FOURIER INVERSION ON A REDUCTIVE SYMMETRIC SPACE

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Contents

1. Introduction

Let X be a semisimple symmetric space. In previous papers, [8] and [9], we have defined an explicit Fourier transform for X and shown that this transform is injective on the space $C_c^\infty(X)$ of compactly supported smooth functions on X. In the present paper, which is a continuation of these papers, we establish an inversion formula for this transform.

More precisely, let $X = G/H$, where G is a connected semisimple real Lie group with an involution σ and H is an open subgroup of the group of elements in G fixed by σ. Let K be a maximal compact subgroup of G invariant under σ; then K acts on X from the left. Let (τ, V_τ) be a finite dimensional unitary representation of K. The Fourier transform $\mathcal{F}$ that we are going to invert is defined as follows, for τ-spherical functions on X, that is, V_τ-valued functions f satisfying $f(kx) = \tau(k)f(x)$ for all $k \in K$, $x \in X$. Related to the (minimal) principal series for X and to τ, there is a family $E(\psi\colon \lambda)$ of Eisenstein integrals on X (cf. [5]). These are sums of τ-spherical joint eigenfunctions for the algebra $\mathbb{D}(X)$ of invariant differential operators on X; they generalize the elementary spherical functions for Riemannian symmetric spaces, as well as Harish-Chandra's Eisenstein integrals associated with a minimal parabolic subgroup of a semisimple Lie group. The Eisenstein integral is linear in the parameter ψ, which belongs to a finite dimensional Hilbert space ${}^\circ\mathcal{C}$ depending on τ, and it is meromorphic in λ, which belongs to the complex linear dual $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ of a maximal abelian subspace $\mathfrak{a}_{\mathrm{q}}$ of $\mathfrak{p} \cap \mathfrak{q}$. Here $\mathfrak{p}$ is the orthocomplement in $\mathfrak{g}$ (the Lie algebra of G) of $\mathfrak{k}$ (the Lie algebra of K), and $\mathfrak{q}$ is the orthocomplement in $\mathfrak{g}$ of $\mathfrak{h}$ (the Lie algebra of H). In [8] we introduced a particular normalization $E^\circ(\psi\colon\lambda)$ of $E(\psi\colon\lambda)$ with the property that as a function of λ it is regular on the set $i\mathfrak{a}^*_{\mathrm{q}}$ of purely imaginary points in $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$. Now $\mathcal{F}f$ is defined as the meromorphic ${}^\circ\mathcal{C}$-valued function on $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ such that

$$\langle \mathcal{F}f(\lambda)|\psi\rangle = \int_X \langle f(x)|E^\circ(\psi\colon\lambda\colon x)\rangle\, dx \tag{1.1}$$

holds for all $\psi \in {}^\circ\mathcal{C}$, $\lambda \in i\mathfrak{a}^*_{\mathrm{q}}$. Here dx is an invariant measure on X, $\langle\cdot|\cdot\rangle$ denotes the sesquilinear inner products on ${}^\circ\mathcal{C}$ and V_τ, and f belongs to the space $C_c^\infty(X\colon\tau)$ of compactly supported smooth τ-spherical functions on X. The Fourier transform on K-finite functions in $C_c^\infty(X)$ can be expressed in terms of the transform $\mathcal{F}$ with suitable τ (see [8, §6]), and an inversion formula for $\mathcal{F}$ thus amounts to an inversion formula for K-finite functions. Expansion over all K-types then yields an inversion formula for all functions in $C_c^\infty(X)$. From now on we shall therefore concentrate on the inversion problem for $\mathcal{F}$ with a fixed K-representation τ.

At first glance a good candidate for the inverse of $\mathcal{F}$ would be the wave packet map $\mathcal{J}$ defined as follows, for φ a ${}^\circ\mathcal{C}$-valued function (of reasonable decay) on $i\mathfrak{a}^*_{\mathrm{q}}$

$$\mathcal{J}\varphi(x) = \int_{i\mathfrak{a}^*_{\mathrm{q}}} E^\circ(\varphi(\lambda)\colon\lambda\colon x)\, d\lambda; \tag{1.2}$$

here $d\lambda$ is a suitably normalized Lebesgue measure on the Euclidean space $i\mathfrak{a}^*_{\mathrm{q}}$. In the case of a Riemannian symmetric space it is indeed true that $\mathcal{J}\mathcal{F} = I$ (cf. [23, Ch. III] and [9, Remark 14.4]), but in general this is not so. In [9] we showed that (taking appropriate closures) the operator $\mathcal{J}\mathcal{F}$ is the orthogonal projection onto a closed subspace of the space $L^2(X\colon\tau)$ of all τ-spherical L^2-functions on X. The subspace is the so-called most continuous part of $L^2(X\colon\tau)$. In general the functions $\mathcal{J}\mathcal{F}f$, $f \in C_c^\infty(X\colon\tau)$, do not belong to $C_c^\infty(X\colon\tau)$; they are smooth functions of L^2-Schwartz type, but not of compact support. A central result in [9] asserts the existence of an invariant differential operator D_0 (depending on τ) on X that is injective as an endomorphism of $C_c^\infty(X)$ and satisfies

$$D_0\mathcal{J}\mathcal{F}f = D_0 f \tag{1.3}$$

for all $f \in C_c^\infty(X\colon\tau)$ (see Theorem 2.1). The injectivity of the Fourier transform is an immediate consequence, but as we do not know an explicit inverse to D_0, (1.3) does not give the inversion formula we want.

The inversion formula that we obtain involves not only the function $\mathcal{F}f$ on $i\mathfrak{a}^*_{\mathrm{q}}$ but also its meromorphic continuation. In order to describe it, we must introduce some more notation. Let Σ denote the system of roots for $\mathfrak{a}_{\mathrm{q}}$ in $\mathfrak{g}$; the corresponding Weyl group W can be realized as the normalizer modulo the centralizer of $\mathfrak{a}_{\mathrm{q}}$ in K. Let Σ^+ be a positive system for Σ and let $A^+_{\mathrm{q}} = \exp \mathfrak{a}^+_{\mathrm{q}}$, where $\mathfrak{a}^+_{\mathrm{q}}$ is the corresponding open Weyl chamber. For simplicity of exposition, we assume in this introduction that the open subset $X_+ = KA_{\mathrm{q}}^+H$ of X is dense (in general a finite and disjoint union of open sets of the form KA_{q}^+wH, $w \in K$, is dense). The normalized Eisenstein integral $E^\circ(\psi\colon\lambda)$ has an expansion (see [10])

$$E^\circ(\psi\colon\lambda\colon x) = \sum_{s\in W} E_+(s\lambda\colon x)C^\circ(s\colon\lambda)\psi, \tag{1.4}$$

valid on X_+, that is a generalization of Harish-Chandra's expansion for the spherical functions on a Riemannian symmetric space. Here $C^\circ(s\colon\lambda)$ is an endomorphism of ${}^\circ\mathcal{C}$, and $E_+(s\lambda\colon x)$ is a linear operator from ${}^\circ\mathcal{C}$ to V_τ. Both of these objects depend meromorphically on λ. For $\psi \in {}^\circ\mathcal{C}$ and λ generic, the function $x \mapsto E_+(\lambda\colon x)\psi$ is defined on X_+ as the unique τ-spherical annihilated by the same ideal of $\mathbb{D}(X)$ as $E^\circ(\psi\colon\lambda)$ and having the leading term $a^{\lambda-\rho}\psi(e)$ in the asymptotic expansion along A_{q}^+. It can be shown (see [5] and [10]) that if $\eta \in \mathfrak{a}_{\mathrm{q}}^*$ is sufficiently antidominant then $\mathcal{F}f(\lambda)$, as well as $E_+(\lambda\colon x)$, are regular for $\lambda \in \eta + i\mathfrak{a}_{\mathrm{q}}^*$. Moreover, these functions of λ have decay properties that allow us to conclude that the expression

$$\mathcal{T}_\eta\mathcal{F}f(x) := |W| \int_{\eta+i\mathfrak{a}_{\mathrm{q}}^*} E_+(\lambda\colon x)\mathcal{F}f(\lambda)\,d\lambda, \tag{1.5}$$

is defined for $x \in X_+$ and (by Cauchy's theorem) independent of η, provided the latter quantity is sufficiently antidominant ($|W|$ is the order of W). We then denote it $\mathcal{T}\mathcal{F}f(x)$ and call it a *pseudo wave packet*. As a function of $x \in X_+$ it is smooth and τ-spherical, and by moving η to infinity one can show that $\mathcal{T}\mathcal{F}f(x)$ vanishes for x outside a set with compact closure in X. Our main result in the present paper is the following (Theorem 4.7).

Theorem 1.1. *Let $f \in C_c^\infty(X\colon\tau)$. Then $\mathcal{T}\mathcal{F}f(x) = f(x)$ for all $x \in X_+$.*

Since X_+ is dense in X this provides the desired inversion formula for $\mathcal{F}$ on $C_c^\infty(X\colon\tau)$. The proof of Theorem 1.1 is carried out in Sections 5-9, but it rests on results from several previous papers. In particular, the papers [11] and [12] have been written primarily for this purpose. We shall now indicate some important steps in the proof. Inserting the expansion (1.4) in (1.2) (for this introduction we disregard the fact that $\lambda \mapsto E_+(\lambda\colon x)$ can be singular for $\lambda \in i\mathfrak{a}_{\mathrm{q}}^*$), and using simple Weyl group transformation properties for the involved functions, one sees that the wave packet $\mathcal{J}\mathcal{F}f$ is identical with $\mathcal{T}_0\mathcal{F}f$, the expression (1.5) for $\eta = 0$. We would like to identify this expression with the pseudo wave packet $\mathcal{T}\mathcal{F}f$, but because there are singularities between $\eta = 0$ and the sufficiently antidominant η, the difference between the two expressions involves residues. In order to study closer these residues we invoke (in Section 5) the residue calculus for root systems that we have developed in [11]. According to this calculus, the difference is a finite sum of expressions of the form

$$\int_{\lambda+i\mathfrak{a}_{F\mathrm{q}}^*} u[\pi E_+(\cdot\colon x)\mathcal{F}f](s\nu)\,d\nu, \tag{1.6}$$

where F is a non-empty subset of the set Δ of simple roots for Σ^+, $\mathfrak{a}_{F\mathrm{q}}^*$ its orthocomplement in $\mathfrak{a}_{\mathrm{q}}^*$, and λ a point in $\mathbb{R}_+F \subset \mathfrak{a}_{F\mathrm{q}}^{*\perp}$. Furthermore, s is an element of W with $s(F) \subset$

Σ^+, π is a suitable polynomial such that $\pi E_+(\cdot : x)\mathcal{F}f$ is regular on a neighborhood of $\mathrm{Ad}(s)(\lambda + \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}})$, and $u \in S(\mathrm{Ad}(s)\mathfrak{a}^{*\perp}_{F\mathrm{q}})$ serves as a constant coefficient differential operator on $\mathrm{Ad}(s)\mathfrak{a}^{*\perp}_{F\mathrm{q}}$. These objects (i.e. λ, s, π and u) can be chosen independently of f and x. We denote by $T_F f(x)$ the sum of all the contributions of the form (1.6) for a given non-empty $F \subset \Delta$. The function $T_F f$ is τ-spherical and smooth on X_+. We now have

$$\mathcal{T}\mathcal{F}f = \mathcal{J}\mathcal{F}f + \sum_{F\subset\Delta, F\neq\emptyset} T_F f = \sum_{F\subset\Delta} T_F f,$$

where we have set $\mathcal{J}\mathcal{F}f = T_\emptyset f$, and the result in Theorem 1.1 can be expressed as follows (Theorem 7.1).

Theorem 1.2. *Let $f \in C_c^\infty(X : \tau)$ and $x \in X_+$. Then*

$$f(x) = \sum_{F\subset\Delta} T_F f(x). \tag{1.7}$$

The main step in the proof of this result consists of establishing the following properties of the operators T_F. In order to simplify the presentation, we assume in the second statement of the following theorem that the map $\lambda \mapsto -\lambda$ belongs to W (see Corollary 10.11 for the general statement).

Theorem 1.3. *The function $T_F f$ on X_+ extends to a smooth function on X, for all $f \in C_c^\infty(X : \tau)$, $F \subset \Delta$. Moreover, the operator $f \mapsto T_F f$ is symmetric, that is,*

$$\int_X \langle T_F f_1(x) | f_2(x)\rangle\, dx = \int_X \langle f_1(x) | T_F f_2(x)\rangle\, dx$$

for all $f_1, f_2 \in C_c^\infty(X : \tau)$.

Theorem 1.3 is first proved under the assumption (which is sufficient to derive Theorem 1.2) that f and f_2 are supported on X_+. This is done (in Section 9) by induction on the number of elements in Δ. The derivation of Theorem 1.2 from Theorem 1.3 is given in Section 7. We shall now outline the proof of Theorem 1.3, which is a central argument for the paper.

We first derive the statements in Theorem 1.3 for $F \neq \Delta$. This is done by a careful analysis of the asymptotic expansion of the integral kernel corresponding to the operator T_F. The principal term in the asymptotic expansion along the standard parabolic subgroup P_F associated with F can be identified in terms of the T_Δ for the Levi subgroup of P_F. Invoking the induction hypothesis and a result from [12] (see Appendix B), the symmetry of T_F is obtained. The smooth extension is a consequence of the symmetry.

Next, we consider the function $g := f - \sum_{F\subset\Delta} T_F f$ on X_+. The statement in Theorem 1.2 is that $g = 0$; we know already that g vanishes outside a set Ω with compact closure in X, since both f and $\mathcal{T}\mathcal{F}f$ have the same property. Knowing also that $T_F f$ extends smoothly to X for $F \neq \Delta$ we are able to deduce that g is annihilated by any invariant differential operator on X that annihilates $T_\Delta f$. Here the result (1.3) from [9] plays an important role. It follows that the annihilator of g in the algebra $\mathbb{D}(X)$ of invariant differential operators on X is a cofinite ideal. Since g is τ-spherical, g is hence analytic on X_+, and since it vanishes outside Ω it must then vanish identically. Equation (1.7) is thus proved for functions supported in X_+. From this the statements of Theorem 1.3 for $F = \Delta$ finally follow (with $\operatorname{supp} f, f_2 \subset X_+$), and the induction is completed.

The part of the proof of Theorem 1.3 outlined above is given in Sections 8-9. In Section 10 we define some generalized Eisenstein integrals and derive a formula for T_F in terms of these. Theorem 1.3 in its full generality follows from this formula.

The inversion formula that we have derived in this paper is an important step towards the Plancherel formula for X. What remains for the Plancherel formula is essentially to identify the contributions $T_F f$ in terms of generalized principal series representations. For example, $T_\Delta f$ should be identified as being in the discrete series for X. These identifications will be given in a sequel [13] to this paper, but since it is an important application we outline the argument here. For $F = \emptyset$ the identification is inherent already in the definition of $\mathcal{F}$ and $\mathcal{J}$ by means of the minimal principal series – an important ingredient is the regularity (from [8]) of the normalized Eisenstein integrals on $i\mathfrak{a}_{\mathrm{q}}^*$. This regularity is, in turn, based on the so-called Maass-Selberg relations from [6], according to which (cf. [9, Prop. 5.3]) the adjoint of the C-function is given by

$$C^\circ(s\colon \lambda)^* = C^\circ(s\colon -\bar{\lambda})^{-1}. \tag{1.8}$$

For the non-minimal principal series analogues of $\mathcal{F}$ and $\mathcal{J}$ have been defined and the Maass-Selberg relations have been generalized, by Carmona and Delorme (see [14], [18], [19], [15]). Using these generalized Maass-Selberg relations we obtain the necessary identifications of $T_F f$ for $F \neq \Delta$. In particular, these functions are tempered. As a consequence of Theorem 1.2 it follows then that $T_\Delta f$ is in the discrete series, and the Plancherel formula is established. A different proof of the Plancherel formula, also based on the generalized Maass-Selberg relations, has been obtained independently and simultaneously by Delorme (see [20]). Later, we have found a proof of these generalized Maass-Selberg relations based on the results of the present paper. This proof will also be given in [13].

For the special case that G/H has but one conjugacy class of Cartan subspaces the Plancherel formula is easier to obtain than by the argument described above. In this case the contributions $T_F f$ for $F \neq \emptyset$ all vanish; we prove this in Section 11, using [25]. Hence in this case we have $\mathcal{J}\mathcal{F} = I$ as in the case of a Riemannian symmetric space (which, in fact, is a subcase).

Another important application of the results presented here is to the Paley-Wiener theorem for τ-spherical functions on X, that is, the description of the range $\mathcal{F}(C_c^\infty(X\colon\tau))$. A conjectural description was given in [9, Remark 21.8], and based on the results of the present paper we are able to prove this conjecture. The first step is given here in Corollary 4.11; the further steps will be given in [13]. The Paley-Wiener theorem for X generalizes Arthur's theorem for G (which is a semisimple symmetric space by itself), [1], the proof of which has been a substantial source of inspiration for the present work. In particular, the inversion formula of Theorem 1.1 is in this special case a consequence of Arthur's result. There are some important differences, however, to Arthur's treatise. First of all, Arthur appeals to Harish-Chandra's Plancherel theorem in his derivation of the Paley-Wiener theorem, whereas eventually we shall derive both theorems from the present results. In this respect our proof is very much in the spirit of that given by Rosenberg and Helgason for the Riemannian symmetric spaces, see [22, Ch. IV, 7]. Secondly, Arthur uses in the inductive argument a lifting theorem due to Casselman (see [1, Thm. II, 4.1]). The use of this result (the proof of which seems as yet unpublished) is here replaced by Theorem 1.3 and the induction of relations of [12], which is explained in Appendix B of this paper.

In the final Section 12 we generalize our inversion formula $\mathcal{T}\mathcal{F}f = f$ to rapidly decreasing functions f on X. The space $\mathcal{S}$ of these functions has been studied, for example, in [21]. For G it is introduced in [29, §7]; it plays an important role in the theory of completions of admissible $(\mathfrak{g}, K)$-modules, developed by Casselman and Wallach (cf. [30, §11], [16]).

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2. Notation and preliminaries

In this paper we use the same notation and basic assumptions as in [9, §§2-3]. In particular, and more generally than what was assumed in the introduction, G is a reductive Lie group of Harish-Chandra's class. As before we write $A_{\mathrm{q}}^+ = \exp \mathfrak{a}_{\mathrm{q}}^+$ where $\mathfrak{a}_{\mathrm{q}}^+$ is an open Weyl chamber in $\mathfrak{a}_{\mathrm{q}}$. The simplifying assumption, that KA_{q}^+H is dense in $X = G/H$, is abandoned. However, the open subset X_+ of X defined by the disjoint union

$$X_+ = \cup_{w\in\mathcal{W}} KA_{\mathrm{q}}^+ wH \tag{2.1}$$

is dense in X (see [9, eq. (2.1)]). The map

$$(k, a, w) \mapsto kw^{-1}awH \tag{2.2}$$

induces a diffeomorphism of $K/(K \cap H \cap M) \times A_{\mathrm{q}}^+ \times \mathcal{W}$ onto X_+. Notice that X_+ does not depend on the choice of the Weyl chamber $\mathfrak{a}_{\mathrm{q}}^+$.

Let (τ, V_τ) be a finite dimensional unitary representation of K, and let $°\mathcal{C} = °\mathcal{C}(\tau)$ be the finite dimensional Hilbert space defined by [9, eq. (5.1)]. For $\psi \in °\mathcal{C}$, $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ and $x \in X$ we define the Eisenstein integral $E(\psi\colon \lambda\colon x) \in V_\tau$ and its normalized version $E°(\psi\colon \lambda\colon x)$ as in [9, §5]. These are τ-spherical functions of x, and they depend meromorphically on λ. We view $E°(\lambda\colon x) := E°(\,\cdot\,\colon \lambda\colon x)$ as an element in $\mathrm{Hom}(°\mathcal{C}, V_\tau)$ and define $E^*(\lambda\colon x) \in \mathrm{Hom}(V_\tau, °\mathcal{C})$, likewise meromorphic in λ, by

$$E^*(\lambda\colon x) = E°(-\bar{\lambda}\colon x)^*, \qquad (x \in X). \tag{2.3}$$

Here the asterisk on the right-hand side indicates that the adjoint has been taken. Then $E^*(\lambda\colon kx) = E^*(\lambda\colon x) \circ \tau(k)^{-1}$ for $k \in K$, and the τ-spherical Fourier transform (1.1) of a function $f \in C_c^\infty(X\colon \tau)$ is conveniently expressed as

$$\mathcal{F}f(\lambda) = \int_X E^*(\lambda\colon x) f(x)\, dx \in °\mathcal{C}. \tag{2.4}$$

In the same spirit we write the definition (1.2) of the wave packet as

$$\mathcal{J}\varphi(x) = \int_{i\mathfrak{a}_{\mathrm{q}}^*} E^\circ(\lambda\colon x)\varphi(\lambda)\, d\lambda \tag{2.5}$$

for $\varphi\colon i\mathfrak{a}_{\mathrm{q}}^* \to {}^\circ\mathcal{C}$ of suitable decay; it is a smooth function of $x \in X$ (see [9, §9]). In these expressions measures are normalized according to [9, §3].

Recall from [9, eq. (5.11)] that there exists a homomorphism μ from $\mathbb{D}(X)$ to the algebra of $\mathrm{End}({}^\circ\mathcal{C})$-valued polynomials on $\mathfrak{a}_{\mathrm{q}}$ such that $DE^\circ(\lambda) = E^\circ(\lambda) \circ \mu(D\colon \lambda)$ for all $D \in \mathbb{D}(X)$. Moreover,

$$\mathcal{F}(Df) = \mu(D)\mathcal{F}f, \qquad D\mathcal{J}\varphi = \mathcal{J}(\mu(D)\varphi) \tag{2.6}$$

for f and φ as above (see [9, Lemmas 6.2, 9.1]).

