent to the existence of a joint distribution calculated in one state, namely $\rho(\alpha)$, and yielding, as marginals, the statistics for the observables $A_1 \otimes A_2$, $A_1 \otimes C_2$, $C_1 \otimes A_2$, $C_1 \otimes C_2$, where

$$C_i = U_i^\dagger \, B_i \, U_i. \tag{4}$$

However, as we have explained for the case of the EPR-experiment, such a joint distribution of four non-commuting observables, yielding the quantum mechanical Born marginals for the pairs of observables, cannot exist in general [5]. Bell inequalities can be violated if there are no restrictions on the state, and the violation of a Bell inequality entails the nonexistence of a joint distribution. Therefore, if $\rho(\alpha)$ is a state such that a Bell inequality can be derived for the observables A_1, C_1, A_2, C_2, then it cannot be the case that A_1 is objective at E_1, A_2 is objective at E_2, B_1 is objective at F_1, and B_2 is objective at F_2.

The argument here completely mimics the earlier Bell argument: the mathematics is the same. The structural identity of the two arguments can clearly be seen from the similarity between Figure 1 and Figure 2. The symbols have different meanings, but the mutual relations are the same. Whereas in the original Bell case locality was at issue, we now find that it must make a difference whether we consider what happens in E_1, e.g., from the perspective of E_2 or from the perspective of F_2. In other words, events are not just there, but are different depending on the hyperplane of which they are considered a part. This result is a lot more perplexing than the original Bell non-locality conclusion! In the Bell case a property was shown to depend on what kind of far-away measurement is made. But since only one such measurement can actually be made, no conflict arises with the uniqueness and objectivity of the property in question. In our new case, however, all the different contexts, i.e. the different hyperplanes, are jointly actual. So, *events cannot in general be unique and objective in themselves* according to this quantum mechanical scheme, but must depend on the hyperplane on which they are considered to lie: a truly amazing conception.

6 Conclusion

The previous section illustrates how mathematical arguments and techniques that are elementary and well-known in themselves can lead to unexpected new results, new ideas, and new directions of research. In the concrete case discussed, indications are that the very concept of an 'event' has to be modified in quantum mechanics. Quite generally, it appears that properties of physical systems are relational in character—that they are *perspective-dependent* [1]. This conclusion is tentative, and further research concerning these issues is needed. It is not the purpose of this article to argue for specific theses concerning the nature of quantum mechanical reality.

Rather, what I wanted to show is how mathematics, by its very nature of a subject without physical content, lends itself to an unlimited variety of applications. As soon as some type of order, structure or regularity is present in an area, mathematics becomes almost automatically useful. Because of the strong internal dynamics of the discipline, and the steady growth of its repertoire, it is not unlikely that some suitable mathematical technique is already available when new fields of physical research are opened up. If not, this will be an impetus to develop new mathematical tools for the purpose at hand.

It would be wrong to summarize this by saying that mathematics is *nothing but* a language, which can be employed in many circumstances. Because mathematics is so expressive and rich in conceptual tools, it transcends the role of just a simple means of description; it can sometimes lead the way in physical research. More accurately, mathematics does not impose directives about how to proceed; rather, the abundance of mathematical instruments makes it possible for researchers of all sorts and conditions to proceed along the ways of their own liking. In this process, mathematics can suggest generalizations and new directions, as illustrated by the cases of holism and relativity. It can also provide new insight by transporting old results to new contexts. It can thus make for 'miraculous' progress, even though its effectiveness is no wonder at all.

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