Theorem 2.1. [9] *There exists an invariant differential operator $D_0 \in \mathbb{D}(X)$ such that $D_0\colon C_c^\infty(X\colon\tau) \to C_c^\infty(X\colon\tau)$ is injective and such that $D_0\mathcal{J}\mathcal{F}f = D_0 f$ for all $f \in C_c^\infty(X\colon\tau)$. In particular, if $\mathcal{F}f = 0$ then $f = 0$.*

Proof. Choose D_0 from the set $\mathbb{D}'_\pi$ defined in [9, Lemma 15.3]. By [9, Thm. 14.1, Prop. 15.2] it has the required properties. The final statement (which is [9, Thm. 15.1]) is an immediate consequence. $\square$

The Eisenstein integrals allow certain asymptotic expansions that we shall now recall (cf. [10]). Let $P \in \mathcal{P}_\sigma^{\min}$ be the σ-minimal parabolic subgroup of G that corresponds to the chosen chamber $\mathfrak{a}_{\mathrm{q}}^+$; then there exists (see [8, §§4-5]), for each $s \in W$, a unique meromorphic $\mathrm{End}({}^\circ\mathcal{C})$-valued function $\lambda \mapsto C^\circ(s\colon\lambda) = C^\circ_{P|P}(s\colon\lambda)$ on $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ (called the normalized C-function) such that

$$E^\circ(\lambda\colon aw)\psi \sim \sum_{s\in W} a^{s\lambda-\rho}[C^\circ(s\colon\lambda)\psi]_w(e)$$

for each $w \in \mathcal{W}$ and all $\lambda \in i\mathfrak{a}_{\mathrm{q}}^*$, as $a \to \infty$ in A_{q}^+. Here $[\,\cdot\,]_w(e) \in V_\tau^{K\cap M\cap wHw^{-1}}$ indicates the evaluation at e of the w-component of the element from ${}^\circ\mathcal{C}$ inside the square brackets (see [8, eqs. (17-18)]). In fact, for $a \in A_{\mathrm{q}}^+$ and $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ generic, there is a converging expansion for $E^\circ(\psi\colon\lambda\colon aw)$ as a function of a on A_{q}^+. This expansion is conveniently expressed by means of the $\mathrm{End}(V_\tau^{K\cap M\cap wHw^{-1}})$-valued functions $\Phi_{P,w}(\lambda\colon\cdot)$ on A_{q}^+ introduced in [10, §10]. Let the function $E_+(\lambda)\colon X_+ \to \mathrm{Hom}({}^\circ\mathcal{C}, V_\tau)$ be defined by

$$E_+(\lambda\colon kawH)\psi = \tau(k)\Phi_{P,w}(\lambda\colon a)[\psi]_w(e) \tag{2.7}$$

for $k \in K$, $a \in A_{\mathrm{q}}^+$, $w \in \mathcal{W}$, $\psi \in {}^\circ\mathcal{C}$. It is easily seen from (2.2) that $E_+(\lambda)$ is well-defined for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$, and that it belongs to the space $C^\infty(X_+\colon\tau)$ of smooth τ-spherical functions on X_+. It satisfies

$$E_+(\lambda\colon aw)\psi \sim a^{\lambda-\rho}\psi_w(e)$$

for $w \in \mathcal{W}$, as $a \to \infty$ in A_{q}^{+}. Furthermore,

$$DE_{+}(\lambda) = E_{+}(\lambda) \circ \mu(D\colon \lambda) \tag{2.8}$$

for $D \in \mathbb{D}(X)$, by [10, Cor. 9.3], and $E_{+}(\lambda)$ depends meromorphically on $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^{*}$ as an element of $C^{\infty}(X_{+}\colon \tau)$. The expression (1.4) now follows from [10, Thm. 11.1]. It will be convenient to rewrite this as follows. Let

$$E_{+,s}(\lambda\colon x) = E_{+}(s\lambda\colon x) \circ C^{\circ}(s\colon \lambda) \in \mathrm{Hom}({}^{\circ}\mathcal{C}, V_{\tau}), \tag{2.9}$$

then

$$E^{\circ}(\lambda\colon x) = \sum_{s \in W} E_{+,s}(\lambda\colon x) \tag{2.10}$$

for $x \in X_{+}$.

The Eisenstein integrals satisfy an invariance property for the action of the Weyl group (see [8, Prop. 4]). Expressed in terms of the notation introduced above it reads

$$E^{\circ}(\lambda\colon x) = E^{\circ}(s\lambda\colon x) \circ C^{\circ}(s\colon \lambda), \qquad E^{*}(s\lambda\colon x) = C^{\circ}(s\colon \lambda) \circ E^{*}(\lambda\colon x) \tag{2.11}$$

for $s \in W$, where the Maass-Selberg relations (1.8) are used in the passage between the two identities. For the Fourier transform of a function $f \in C_c^{\infty}(X\colon \tau)$ the property (2.11) implies that

$$\mathcal{F}f(s\lambda) = C^{\circ}(s\colon \lambda)\mathcal{F}f(\lambda). \tag{2.12}$$

3. The singular hyperplanes

In this section we study the singular set for the normalized Eisenstein integral $E^{\circ}(\lambda\colon x)$, as a function of λ. Our aim is to prove that $E^{\circ}(\lambda\colon x)$ is singular only along real root hyperplanes in $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^{*}$, that is, hyperplanes of the form $\{\lambda \mid \langle \lambda, \alpha \rangle = c\}$ with $\alpha \in \Sigma$ and $c \in \mathbb{R}$. Part of the proof will, however, be deferred to an Appendix.

For $S \subset \mathfrak{a}_{\mathrm{q}\mathbb{C}}^{*} \setminus \{0\}$ we denote by $\Pi_S = \Pi_S(\mathfrak{a}_{\mathrm{q}})$ the set of complex polynomials on $\mathfrak{a}_{\mathrm{q}}$ which are products of affine functions of the form $\lambda \mapsto \langle \lambda, \xi \rangle - c$ with $\xi \in S$ and $c \in \mathbb{C}$. We agree that $1 \in \Pi_S$. For $S \subset \mathfrak{a}_{\mathrm{q}}^{*} \setminus \{0\}$ we define $\Pi_{S,\mathbb{R}} \subset \Pi_S$ similarly, but with $c \in \mathbb{R}$.

For $R \in \mathbb{R}$ we define

$$\mathfrak{a}_{\mathrm{q}}^{*}(P, R) := \{\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^{*} \mid \mathrm{Re}\langle \lambda, \alpha \rangle < R \text{ for } \alpha \in \Sigma^{+}\} \tag{3.1}$$

and denote by $\bar{\mathfrak{a}}_{\mathrm{q}}^{*}(P, R)$ the closure of this set.

Proposition 3.1. *Let $R \in \mathbb{R}$. There exists $p \in \Pi_{\Sigma,\mathbb{R}}$ such that the map $\lambda \mapsto p(\lambda)E^{*}(\lambda) \in C^{\infty}(X\colon \tau)$ is holomorphic on an open neighborhhod of the set $\bar{\mathfrak{a}}_{\mathrm{q}}^{*}(P, R)$. Moreover, $\lambda \mapsto p(\lambda)\mathcal{F}f(\lambda) \in {}^{\circ}\mathcal{C}$ is holomorphic on this neighborhood for all $f \in C_c^{\infty}(X\colon \tau)$.*

Proof. We must prove, for each R, the existence of $p \in \Pi_{\Sigma,\mathbb{R}}$ such that $\lambda \mapsto p(\lambda)E^{\circ}(\lambda\colon x)$ is holomorphic on

$$\{\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^{*} \mid \mathrm{Re}\langle \lambda, \alpha \rangle > -R \text{ for } \alpha \in \Sigma^{+}\}.$$

It is known (from [5], see [8, Lemma 14]) that there exists $p \in \Pi_{\Sigma}$ with this property. It remains to be seen that the singularities of $E^{\circ}(\lambda\colon x)$ are along *real* root hyperplanes. The main step is contained in the following lemma, in which notation is as in [9, §2].

Lemma 3.2. *Let $\xi \in \hat{M}_H$. There exists, for each $R \in \mathbb{R}$, a polynomial $p \in \Pi_{\Sigma,\mathbb{R}}$ such that the map $\lambda \mapsto p(\lambda)j(P\colon\xi\colon\lambda)\eta \in C^{-\infty}(K\colon\xi)$ is holomorphic on $\mathfrak{a}_{\mathrm{q}}^*(P,R)$, for each $\eta \in V(\xi)$.*

Proof. See Appendix A. $\square$

It follows immediately from Lemma 3.2 and [8, eq. (25)] that $E(\psi\colon\lambda)$ is singular only along real root hyperplanes for all $\psi \in {}^\circ\mathcal{C}$. In order to establish the corresponding result for the normalized Eisenstein integrals, we recall that the standard intertwining operator $A(Q'\colon Q\colon\xi\colon\lambda)$ is singular only on real root hyperplanes for all $Q, Q' \in \mathcal{P}$ (see [24, Thm. 6.6]). The same holds for the inverse of the operator (cf. [9, proof of Lemma 20.3]). Moreover, by Lemma 3.2, also the operator $B(Q'\colon Q\colon\xi\colon\lambda) \in \operatorname{End} V(\xi)$ defined by [4, Prop. 6.1], as well as its inverse, is singular only along real root hyperplanes (cf. also [9, proof of Lemma 20.5]). Finally, it then follows from [8, Lemma 3 and eqs. (47), (49)] that the normalized Eisenstein integral has only real root hyperplane singularities. This completes the proof of Proposition 3.1. $\square$

Let $\pi \in \Pi_\Sigma$ be the polynomial defined in [9, eq. (8.1)]. It is characterized (up to a constant multiple) by being minimal subject to the condition that $\lambda \mapsto \pi(\lambda)E^*(\lambda)$ is holomorphic on $\mathfrak{a}_{\mathrm{q}}^*(P,0)$, and hence also on $\mathfrak{a}_{\mathrm{q}}^*(P,\epsilon)$ for some $\epsilon > 0$, cf. [9, Lemma 8.1]. Hence by Proposition 3.1 we have $\pi \in \Pi_{\Sigma,\mathbb{R}}$. The map $\lambda \mapsto \pi(\lambda)\mathcal{F}f(\lambda)$ is holomorphic on $\mathfrak{a}_{\mathrm{q}}^*(P,\epsilon)$ for all $f \in C_c^\infty(X\colon\tau)$.

The function $\lambda \mapsto E_+(\lambda\colon x)$, defined in the previous section, has a singular set which is similar to that of $E^*(\lambda)$:

Lemma 3.3. *There exists, for each $R \in \mathbb{R}$, a polynomial $p_R \in \Pi_{\Sigma,\mathbb{R}}$ such that $\lambda \mapsto p_R(\lambda)E_+(\lambda\colon x)$ is holomorphic on a neighborhood of $\bar{\mathfrak{a}}_{\mathrm{q}}^*(P,R)$, for all $x \in X_+$.*

Proof. See [10, Thm. 9.1, Prop. 9.4]. $\square$

4. Pseudo wave packets

Let $\varphi\colon \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \to {}^\circ\mathcal{C}$. For $\eta \in \mathfrak{a}_{\mathrm{q}}^*$ we define a V_τ-valued function on X_+ by

$$\mathcal{T}_\eta\varphi(x) = |W| \int_{\eta+i\mathfrak{a}_{\mathrm{q}}^*} E_+(\lambda\colon x)\varphi(\lambda)\,d\lambda, \tag{4.1}$$

provided the integral converges. We shall see that this is the case when $\varphi = \mathcal{F}f$ for $f \in C_c^\infty(X\colon\tau)$. First we need an estimate of $E_+(\lambda\colon x)$ as a function of λ. For $u \in U(\mathfrak{g})$ and f a smooth function on X, we denote by $f(u;x)$ the value at x of the function obtained from f by application of u from the left.

Lemma 4.1. *Let $R \in \mathbb{R}$ and let p_R be as in Lemma 3.3. There exists for each $u \in U(\mathfrak{g})$ a constant $d \in \mathbb{N}$ with the following property. Let $\omega \subset \bar{\mathfrak{a}}_{\mathrm{q}}^*(P,R)$ and $\Omega \subset X_+$ be compact sets. Then*

$$\sup_{x\in\Omega,\lambda\in\omega+i\mathfrak{a}_{\mathrm{q}}^*} (1+|\lambda|)^{-d}\|p_R(\lambda)E_+(\lambda\colon u;x)\| < \infty. \tag{4.2}$$

Proof. By sphericality it suffices to prove this result for the case that Ω is contained in $A_{\mathrm{q}}^{\mathrm{reg}}$, the set of regular points in A_{q}. By the infinitesimal Cartan decomposition $\mathfrak{g} =$

$\mathfrak{k} + \mathfrak{a}_q + \mathrm{Ad}(a)\mathfrak{h}$, for $a \in A_q^{\mathrm{reg}}$, we may as well assume that $u \in U(\mathfrak{a}_q)$ (use [2, Lemma 3.2]). For the present Ω and u, the function $E_+(\lambda\colon u; a)$, $a \in \Omega$, may be computed by termwise differentiation of the power series [10, eq. (15)] that defines the functions $\Phi_{P,w}(\lambda\colon a)$ in (2.7). The coefficient $\Gamma_\nu(\lambda)$ in this series is thus replaced by $\Gamma'_\nu(\lambda) = p(\lambda - \nu)\Gamma_\nu(\lambda)$, with p a polynomial depending on u. Let d be the degree of p; then there exists a constant $C > 0$ such that $|p(\lambda - \nu)| \leq C(1 + |\nu|)^d(1 + |\lambda|)^d$ for all $\lambda \in \mathfrak{a}^*_{q\mathbb{C}}$ and all $\nu \in \mathbb{N}\Delta$. It follows that the coefficient $\Gamma'_\nu(\lambda)$ of the differentiated series satisfies an estimate analogous to the estimate for $\Gamma_\nu(\lambda)$ in [10, Thm. 7.4]. The desired estimate is now obtained as in [10, Thm. 9.1]. $\square$

Lemma 4.2. *Let $R \in \mathbb{R}$, let $\omega \subset \mathfrak{a}_q^*$ be open and contained in $\mathfrak{a}_q^*(P, R)$ and let $p \in \Pi_{\Sigma,\mathbb{R}}$. Let φ be a meromorphic ${}^\circ\mathcal{C}$-valued function on $\omega + i\mathfrak{a}_q^*$ with the following property: The map $\lambda \mapsto p(\lambda)\varphi(\lambda) \in {}^\circ\mathcal{C}$ is holomorphic on $\omega + i\mathfrak{a}_q^*$ and satisfies*

$$\sup_{\lambda \in \omega + i\mathfrak{a}_q^*} (1 + |\lambda|)^n \|p(\lambda)\varphi(\lambda)\| < \infty \tag{4.3}$$

for all $n \in \mathbb{N}$. Let $\eta \in \omega$ with $p(\eta) \neq 0$, and assume in addition that $p_R(\eta) \neq 0$, where p_R is as in Lemma 3.3. Then the integral in (4.1) converges absolutely. The V_τ-valued function $\mathcal{T}_\eta\varphi$ on X_+ is τ-spherical and smooth, and it is locally independent of η. Moreover,

$$D\mathcal{T}_\eta\varphi = \mathcal{T}_\eta(\mu(D)\varphi) \tag{4.4}$$

for $D \in \mathbb{D}(G/H)$.

Proof. It follows from (4.2) and (4.3) that

$$\sup_{\lambda \in \eta + i\mathfrak{a}_q^*} (1 + |\lambda|)^n \|E_+(\lambda\colon u; x)\varphi(\lambda)\| < \infty \tag{4.5}$$

with a bound that is locally uniform in η. The convergence and the smoothness of (4.1) follows immediately. The local independence on η results from a standard application of Cauchy's theorem, and (4.4) is a consequence of (2.8). $\square$

In order to see that the Fourier transform of a compactly supported smooth function satisfies the required estimates (4.3) we first recall the estimate for the Eisenstein integrals in the following lemma. For $M > 0$, let $B_M \subset \mathfrak{a}_q$ be the closed ball of radius M, and let

$$X_M = K \exp B_M H \subset X$$

and $C^\infty_M(X\colon\tau) = \{f \in C^\infty_c(X\colon\tau) \mid \operatorname{supp} f \subset X_M\}$.

Lemma 4.3. *Let $R \in \mathbb{R}$ and let p be as in Proposition 3.1. Let $u \in U(\mathfrak{g})$. There exists a constant $N \in \mathbb{N}$ such that*

$$\sup_{x \in X_M, \lambda \in \mathfrak{a}_q^*(P,R)} (1 + |\lambda|)^{-N} e^{-M|\operatorname{Re}\lambda|} \|p(\lambda)E^*(\lambda\colon u; x)\| < \infty$$

for all $M > 0$.

Proof. See [5, Prop. 10.3, Cor. 16.2] and [8, eq. (52)]. $\square$

Lemma 4.4. *Let $R \in \mathbb{R}$ and let $p \in \Pi_{\Sigma,\mathbb{R}}$ be as in Proposition 3.1. There exists for each $M > 0$ and for each $n \in \mathbb{N}$ a continuous seminorm ν on $C_M^\infty(X{:}\,\tau)$ such that*

$$\|p(\lambda)\mathcal{F}f(\lambda)\| \leq (1+|\lambda|)^{-n} e^{M|\operatorname{Re}\lambda|}\nu(f) \tag{4.6}$$

for all $\lambda \in \mathfrak{a}_{\mathrm{q}}^*(P, R)$, $f \in C_M^\infty(X{:}\,\tau)$.

Proof. This follows from Lemma 4.3 in the same manner as [9, Prop. 8.3]. □

Let $\omega \subset \mathfrak{a}_{\mathrm{q}}^*$ be open and bounded, and choose $R \in \mathbb{R}$ such that $\omega \subset \mathfrak{a}_{\mathrm{q}}^*(P, R)$. It follows from Lemma 4.4 that the functions $\varphi = \mathcal{F}f$ satisfy (4.3). Hence the results of Lemma 4.2 hold for these functions. Notice that it easily follows from (4.6) and (4.2) that $f \mapsto \mathcal{T}_\eta \mathcal{F} f$ is a continuous linear operator from $C_c^\infty(X{:}\,\tau)$ to $C^\infty(X_+{:}\,\tau)$, for generic $\eta \in \mathfrak{a}_{\mathrm{q}}^*$.

Let $\pi \in \Pi_\Sigma$ be as in the text preceding Lemma 3.3. We define the space $\mathcal{P}(X{:}\,\tau)$ as the space of meromorphic functions $\varphi\colon \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \to {}^\circ\mathcal{C}$ having the following properties (a)-(b).

(a) $\varphi(s\lambda) = C^\circ(s{:}\,\lambda)\varphi(\lambda)$ for all $s \in W$, $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$.

(b) There exists a constant $\epsilon > 0$ such that $\pi\varphi$ is holomorphic on $\mathfrak{a}_{\mathrm{q}}^*(P, \epsilon)$; moreover, for every compact set $\omega \subset \mathfrak{a}_{\mathrm{q}}^*(P, \epsilon) \cap \mathfrak{a}_{\mathrm{q}}^*$ and all $n \in \mathbb{N}$,

$$\sup_{\lambda \in \omega + i\mathfrak{a}_{\mathrm{q}}^*} (1+|\lambda|)^n \|\pi(\lambda)\varphi(\lambda)\| < \infty. \tag{4.7}$$

Furthermore, for $M > 0$, we define $\mathcal{P}_M(X{:}\,\tau)$ to be the subspace of $\mathcal{P}(X{:}\,\tau)$ consisting of the functions φ that also satisfy the following condition (c).

(c) For every strictly antidominant $\eta \in \mathfrak{a}_{\mathrm{q}}^*$ there exist constants $t_\eta, C_\eta > 0$ such that

$$\|\varphi(\lambda)\| \leq C_\eta (1+|\lambda|)^{-\dim \mathfrak{a}_{\mathrm{q}} - 1} e^{tM|\eta|}$$

for all $t \geq t_\eta$ and $\lambda \in t\eta + i\mathfrak{a}_{\mathrm{q}}^*$.

Notice that the Fourier transform $\mathcal{F}$ maps $C_M^\infty(X{:}\,\tau)$ into $\mathcal{P}_M(X{:}\,\tau)$, by (2.12) and Lemma 4.4. It follows from Lemma 4.2 that if $\varphi \in \mathcal{P}(X{:}\,\tau)$ then $\mathcal{T}_\eta\varphi$ is well defined for all generic η in $\mathfrak{a}_{\mathrm{q}}^*(P, 0) \cap \mathfrak{a}_{\mathrm{q}}^*$.

Lemma 4.5. *Let $\varphi \in \mathcal{P}(X{:}\,\tau)$. Then $\mathcal{T}_\eta\varphi$ is defined for η regular and sufficiently close to 0 in $\mathfrak{a}_{\mathrm{q}}^*$. Moreover, the wave packet (2.5) is defined and satisfies*

$$\mathcal{J}\varphi = \frac{1}{|W|} \sum_{s \in W} \mathcal{T}_{s\eta}\varphi \tag{4.8}$$

for η regular and sufficiently close to 0. If $\lambda \mapsto E_+(\lambda{:}\,x)\varphi(\lambda)$ is regular along $i\mathfrak{a}_{\mathrm{q}}^*$, then $\mathcal{T}_0\varphi$ is defined and $\mathcal{J}\varphi = \mathcal{T}_0\varphi$.

Proof. Fix $R > 0$ and let p_R be as in Lemma 3.3. Since $p_R \in \Pi_{\Sigma,\mathbb{R}}$ there exists a W-invariant open neighborhood ω of 0, such that (4.7) holds and p_R has no zeros in $\omega \cap \mathfrak{a}_{\mathrm{q}}^{*\mathrm{reg}}$. Moreover, by [9, Lemma 8.1 (a)] we may assume that π has no zeros in ω. For $\eta \in \omega \cap \mathfrak{a}_{\mathrm{q}}^{*\mathrm{reg}}$ the pseudo-wave packets $\mathcal{T}_{s\eta}\varphi$, $s \in W$, are well defined in view of (4.7) and Lemma 4.2.

It follows from (2.5), the estimate [9, (8.2)] and Cauchy's theorem that

$$\mathcal{J}\varphi(x) = \int_{\eta + i\mathfrak{a}_{\mathrm{q}}^*} E^\circ(\lambda : x)\varphi(\lambda)\, d\lambda$$

for η sufficiently close to 0. The result, (4.8), now easily follows from insertion of (2.10) and (2.9) in this expression.

Moreover, if $\lambda \mapsto E_+(\lambda : x)\varphi(\lambda)$ is regular along $i\mathfrak{a}_{\mathrm{q}}^*$, then it follows from (4.2) and (4.7) that (4.5) holds uniformly for η in a neighborhood of 0. It then follows as in Lemma 4.2 that $\mathcal{T}_\eta\varphi$ is defined and independent of η, for all η in a neighborhood of 0. Hence $\mathcal{J}\varphi = \mathcal{T}_0\varphi$ follows from (4.8). $\square$

Choose $R < 0$ such that $\pi(\lambda) \neq 0$ for $\lambda \in \mathfrak{a}_{\mathrm{q}}^*(P, R)$, and let $\eta \in \mathfrak{a}_{\mathrm{q}}^*(P, R)$. Let $\varphi \in \mathcal{P}(X : \tau)$. We define $\mathcal{T}\varphi := \mathcal{T}_\eta\varphi$ and call this function on X_+ the *pseudo wave packet* formed by φ. It is independent of the choices of R and η, by the statement of local independence in Lemma 4.2.

Lemma 4.6. *Let $M > 0$ and $\varphi \in \mathcal{P}_M(X : \tau)$. The pseudo wave packet $\mathcal{T}\varphi$ is a smooth τ-spherical function on X_+. The set $\{x \in X_+ \mid \mathcal{T}\varphi(x) \neq 0\}$ is contained in X_M.*

Proof. The first statement is immediate from Lemma 4.2. Let $x \in X_+ \setminus X_M$. We claim that $\mathcal{T}\varphi(x) = 0$. Let $x = kawH$, where $k \in K$, $a \in A_{\mathrm{q}}^+$, $w \in \mathcal{W}$, then $|\log a| > M$. Since the inner product on $\mathfrak{a}_{\mathrm{q}}^*$ is the dual to that on $\mathfrak{a}_{\mathrm{q}}$, we may fix $\eta \in \mathfrak{a}_{\mathrm{q}}^*$, strictly antidominant, such that $|\eta| = 1$ and $\eta(\log a) < -M$. Then $\mathcal{T}\varphi = \mathcal{T}_{t\eta}\varphi$ for $t \in \mathbb{R}$ sufficiently large. The estimate

$$\|E_+(t\eta + \lambda : x)\| \leq Ca^{t\eta}, \qquad (\lambda \in i\mathfrak{a}_{\mathrm{q}}^*,\ t \gg 0)$$

follows from [10, Thm. 9.1]. Hence

$$\|\mathcal{T}\varphi(x)\| \leq |W| \int_{t\eta + i\mathfrak{a}_{\mathrm{q}}^*} Ca^{t\eta}\|\varphi(\lambda)\|\, d\lambda \leq C' a^{t\eta} e^{Mt},$$

by (c), and we conclude by taking the limit as $t \to \infty$ that $\mathcal{T}\varphi(x) = 0$. $\square$

We can now state the main result of this paper, the inversion formula for the τ-spherical Fourier transform.

Theorem 4.7. *Let $f \in C_c^\infty(X : \tau)$. Then $\mathcal{T}\mathcal{F}f(x) = f(x)$ for all $x \in X_+$.*

The proof will be given in the course of the next 5 sections. In the proof we shall use the following result, which is a consequence of Theorem 2.1 and its proof.

Lemma 4.8. *There exists $D_0 \in \mathbb{D}(X)$ such that $\det \mu(D_0) \neq 0$ and such that*

$$D_0\mathcal{T}\varphi(x) = D_0\mathcal{J}\varphi(x) \tag{4.9}$$

for all $x \in X_+$, $\varphi \in \mathcal{P}(X : \tau)$. For every $M > 0$ and every $\varphi \in \mathcal{P}_M(X : \tau)$, the function $D_0\mathcal{T}\varphi$ on X_+ has a smooth extension to a function in $C_M^\infty(X : \tau)$; the Fourier transform of this extension is given by

$$\mathcal{F}D_0\mathcal{T}\varphi = \mu(D_0)\varphi. \tag{4.10}$$

Moreover, for every $f \in C_c^\infty(X\colon\tau)$ *we have* $D_0\mathcal{T}\mathcal{F}f = D_0 f$.

Proof. Let $p_0 \in \Pi_{\Sigma,\mathbb{R}}$ be given by Lemma 3.3 with $R = 0$, and let $D_0 \in \mathbb{D}_{p_0\pi}$, cf. [9, Defn. 10.3 and Cor. 10.4]. Then $\det \mu(D_0) \neq 0$ and $p_0\pi$ divides $\mu(D_0)$ in $S(\mathfrak{a}_q) \otimes \mathrm{End}({}^\circ\mathcal{C})$. Moreover, let $\varphi \in \mathcal{P}(X\colon\tau)$ and put $\tilde\varphi = \mu(D_0)\varphi$. Then $\tilde\varphi \in \mathcal{P}(X\colon\tau)$ (use [9, Eqn. (5.13)]), and $E_+(\lambda\colon x)\tilde\varphi(\lambda)$ satisfies an estimate of the form (4.5) for all $\eta \in \bar{\mathfrak{a}}_q^*(P,0) \cap \mathfrak{a}_q^*$. We infer as in Lemma 4.2 that $\mathcal{T}_\eta\tilde\varphi$ is defined and equal to $\mathcal{T}\tilde\varphi$ for all $\eta \in \bar{\mathfrak{a}}_q^*(P,0) \cap \mathfrak{a}_q^*$. By Lemma 4.5 we conclude then that $\mathcal{T}\tilde\varphi = \mathcal{J}\tilde\varphi$ on X_+, and (4.9) follows from (2.6), (4.4).

By a standard application of Cauchy's integral formula the restriction of φ to the Euclidean space $i\mathfrak{a}_q^*$ is a ${}^\circ\mathcal{C}$-valued Schwartz function. Therefore, by [9, Thm. 16.4], the wave packet $\mathcal{J}\varphi$ belongs to the Schwartz space $\mathcal{C}(X\colon\tau)$ (see [9, §6]) and its Fourier transform $\mathcal{F}\mathcal{J}\varphi$ equals φ by [9, Thm. 16.6]. Assume now that $\varphi \in \mathcal{P}_M(X\colon\tau)$. Then $D_0\mathcal{T}\varphi$ has a smooth extension to a function in $C_M^\infty(X\colon\tau)$, by (4.9) and Lemma 4.6. Moreover,

$$\mathcal{F}D_0\mathcal{T}\varphi = \mathcal{F}D_0\mathcal{J}\varphi = \mu(D_0)\mathcal{F}\mathcal{J}\varphi = \mu(D_0)\varphi,$$

where the second equality is a consequence of [9, Lemma 6.2]. This establishes (4.10).

Let $f \in C_M^\infty(X\colon\tau)$ and put $\varphi = \mathcal{F}f$. Then $\varphi \in \mathcal{P}_M(X\colon\tau)$, and it follows from the previous statements that $D_0\mathcal{T}\mathcal{F}f \in C_M^\infty(X\colon\tau)$ and $\mathcal{F}D_0\mathcal{T}\mathcal{F}f = \mu(D_0)\mathcal{F}f = \mathcal{F}(D_0 f)$. Since $\mathcal{F}$ is injective (cf. Theorem 2.1), the final statement follows. $\square$

Corollary 4.9. *Let* $M > 0$ *and* $\varphi \in \mathcal{P}_M(X\colon\tau)$. *Assume that* $\mathcal{T}\varphi$ *has a smooth extension to* X. *Then this extension belongs to* $C_M^\infty(X\colon\tau)$ *and its Fourier transform is given by* $\mathcal{F}\mathcal{T}\varphi = \varphi$.

Proof. It follows from Lemma 4.6 that the extension of $\mathcal{T}\varphi$ belongs to $C_M^\infty(X\colon\tau)$. Hence its Fourier transform $\mathcal{F}\mathcal{T}\varphi$ makes sense, and we obtain from (4.10) that $\mu(D_0)\mathcal{F}\mathcal{T}\varphi = \mu(D_0)\varphi$. Since $\det \mu(D_0) \neq 0$ it follows immediately that $\mathcal{F}\mathcal{T}\varphi = \varphi$. $\square$

Corollary 4.10. *Let* $f \in C_c^\infty(X\colon\tau)$ *and assume that* $\mathcal{T}\mathcal{F}f$ *has a smooth extension to* X. *Then this extension equals* f.

Proof. There exists a constant $M > 0$ such that $f \in C_M^\infty(X\colon\tau)$. Now $\mathcal{F}f \in \mathcal{P}_M(X\colon\tau)$, hence it follows from Corollary 4.9 that $\mathcal{T}\mathcal{F}f \in C_c^\infty(X\colon\tau)$ and that $\mathcal{F}\mathcal{T}\mathcal{F}f = \mathcal{F}f$. Since $\mathcal{F}$ is injective we conclude that $\mathcal{T}\mathcal{F}f = f$. $\square$

The preceding corollaries have been established without use of Theorem 4.7. On the other hand, it follows from the conclusion of this theorem that $\mathcal{T}\mathcal{F}f$ really has a smooth extension to X, for all $f \in C_c^\infty(X\colon\tau)$. Thus we obtain from Theorem 4.7 and Corollary 4.9 the following (weak) Paley-Wiener theorem:

Corollary 4.11. *Let* $M > 0$ *and let* $\mathcal{P}'_M(X\colon\tau)$ *denote the set of functions* $\varphi \in \mathcal{P}_M(X\colon\tau)$ *for which* $\mathcal{T}\varphi$ *has a smooth extension to* X. *The Fourier transform* $\mathcal{F}$ *maps* $C_M^\infty(X\colon\tau)$ *bijectively onto* $\mathcal{P}'_M(X\colon\tau)$*; the inverse map is given by* $\mathcal{T}$ *followed by the extension to* X.

5. Residue operators

In order to study closer the pseudo wave packets $\mathcal{T}\mathcal{F}f$ we apply the residue calculus from [11]. We first recall some basic notions from this reference. A subset of $V = \mathfrak{a}_q^*$ of the form $H_{\alpha,s} := \{\lambda \in \mathfrak{a}_{\mathrm{q}}^* \mid \langle \alpha, \lambda\rangle = s\}$ for some $\alpha \in \mathfrak{a}_{\mathrm{q}}^* \setminus \{0\}$ and $s \in \mathbb{R}$ is called an affine hyperplane; if $\alpha \in \Sigma$ it is called an affine root hyperplane. A locally finite collection of affine hyperplanes in V is called an *affine hyperplane configuration*; if it consists of affine root hyperplanes it is said to be Σ-*admissible*. Moreover, if its elements are given as above, with $\alpha \in \Sigma^+$ and with a uniform lower bound on s, then it is said to be P-*bounded*.

Let $\mathcal{H}$ be an affine hyperplane configuration in $\mathfrak{a}_{\mathrm{q}}^*$, and let $d\colon \mathcal{H} \to \mathbb{N}$ be a map. For any compact set $\omega \subset \mathfrak{a}_{\mathrm{q}}^*$ we denote by $\pi_{\omega,d}$ the polynomial on $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ given by the product of the functions $(\langle \alpha, \cdot\rangle - s)^{d(H_{\alpha,s})}$, where $H_{\alpha,s}$ is any hyperplane that belongs to $\mathcal{H}$ and meets ω. We then denote by $\mathcal{M}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}, d)$ the space of meromorphic functions $\varphi\colon \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \to \mathbb{C}$ for which $\pi_{\omega,d}\varphi$ is holomorphic on a neighborhood of $\omega + i\mathfrak{a}_{\mathrm{q}}^*$, for all compact sets $\omega \subset \mathfrak{a}_{\mathrm{q}}^*$. Furthermore, we denote by $\mathcal{P}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}, d)$ the subspace of those $\varphi \in \mathcal{M}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}, d)$ for which

$$\sup_{\lambda \in \omega + i\mathfrak{a}_{\mathrm{q}}^*} (1 + |\lambda|)^n |\pi_{\omega,d}(\lambda)\varphi(\lambda)| < \infty$$

for all ω and all $n \in \mathbb{N}$. The unions over all $d\colon \mathcal{H} \to \mathbb{N}$ of these spaces are denoted

$$\mathcal{M}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}) := \cup_d \mathcal{M}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}, d), \qquad \mathcal{P}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}) := \cup_d \mathcal{P}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}, d).$$

Let L be an affine subspace of $\mathfrak{a}_{\mathrm{q}}^*$, that is, $L = \lambda + V_L$ where $\lambda \in \mathfrak{a}_{\mathrm{q}}^*$ and V_L is a linear subspace of $V = \mathfrak{a}_{\mathrm{q}}^*$. The set $L \cap V_L^\perp$ consists of a single point $c(L)$, called the central point in L. The map $\lambda \mapsto c(L) + \lambda$ is a bijection of V_L onto L; via this map we can view L as a linear space. The set

$$\mathcal{H}_L = \{H \cap L \mid H \in \mathcal{H},\ \emptyset \subsetneq H \cap L \subsetneq H\}$$

is an affine hyperplane configuration in L. We may then define the sets $\mathcal{M}(L, \mathcal{H}_L)$ and $\mathcal{P}(L, \mathcal{H}_L)$ similarly as $\mathcal{M}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H})$ and $\mathcal{P}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H})$ above; they consist of functions that are meromorphic on the complexification $L_\mathbb{C} = c(L) + (V_L)_\mathbb{C}$ of L. If $d\colon \mathcal{H} \to \mathbb{N}$ we denote by $q_{L,d}$ the polynomial on $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ given by the product of the functions $(\langle \alpha, \cdot\rangle - s)^{d(H_{\alpha,s})}$, where $H_{\alpha,s}$ is any hyperplane that belongs to $\mathcal{H}$ and contains L. The restriction $(q_{L,d}\varphi)|_{L_\mathbb{C}}$ then makes sense and belongs to $\mathcal{M}(L, \mathcal{H}_L)$, for all $\varphi \in \mathcal{M}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}, d)$. More generally, a linear map

$$R\colon \mathcal{M}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}) \to \mathcal{M}(L, \mathcal{H}_L)$$

is called a *Laurent operator* if there exists, for each $d\colon \mathcal{H} \to \mathbb{N}$, an element u_d in the symmetric algebra $S(V_L^\perp)$ of $V_L^\perp$ such that, for every $\varphi \in \mathcal{M}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}, d)$, the image $R\varphi$ is the restriction to $L_\mathbb{C}$ of $u_d(q_{L,d}\varphi)$. The space of Laurent operators from $\mathcal{M}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H})$ to $\mathcal{M}(L, \mathcal{H}_L)$ is denoted $\mathrm{Laur}(\mathfrak{a}_{\mathrm{q}}^*, L, \mathcal{H})$. A Laurent operator $R \in \mathrm{Laur}(\mathfrak{a}_{\mathrm{q}}^*, L, \mathcal{H})$ automatically maps $\mathcal{P}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H})$ into $\mathcal{P}(L, \mathcal{H}_L)$ (cf. [11, Lemma 1.10]).

Let $\mathcal{H}$ denote the set of affine hyperplanes in $\mathfrak{a}_{\mathrm{q}}^*$ along which $\lambda \mapsto E_+(\lambda\colon x)$ or $\lambda \mapsto E^*(\lambda\colon x)$ is singular, for some x (in X_+, resp. X). It follows from Proposition 3.1 and Lemma 3.3 that $\mathcal{H}$ is a P-bounded Σ-admissible hyperplane configuration. Moreover, by Lemmas 4.1 and 4.4 the functions $\lambda \mapsto E_+(\lambda\colon x)\mathcal{F}f(\lambda)$, where $f \in C_c^\infty(X\colon \tau)$ and $x \in X_+$, belong to the space $\mathcal{P}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}) \otimes V_\tau$.

Let $\mathcal{R}$ denote the set of root spaces in $\mathfrak{a}_{\mathrm{q}}$, that is, the set of all subspaces $\mathfrak{b} \subset \mathfrak{a}_{\mathrm{q}}$ of the form $\mathfrak{b} = \alpha_1^{-1}(0) \cap \cdots \cap \alpha_l^{-1}(0)$ with $\alpha_1, \dots, \alpha_l \in \Sigma$, and for $\mathfrak{b} \in \mathcal{R}$ let

$$\mathrm{sing}(\mathfrak{b}, \Sigma) = \cup_{\alpha \in \Sigma, \alpha|\mathfrak{b} \neq 0} \mathfrak{b} \cap \alpha^{-1}(0), \qquad \mathrm{reg}(\mathfrak{b}, \Sigma) = \mathfrak{b} \setminus \mathrm{sing}(\mathfrak{b}, \Sigma).$$

Furthermore, let $\mathcal{P}(\mathfrak{b})$ denote the set of chambers in $\mathfrak{b}$, that is, the connected components of $\mathrm{reg}(\mathfrak{b}, \Sigma)$, and let $\mathcal{P} = \cup_{\mathfrak{b} \in \mathcal{R}} \mathcal{P}(\mathfrak{b})$. There is a natural 1-1 correspondence between the set $\mathcal{P}_\sigma$ of all $\sigma\theta$-stable parabolic subgroups of G, containing A_{q}, and $\mathcal{P}$. Thus, a parabolic subgroup $Q \in \mathcal{P}_\sigma$ with σ-split component $\exp \mathfrak{a}_{Q\mathrm{q}}$ corresponds to the element $\mathfrak{a}_{Q\mathrm{q}}^+ \in \mathcal{P}(\mathfrak{a}_{Q\mathrm{q}})$ on which its roots are positive (in particular, elements in $\mathcal{P}_\sigma^{\min}$ correspond to chambers in $\mathfrak{a}_{\mathrm{q}}$).

Let $\Delta \subset \Sigma$ denote the set of simple roots for Σ^+, and let $F \subset \Delta$. Let $\mathfrak{a}_{F\mathrm{q}} = \cap_{\alpha \in F} \alpha^{-1}(0) \in \mathcal{R}$, and let $\mathfrak{a}_{F\mathrm{q}}^+ \in \mathcal{P}(\mathfrak{a}_{F\mathrm{q}})$ be the chamber on which the roots in $\Delta \setminus F$ are positive. This chamber corresponds to a $\sigma\theta$-stable standard parabolic subgroup which we denote P_F (see [4, §2]). Furthermore, we denote by W_F the subgroup of W generated by the reflections in the elements of F, and by W^F the set $\{s \in W \mid s(F) \subset \Sigma^+\}$, which is a set of representatives for the quotient W/W_F.

For $\mathfrak{b} \in \mathcal{R}$ we identify the dual space $\mathfrak{b}^*$ with a subspace of $\mathfrak{a}_{\mathrm{q}}^*$ by means of the extended Killing form B.

Let t be a W-invariant residue weight for Σ, that is, a map from $\mathcal{P}$ to $[0; 1]$ such that $\sum_{Q \in \mathcal{P}(\mathfrak{a})} t(Q) = 1$ for all $\mathfrak{a} \in \mathcal{R}$, and $t(wQ) = t(Q)$ for all $Q \in \mathcal{P}$, $w \in W$. Starting from the data Σ, P, t we defined, in [11, §3.4], for each subset $F \subset \Delta$ and every $\lambda \in \mathfrak{a}_{F\mathrm{q}}^{*\perp}$ a *universal Laurent operator* $\mathrm{Res}^{P,t}_{\lambda + \mathfrak{a}_{F\mathrm{q}}^*}$. This operator encodes the procedure of taking a residue along the affine subspace $\lambda + \mathfrak{a}_{F\mathrm{q}\mathbb{C}}^*$ of $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$; it induces a Laurent operator (denoted by the same symbol) $\mathrm{Res}^{P,t}_{\lambda + \mathfrak{a}_{F\mathrm{q}}^*} \in \mathrm{Laur}(\mathfrak{a}_{\mathrm{q}}^*, \lambda + \mathfrak{a}_{F\mathrm{q}}^*, \mathcal{H}')$, for each Σ-admissible hyperplane configuration $\mathcal{H}'$. Define

$$\Lambda(F) := \{\lambda \in \mathfrak{a}_{F\mathrm{q}}^{*\perp} \mid \mathrm{Res}^{P,t}_{\lambda + \mathfrak{a}_{F\mathrm{q}}^*}(\varphi \circ s) \neq 0 \text{ for some } s \in W^F, \varphi \in \mathcal{M}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H})\}. \tag{5.1}$$

Then by [11, Cor 3.18] this set is finite and contained in $-\mathbb{R}_+ F$, the negative of the cone spanned by F. Moreover, from the same reference it follows, for $\eta \in \mathfrak{a}_{\mathrm{q}}^*$ sufficiently anti-dominant and for ε_F a point in the chamber $\mathfrak{a}_{F\mathrm{q}}^{*+}$ sufficiently close to the origin (and $\varepsilon_\Delta = 0$), that

$$\int_{\eta + i\mathfrak{a}_{\mathrm{q}}^*} \varphi(\lambda)\, d\lambda = \sum_{F \subset \Delta} t(\mathfrak{a}_{F\mathrm{q}}^+) \sum_{\lambda \in \Lambda(F)} \int_{\lambda + \varepsilon_F + i\mathfrak{a}_{F\mathrm{q}}^*} \mathrm{Res}^{P,t}_{\lambda + \mathfrak{a}_{F\mathrm{q}}^*}\Big(\sum_{s \in W^F} \varphi \circ s\Big)\, d\mu_F \tag{5.2}$$

for all $\varphi \in \mathcal{P}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H})$. Here $d\lambda$ denotes the choice of Lebesgue measure on the real linear space $i\mathfrak{a}_{\mathrm{q}}^*$, specified in [9, §3], as well as its translation to $\eta + i\mathfrak{a}_{\mathrm{q}}^*$. Furthermore, $d\mu_F := d\mu_{\mathfrak{a}_{F\mathrm{q}}^*}$ denotes a compatible choice of Lebesgue measure on the subspace $i\mathfrak{a}_{F\mathrm{q}}^*$ of $i\mathfrak{a}_{\mathrm{q}}^*$, as well as its translation to $\lambda + \varepsilon_F + i\mathfrak{a}_{F\mathrm{q}}^*$. The required compatibility is as follows. Let $(i\mu, i\nu) \mapsto cB(\mu, \nu)$ be the positive definite inner product on the real linear space $i\mathfrak{a}_{\mathrm{q}}^*$, with respect to which the normalized Lebesgue measure is $d\lambda$. Then $d\mu_F$ is normalized with respect to the restriction of this inner product. In particular, $d\lambda = d\mu_\emptyset = d\mu_{\mathfrak{a}_q^*}$. Moreover, if $\mathfrak{a}_{\Delta q} = \{0\}$, so that $\lambda + \varepsilon_\Delta + i\mathfrak{a}_{\Delta q}^*$ just consists of the point λ, then the integral $\int_{\lambda+\varepsilon_\Delta+i\mathfrak{a}_{\Delta q}^*} d\mu_\Delta$ in (5.2) represents evaluation in λ, for each $\lambda \in \Lambda(\Delta)$.

Applying the identity (5.2) on components we generalize it to V_τ-valued functions; hence in particular the identity holds for $\varphi(\lambda) = E_+(\lambda : x)\mathcal{F}f(\lambda)$, where $f \in C_c^\infty(X : \tau)$ and $x \in X_+$. We conclude

$$\mathcal{T}\mathcal{F}f(x) = |W| \sum_{F\subset\Delta} t(\mathfrak{a}_{Fq}^+) \sum_{\lambda\in\Lambda(F)} \int_{\lambda+\varepsilon_F+i\mathfrak{a}_{Fq}^*} \mathrm{Res}_{\lambda+\mathfrak{a}_{Fq}^*}^{P,t} \Big(\sum_{s\in W^F} E_+(s\,\cdot : x)\mathcal{F}f(s\,\cdot)\Big)\, d\mu_{\mathfrak{a}_{Fq}^*}. \tag{5.3}$$

The estimate that ensures the convergence of the integral over $\lambda + \varepsilon_F + i\mathfrak{a}_{Fq}^*$ follows from estimates (4.2), (4.6) by general properties of the Laurent operators (see [11, Lemma 1.11]): Let $n \in \mathbb{N}$. There exists, for each $u \in U(\mathfrak{g})$, a constant $C > 0$ such that

$$\| \mathrm{Res}_{\lambda+\mathfrak{a}_{Fq}^*}^{P,t} \big(E_+(s\,\cdot : u; x)\mathcal{F}f(s\,\cdot)\big)(\lambda + \varepsilon_F + i\nu)\| \le C(1 + |\nu|)^{-n} \tag{5.4}$$

for all $\nu \in \mathfrak{a}_{Fq}^*$, $\lambda \in \Lambda(F)$, $s \in W^F$. The constant C is locally uniform in $x \in X_+$. It follows that the integral over $\lambda+\varepsilon_F+i\mathfrak{a}_{Fq}^*$ in (5.3) is a smooth function of x. The constant C is also locally uniform in ε_F and in $f \in C_c^\infty(X : \tau)$ (cf. (4.6)). Thus, each integral in (5.3) represents a continuous linear map from $C_c^\infty(X : \tau)$ to $C^\infty(X_+ : \tau)$.

We now define for each $F \subset \Delta$ a continuous linear operator T_F^t from $C_c^\infty(X : \tau)$ to $C^\infty(X_+ : \tau)$ by

$$\mathrm{T}_F^t f(x) = |W|\, t(\mathfrak{a}_{Fq}^+) \sum_{\lambda\in\Lambda(F)} \int_{\lambda+\varepsilon_F+i\mathfrak{a}_{Fq}^*} \mathrm{Res}_{\lambda+\mathfrak{a}_{Fq}^*}^{P,t} \Big(\sum_{s\in W^F} E_+(s\,\cdot : x)\mathcal{F}f(s\,\cdot)\Big)\, d\mu_{\mathfrak{a}_{Fq}^*}; \tag{5.5}$$

as mentioned convergence follows from (5.4). Then

$$\mathcal{T}\mathcal{F}f = \sum_{F\subset\Delta} \mathrm{T}_F^t f. \tag{5.6}$$

The operator T_F^t is independent of the choice of ε_F (provided the latter is sufficiently close to 0). We also define the kernel $\mathrm{K}_F^t(\nu : x : y) \in \mathrm{End}(V_\tau)$ for $\nu \in \mathfrak{a}_{Fq\mathbb{C}}^*$, $x \in X_+$, $y \in X$ by

$$\mathrm{K}_F^t(\nu : x : y) = \sum_{\lambda\in\Lambda(F)} \mathrm{Res}_{\lambda+\mathfrak{a}_{Fq}^*}^{P,t} \Big(\sum_{s\in W^F} E_+(s\,\cdot : x) \circ E^*(s\,\cdot : y)\Big)(\lambda + \nu). \tag{5.7}$$

Clearly this is smooth as a function of $(x, y) \in X_+ \times X$ and meromorphic as a function of ν. Note that by (2.9) and (2.11) we can rewrite the expression (5.7) as follows

$$\mathrm{K}_F^t(\nu : x : y) = \sum_{\lambda\in\Lambda(F)} \mathrm{Res}_{\lambda+\mathfrak{a}_{Fq}^*}^{P,t} \Big(\sum_{s\in W^F} E_{+,s}(\,\cdot : x) \circ E^*(\,\cdot : y)\Big)(\lambda + \nu). \tag{5.8}$$

Lemma 5.1. *Let $F \subset \Delta$, $u \in U(\mathfrak{g})$, $x \in X_+$, and $f \in C_c^\infty(X\colon\tau)$. Then*

$$\sup_{\nu \in i\mathfrak{a}_{F\mathrm{q}}^*} (1+|\nu|)^n \| \int_X \mathrm{K}_F^t(\varepsilon_F + \nu\colon u; x\colon y) f(y)\, dy \| < \infty \tag{5.9}$$

for each $n \in \mathbb{N}$. The bound is locally uniform in x, ε_F and f. Moreover,

$$\mathrm{T}_F^t f(u;x) = |W|\, t(\mathfrak{a}_{F\mathrm{q}}^+) \int_{\varepsilon_F + i\mathfrak{a}_{F\mathrm{q}}^*} \int_X \mathrm{K}_F^t(\,\cdot\, \colon u; x\colon y) f(y)\, dy\, d\mu_{\mathfrak{a}_{F\mathrm{q}}^*}. \tag{5.10}$$

Proof. Let $R \in \mathbb{R}$ and let $\omega \subset \mathfrak{a}_\mathrm{q}^*(P,R)$ be compact. It follows from Lemmas 4.1 and 4.3 that there exist $N \in \mathbb{N}$ and $C > 0$ such that

$$\|p_R(\lambda) p(\lambda) E_+(\lambda\colon u; x) \circ E^*(\lambda\colon y)\| \le C(1+|\lambda|)^N$$

for all $\lambda \in \omega + i\mathfrak{a}_\mathrm{q}^*$. Moreover, this estimate holds locally uniformly in $x \in X_+$ and $y \in X$. From [11, Lemma 1.11] we obtain a similar estimate for all derivatives with respect to λ of the expression inside $\|\cdot\|$. This implies that for $f \in C_c^\infty(X\colon\tau)$ the expression

$$\int_X p_R(\lambda) p(\lambda) E_+(\lambda\colon u; x) \circ E^*(\lambda\colon y) f(y)\, dy$$

can be differentiated with respect to λ before the integration over y. It follows that

$$\int_X \mathrm{K}_F^t(\nu\colon u; x\colon y) f(y)\, dy = \sum_{\lambda \in \Lambda(F)} \mathrm{Res}^{P,t}_{\lambda + \mathfrak{a}_{F\mathrm{q}}^*} \Big(\sum_{s \in W^F} E_+(s\,\cdot\,\colon u; x) \circ \mathcal{F}f(s\,\cdot\,) \Big)(\lambda + \nu) \tag{5.11}$$

and (5.9) is obtained from (5.4), with the stated uniformity. It follows from (5.4) that differentiations with respect to x can be carried under the integral sign in (5.5). Then (5.10) follows from this equation and (5.11). $\square$

Consider the operator T_F^t for $F = \emptyset$. We have $\mathfrak{a}_{\emptyset\mathrm{q}} = \mathfrak{a}_\mathrm{q}$ and $W^\emptyset = W$. Since all the chambers of $\mathfrak{a}_\mathrm{q}$ are conjugate, a Weyl invariant residue weight necessarily takes the value $1/|W|$ on each chamber. Moreover $\Lambda(\emptyset) = \{0\}$ and $\mathrm{Res}^{P,t}_{\mathfrak{a}_{\emptyset\mathrm{q}}}$ is the identity operator. Hence

$$\mathrm{T}_\emptyset^t f(x) = \int_{\varepsilon_\emptyset + i\mathfrak{a}_\mathrm{q}^*} \sum_{s \in W} E_+(s\lambda\colon x) \mathcal{F}f(s\lambda)\, d\lambda = \frac{1}{|W|} \sum_{s \in W} \mathcal{T}_{s\varepsilon_\emptyset} \mathcal{F}f(x) = \mathcal{J}\mathcal{F}f(x) \tag{5.12}$$

by Lemma 4.5, since the expression is independent of the choice of $\varepsilon_\emptyset$. Moreover, by (2.10) and (2.11)

$$\mathrm{K}_\emptyset^t(\nu\colon x\colon y) = \sum_{s \in W} E_+(s\nu\colon x) \circ E^*(s\nu\colon x) = E^\circ(\nu\colon x) \circ E^*(\nu\colon y). \tag{5.13}$$

Remark 5.2. Consider the special case when G is compact. In this case $K = G$ and $\mathfrak{a}_\mathrm{q} = \{0\}$. It is easily seen from the definitions that ${}^\circ\mathcal{C} = C^\infty(X\colon\tau)$ and that the Eisenstein integral $E(x) = E^\circ(x)\colon {}^\circ\mathcal{C} \to V_\tau$ is the evaluation at x, for each $x \in X$. In particular, $E(e)$ is an isomorphism of ${}^\circ\mathcal{C}$ onto V_τ^H. It follows easily that $E(e) \circ E^*(e) \in \mathrm{End}(V_\tau)$ is the orthogonal projection $\mathrm{P}_H\colon V_\tau \to V_\tau^H$. Then $E(x) \circ E^*(y) = \tau(x) \circ \mathrm{P}_H \circ \tau(y^{-1})$ for $x, y \in G$ by sphericality, and it follows that the kernel $\mathrm{K}_F^t(x\colon y)$ for $F = \Delta = \emptyset$ is given by the same expression $\mathrm{K}_F^t(x\colon y) = \tau(x) \circ \mathrm{P}_H \circ \tau(y^{-1})$.

6. Some properties of the residue operators

Let $F \subset \Delta$ and let t be a W-invariant residue weight. We shall determine some further properties of the operator T_F^t and its kernel K_F^t.

Lemma 6.1. *Let $\omega \subset \mathfrak{a}_{F\mathrm{q}}^*$ be bounded. There exists a polynomial $q \in \Pi_{\Sigma,\mathbb{R}}$ with non-trivial restriction to $\mathfrak{a}_{F\mathrm{q}}^*$, for every $u, u' \in U(\mathfrak{g})$ a number $N \in \mathbb{N}$, and for all $x \in X_+$, $y \in X$ a constant $C > 0$, locally uniform in x, y, such that*

$$\|q(\nu)\mathrm{K}_F^t(\nu\colon u; x\colon u'; y)\| \leq C(1 + |\nu|)^N$$

for all $\nu \in \omega + i\mathfrak{a}_{F\mathrm{q}}^*$.

Proof. This follows from Lemmas 4.1, 4.3 and [11, Lemma 1.11]. $\square$

Lemma 6.2. *Let $F, F' \subset \Delta$ and assume that $\mathfrak{a}_{F'\mathrm{q}} = w\mathfrak{a}_{F\mathrm{q}}$ for some $w \in W$. Then*

$$\mathrm{K}_{F'}^t(w\nu\colon x\colon y) = \mathrm{K}_F^t(\nu\colon x\colon y) \tag{6.1}$$

for all generic $\nu \in \mathfrak{a}_{F\mathrm{q}\mathbb{C}}^*$, and all $x \in X_+$, $y \in X$.

Proof. The set wF is a basis for the root system spanned by F', hence there exists $w' \in W_{F'}$ such that $w'wF = F'$. Since w' acts trivially on $\mathfrak{a}_{F'\mathrm{q}}$ we may thus assume that $wF = F'$. Notice that then $s \mapsto sw^{-1}$ is a bijection of W^F onto $W^{F'}$. It follows from [11, Prop. 3.10] that

$$\mathrm{Res}^{P,t}_{w\lambda + i\mathfrak{a}_{F'\mathrm{q}}^*}(\varphi \circ w^{-1}) = \mathrm{Res}^{P,t}_{\lambda + i\mathfrak{a}_{F\mathrm{q}}^*}(\varphi) \circ w^{-1}$$

for all $\lambda \in \mathfrak{a}_{F\mathrm{q}}^{*\perp}$ and $\varphi \in \mathcal{M}(\mathfrak{a}_\mathrm{q}^*, \mathcal{H})$. Hence $\Lambda(F') = w\Lambda(F)$ and (6.1) follows easily from (5.7). $\square$

Lemma 6.3. *Let $F \subset \Delta$ and let $\nu \in \mathfrak{a}_{F\mathrm{q}\mathbb{C}}^*$ be such that $\mathrm{K}_F^t(\,\cdot\,\colon x\colon y)$ is regular at ν. The set*

$$I := \{D \in \mathbb{D}(X) \mid D\mathrm{K}_F^t(\nu\colon\cdot\,\colon y) = 0, \forall y \in X\}$$

is an ideal in $\mathbb{D}(X)$ of finite codimension.

Proof. From (2.8) we obtain

$$D\mathrm{K}_F^t(\nu\colon x\colon y) = \sum_{\lambda \in \Lambda(F)} \mathrm{Res}^{P,t}_{\lambda + \mathfrak{a}_{F\mathrm{q}}^*}\big(E_+(\,\cdot\,\colon x) \circ \mu(D\colon\cdot\,) \circ E^*(\,\cdot\,\colon y)\big)(\lambda + \nu)$$

for the action of D in the variable x. The endomorphisms $\mu(D\colon\lambda)$ of ${}^\circ\mathcal{C}$ are simultaneously diagonalizable for all $D \in \mathbb{D}(X)$, $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ (see [8, Lemma 4]). Let $\gamma_i(D\colon\lambda)$, $i = 1, \ldots, m$ be the eigenvalues, and let $I_{i,\lambda} \subset \mathbb{D}(X)$ for $i = 1, \ldots, m$, $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$, be the ideal generated by all elements of the form $D - \gamma_i(D\colon\lambda)$ where $D \in \mathbb{D}(X)$. This is a finitely generated ideal of codimension 1. Let $\lambda \in \Lambda(F)$. If k is sufficiently large then the polynomial $\mu(D)$ vanishes at $\lambda + \nu$ to sufficiently high order, for $D \in \prod_{i=1}^m (I_{i,\lambda+\nu})^k$. Hence, for sufficiently large k,

$$\mathrm{Res}^{P,t}_{\lambda + \mathfrak{a}_{F\mathrm{q}}^*}\big(E_+(\,\cdot\,\colon x) \circ \mu(D\colon\cdot\,) \circ E^*(\,\cdot\,\colon y)\big)(\lambda + \nu) = 0,$$

and thus $I \supset \prod_{\lambda \in \Lambda(F)} \prod_{i=1}^m (I_{i,\lambda+\nu})^k$. The latter ideal is cofinite, since it is a product of finitely generated cofinite ideals, so I is cofinite. $\square$

Corollary 6.4. *Let F, ν and I be as in Lemma 6.3. A function in $C^\infty(X_+\colon\tau)$ or $C^\infty(X\colon\tau)$ that is annihilated by I is real analytic.*

Proof. This is a standard application of the elliptic regularity theorem (see [27, p. 310]). $\square$

In particular, for generic ν, the functions $x \mapsto \mathrm{K}^t_F(\nu\colon x\colon y)$ (for $y \in X$) are real analytic on X_+. A similar argument shows that $y \mapsto \mathrm{K}^t_F(\nu\colon x\colon y)$ is real analytic on X (for $x \in X_+$).

Let $C_c^\infty(X_+\colon\tau)$ be the space of functions in $C^\infty(X_+\colon\tau)$ that are supported by a compact subset of X_+. The rest of this section is devoted to the determination of the adjoint of the operator $\mathrm{T}^t_F\colon C_c^\infty(X\colon\tau) \to C^\infty(X_+\colon\tau)$ with respect to the sesquilinear form

$$\langle f|g\rangle := \int_{X_+} \langle f(x)|g(x)\rangle\, dx$$

on $C_c^\infty(X\colon\tau) \times C_c^\infty(X_+\colon\tau)$. The following definitions and lemmas will be helpful.

Define

$$E_+^*(\lambda\colon x) = E_+(-\bar\lambda\colon x)^*$$

for $x \in X_+$, in analogy with (2.3). Furthermore, let

$$E_{+,s}^*(\lambda\colon x) = E_{+,s}(-\bar\lambda\colon x)^* = C^\circ(s\colon\lambda)^{-1} \circ E_+^*(s\lambda\colon x),$$

cf. (2.9) and (1.8), then

$$E^*(\lambda\colon x) = \sum_{s\in W} E_{+,s}^*(\lambda\colon x) \tag{6.2}$$

for $x \in X_+$.

Lemma 6.5. *Let $F \subset \Delta$ and let $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$ be generic. Then*

$$\mathrm{K}^t_F(\nu\colon x\colon y)^* = \sum_{\lambda\in\Lambda(F)} \mathrm{Res}^{P,t}_{\lambda+\mathfrak{a}^*_{F\mathrm{q}}}\Big(\sum_{s\in W^F} E^\circ(-s\,\cdot\,\colon y) \circ E_+^*(-s\,\cdot\,\colon x)\Big)(\lambda+\bar\nu) \tag{6.3}$$

$$= \sum_{\lambda\in\Lambda(F)} \mathrm{Res}^{P,t}_{\lambda+\mathfrak{a}^*_{F\mathrm{q}}}\Big(E^\circ(-\,\cdot\,\colon y) \circ \sum_{s\in W^F} E_{+,s}^*(-\,\cdot\,\colon x)\Big)(\lambda+\bar\nu) \tag{6.4}$$

for $x \in X_+$, $y \in X$.

Proof. The Laurent operators $\mathrm{Res}^{P,t}_{\lambda+\mathfrak{a}^*_{F\mathrm{q}}}$ are real (see [11, Thm. 1.13]). It follows easily that

$$\mathrm{Res}^{P,t}_{\lambda+\mathfrak{a}^*_{F\mathrm{q}}}(\varphi)^\vee = \mathrm{Res}^{P,t}_{\lambda+\mathfrak{a}^*_{F\mathrm{q}}}(\varphi^\vee) \tag{6.5}$$

for all $\varphi \in \mathcal{M}(\mathfrak{a}^*_\mathrm{q}, \mathcal{H})$, where $\varphi^\vee\colon \nu \mapsto \overline{\varphi(\bar\nu)}$. The identity (6.5) generalizes to $\mathrm{End}(V_\tau)$-valued functions φ if we replace the definition of $\varphi^\vee$ by $\varphi^\vee\colon \nu \mapsto \varphi(\bar\nu)^*$. We apply (6.5) with

$$\varphi(\nu) = \sum_{s\in W^F} E_+(s\nu\colon x) \circ E^*(s\nu\colon y).$$

Then

$$\varphi^\vee(\nu) = \sum_{s\in W^F} E^\circ(-s\nu\colon y)\circ E^*_+(-s\nu\colon x);$$

for $\lambda \in \Lambda(F)$ we thus obtain

$$[\mathrm{Res}^{P,t}_{\lambda+\mathfrak{a}^*_{F\mathrm{q}}}(\varphi)(\lambda+\nu)]^* = \mathrm{Res}^{P,t}_{\lambda+\mathfrak{a}^*_{F\mathrm{q}}}(\varphi^\vee)(\lambda+\bar\nu).$$

Applying this termwise to (5.7) we obtain (6.3), and using (2.11) we then obtain (6.4). □

For $f \in C^\infty_c(X_+\colon\tau)$ we define, in analogy with (2.4):

$$\mathcal{F}_+f(\lambda) = \int_{X_+} E^*_+(\lambda\colon x)f(x)\,dx \in {}^\circ\mathcal{C}, \tag{6.6}$$

for generic $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$. Then $\mathcal{F}_+f$ is a ${}^\circ\mathcal{C}$-valued meromorphic function on $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$.

Lemma 6.6. *Let $R \in \mathbb{R}$ and let p_R be as in Lemma 3.3. Let $\omega \subset \mathfrak{a}^*_{\mathrm{q}}(P,R)$ be compact. Then*

$$\sup_{\lambda\in\omega+i\mathfrak{a}^*_{\mathrm{q}}} (1+|\lambda|)^n \|p_R(\lambda)\mathcal{F}_+f(-\lambda)\| < \infty$$

for all $n \in \mathbb{N}$, $f \in C^\infty_c(X_+\colon\tau)$.

Proof. This follows from Lemma 4.1 in the same manner as [9, Prop. 8.3]. □

Lemma 6.7. *Let $F \subset \Delta$ and let ω and q be as in Lemma 6.1. Let $u \in U(\mathfrak{g}), n \in \mathbb{N}$. Then*

$$\sup_{\nu\in\omega+i\mathfrak{a}^*_{\mathrm{q}}} (1+|\nu|)^n |\,q(\nu)\,|\ \|\int_{X_+} \mathrm{K}^t_F(\nu\colon x\colon u;y)^* g(x)\,dx\| < \infty \tag{6.7}$$

for every $g \in C^\infty_c(X_+\colon\tau)$ and all $y \in X$, with a bound that is locally uniform in g and y. If $g \in C^\infty_c(X_+\colon\tau)$, then the function

$$y \mapsto \mathrm{S}^t_F g(y) := |W|\,t(\mathfrak{a}^+_{F\mathrm{q}}) \int_{-\varepsilon_F+i\mathfrak{a}^*_{F\mathrm{q}}} \int_{X_+} \mathrm{K}^t_F(\,\cdot\,\colon x\colon y)^* g(x)\,dx\,d\mu_{\mathfrak{a}^*_{F\mathrm{q}}} \in V_\tau \tag{6.8}$$

belongs to $C^\infty(X\colon\tau)$ and is independent of the choice (sufficiently close to 0) of ε_F.

Proof. Let $n \in \mathbb{N}$ and $u \in U(\mathfrak{g})$. In analogy with (5.4) it follows from Lemmas 4.3 and 6.6 that

$$\|q(\nu)\,\mathrm{Res}^{P,t}_{\lambda+\mathfrak{a}^*_{F\mathrm{q}}}\big(E^\circ(-s\,\cdot\colon u;y)\circ\mathcal{F}_+g(-s\,\cdot)\big)(\lambda+\nu)\| \le C(1+|\nu|)^{-n} \tag{6.9}$$

for all $\nu \in \omega+i\mathfrak{a}^*_{F\mathrm{q}}$, $\lambda \in \Lambda(F)$, $s \in W^F$, with a constant C locally uniform in $g \in C^\infty_c(X_+\colon\tau)$ and $y \in X$. From Lemma 6.5 we obtain, as in the proof of Lemma 5.1,

$$\int_{X_+} \mathrm{K}^t_F(\nu\colon x\colon y)^* g(x)\,dx = \sum_{\lambda\in\Lambda(F)} \mathrm{Res}^{P,t}_{\lambda+\mathfrak{a}^*_{F\mathrm{q}}}\Big(\sum_{s\in W^F} E^\circ(-s\,\cdot\,\colon y)\circ\mathcal{F}_+g(-s\,\cdot)\Big)(\lambda+\bar\nu)$$

and the estimate (6.7) follows from (6.9). The final statement of the lemma is an immediate consequence. □

Let $F \subset \Delta$ and let $F' \subset \Delta$ be given by $F' = -w_0F$, where w_0 denotes the longest element in W (with respect to Δ). Then $-\mathfrak{a}^+_{F\mathrm{q}} = w_0\mathfrak{a}^+_{F'\mathrm{q}}$. Recall that a residue weight t is called *even* if $t(Q) = t(-Q)$ for all $Q \in \mathcal{P}$. If t is even (and Weyl invariant) then $t(\mathfrak{a}^+_{F\mathrm{q}}) = t(\mathfrak{a}^+_{F'\mathrm{q}})$.

Lemma 6.8. *Let $F \subset \Delta$ and $F' = -w_0 F$. Assume that t is even (in addition to being W-invariant). Define $\mathrm{S}^t_{F'}$ as in Lemma 6.7. Then*

$$\langle \mathrm{T}^t_F f | g \rangle = \langle f | \mathrm{S}^t_{F'} g \rangle$$

for all $f \in C_c^\infty(X{:}\,\tau)$ and $g \in C_c^\infty(X_+{:}\,\tau)$.

Proof. By (5.9), (5.10) and Fubini's theorem,

$$\begin{aligned}\langle \mathrm{T}^t_F f | g \rangle &= |W|\, t(\mathfrak{a}^+_{F\mathrm{q}}) \int_{X_+} \int_{\varepsilon_F + i\mathfrak{a}^*_{F\mathrm{q}}} \int_X \langle \mathrm{K}^t_F(\nu{:}\, x{:}\, y) f(y) | g(x) \rangle \, dy \, d\nu \, dx \\ &= |W|\, t(\mathfrak{a}^+_{F\mathrm{q}}) \int_{\varepsilon_F + i\mathfrak{a}^*_{F\mathrm{q}}} \int_{X_+} \int_X \langle \mathrm{K}^t_F(\nu{:}\, x{:}\, y) f(y) | g(x) \rangle \, dy \, dx \, d\nu. \qquad (6.10)\end{aligned}$$

Similarly, by (6.8) and (6.7),

$$\begin{aligned}\langle f | \mathrm{S}^t_{F'} g \rangle &= |W|\, t(\mathfrak{a}^+_{F'\mathrm{q}}) \int_X \int_{-\varepsilon_{F'} + i\mathfrak{a}^*_{F'\mathrm{q}}} \int_{X_+} \langle f(y) | \mathrm{K}^t_{F'}(\nu'{:}\, x{:}\, y)^* g(x) \rangle \, dx \, d\nu' \, dy \\ &= |W|\, t(\mathfrak{a}^+_{F'\mathrm{q}}) \int_{-\varepsilon_{F'} + i\mathfrak{a}^*_{F'\mathrm{q}}} \int_X \int_{X_+} \langle f(y) | \mathrm{K}^t_{F'}(\nu'{:}\, x{:}\, y)^* g(x) \rangle \, dx \, dy \, d\nu'.\end{aligned}$$

We have $\mathfrak{a}_{F'\mathrm{q}} = w_0 \mathfrak{a}_{F\mathrm{q}}$, and we may assume that $-\varepsilon_{F'} = w_0 \varepsilon_F$. Hence by the change of variables $\nu' = w_0 \nu$ and by Lemma 6.2

$$\langle f | \mathrm{S}^t_{F'} g \rangle = |W|\, t(\mathfrak{a}^+_{F\mathrm{q}}) \int_{\varepsilon_F + i\mathfrak{a}^*_{F\mathrm{q}}} \int_X \int_{X_+} \langle \mathrm{K}^t_F(\nu{:}\, x{:}\, y) f(y) | g(x) \rangle \, dx \, dy \, d\nu. \qquad (6.11)$$

Finally, the expressions (6.10) and (6.11) are equal, since the order of the inner integrals can be interchanged by continuity of the integrands, cf. Lemma 6.1. □

7. Main results

With the notation introduced in Section 5 we can rewrite our main result as follows. By (5.6) the following theorem is equivalent with Theorem 4.7.

Theorem 7.1. *Let t be a W-invariant residue weight for Σ. Then*

$$f(x) = \sum_{F \subset \Delta} \mathrm{T}^t_F f(x) \qquad (7.1)$$

for all $f \in C_c^\infty(X{:}\,\tau)$ and $x \in X_+$.

When stated as in (7.1) the inversion formula depends on the choice of a residue weight. We shall see in [13] how this dependence can be eliminated from the formula.

The proof of Theorem 7.1 is based on the following result, which is the second main result of our paper.

Theorem 7.2. *Let t be a W-invariant even residue weight for Σ and let $F \subset \Delta$. The* $\mathrm{End}(V_\tau)$*-valued kernel defined in* (5.7) *satisfies the following property of symmetry*

$$\mathrm{K}^t_F(\nu : x : y)^* = \mathrm{K}^t_F(-\bar{\nu} : y : x) \tag{7.2}$$

for all $x, y \in X_+$ and generic $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$.

For $F = \emptyset$ this result is a direct consequence of (5.13) and (2.3). For the case of a general set F it will be proved in the course of the following two sections.

The symmetry of the kernel K^t_F is related to a similar property of symmetry for the operator T^t_F. The following lemma will be used in an inductive argument in the proof of Theorem 7.2.

Lemma 7.3. *Let t be a W-invariant even residue weight for Σ, and let $F \subset \Delta$. Assume that K^t_F is symmetric, i.e.* (7.2) *holds for $x, y \in X_+$ and $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$ generic. Then the following holds. Let $g \in C^\infty_c(X_+ : \tau)$.*

(i) *The function $x \mapsto \mathrm{T}^t_F g(x)$, $X_+ \to V_\tau$, extends to a smooth τ-spherical function on X.*

(ii) *Let $F' = -w_0 F \subset \Delta$. Then $x \mapsto \mathrm{T}^t_{F'} g(x)$ extends to a smooth τ-spherical function on X, and $\langle \mathrm{T}^t_F f | g \rangle = \langle f | \mathrm{T}^t_{F'} g \rangle$ for all $f \in C^\infty_c(X : \tau)$.*

Later (after Theorem 7.2 has been proved) we shall see that (i), (ii) actually hold with $g \in C^\infty_c(X : \tau)$ (see Corollary 10.11).

Proof. It follows from (7.2) that the operator S^t_F defined in Lemma 6.7 is identical with the restriction of T^t_F to $C^\infty_c(X_+ : \tau)$ (apply the substitution of variables $\nu \to -\bar{\nu}$ in the outer integral). Hence (i) follows from this lemma. Notice that the symmetry of K^t_F expressed in (7.2) implies that $\mathrm{K}^t_{F'}$ satisfies the same kind of symmetry (by Lemma 6.2), and hence (i) holds for $\mathrm{T}^t_{F'} g$ as well. Now (ii) follows from Lemma 6.8. $\square$

We shall now derive Theorem 7.1 from Theorem 7.2.

Proof of Theorem 7.1. We assume that Theorem 7.2 holds. We see from (5.6) that Theorem 7.1 is equivalent with Theorem 4.7, in which the residue weight t is absent, and we may therefore assume that t is even (cf. [11, Example 3.3]).

Let first $f \in C^\infty_c(X_+ : \tau)$. Since (7.2) holds by assumption, it follows by application of Lemma 7.3 (i) that $\mathrm{T}^t_F f \in C^\infty(X : \tau)$ for all $F \subset \Delta$. Hence $\mathcal{T}\mathcal{F} f \in C^\infty(X : \tau)$ by (5.6), and Corollary 4.10 shows that $\mathcal{T}\mathcal{F} f = f$.

Let now $f \in C^\infty_c(X : \tau)$, and let $g \in C^\infty_c(X_+ : \tau)$ be arbitrary. Then from (7.2) together with Lemma 7.3 (ii) it follows that $\langle \mathrm{T}^t_F f | g \rangle = \langle f | \mathrm{T}^t_{F'} g \rangle$. Since $F \mapsto F'$ is a bijection of the set of subsets of Δ, we conclude by summation and application of (5.6) that $\langle \mathcal{T}\mathcal{F} f | g \rangle = \langle f | \mathcal{T}\mathcal{F} g \rangle$. The expression on the right-hand side of the latter equation equals $\langle f | g \rangle$ by the first part of the proof. We conclude (cf. [9, Lemma 11.3]) that $\mathcal{T}\mathcal{F} f = f$ on X_+. $\square$

In the proof of Theorem 7.2 we shall need the reformulation of (7.2) given in the following lemma.

Lemma 7.4. *Let $x, y \in X_+$ and $F \subset \Delta$. Let $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$ be generic (or more precisely, such that $\mathrm{K}^t_F(\nu\colon x\colon y)$ and $\mathrm{K}^t_F(-\bar\nu\colon y\colon x)$ are both regular at ν). Then (7.2) holds if and only if*

$$\sum_{\lambda\in\Lambda(F)} \mathrm{Res}^{P,t}_{\lambda+\mathfrak{a}^*_{F\mathrm{q}}}\Big(\sum_{s\in W^F} E_{+,s}(\,\cdot:x)\circ E^*(\,\cdot:y)\Big)(\lambda+\nu)$$
$$= \sum_{\lambda\in\Lambda(F)} \mathrm{Res}^{P,t}_{\lambda+\mathfrak{a}^*_{F\mathrm{q}}}\Big(E^\circ(-\,\cdot:x)\circ \sum_{s\in W^F} E^*_{+,s}(-\,\cdot:y)\Big)(\lambda-\nu). \tag{7.3}$$

In particular, if $F=\Delta$ and $\mathfrak{a}_{\Delta\mathrm{q}}=\{0\}$, then this identity simplifies to the following identity in V_τ

$$\sum_{\lambda\in\Lambda(\Delta)} \mathrm{Res}^{P,t}_{\lambda}\big(E_+(\,\cdot:x)\circ E^*(\,\cdot:y)\big) = \sum_{\lambda\in\Lambda(\Delta)} \mathrm{Res}^{P,t}_{\lambda}\big(E^\circ(-\,\cdot:x)\circ E^*_+(-\,\cdot:y)\big). \tag{7.4}$$

Proof. By means of (5.8) and (6.4) the two sides of (7.3) are identified as $\mathrm{K}^t_F(\nu\colon x\colon y)$ and $\mathrm{K}^t_F(-\bar\nu\colon y\colon x)^*$, respectively. For $F=\Delta$ we have $W^F=\{1\}$, so that (7.3) simplifies to (7.4). $\square$

Lemma 7.5. *Let $x, y \in X_+$, $F\subset\Delta$. Let Λ be any finite subset of $\mathfrak{a}^{*\perp}_{F\mathrm{q}}$ containing $\Lambda(F)$. Then the identity (7.3) is equivalent to each of the identities resulting from it by replacing $\Lambda(F)$ by Λ on either one or both sides.*

Proof. It suffices to show that the residues

$$\mathrm{Res}^{P,t}_{\lambda+\mathfrak{a}^*_{F\mathrm{q}}}\big(E_{+,s}(\,\cdot:x)\circ E^*(\,\cdot:y)\big)$$

and

$$\mathrm{Res}^{P,t}_{\lambda+\mathfrak{a}^*_{F\mathrm{q}}}\big(E^\circ(-\cdot:x)\circ E^*_{+,s}(-\cdot:y)\big)$$

vanish for $\lambda\in\mathfrak{a}^{*\perp}_{F\mathrm{q}}\setminus\Lambda(F)$ and $s\in W^F$. We note that

$$E_{+,s}(\mu\colon x)\circ E^*(\mu\colon y) = E_+(s\mu\colon x)\circ E^*(s\mu\colon y)$$

and

$$E^\circ(-\mu\colon x)\circ E^*_{+,s}(-\mu\colon y) = (E_+(s\bar\mu\colon x)\circ E^*(s\bar\mu\colon y))^*.$$

It is easily seen that the functions $\mu\mapsto E_+(\mu\colon x)\circ E^*(\mu\colon y)$ and $\mu\mapsto (E_+(\bar\mu\colon x)\circ E^*(\bar\mu\colon y))^*$ both belong to $\mathcal{M}(\mathfrak{a}^*_{\mathrm{q}},\mathcal{H})\otimes\mathrm{End}(V_\tau)$, with $\mathcal{H}$ defined as in the beginning of Section 5. The assertion now follows from the definition of $\Lambda(F)$, see (5.1). $\square$

8. Application of an asymptotic result

In this section we shall apply the theory of [12], as outlined in Appendix B, in order to prove the following Proposition 8.2, which will be crucial for the induction in the following section.

In order to prepare for the mentioned induction we must introduce some notation. Let $Q = M_Q A_Q N_Q \in \mathcal{P}_\sigma$ with the indicated Langlands decomposition. For each element $v \in N_K(\mathfrak{a}_\mathrm{q})$ we consider the symmetric space

$$X_{Q,v} = M_Q / M_Q \cap vHv^{-1};$$

this is a reductive symmetric space of Harish-Chandra's class, and its vectorial part is trivial. If $F \subset \Delta$ and Q is the corresponding standard σ-parabolic subgroup P_F, then we write $M_F = M_Q$ and $X_{F,v} = X_{Q,v}$. In particular, $P_\emptyset$ is the fixed σ-minimal parabolic subgroup P, and $X_{\emptyset,v}$ is the compact symmetric space $M/M \cap vHv^{-1}$.

Let $F \subset \Delta$ be fixed. For the symmetric space $X_{F,v}$ the role of $\mathfrak{a}_\mathrm{q}$ is played by the orthocomplement $\mathfrak{a}_{F\mathrm{q}}^\perp$ of $\mathfrak{a}_{F\mathrm{q}}$ in $\mathfrak{a}_\mathrm{q}$, that is, this is a maximal abelian subspace of $\mathfrak{m}_F \cap \mathfrak{p} \cap \mathrm{Ad}\, v(\mathfrak{q})$. As before we identify the elements of the dual $\mathfrak{a}_{F\mathrm{q}}^{*\perp}$ with the linear forms on $\mathfrak{a}_\mathrm{q}$ that vanish on $\mathfrak{a}_{F\mathrm{q}}$. Then $\Sigma_F^+ = \Sigma^+ \cap \mathfrak{a}_{F\mathrm{q}}^{*\perp}$ is a positive system for $\Sigma(\mathfrak{a}_{F\mathrm{q}}^\perp, \mathfrak{m}_F)$, and F is the corresponding set of simple roots. We denote by ${}^*P$ the parabolic subgroup $M_F \cap P$ of M_F; it is the analog for $X_{F,v}$ of P. In the following, when we consider Eisenstein integrals on $X_{F,v}$, we relate them to Σ_F^+ and ${}^*P$, and consider these latter data as fixed. Similarly, the open chamber $\mathfrak{a}_{F\mathrm{q}}^{\perp+}$ is defined relative to Σ_F^+.

As in [8, Sect. 8] we fix a set $\mathcal{W}_F \subset N_{M_F \cap K}(\mathfrak{a}_\mathrm{q})$ of representatives for the two-sided quotient $Z_{M_F\cap K}(\mathfrak{a}_\mathrm{q}) \backslash N_{M_F \cap K}(\mathfrak{a}_\mathrm{q}) / N_{M_F\cap K\cap H}(\mathfrak{a}_\mathrm{q})$. The set $\mathcal{W}_F$ is the analog for $X_F = M_F/M_F \cap H$ of the set $\mathcal{W} \subset N_K(\mathfrak{a}_\mathrm{q})$; we recall that the latter set has been chosen as a set of representatives for $Z_K(\mathfrak{a}_\mathrm{q})\backslash N_K(\mathfrak{a}_\mathrm{q})/N_{K\cap H}(\mathfrak{a}_\mathrm{q})$ (or, equivalently, for $W/W_{K\cap H}$). We define

$$^\circ\mathcal{C}_F = \oplus_{w\in\mathcal{W}_F} C^\infty(M/M\cap wHw^{-1} \colon \tau_M),$$

where $\tau_M = \tau|_{M\cap K}$. Then $^\circ\mathcal{C}_F$ is the analog for X_F of the space $^\circ\mathcal{C}$, which, we recall, is given by

$$^\circ\mathcal{C} = \oplus_{w\in\mathcal{W}} C^\infty(M/M\cap wHw^{-1} \colon \tau_M). \tag{8.1}$$

In particular, the Eisenstein integrals on X_F, $E(X_F \colon \psi \colon \lambda) \in C^\infty(X_F \colon \tau|_{M_F\cap K})$, are parametrized by $\psi \in {}^\circ\mathcal{C}_F$ and $\lambda \in \mathfrak{a}_{F\mathrm{q}}^{*\perp}$.

More generally, for each $v \in N_K(\mathfrak{a}_\mathrm{q})$ we fix a set $\mathcal{W}_{F,v} \subset N_{M_F\cap K}(\mathfrak{a}_\mathrm{q})$ of representatives for $Z_{M_F\cap K}(\mathfrak{a}_\mathrm{q})\backslash N_{M_F\cap K}(\mathfrak{a}_\mathrm{q})/N_{M_F\cap K\cap vHv^{-1}}(\mathfrak{a}_\mathrm{q})$; then $\mathcal{W}_{F,v}$ plays the role for $X_{F,v}$ of $\mathcal{W}$. We put

$$^\circ\mathcal{C}_{F,v} = \oplus_{w\in\mathcal{W}_{F,v}} C^\infty(M/M\cap wvHv^{-1}w^{-1} \colon \tau_M); \tag{8.2}$$

this is the analog for $X_{F,v}$ of (8.1). Then we have the Eisenstein integrals $E(X_{F,v} \colon \psi \colon \lambda)$ on $X_{F,v}$, where $\psi \in {}^\circ\mathcal{C}_{F,v}$, $\lambda \in \mathfrak{a}_{F\mathrm{q}}^{*\perp}$. Similarly we introduce $E^\circ(X_{F,v} \colon \psi \colon \lambda)$, $E^*(X_{F,v} \colon \psi \colon \lambda)$, $E_+(X_{F,v} \colon \psi \colon \lambda)$ and $E_+^*(X_{F,v} \colon \psi \colon \lambda)$. The latter two functions are defined on

$$X_{F,v,+} = \cup_{w\in\mathcal{W}_{F,v}} (M_F\cap K)\,{}^*A_{F\mathrm{q}}^+ w(M_F \cap vHv^{-1})$$

where ${}^*A^+_{F\mathrm{q}} = \exp \mathfrak{a}^{\perp+}_{F\mathrm{q}}$.

Let ${}^F\mathcal{W}$ be a (fixed) subset of $N_K(\mathfrak{a}_\mathrm{q})$ that is a complete set of representatives for $W_F\backslash W/W_{K\cap H}$. The proof of the following result is straightforward.

Lemma 8.1. *The union*

$$\cup_{v\in {}^F\mathcal{W}} \mathcal{W}_{F,v}\, v \tag{8.3}$$

in $N_K(\mathfrak{a}_\mathrm{q})$ is disjoint and forms a complete set of representatives for $W/W_{K\cap H}$.

In the following we shall assume (with F fixed), that $\mathcal{W}$ has been chosen such that it equals the set (8.3). Since the basic definitions, for example of the Eisenstein integrals, are essentially independent of the choice of $\mathcal{W}$ (cf. [8, eq. (27)]), this assumption is harmless (although in general it cannot be realized simultaneously for all F). Then, corresponding to the injection of $\mathcal{W}_{F,v}$ in $\mathcal{W}$ by (8.3) and the assumption just made, there is a natural injection $\mathrm{i}_{F,v}$ of ${}^\circ\mathcal{C}_{F,v}$ into ${}^\circ\mathcal{C}$, simply given by the identity on each component of (8.2). We denote by $\mathrm{pr}_{F,v}$ the corresponding orthogonal projection of ${}^\circ\mathcal{C}$ onto ${}^\circ\mathcal{C}_{F,v}$. It follows from Lemma 8.1 that

$${}^\circ\mathcal{C} = \oplus_{v\in {}^F\mathcal{W}}\ \mathrm{i}_{F,v}({}^\circ\mathcal{C}_{F,v}). \tag{8.4}$$

Given a residue weight t for Σ, we define a residue weight ${}^*t$ for Σ_F as in [11, Sect. 3.6]. Let $\lambda \in \mathfrak{a}^{*\perp}_{F\mathrm{q}}$ and $\varphi \in \mathcal{M}(\mathfrak{a}^*_\mathrm{q}, \mathcal{H})$, where $\mathcal{H}$ is any Σ-admissible hyperplane configuration in $\mathfrak{a}^*_\mathrm{q}$. Then, according to Lemma B.5 with V and L as described below the proof of the lemma, the function $z \mapsto \varphi(\nu + z)$ on $\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}$ belongs to $\mathcal{M}(\lambda, \Sigma_F)$, for generic $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$. Moreover, according to Remark B.4, the universal residue operator $\mathrm{Res}^{{}^*P,{}^*t}_\lambda$ on $\mathfrak{a}^{*\perp}_{F\mathrm{q}}$ can be identified with an element in $\mathcal{M}(\lambda, \Sigma_F)^*_{\mathrm{laur}}$. Then, by Lemma B.5 the function $\nu \mapsto \mathrm{Res}^{{}^*P,{}^*t}_\lambda[\varphi(\nu + \cdot\,)]$ is meromorphic on $\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$. It now follows from [11, Thm. 3.14] that

$$[\mathrm{Res}^{P,t}_{\lambda+\mathfrak{a}^*_{F\mathrm{q}}}\varphi](\nu + \lambda) = \mathrm{Res}^{{}^*P,{}^*t}_\lambda[\varphi(\nu + \cdot\,)] \tag{8.5}$$

as an identity of meromorphic functions in ν.

Proposition 8.2. *Fix $F \subset \Delta$ and assume that for each $v \in {}^F\mathcal{W}$ the identity* (7.4) *in Lemma 7.4 holds for the symmetric space $X_{F,v}$, for all $x, y \in X_{F,v,+}$. Then* (7.2) *holds (for the symmetric space X) for all $x, y \in X_+$ and generic $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$.*

In particular, if $F = \emptyset$, then the hypothesis in Proposition 8.2 amounts to the symmetry, for each $v \in \mathcal{W}$, of the kernel $\mathrm{K}^t_\emptyset(X_{\emptyset,v}\colon m\colon m')$ for the compact symmetric space $X_{\emptyset,v} = M/M\cap vHv^{-1}$. This hypothesis is easily seen to be fulfilled (cf. Remark 5.2). The conclusion, on the other hand, is the symmetry of the kernel $\mathrm{K}^t_\emptyset(\nu\colon x\colon y)$ for X; this symmetry was however already verified below Theorem 7.2.

Proof. Let $v \in {}^F\mathcal{W}$. The assumption (7.4) for $X_{F,v}$ reads

$$\begin{aligned}\sum_{\lambda\in\Lambda(X_{F,v},F)} &\mathrm{Res}^{{}^*P,{}^*t}_\lambda\big(E_+(X_{F,v}\colon \cdot\colon x)\circ E^*(X_{F,v}\colon \cdot\colon y)\big)\\ &= \sum_{\lambda\in\Lambda(X_{F,v},F)} \mathrm{Res}^{{}^*P,{}^*t}_\lambda\big(E^\circ(X_{F,v}\colon -\cdot\colon x)\circ E^*_+(X_{F,v}\colon -\cdot\colon y)\big)\end{aligned} \tag{8.6}$$

for all $x, y \in X_{F,v,+}$. Here $\Lambda(X_{F,v}, F)$ is the analog for $X_{F,v}$ of $\Lambda(\Delta)$ (see (5.1)), that is,

$$\Lambda(X_{F,v}, F) = \{\lambda \in \mathfrak{a}_{F\mathrm{q}}^{*\perp} \mid \mathrm{Res}_\lambda^{*P,*t} \varphi \neq 0 \text{ for some } \varphi \in \mathcal{M}(\mathfrak{a}_{F\mathrm{q}}^{*\perp}, \mathcal{H}_{F,v})\} \tag{8.7}$$

where $\mathcal{H}_{F,v}$ is the set of affine hyperplanes in $\mathfrak{a}_{F\mathrm{q}}^{*\perp}$ along which $\lambda \mapsto E_+(X_{F,v}\colon \cdot : x)$ or $\lambda \mapsto E^*(X_{F,v}\colon \cdot : y)$ is singular for some x, y.

Note that, by Lemma 7.5, an equivalent form of the identity (8.6) is obtained if we replace on both sides the set of summation $\Lambda(X_{F,v}, F)$ by any finite subset Λ of $\mathfrak{a}_{F\mathrm{q}}^{*\perp}$ that contains $\Lambda(X_{F,v}, F)$. Likewise, in order to prove (7.3) (which, by Lemma 7.4, is sufficient for our goal) it suffices to prove this identity with $\Lambda(F)$ replaced on both sides by any finite subset Λ of $\mathfrak{a}_{F\mathrm{q}}^{*\perp}$ that contains $\Lambda(F)$. We shall apply these observations with the following set Λ:

$$\Lambda = [\cup_{v \in {}^F\mathcal{W}} \Lambda(X_{F,v}, F)] \cup \Lambda(F). \tag{8.8}$$

We shall now apply the induction of relations of Appendix B. We first apply it in the version of Theorem B.6. According to the discussion before (8.5), the linear functional

$$\mathcal{L}\colon \varphi \mapsto \sum_{\lambda \in \Lambda} \mathrm{Res}_\lambda^{*P,*t} \varphi$$

on $\mathcal{M}(\mathfrak{a}_{F\mathrm{q}\mathbb{C}}^{*\perp}, \Sigma_F)$ is a Laurent functional in $\mathcal{M}(\mathfrak{a}_{F\mathrm{q}\mathbb{C}}^{*\perp}, \Sigma_F)^*_{\mathrm{laur}}$. We define the Laurent functionals $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{M}(\mathfrak{a}_{F\mathrm{q}\mathbb{C}}^{*\perp}, \Sigma_F)^*_{\mathrm{laur}}$ by $\mathcal{L}_1(\varphi) = \mathcal{L}(\varphi(-\,\cdot))$ and $\mathcal{L}_2 = \mathcal{L}$. For fixed $y \in X_{F,v,+}$ and $a \in V_\tau$, we define the functions $\phi_1, \phi_2\colon \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \to {}^\circ\mathcal{C}_{F,v}$ by $\phi_1(\nu + \lambda) = E_+^*(X_{F,v}\colon \lambda\colon y)a$ and $\phi_2(\nu + \lambda) = E^*(X_{F,v}\colon \lambda\colon y)a$, for generic $\lambda \in \mathfrak{a}_{F\mathrm{q}\mathbb{C}}^{*\perp}$ and $\nu \in \mathfrak{a}_{F\mathrm{q}\mathbb{C}}^*$. Then ϕ_1 and ϕ_2 belong to $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \Sigma) \otimes {}^\circ\mathcal{C}_{F,v}$. From (8.6) we now obtain, by applying Theorem B.6 with $\mathcal{L}_1, \mathcal{L}_2, \phi_1, \phi_2$ as mentioned, that

$$\begin{aligned}\sum_{\lambda \in \Lambda} &\mathrm{Res}_\lambda^{*P,*t}\big(\sum_{s \in W^F} E_{+,s}(\nu + \,\cdot : x) \circ \mathrm{i}_{F,v} \circ E^*(X_{F,v}\colon \cdot : y)\big) \\ &= \sum_{\lambda \in \Lambda} \mathrm{Res}_\lambda^{*P,*t}\big(E^\circ(\nu - \,\cdot : x) \circ \mathrm{i}_{F,v} \circ E_+^*(X_{F,v}\colon - \cdot : y)\big)\end{aligned} \tag{8.9}$$

for all $y \in X_{F,v,+}$, $x \in X_+$ and generic $\nu \in \mathfrak{a}_{F\mathrm{q}\mathbb{C}}^*$.

We apply the induction of relations once more, this time in the dual version Corollary B.7, and obtain with $x \in X_+$ fixed,

$$\begin{aligned}\sum_{\lambda \in \Lambda} &\mathrm{Res}_\lambda^{*P,*t}\big(\sum_{s \in W^F} E_{+,s}(\nu + \,\cdot : x) \circ \mathrm{i}_{F,v} \circ \mathrm{pr}_{F,v} \circ E^*(\nu + \,\cdot : y)\big) \\ &= \sum_{\lambda \in \Lambda} \mathrm{Res}_\lambda^{*P,*t}\big(E^\circ(\nu - \,\cdot : x) \circ \mathrm{i}_{F,v} \circ \mathrm{pr}_{F,v} \circ \sum_{s \in W^F} E_{+,s}^*(\nu - \,\cdot : y)\big)\end{aligned}$$

for all $y \in X_+$ and generic $\nu \in \mathfrak{a}_{F\mathrm{q}\mathbb{C}}^*$. Summing over $v \in {}^F\mathcal{W}$, cf. (8.4), we obtain

$$\begin{aligned}\sum_{\lambda \in \Lambda} &\mathrm{Res}_\lambda^{*P,*t}\big(\sum_{s \in W^F} E_{+,s}(\nu + \,\cdot : x) \circ E^*(\nu + \,\cdot : y)\big) \\ &= \sum_{\lambda \in \Lambda} \mathrm{Res}_\lambda^{*P,*t}\big(E^\circ(\nu - \,\cdot : x) \circ \sum_{s \in W^F} E_{+,s}^*(\nu - \,\cdot : y)\big).\end{aligned}$$

By (8.5) we can replace the residue operators $\mathrm{Res}_{\lambda}^{*P,*t}$ by $\mathrm{Res}_{\lambda+\mathfrak{a}_{F\mathrm{q}}^*}^{P,t}$ and, as remarked above, Λ by $\Lambda(F)$. We thus obtain the desired identity (7.3). $\square$

9. Proof of Theorem 7.2

The proof is by induction on the rank of the root system Σ. We assume the statement of the theorem holds for all reductive symmetric spaces for which the corresponding root system is of lower rank than Σ (this is definitely true if the rank of Σ is zero). Then the hypothesis in Proposition 8.2 is valid for all $F \subsetneq \Delta$, and we conclude that (7.2) holds for such F. Hence the statements in Lemma 7.3 are valid for all $F \subsetneq \Delta$. In order to complete the proof we must establish (7.2) for $F = \Delta$.

Let Υ denote the set of continuous homomorphisms $\chi: G \to \mathbb{R}_+$ for which $\chi(h) = 1$ for all $h \in H$, and let

$$^\circ G = \bigcap_{\chi \in \Upsilon} \chi^{-1}(1),$$

then $H \subset {}^\circ G$, and the pair $({}^\circ G, H)$ satisfies the same general assumptions as we have required for (G, H). Moreover, we have

$$G/H \simeq {}^\circ G/H \times A_{\Delta\mathrm{q}}.$$

Let $x, y \in {}^\circ G/H$, $a, b \in A_{\Delta\mathrm{q}}$, then it follows easily from the definitions that

$$\mathrm{K}_\Delta^t(\nu: xa: yb) = (ab^{-1})^\nu\, {}^\circ\mathrm{K}_\Delta^t(x: y) \tag{9.1}$$

for $\nu \in \mathfrak{a}_{\Delta\mathrm{q}\mathbb{C}}^*$, where ${}^\circ\mathrm{K}_\Delta^t$ is the kernel defined as K_Δ^t, but on ${}^\circ G/H$. It follows that in order to establish (7.2) it suffices to consider ${}^\circ\mathrm{K}_\Delta^t$ on ${}^\circ G/H$; in other words, we may assume that $\mathfrak{a}_{\Delta\mathrm{q}} = \{0\}$. Moreover, we may assume that X is not compact, since otherwise the symmetry of $K_\Delta(x, y)$ follows easily from Remark 5.2.

Let $f \in C_c^\infty(X_+: \tau)$ be fixed and consider the function $g := f - \mathcal{T}\mathcal{F}f$ on X_+. We shall first prove that $g = 0$ on X_+, as would follow from Theorem 7.1 if we could use it at this stage. Afterwards we derive Theorem 7.2. Notice that g vanishes outside a bounded subset of X_+, since f has compact support and Lemma 4.6 applies to $\mathcal{T}\mathcal{F}f$.

In Lemma 6.3 take $F = \Delta$ and let D belong to the corresponding ideal I (the parameter ν is not present because of our assumption that $\mathfrak{a}_{\Delta\mathrm{q}} = \{0\}$). Then $D\mathrm{T}_\Delta^t f = 0$. Hence

$$Dg = D(f - \mathcal{T}\mathcal{F}f) = D(f - \sum_{F \subsetneq \Delta} T_F f),$$

and it follows from Lemma 7.3 (i) that Dg extends to a smooth function on X. Moreover, it has compact support because as mentioned g has bounded support on X_+. Thus $Dg \in C_c^\infty(X: \tau)$. Let $D_0 \in \mathbb{D}(X)$ be as in Lemma 4.8. Then $D_0 g = D_0(f - \mathcal{T}\mathcal{F}f) = 0$ on X_+, and hence, since $\mathbb{D}(X)$ is commutative, $D_0 D g = 0$. As D_0 is injective we conclude that $Dg = 0$. Thus g is annihilated by I, and we conclude from Corollary 6.4 that g is real analytic on X_+. However, we saw that g has bounded support, hence $g = 0$ on X_+ as claimed.

From the above it follows that the identity $f = \sum_F \mathrm{T}_F^t f$ holds on X_+ for all $f \in C_c^\infty(X_+ \colon \tau)$. Isolating $\mathrm{T}_\Delta^t f$ and applying Lemma 7.3 (ii) for all $F \subsetneq \Delta$ we obtain that the identity

$$\langle \mathrm{T}_\Delta^t f | g \rangle = \langle f | \mathrm{T}_\Delta^t g \rangle \tag{9.2}$$

holds for all $f, g \in C_c^\infty(X_+ \colon \tau)$. Hence we conclude from Lemma 9.1 below that (7.2) holds for $F = \Delta$. This completes the proof of Theorem 7.2. □

Lemma 9.1. *Let t be a W-invariant even residue weight. Assume that $\mathfrak{a}_{\Delta \mathrm{q}} = \{0\}$ and that* (9.2) *holds for all $f, g \in C_c^\infty(X_+ \colon \tau)$. Then*

$$\mathrm{K}_\Delta^t(x \colon y)^* = \mathrm{K}_\Delta^t(y \colon x) \tag{9.3}$$

for all $x, y \in X_+$.

Proof. Note that $\Delta' = \Delta$. We conclude from (9.2), Lemma 6.8 and [9, Lemma 11.3] that $\mathrm{T}_\Delta^t g(z) = \mathrm{S}_\Delta^t g(z)$ for all $z \in X_+$, $g \in C_c^\infty(X_+ \colon \tau)$, that is

$$\int_{X_+} \mathrm{K}_\Delta^t(z \colon y) g(y)\, dy = \int_{X_+} \mathrm{K}_\Delta^t(x \colon z)^* g(x)\, dx.$$

Now (9.3) follows by means of [9, Lemma 11.3]. □

10. A product formula for the residue kernels

Fix a subset $F \subset \Delta$ and an even and W-invariant residue weight $t \in \mathrm{WT}(\Sigma)$. Furthermore, fix an element $\nu \in \mathfrak{a}_{F\mathrm{q}\mathbb{C}}^*$ for which the kernel $K(\nu \colon x \colon y) := \mathrm{K}_F^t(\nu \colon x \colon y) \in \mathrm{End}(V_\tau)$ is regular. This kernel is real analytic as function of (x, y) in $X_+ \times X$. However, as we have seen in Theorem 7.2 that

$$K(\nu \colon x \colon y) = K(-\bar{\nu} \colon y \colon x)^* \tag{10.1}$$

for all $x, y \in X_+$, it follows that $(x, y) \mapsto K(\nu \colon x \colon y)$ extends real analytically to $(X \times X_+) \cup (X_+ \times X)$. Let

$$\mathcal{C}_\nu = \mathrm{Span}\{K(\nu \colon \cdot \colon y)v \mid y \in X_+, v \in V_\tau\} \subset C^\infty(X \colon \tau). \tag{10.2}$$

Lemma 10.1. *The space $\mathcal{C}_\nu$ is finite dimensional and consists of real analytic $\mathbb{D}(G/H)$-finite functions.*

Proof. It was seen below (10.1) that $x \mapsto K(\nu \colon x \colon y)$ is real analytic on X for $y \in X_+$. The functions in $\mathcal{C}_\nu$ are annihilated by a cofinite ideal in $\mathbb{D}(X)$ by Lemma 6.3; from this the finite dimensionality follows as in [3, Lemma 3.9]. □

Lemma 10.2. *The function* $(x, y) \mapsto K(\nu : x : y) = \mathrm{K}_F^t(\nu : x : y) \in \mathrm{End}(V_\tau)$ *extends to a real analytic function on* $X \times X$. *It satisfies* (10.1) *for all* $x, y \in X$.

Proof. For $x \in X_+$, $v \in V_\tau$ we define the linear functional $\xi_{x,v} \in \mathcal{C}_\nu^*$ by $\xi_{x,v}(f) = \langle f(x) | v \rangle$. If an element of $\mathcal{C}_\nu$ is annihilated by all $\xi_{x,v}$, then this element is zero. It follows (by the finite dimensionality of $\mathcal{C}_\nu$) that the $\xi_{x,v}$ span $\mathcal{C}_\nu^*$. Let $n = \dim \mathcal{C}_\nu$. Then there exists a collection $(x_1, v_1), \dots, (x_n, v_n) \in X_+ \times V_\tau$, such that the ξ_{x_j, v_j} form a basis for $\mathcal{C}_\nu^*$. Let $f_1, \dots, f_n$ be the dual basis for $\mathcal{C}_\nu$. Then

$$f = \sum_{j=1}^n \langle f(x_j) | v_j \rangle f_j$$

for all $f \in \mathcal{C}_\nu$. In particular,

$$K(\nu : x : y)v = \sum_{j=1}^n \langle K(\nu : x_j : y)v | v_j \rangle f_j(x) \tag{10.3}$$

for $x \in X$, $y \in X_+$ and $v \in V_\tau$. The right hand side of (10.3) is real analytic on $X \times X$ since $y \mapsto K(\nu : x_j : y)$ and f_j are both real analytic on X.

The identity (10.1) is valid on $X \times X$ by continuity. $\square$

Define $\mathbf{e}(x) = \mathbf{e}_\nu(x) \in \mathrm{Hom}(\mathcal{C}_\nu, V_\tau)$, for $x \in X$, by

$$\mathbf{e}(x)u = u(x); \tag{10.4}$$

then $\mathbf{e}$ is spherical for the $\tau \otimes 1$-action of K on $\mathrm{Hom}(\mathcal{C}_\nu, V_\tau) = V_\tau \otimes \mathcal{C}_\nu^*$, and it is a real analytic function of x.

Lemma 10.3. *Assume in addition that* $\nu \in i\mathfrak{a}_{F\mathrm{q}}^*$, *and let a Hilbert space structure* $\langle \cdot | \cdot \rangle_{\mathcal{C}_\nu}$ *on the finite dimensional space* $\mathcal{C}_\nu$ *be given. Then there exists a unique endomorphism* α *of* $\mathcal{C}_\nu$ *such that*

$$K(\nu : x : y) = \mathbf{e}(x) \circ \alpha \circ \mathbf{e}(y)^* \tag{10.5}$$

for all $x, y \in X_+$. *Moreover,* α *is selfadjoint and bijective.*

Proof. Let $(x_1, v_1), \dots, (x_n, v_n) \in X_+ \times V_\tau$ and $f_1, \dots, f_n \in \mathcal{C}_\nu$ be as in the proof of Lemma 10.2. Define

$$\alpha f(x) = \sum_{j=1}^n \langle f | K(\nu : \cdot : x_j) v_j \rangle_{\mathcal{C}_\nu} \, f_j(x)$$

for $f \in \mathcal{C}_\nu$, $x \in X_+$, then $\alpha f \in \mathcal{C}_\nu$ and $\alpha \in \mathrm{End}(\mathcal{C}_\nu)$. Moreover for $x, y \in X_+$, $v \in V_\tau$

$$\begin{aligned}
\mathbf{e}(x)\alpha\mathbf{e}(y)^* v = (\alpha \mathbf{e}(y)^* v)(x) &= \sum_{j=1}^n \langle \mathbf{e}(y)^* v | K(\nu : \cdot : x_j) v_j \rangle_{\mathcal{C}_\nu} f_j(x) \\
&= \sum_{j=1}^n \langle v | \mathbf{e}(y) K(\nu : \cdot : x_j) v_j \rangle f_j(x) = \sum_{j=1}^n \langle v | K(\nu : y : x_j) v_j \rangle f_j(x) \\
&= \sum_{j=1}^n \langle K(-\bar{\nu} : x_j : y) v | v_j \rangle f_j(x);
\end{aligned}$$

in the last equality we have used (10.1). Since $-\bar{\nu} = \nu$, it follows from (10.3) that the latter expression equals $K(\nu\colon x\colon y)v$. This shows (10.5), that is, the existence of α has been established.

Assume that $\mathbf{e}(x) \circ \beta \circ \mathbf{e}(y)^* = 0$ for all $x, y \in X_+$, for some operator $\beta \in \mathrm{End}(\mathcal{C}_\nu)$. By (10.4) this means that $(\beta \circ \mathbf{e}(y)^*)(x) = 0$ for all x, y, hence $\beta \circ \mathbf{e}(y)^* = 0$. Taking adjoints we conclude that $\mathbf{e}(y) \circ \beta^* = 0$ for all y, hence $\beta^* = 0$ by (10.4). Thus $\beta = 0$. The uniqueness of α follows.

That α is selfadjoint is an immediate consequence of (10.1) and (10.5), by means of the uniqueness just established. We have $K(\nu\colon \cdot : y)v = \alpha \mathbf{e}(y)^* v$, so if α was not surjective a contradiction with the definition of $\mathcal{C}_\nu$ would arise. Hence the bijectivity of α. $\square$

Remark 10.4. Let $F = \emptyset$ in Lemma 10.3. If $\langle \nu, \beta \rangle \neq 0$ for all $\beta \in \Sigma$ then it follows easily from (5.13) and [9, Lemma 16.14] that $\psi \mapsto E^\circ(\psi\colon \nu)$ is a linear bijection of ${}^\circ\mathcal{C}$ onto $\mathcal{C}_\nu$. Moreover, if $\mathcal{C}_\nu$ is given the Hilbert structure so that this is a unitary isomorphism, then (5.13) shows that α is the identity operator.

Remark 10.5. Let $F = \Delta$ and assume that $\mathfrak{a}_{\Delta \mathrm{q}} = \{0\}$. In this case we denote the space $\mathcal{C}_\nu$ defined in (10.2) by $\mathcal{C}_\Delta$. It will be shown in [13] that $\mathcal{C}_\Delta$ is contained in $L^2(X\colon \tau)$ (as the discrete series). It will then be natural to use for $\langle \cdot | \cdot \rangle_{\mathcal{C}_\nu}$ in Lemma 10.3 the inherited Hilbert structure. Then $\mathbf{e}$ is square integrable on X, and it follows from (10.5) that $\mathrm{K}^t_\Delta(x\colon y) = \mathbf{e}(x) \circ \alpha \circ \mathbf{e}(y)^*$ is the kernel of an integral operator on $L^2(X\colon \tau)$. It is easily seen from the definition (10.4) of $\mathbf{e}$ that this integral operator is the orthogonal projection onto $\mathcal{C}_\Delta$ followed by α. However, it will also be shown in [13] that T_Δ is the restriction to $C_c^\infty(X\colon \tau)$ of the orthogonal projection of $L^2(X\colon \tau)$ onto $\mathcal{C}_\Delta$; by (5.10) this orthogonal projection is the integral operator with kernel $|W|\,\mathrm{K}^t_\Delta(x\colon y)$. We conclude that with the present choice of Hilbert structure on $\mathcal{C}_\Delta$ then α is $|W|^{-1}$ times the identity operator.

For $F \neq \Delta$ the product formula for $K(\nu\colon x\colon y) = \mathrm{K}^t_F(\nu\colon x\colon y)$ obtained in Lemma 10.3 has the drawback that its dependence on ν is obscure. Moreover, it is only valid under the assumption that $\nu \in i\mathfrak{a}^*_{F\mathrm{q}}$. We shall now give a different construction of a product formula which does not have these disadvantages.

Fix $F \subset \Delta$ and $v \in {}^F\mathcal{W}$ (see Section 8), and let $K(m\colon m') = \mathrm{K}^{*t}_F(X_{F,v}\colon m\colon m')$, $m, m' \in X_{F,v}$, be the analog for $X_{F,v}$ of the kernel K^t_Δ on X. Using the symmetry of this kernel we have (cf. (10.1), (6.3))

$$K(m\colon m') = \sum_{\lambda \in \Lambda(X_{F,v}, F)} \mathrm{Res}^{*P,*t}_\lambda [E^\circ(X_{F,v}\colon - \cdot : m) \circ E^*_+(X_{F,v}\colon - \cdot : m')] \tag{10.6}$$

for $m \in X_{F,v}$, $m' \in X_{F,v,+}$. Let the space $\mathcal{C}_{F,v} \subset C^\infty(X_{F,v}\colon \tau)$ be defined as (10.2), but for $K(m\colon m')$, that is,

$$\mathcal{C}_{F,v} = \mathrm{Span}\{K(\cdot : m')v_0 \mid m' \in X_{F,v,+}, v_0 \in V_\tau\}; \tag{10.7}$$

it is thus the analog for $X_{F,v}$ of the space $\mathcal{C}_\Delta$ discussed in Remark 10.5. Let $\psi \in \mathcal{C}_{F,v}$; then

$$\psi(m) = \sum_{j=1}^{k} K(m\colon m_j)v_j \tag{10.8}$$

for some pairs $(m_j, v_j) \in X_{F,v,+} \times V_\tau$. Equivalently

$$\psi(m) = \sum_{\lambda \in \Lambda(X_{F,v},F)} \operatorname{Res}_\lambda^{*P,*t}[E^\circ(X_{F,v}\colon -\,\cdot : m)\Phi(\,\cdot\,)] \tag{10.9}$$

where

$$\Phi(\lambda) = \sum_{j=1}^k E_+^*(X_{F,v}\colon -\lambda\colon m_j)v_j \in {}^\circ\mathcal{C}_{F,v}, \qquad (\lambda \in \mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}). \tag{10.10}$$

Let ν be a generic element in $\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$ and consider the V_τ-valued function on X given by

$$x \mapsto \sum_{\lambda \in \Lambda(X_{F,v},F)} \operatorname{Res}_\lambda^{*P,*t}[E^\circ(\nu - \,\cdot : x) \circ \mathrm{i}_{F,v}\, \Phi(\,\cdot\,)]; \tag{10.11}$$

it clearly belongs to $C^\infty(X\colon \tau)$ and depends meromorphically on $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$ (cf. Lemma B.5).

Lemma 10.6. *The expression in* (10.11) *is independent of the choice of the pairs* (m_j, v_j) *that represent* ψ *in* (10.8). *It depends linearly on* $\psi \in \mathcal{C}_{F,v}$. *Moreover, it remains unchanged if we replace the set of summation* $\Lambda(X_{F,v}, F)$ *by any finite subset of* $\mathfrak{a}^{*\perp}_{F\mathrm{q}}$ *containing* $\Lambda(X_{F,v}, F)$.

Proof. For the first statement it suffices to prove that (10.11) represents the trivial function if $\psi = 0$. The latter assumption amounts to

$$\sum_{\lambda \in \Lambda} \operatorname{Res}_\lambda^{*P,*t}[E^\circ(X_{F,v}\colon -\,\cdot : m)\Phi(\,\cdot\,)] = 0 \tag{10.12}$$

for all $m \in X_{F,v}$, where $\Lambda = \Lambda(X_{F,v}, F)$.

We shall now apply the induction of relations of Appendix B. We define a Laurent functional $\mathcal{L}_1 \in \mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F)^*_{\mathrm{laur}}$ by

$$\mathcal{L}_1(\varphi) = \sum_{\lambda \in \Lambda} \operatorname{Res}_\lambda^{*P,*t} \varphi(-\,\cdot\,), \qquad (\varphi \in \mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F)).$$

Applying Theorem B.6 with $\mathcal{L}_2 = 0$ and $\phi_1(\nu+\lambda) = \Phi(-\lambda)$ for generic $\lambda \in \mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}$, $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$, we conclude that

$$\sum_{\lambda \in \Lambda} \operatorname{Res}_\lambda^{*P,*t}[E^\circ(\nu - \,\cdot : x) \circ \mathrm{i}_{F,v}\, \Phi(\,\cdot\,)] = 0 \tag{10.13}$$

for all $x \in X_+$. By continuity, (10.13) holds for all $x \in X$. Thus indeed (10.11) represents the trivial function if $\psi = 0$.

Let $\psi = \alpha'\psi' + \alpha''\psi''$, where $\alpha', \alpha'' \in \mathbb{C}$, $\psi', \psi'' \in \mathcal{C}_{F,v}$, and let ψ', ψ'' be represented as in (10.8) with pairs $(m_j', v_j')_{j=1,\dots,k'}$ and $(m_j'', v_j'')_{j=1,\dots,k''}$, respectively. Then ψ is represented by (10.8) with $k = k' + k''$, $(m_j, v_j) = (m_j', \alpha' v_j')$ for $j = 1, \dots, k'$, and $(m_{k'+j}, v_{k'+j}) = (m_j'', \alpha'' v_j'')$ for $j = 1, \dots, k''$. The corresponding functions Φ, Φ' and Φ'' in (10.10) are then related by $\Phi = \alpha'\Phi' + \alpha''\Phi''$, and the similar relation then holds for the functions in (10.11). This proves the linear dependence of (10.11) on ψ.

To establish the final claim we must prove that (10.13) holds whenever $\Lambda \subset \mathfrak{a}^{*\perp}_{F\mathrm{q}}$ is finite and disjoint from $\Lambda(X_{F,v}, F)$. This follows by the same argument as above; indeed (10.12) holds for such sets Λ since all its terms vanish by the definition of $\Lambda(X_{F,v}, F)$ (see (8.7) and the proof of Lemma 7.5). $\square$

Definition 10.7. We denote by $E^\circ_{F,v}(\nu\colon x) \in \mathrm{Hom}(\mathcal{C}_F, V_\tau)$ the operator that takes $\psi \in \mathcal{C}_{F,v}$ to the element of V_τ given by (10.11). The functions $E^\circ_{F,v}(\psi\colon \nu) := E^\circ_{F,v}(\nu)\psi \in C^\infty(X\colon\tau)$, for $\psi \in \mathcal{C}_{F,v}$ and generic $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$, are called *generalized Eisenstein integrals.* Furthermore, we define the finite dimensional vector space $\mathcal{C}_F$ as the formal direct sum

$$\mathcal{C}_F = \oplus_{v\in {}^F\mathcal{W}}\, \mathcal{C}_{F,v},$$

and we define $E^\circ_F(\nu\colon x) \in \mathrm{Hom}(\mathcal{C}_F, V_\tau)$ by

$$E^\circ_F(\nu\colon x)\psi = \sum_{v\in {}^F\mathcal{W}} E^\circ_{F,v}(\nu\colon x)\psi_v$$

for $\psi = \sum_{v\in {}^F\mathcal{W}} \psi_v \in \mathcal{C}_F$. The functions $E^\circ_F(\psi\colon\nu) := E^\circ_F(\nu)\psi \in C^\infty(X\colon\tau)$, $\psi \in \mathcal{C}_F$, are also called *generalized Eisenstein integrals.*

The generalized Eisenstein integral $E^\circ_F(\psi\colon\nu\colon x)$ depends meromorphically on $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$. Notice that for $F = \emptyset$ we obtain, by application of Remark 5.2 to the symmetric space $M/M\cap vHv^{-1}$, that $\mathcal{C}_{\emptyset,v} = C^\infty(M/M\cap vHv^{-1}\colon\tau_M)$. Hence $\mathcal{C}_\emptyset = {}^\circ\mathcal{C}$ (cf. (8.1)). Moreover, in this case the generalized Eisenstein integral $E^\circ_\emptyset(\psi\colon\nu)$ coincides with the normalized Eisenstein integral $E^\circ(\psi\colon\nu)$.

Arguing as in the proof of Lemma 6.3 we see that $E^\circ_F(\nu\colon x)$ is annihilated by an ideal of finite codimension in $\mathbb{D}(G/H)$ (the product over $v \in {}^F\mathcal{W}$, $\lambda \in \Lambda(X_{F,v}, F)$ and $i = 1, \dots, m$ of the ideals $(I_{i,\nu-\lambda})^k \subset \mathbb{D}(G/H)$ for k sufficiently large).

Both factors $E^\circ(\,\cdot : x)$ and $E^*_+(X_{F,v}\colon \cdot : m_j)$ in (10.11) allow suitable estimates. It follows that the generalized Eisenstein integral $E^\circ_F(\psi\colon\nu\colon x)$ allows an estimate of the following form. Let $\Sigma_r(F)$ denote the set of non-zero restrictions to $\mathfrak{a}_{F\mathrm{q}}$ of roots from Σ, and define the set $\Pi_{\Sigma_r(F),\mathbb{R}}(\mathfrak{a}_{F\mathrm{q}})$ of polynomials on $\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$ similarly as the set $\Pi_{S,\mathbb{R}}$ was defined in Section 3.

Lemma 10.8. *Let $\omega \subset \mathfrak{a}^*_{F\mathrm{q}}$ be compact. There exists a polynomial $p \in \Pi_{\Sigma_r(F),\mathbb{R}}(\mathfrak{a}_{F\mathrm{q}})$, for every $u \in U(\mathfrak{g})$ a number $N \in \mathbb{N}$, and for every $x \in X$, $\psi \in \mathcal{C}_F$, a constant C such that*

$$\|p(\nu)E^\circ_F(\psi\colon\nu\colon u; x)\| \le C(1 + |\nu|)^N$$

for all $\nu \in \omega + i\mathfrak{a}^*_{F\mathrm{q}}$. The constant C can be chosen locally uniformly in x.

Proof. This follows from the estimates in Lemmas 4.1, 4.3 and [11, Lemma 1.11]. □

We fix a Hilbert space structure on $\mathcal{C}_{F,v}$ for each $v \in {}^F\mathcal{W}$, and equip $\mathcal{C}_F$ with the direct sum Hilbert space structure. Let

$$E^*_F(\nu\colon x) = E^\circ_F(-\bar\nu\colon x)^* \in \mathrm{Hom}(V_\tau, \mathcal{C}_F).$$

For each $v \in {}^F\mathcal{W}$, let $\alpha_{F,v} \in \mathrm{End}(\mathcal{C}_{F,v})$ denote the operator given by Lemma 10.3 for the kernel $\mathrm{K}^{*t}_F(X_{F,v}\colon m\colon m')$ for $X_{F,v}$ and the given Hilbert structure on $\mathcal{C}_{F,v}$, and let $\alpha_F \in \mathrm{End}(\mathcal{C}_F)$ be given by $[\alpha_F\psi]_v = \alpha_{F,v}\psi_v$ for $v \in {}^F\mathcal{W}$.

Proposition 10.9. *Let $x \in X_+$, $y \in X$. Then*

$$\begin{aligned} &E^\circ_F(\nu\colon x) \circ \alpha_F \circ E^*_F(\nu'\colon y) \\ &\quad = \sum_{\lambda \in \Lambda(F)} \mathrm{Res}^{P,t}_{\lambda + \mathfrak{a}^*_{F\mathrm{q}}} \Big(\sum_{s \in W^F} E_{+,s}(\nu + \,\cdot\,\colon x) \circ E^*(\nu' + \,\cdot\,\colon y) \Big)(\lambda), \end{aligned} \tag{10.14}$$

as an identity between meromorphic functions in $\nu, \nu' \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$. In particular,

$$E^\circ_F(\nu\colon x) \circ \alpha_F \circ E^*_F(\nu\colon y) = \mathrm{K}^t_F(\nu\colon x\colon y). \tag{10.15}$$

We remark that by application of Remark 10.5 to each of the symmetric spaces $X_{F,v}$, it follows (from results to be seen in [13]) that $\mathcal{C}_F$ can be equipped with a natural Hilbert space structure with respect to which α_F is a constant times the identity operator.

Proof. Let $v \in {}^F\mathcal{W}$. For $m \in X_{F,v}$ we denote by $\mathbf{e}(X_{F,v}\colon m)$ the linear map $\mathcal{C}_{F,v} \to V_\tau$ given by evaluation at m; this is the analog of (10.4) for $X_{F,v}$. By the definition of $\alpha_{F,v}$ we have

$$\mathrm{K}^{*t}_F(X_{F,v}\colon m\colon m') = \mathbf{e}(X_{F,v}\colon m) \circ \alpha_{F,v} \circ \mathbf{e}(X_{F,v}\colon m')^*,$$

for $m \in X_{F,v}$, $m' \in X_{F,v,+}$. Thus, for $v_0 \in V_\tau$,

$$\mathrm{K}^{*t}_F(X_{F,v}\colon m\colon m')v_0 = [\alpha_{F,v} \circ \mathbf{e}(X_{F,v}\colon m')^* v_0](m). \tag{10.16}$$

Let $\psi(m) = [\alpha_{F,v} \circ \mathbf{e}(X_{F,v}\colon m')^* v_0](m)$, then (10.16) is an expression for ψ of the form (10.8). The function Φ in (10.10) is then given by $\Phi(\lambda) = E^*_+(X_{F,v}\colon -\lambda\colon m')v_0$. By the definition of $E^\circ_{F,v}(\nu'\colon y)$ (cf. (10.11)) we then obtain

$$\begin{aligned} &E^\circ_{F,v}(\nu'\colon y) \circ \alpha_{F,v} \circ \mathbf{e}(X_{F,v}\colon m')^* \\ &\quad = \sum_{\lambda \in \Lambda(X_{F,v},F)} \mathrm{Res}^{*P,*t}_\lambda [E^\circ(\nu' - \,\cdot\,\colon y) \circ \mathrm{i}_{F,v} \circ E^*_+(X_{F,v}\colon - \,\cdot\,\colon m')] \end{aligned} \tag{10.17}$$

for $y \in X$, $\nu' \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$. Recall from Lemma 10.6 that the above expression remains unchanged if we replace in it $\Lambda(X_{F,v}, F)$ by a larger finite set Λ; as in the proof of Proposition 8.2 we take as Λ the set given by (8.8).

For the moment, we fix a generic element $\nu' \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$; we insert $-\bar{\nu}'$ for ν' in (10.17). Taking adjoints as in the proof of Lemma 6.5 and applying the resulting operator to an arbitrary vector $a \in V_\tau$ we obtain (recall from Lemma 10.3 that $\alpha_{F,v}$ is selfadjoint)

$$\begin{aligned} &\mathbf{e}(X_{F,v}\colon m') \circ \alpha_{F,v} \circ E^*_{F,v}(\nu'\colon y)a \\ &\quad = \sum_{\lambda \in \Lambda} \mathrm{Res}^{*P,*t}_\lambda [E_+(X_{F,v}\colon \,\cdot\,\colon m') \circ \mathrm{pr}_{F,v} \circ E^*(\nu' + \,\cdot\,\colon y)a]. \end{aligned} \tag{10.18}$$

On the left-hand side of this equation we have the element $\alpha_{F,v} \circ E^*_{F,v}(\nu'\colon y)a$ from $\mathcal{C}_{F,v}$ evaluated at $m' \in X_{F,v,+}$. By the definition of the space $\mathcal{C}_{F,v}$ (cf. (10.9)) this evaluated element has the form

$$\begin{aligned} &[\alpha_{F,v} \circ E^*_{F,v}(\nu'\colon y)a](m') \\ &\quad = \sum_{\lambda \in \Lambda(X_{F,v},F)} \mathrm{Res}^{*P,*t}_\lambda [E^\circ(X_{F,v}\colon - \,\cdot\,\colon m') \circ \sum_{j=1}^k E^*_+(X_{F,v}\colon - \,\cdot\,\colon m_j)v_j] \end{aligned} \tag{10.19}$$

for some $m_1, \dots, m_k \in X_{F,v,+}$ and $v_1, \dots, v_k \in V_\tau$ (depending on v, ν', y, a). In particular we have an identity between the right-hand sides of (10.18) and (10.19), for all $m' \in X_{F,v,+}$. To this identity we shall now apply the induction of relations of Appendix B. Let the Laurent functionals $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F)^*_{\mathrm{laur}}$ be defined by

$$\mathcal{L}_1\varphi = \sum_{\lambda \in \Lambda(X_{F,v},F)} \mathrm{Res}^{*P,*t}_\lambda(\varphi(-\,\cdot\,)), \qquad \mathcal{L}_2\varphi = \sum_{\lambda\in\Lambda} \mathrm{Res}^{*P,*t}_\lambda(\varphi)$$

for $\varphi \in \mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F)$. Moreover, let $\phi_1, \phi_2 \in \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \Sigma_F) \otimes {}^\circ\mathcal{C}_{F,v}$ be the meromorphic functions defined by

$$\phi_1(\nu+\lambda) = \sum_{j=1}^k E^*_+(X_{F,v} \colon \lambda \colon m_j)v_j, \qquad \phi_2(\nu+\lambda) = \mathrm{pr}_{F,v} \circ E^*(\nu'+\lambda \colon y)a$$

for generic $\lambda \in \mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}$ and $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$.

Applying Theorem B.6 to the identity between the right-hand sides of (10.18) and (10.19), with $\mathcal{L}_1, \mathcal{L}_2, \phi_1, \phi_2$ as above, we conclude that

$$\begin{aligned}
&\sum_{\lambda\in\Lambda} \mathrm{Res}^{*P,*t}_\lambda\Big[\sum_{s\in W^F} E_{+,s}(\nu + \,\cdot\, \colon x) \circ \mathrm{i}_{F,v} \circ \mathrm{pr}_{F,v} \circ E^*(\nu' + \,\cdot\, \colon y)a\Big] \\
&= \sum_{\lambda\in\Lambda(X_{F,v},F)} \mathrm{Res}^{*P,*t}_\lambda\Big[E^\circ(\nu - \,\cdot\, \colon x) \circ \mathrm{i}_{F,v} \circ \sum_{j=1}^k E^*_+(X_{F,v} \colon -\,\cdot\, \colon m_j)v_j\Big]
\end{aligned}$$

for all $x \in X_+$ and generic $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$. On the other hand, by the definition (10.11) of $E^\circ_{F,v}(\nu \colon x)$, it follows from (10.19) that

$$\begin{aligned}
&E^\circ_{F,v}(\nu \colon x)[\alpha_{F,v} \circ E^*_{F,v}(\nu' \colon y)a] \\
&= \sum_{\lambda\in\Lambda(X_{F,v},F)} \mathrm{Res}^{*P,*t}_\lambda\Big[E^\circ(\nu - \,\cdot\, \colon x) \circ \mathrm{i}_{F,v} \circ \sum_{j=1}^k E^*_+(X_{F,v} \colon -\,\cdot\, \colon m_j)v_j\Big],
\end{aligned}$$

and so we conclude that

$$\begin{aligned}
&\sum_{\lambda\in\Lambda} \mathrm{Res}^{*P,*t}_\lambda\Big[\sum_{s\in W^F} E_{+,s}(\nu + \,\cdot\, \colon x) \circ \mathrm{i}_{F,v} \circ \mathrm{pr}_{F,v} \circ E^*(\nu' + \,\cdot\, \colon y)\Big] \\
&\qquad = E^\circ_{F,v}(\nu \colon x) \circ \alpha_{F,v} \circ E^*_{F,v}(\nu' \colon y).
\end{aligned}$$

In this latter expression we may replace the residue operators $\mathrm{Res}^{*P,*t}_\lambda$ by $\mathrm{Res}^{P,t}_{\lambda+\mathfrak{a}^*_{F\mathrm{q}}}$ (cf. (8.5)), and we may shrink the set of summation to $\Lambda(F)$, since the extra terms vanish, by the definition (5.5) of the latter set (see the proof of Lemma 7.5). Summing over $v \in {}^F\mathcal{W}$ (cf. (8.4)), we finally obtain (10.14). The expression (10.15) is obtained by taking $\nu = \nu'$. $\square$

We can now sharpen the estimate for K^t_F in Lemma 5.1 so that it is valid on $X \times X$.

Corollary 10.10. *Let $\omega \subset \mathfrak{a}_{F\mathrm{q}}^*$ be compact. There exists a polynomial $q \in \Pi_{\Sigma_r(F),\mathbb{R}}(\mathfrak{a}_{F\mathrm{q}})$ on $\mathfrak{a}_{F\mathrm{q}\mathbb{C}}^*$, for every $u, u' \in U(\mathfrak{g})$ a number $N \in \mathbb{N}$, and for every $x, y \in X$ a constant $C > 0$ such that*

$$\|q(\nu)\mathrm{K}_F^t(\nu\colon u; x\colon u'; y)\| \leq C(1 + |\nu|)^N$$

for all $\nu \in \omega + i\mathfrak{a}_{F\mathrm{q}}^*$. The constant C can be chosen locally uniformly in x and y.

Proof. Immediate from (10.15) and Lemma 10.8. □

Corollary 10.11. *Let $t \in \mathrm{WT}(\Sigma)$ be even and W-invariant, and let $F \subset \Delta$. Then $\mathrm{T}_F^t f$ extends to a smooth function on X for every $f \in C_c^\infty(X\colon\tau)$. Moreover, $f \mapsto \mathrm{T}_F^t f$ is a continuous operator from $C_c^\infty(X\colon\tau)$ to $C^\infty(X\colon\tau)$. Finally, if F' is defined as in Lemma 7.3 (ii), then $\langle \mathrm{T}_F^t f | g\rangle = \langle f | \mathrm{T}_{F'}^t g\rangle$ for all $f, g \in C_c^\infty(X\colon\tau)$.*

Proof. It follows from Corollary 10.10 in the same manner as [9, Prop. 8.3] that (5.9) holds for $x \in X$, $f \in C_c^\infty(X\colon\tau)$, with similar uniformity as stated in Lemma 5.1. Then (5.10) shows that $\mathrm{T}_F^t f$ extends. The final statement now follows from Lemma 7.3 (ii) by continuity. □

11. Application: The Plancherel formula for one conjugacy class of Cartan subspaces.

Recall that the reductive symmetric space is said to have one conjugacy class of Cartan subspaces if all the Cartan subspaces of $\mathfrak{q}$ are conjugate under H.

Lemma 11.1. *If X has one conjugacy class of Cartan subspaces then so has $X_{F,v}$ for every $F \subset \Delta$ and $v \in N_K(\mathfrak{a}_\mathrm{q})$.*

Proof. We first notice that $X_{F,v}$ has only one conjugacy class of Cartan subspaces if and only if the same holds for X_F. Indeed, conjugation by v provides a bijection from the set of Cartan subspaces for X_F to the set of Cartan subspaces for $X_{F,v}$. We may therefore assume that $v = e$.

Let $\mathfrak{b} \subset \mathfrak{q}$ be a Cartan subspace with $\mathfrak{a}_\mathrm{q} = \mathfrak{b} \cap \mathfrak{p}$. Then $\mathfrak{b}$ is θ-invariant and maximally split. It is also a Cartan subspace for the pair $(\mathfrak{m}_{1F}, \mathfrak{m}_{1F} \cap \mathfrak{h})$ (where $\mathfrak{m}_{1F} = \mathfrak{m}_F + \mathfrak{a}_F$). Let $\mathfrak{b}'$ be an arbitrary Cartan subspace for this pair; it is sufficient to prove that $\mathfrak{b}'$ is conjugate to $\mathfrak{b}$ under $M_F \cap H$. Moreover, we may assume that $\mathfrak{b}'$ is θ-invariant. Since $\mathfrak{b}'$ has the same dimension as $\mathfrak{b}$ and is contained in $\mathfrak{q}$, it is a Cartan subspace for $(\mathfrak{g}, \mathfrak{h})$, and therefore it is conjugate to $\mathfrak{b}$ under H. It follows that $\mathfrak{b}'$ is a maximally split Cartan subspace for $(\mathfrak{g}, \mathfrak{h})$, by conjugacy. Thus, $\mathfrak{b}$ and $\mathfrak{b}'$ are also maximally split Cartan subspaces for $(\mathfrak{m}_{1F}, \mathfrak{m}_{1F} \cap \mathfrak{h})$; from this it follows that they are conjugate under $M_F \cap H$. □

In what follows we assume that G is linear, in order to be able to apply [25, Thm.] (see, however, [25, p. 388, (i)]).

Lemma 11.2. *If X is not compact and has one conjugacy class of Cartan subspaces then the discrete series for X is empty.*

Proof. If the discrete series is not empty there is a compact Cartan subspace according to [25, Thm.]. If all Cartan subspaces are conjugate this would then imply that all Cartan subspaces are compact, which is only possible if X is compact. □

Theorem 11.3. *If X has one conjugacy class of Cartan subspaces then $\mathrm{K}_F^t = 0$ for all $F \neq \emptyset$. Moreover, in that case*

$$\mathcal{J}\mathcal{F} = I. \tag{11.1}$$

Proof. The proof of the first statement is by induction on the rank of Σ. The second statement, the identity (11.1), is an immediate consequence, in view of Theorem 7.1. Assume that the first statement is true for all reductive symmetric spaces for which the corresponding root system has lower rank than Σ. Let $F \subset \Delta$, $F \neq \emptyset, \Delta$, and consider the generalized Eisenstein integral as defined in Definition 10.7. The induction hypothesis implies that the kernel $K(m\colon m')$ in (10.6) vanishes. Hence $\mathcal{C}_{F,v} = \{0\}$ for all $v \in \mathcal{W}^F$, and it follows from (10.15) that $\mathrm{K}_F^t = 0$. As an immediate consequence we have $\mathrm{T}_F^t = 0$.

It remains to prove that $\mathrm{K}_\Delta^t = 0$ if $\Delta \neq \emptyset$. By (9.1) we may assume that $A_{\Delta\mathrm{q}} = \{0\}$. By the inversion formula (Theorem 7.1) and (5.12) we have $\mathrm{T}_\Delta^t f = f - \mathcal{J}\mathcal{F}f$ for all $f \in C_c^\infty(X\colon\tau)$, hence $\mathrm{T}_\Delta^t f$ belongs to the Schwartz space of X (cf. [7, Thm. 1]). However, as $\mathrm{T}_\Delta^t f$ is annihilated by a cofinite ideal in $\mathbb{D}(G/H)$ (cf. Lemma 6.3) it then follows that $\mathrm{T}_\Delta^t f$ belongs to the discrete part of $L^2(G/H) \otimes V_\tau$ (since it generates a subrepresentation of finite length). Now $\mathrm{T}_\Delta^t = 0$ by Lemma 11.2 and it follows from (5.10) that $\mathrm{K}_\Delta^t = 0$. □

12. Application: The Fourier transform of rapidly decreasing functions

The Fourier transform $\mathcal{F}$ is injective when defined on $C_c^\infty(X\colon\tau)$ (cf. Theorem 2.1). On the other hand, its extension to the L^2-type Schwartz space $\mathcal{C}(X\colon\tau)$ (see [9, Sect. 6]) will in general not be injective because of the possible presence of non-trivial discrete or intermediate series. In this section we extend the injectivity to a certain function space $\mathcal{S}(X\colon\tau)$ that lies between $C_c^\infty(X\colon\tau)$ and $\mathcal{C}(X\colon\tau)$. We also extend our Fourier inversion formula to this space.

Let $\|\cdot\|$ be the function on X defined by $\|kaH\| = e^{|\log a|}$ for $k \in K$, $a \in A_\mathrm{q}$; then $\|x\| \geq 1$. We define $\|f\|_r = \sup_{x\in X} \|x\|^{-r}|f(x)|$ for $r \in \mathbb{R}$, $f \in C(X)$. The space

$$C_r(X) = \{f \in C(X) \mid \|f\|_r < \infty\}$$

is a Banach space, invariant under the left regular representation of G. The Fréchet space of smooth vectors for this representation is given by

$$C_r^\infty(X) = \{f \in C^\infty(X) \mid f(u;\,\cdot\,) \in C_r(X), \forall u \in U(\mathfrak{g})\}$$

with the continuous seminorms $f \mapsto \nu_{r,u}(f) := \|f(u;\,\cdot\,)\|_r$, $u \in U(\mathfrak{g})$. Clearly $C_c^\infty(X) \subset C_r^\infty(X)$ with continuous inclusion. We define

$$\mathcal{S}(X) = \cap_{r\in\mathbb{R}}\, C_r^\infty(X)$$

and provide this space with the seminorms $\nu_{u,r}$, $u \in U(\mathfrak{g})$, $r \in \mathbb{R}$. It follows easily that $\mathcal{S}(X)$ is a Fréchet space, and that the inclusion map $C_c^\infty(X) \to \mathcal{S}(X)$ is continuous. Following [29, 7.1.2] we call $\mathcal{S}(X)$ the space of *rapidly decreasing* functions on X.

Lemma 12.1. *The subspace $C_c^\infty(X)$ is dense in $\mathcal{S}(X)$.*

Proof. We shall prove the following statement, from which the density in $\mathcal{S}(X)$ immediately follows. Let $f \in C^\infty(X)$. There exists a sequence $f_n \in C_c^\infty(X)$ with the following property. Let $r \in \mathbb{R}$ and assume that $f \in C_r^\infty(X)$. Then $f_n \to f$ in $C_{r+s}^\infty(X)$ for all $s > 0$.

Let $\{\psi_t\} \subset C_c^\infty(X)$, $t > 0$, be as in [3, Lemma 2.2] for some $\epsilon > 0$. Fix $s > 0$ and $u \in U(\mathfrak{g})$. We have $\psi_t(x) = 1$ for $\|x\| \leq e^t$ and $\sup_{x\in X, t>0} |\psi_t(u;x)| < \infty$. Hence

$$\nu_{u,s}(\psi_t - 1) = \sup_{\|x\|\geq e^t} \|x\|^{-s}|(\psi_t - 1)(u;x)| \leq Ce^{-ts}.$$

We conclude that $\psi_t \to 1$ in $C_s^\infty(X)$ as $t \to \infty$. Let $f_n = \psi_n f \in C_c^\infty(X)$. The proof is completed by the observation that pointwise multiplication is continuous from $C_s^\infty(X) \times C_r^\infty(X)$ to $C_{s+r}^\infty(X)$. The latter is readily seen from the Leibniz rule. $\square$

In [21, p. 134] the term *zero Schwartz space* is used for $\mathcal{S}(X)$ because it is the intersection of the L^p-type Schwartz spaces $\mathcal{C}^p(X)$, $p > 0$. Let $C_r(X\colon\tau)$, $C_r^\infty(X\colon\tau)$ and $\mathcal{S}(X\colon\tau)$ denote corresponding spaces of τ-spherical functions, then

$$\mathcal{S}(X\colon\tau) = \cap_{r\in\mathbb{R}} C_r^\infty(X\colon\tau),$$

and we have the continuous inclusions

$$C_c^\infty(X\colon\tau) \subset \mathcal{S}(X\colon\tau) \subset \mathcal{C}(X\colon\tau).$$

Lemma 12.2. *Let $R \in \mathbb{R}$, let p be as in Proposition 3.1 and let $\omega \subset \mathfrak{a}_{\mathrm{q}}^*(P, R)$ be open and bounded. There exists $r \in \mathbb{R}$ such that the integral (2.4) that defines the Fourier transform $\mathcal{F}f(\lambda)$ converges for all $f \in C_r(X\colon\tau)$ and generic $\lambda \in \omega + i\mathfrak{a}_{\mathrm{q}}^*$. The Fourier transform is a meromorphic ${}^\circ\mathcal{C}$-valued function of λ, and there exist constants $N \in \mathbb{N}$ and $C > 0$ such that*

$$\|p(\lambda)\mathcal{F}f(\lambda)\| \leq C(1 + |\lambda|)^N \|f\|_r \tag{12.1}$$

for all $\lambda \in \omega + i\mathfrak{a}_{\mathrm{q}}^*$, $f \in C_r(X\colon\tau)$. Moreover, for each $n \in \mathbb{N}$ there exists a continuous seminorm ν on $C_r^\infty(X)$ such that

$$\|p(\lambda)\mathcal{F}f(\lambda)\| \leq (1 + |\lambda|)^{-n}\nu(f) \tag{12.2}$$

for all $\lambda \in \omega + i\mathfrak{a}_{\mathrm{q}}^*$, $f \in C_r^\infty(X\colon\tau)$.

Proof. We note that the estimate of the normalized Eisenstein integral stated in Lemma 4.3 can be sharpened as follows, by the same references as given in the proof. There exists $r_0 \in \mathbb{R}$ and for every $u \in U(\mathfrak{g})$ an integer $N \geq 0$ such that

$$\sup_{x\in X, \lambda\in\mathfrak{a}_{\mathrm{q}}^*(P,R)} (1 + |\lambda|)^{-N} \|x\|^{-r_0 - |\operatorname{Re}\lambda|} \|p(\lambda)E^*(\lambda\colon u; x)\| < \infty. \tag{12.3}$$

We take $u = 1$. Then, for all $r \in \mathbb{R}$

$$\|p(\lambda)\mathcal{F}f(\lambda)\| = \|\int_X p(\lambda)E^*(\lambda\colon x)f(x)\,dx\| \leq C(1 + |\lambda|)^N \|f\|_r \int_X \|x\|^{r_0 + |\operatorname{Re}\lambda| + r}\,dx$$

where C is the supremum in (12.3). Since $\|x\|^{-m}$ is integrable on X for m sufficiently large (cf. [9, eq. (3.1)]), we have $\int_X \|x\|^{r_0+|\operatorname{Re}\lambda|+r}\,dx < \infty$ for $-r$ sufficiently large. The statements up to and including (12.1) follow. The statement concerning (12.2) is obtained from (12.1) in the same manner as [9, Prop. 8.3]. $\square$

It follows from Lemma 12.2 that for all generic $\eta \in \mathfrak{a}_{\mathrm{q}}^*$ there exists a real number r such that the Fourier transform $\mathcal{F}f$ is defined and meromorphic in a neighborhood of $\eta + i\mathfrak{a}_{\mathrm{q}}^*$ for all $f \in C_r(X\colon\tau)$. It then follows from (12.2) and Lemma 4.2 that $\mathcal{T}_\eta \mathcal{F}f$ is well defined and belongs to $C^\infty(X_+\colon\tau)$ for all $f \in C_r^\infty(X\colon\tau)$. Moreover, the map $f \mapsto \mathcal{T}_\eta\mathcal{F}f$ is a continuous linear operator from $C_r^\infty(X\colon\tau)$ to $C^\infty(X_+\colon\tau)$ (cf. (12.2) and (4.2)).

Proposition 12.3. *Let $R < 0$ be such that $\pi(\lambda) \neq 0$ for all $\lambda \in \mathfrak{a}_{\mathrm{q}}^*(P,R)$, and let $\eta \in \mathfrak{a}_{\mathrm{q}}^*(P,R)$. There exists $r \in \mathbb{R}$ such that if $f \in C_r^\infty(X\colon\tau)$ and $\mathcal{F}f = 0$ on $\eta + i\mathfrak{a}_{\mathrm{q}}^*$, then $f = 0$.*

Proof. For $f \in C_c^\infty(X\colon\tau)$ we have $\mathcal{T}_\eta\mathcal{F}f = \mathcal{T}\mathcal{F}f$, by the definition of the pseudo wave packet $\mathcal{T}\mathcal{F}f$, and hence $\mathcal{T}_\eta\mathcal{F}f = f$ on X_+ by Theorem 4.7.

Let ω be a bounded neighborhood of η and let $r \in \mathbb{R}$ be as in Lemma 12.2. Let $r' < r$. Then for $f \in C_{r'}^\infty(X\colon\tau)$ there exists, according to the proof of Lemma 12.1, a sequence $f_n \in C_c^\infty(X\colon\tau)$ such that $f_n \to f$ in $C_r^\infty(X\colon\tau)$. Since $\mathcal{T}_\eta\mathcal{F}f_n = f_n$ on X_+ we conclude by continuity that

$$\mathcal{T}_\eta\mathcal{F}f = f \tag{12.4}$$

for $f \in C_{r'}^\infty(X\colon\tau)$. In particular, if $\mathcal{F}f = 0$ on $\eta + i\mathfrak{a}_{\mathrm{q}}^*$ then $f = 0$. $\square$

Lemma 12.4. *The integral (2.4) that defines the Fourier transform $\mathcal{F}f(\lambda)$ converges for all $f \in \mathcal{S}(X\colon\tau)$ and generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$; it is a meromorphic ${}^\circ\mathcal{C}$-valued function of λ.*

Moreover, let $R \in \mathbb{R}$ and let p be as in Proposition 3.1. Then for each compact set $\omega \subset \mathfrak{a}_{\mathrm{q}}^*(P,R)$ and each $n \in \mathbb{N}$ there exists a continuous seminorm ν on $\mathcal{S}(X\colon\tau)$ such that

$$\|p(\lambda)\mathcal{F}f(\lambda)\| \leq (1+|\lambda|)^{-n}\nu(f)$$

for all $\lambda \in \omega + i\mathfrak{a}_{\mathrm{q}}^*$, $f \in \mathcal{S}(X\colon\tau)$.

Proof. This is immediate from Lemma 12.2. $\square$

In particular, $\mathcal{F}f$ belongs to the space $\mathcal{P}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}) \otimes {}^\circ\mathcal{C}$ (see Section 5) for all $f \in \mathcal{S}(X\colon\tau)$, and if $\mathcal{P}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H})$ is topologized as in [11, Sect. 1.5], then the final estimate in Lemma 12.4 amounts to the continuity of the map $\mathcal{F}\colon \mathcal{S}(X\colon\tau) \to \mathcal{P}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}) \otimes {}^\circ\mathcal{C}$. The following theorem is an immediate consequence of Proposition 12.3.

Theorem 12.5. *The map $\mathcal{F}\colon \mathcal{S}(X\colon\tau) \to \mathcal{P}(\mathfrak{a}_{\mathrm{q}}^*, \mathcal{H}) \otimes {}^\circ\mathcal{C}$ is injective.*

We can also write down the inversion formula for the Fourier transform on $\mathcal{S}(X\colon\tau)$. The function $\mathcal{T}_\eta\mathcal{F}f \in C^\infty(X_+\colon\tau)$ is defined for all $f \in \mathcal{S}(X\colon\tau)$ and all generic $\eta \in \mathfrak{a}_{\mathrm{q}}^*$ by the remarks preceding Proposition 12.3. As usual we define the pseudo wave packet $\mathcal{T}\mathcal{F}f$ as $\mathcal{T}_\eta\mathcal{F}f$ for η sufficiently antidominant; it is independent of η by Lemma 4.2. Then (12.4) implies the following.

Theorem 12.6. *Let $f \in \mathcal{S}(X\colon\tau)$. Then $\mathcal{T}\mathcal{F}f(x) = f(x)$ for all $x \in X_+$.*

The space $\mathcal{S}(X)$ is contained in $L^2(X)$, and hence the L^2-Fourier transform $\mathfrak{F}$ defined in [9, §18] can be applied to functions in $\mathcal{S}(X)$. Recall that $\mathfrak{F}$ is defined by continuous extension of the map $f \in C_c^\infty(X) \mapsto \hat{f}(\xi,\lambda) \in L^2(K\colon\xi) \otimes V(\xi)^*$, where $\hat{f}(\xi,\lambda)$ is defined in [9, §4], for $\xi \in \hat{M}_H$ and generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. In [9, Thm. 15.5] we saw that the injectivity of the τ-spherical Fourier transform $\mathcal{F}$ on $C_c^\infty(X\colon\tau)$ for all τ implies injectivity of $\mathfrak{F}$ on $C_c^\infty(X)$. The same proof applies to $\mathcal{S}(X)$, and we conclude:

Corollary 12.7. *Let $f \in \mathcal{S}(X)$. If $\mathfrak{F}f = 0$ then $f = 0$.*

Notice that in the case of a group considered as a symmetric space the injectivity of the Fourier transform on $\mathcal{S}(X)$ (as well as on $C_c^\infty(X)$) is a consequence of Harish-Chandra's subquotient theorem together with the abstract Plancherel formula. There exists a generalized subquotient (in fact, subrepresentation) theorem for reductive symmetric spaces (see [17, Thm. 1]), but it does not allow one to conclude similarly the injectivity, because in general, for special values of λ, there are H-fixed distribution vectors in the σ-minimal principal series other than those used to define the Fourier transform.

Appendix A. On the functional equation for spherical distributions

The purpose of this appendix is to give a proof of Lemma 3.2. If it were not for the assertion that the polynomial p is real, this lemma would be an immediate consequence of [5, Thm. 9.1]. The additional assertion can be derived from [26, Thm. 11.4] if it is assumed that the identity component of G is linear (which is a general assumption in [26]). In order to cover the generality of the present paper, and for convenience, a proof based on [5] is given below. We shall follow the proof of [5, Thm. 9.1], and indicate where the arguments have to be sharpened in order to obtain the extra assertion. In particular, we use in this appendix the notation from [5].

Let $\mathbf{S} \subset \mathfrak{a}_{\mathrm{q}}^* \setminus \{0\}$ be as in [5, §7]. There it is stated that $\mathbf{S} \subset \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$, but it is obvious that one can take $\mathbf{S} \subset \mathfrak{a}_{\mathrm{q}}^*$. In [5, p. 356] the concept of S-polynomial growth of a function on $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ is defined for finite sets $S \subset \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \setminus \{0\}$. If $S \subset \mathfrak{a}_{\mathrm{q}}^*$ we define $(S,\mathbb{R})$-polynomial growth similarly, but with $\Pi_{S,\mathbb{R}}$ instead of Π_S. We shall establish Lemma 3.2 by means of the following Theorem A.1, which improves the functional equation for $j(\xi\colon\lambda)$, [5, Thm. 9.3], exactly in the way needed. Let (π, F) be a finite dimensional irreducible representation of G that is both K-spherical and H-spherical (i.e., it has both a non-trivial K-fixed vector and a non-trivial H-fixed vector). Then this representation has a lowest weight $\mu \in \mathfrak{a}_{\mathrm{q}}^*$ (with respect to P), which belongs to the set $\Lambda(\mathfrak{a}_{\mathrm{q}})$ (see [5, p. 354]).

Let $\xi \in \hat{M}_H$ $(= \hat{M}_{\mathbf{ps}}$ in the notation of [5]). In [5, p. 365] a differential operator

$$D_\mu(\xi\colon\lambda)\colon C^{-\infty}(P\colon\xi\colon\lambda+\mu) \to C^{-\infty}(P\colon\xi\colon\lambda)$$

is defined for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$, and it is asserted in [5, Lemma 9.2] that the map $\lambda \mapsto q(\lambda)D_\mu(\xi\colon\lambda)$ is polynomial on $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ for a suitable $q \in \Pi_{\mathbf{S}}(\mathfrak{a}_{\mathrm{q}})$. Going through the proof of the cited lemma one sees that q can be taken in $\Pi_{\mathbf{S},\mathbb{R}}(\mathfrak{a}_{\mathrm{q}})$, if the polynomial q in [5, Prop. 8.3] can be taken from $\Pi_{\mathbf{S},\mathbb{R}}(\mathfrak{a}_{\mathrm{q}})$. The latter polynomial is constructed by means of [5, Lemma 7.2], in which q can be taken from $\Pi_{\mathbf{S},\mathbb{R}}(\mathfrak{a}_{\mathrm{q}})$ provided η_1, η_2 belong to the real span of $\Sigma(\mathfrak{g}, \mathfrak{j})$. In the application of [5, Lemma 7.2] on [5, p. 361] we do have this property of η_1, η_2, and hence indeed we see that we can take $q \in \Pi_{\mathbf{S},\mathbb{R}}(\mathfrak{a}_{\mathrm{q}})$ in both [5, Prop. 8.3] and [5, Lemma 9.2].

Theorem A.1. *There exists a rational* $\mathrm{End}(V(\xi))$*-valued function* $\lambda \mapsto R_\mu(\xi : \lambda)$ *on* $\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ *of* $(\mathbf{S}, \mathbb{R})$*-polynomial growth such that*

$$j(P : \xi : \lambda) = D_\mu(\xi : \lambda) \circ j(P : \xi : \lambda + \mu) \circ R_\mu(\xi : \lambda).$$

Before giving the proof of Theorem A.1 we notice that based on it and the previous remark about [5, Lemma 9.2] we can repeat the proof of [5, Thm. 9.1] and obtain $(\Sigma, \mathbb{R})$-polynomial growth in the latter result. Thus Lemma 3.2 follows from Theorem A.1

Proof. The proof of [5, Thm. 9.3] is given on [5, p. 369]. For the improvement asserted in Theorem A.1 we must establish that the polynomials q in [5, Lemma 9.9] and q_1, q_2 in [5, Prop. 9.11] can be taken in $\Pi_{\mathbf{S},\mathbb{R}}(\mathfrak{a}_{\mathrm{q}})$. We have already seen that this is the case for q, and we thus turn to the proof of [5, Prop. 9.11], which is based on [5, Lemma 9.13]. The latter result can be improved as in the following Lemma A.2. The claimed improvement of [5, Prop. 9.11], that $q_1, q_2 \in \Pi_{\mathbf{S},\mathbb{R}}(\mathfrak{a}_{\mathrm{q}})$, then follows immediately as on [5, p. 372]. □

Let $Q \in \mathcal{P}_\sigma^{\min}$ and assume that $\mu \in \Lambda(\mathfrak{a}_{\mathrm{q}})$ is Q-dominant. Let $\psi_\mu(Q : \xi) : \mathfrak{a}^*_{\mathrm{q}\mathbb{C}} \to \mathrm{End}(V(\xi, 1))$ be the rational function in [5, Lemma 9.13]. Its exact definition will be recalled in the following proof.

Lemma A.2. *There exist polynomials* $q_1, q_2 \in \Pi_{\Sigma,\mathbb{R}}(\mathfrak{a}_{\mathrm{q}})$ *and a constant* $c \neq 0$ *such that*

$$\psi_\mu(Q : \xi : \lambda) = c\, \frac{q_1(\lambda)}{q_2(\lambda)} I_{V(\xi,1)} \tag{A.1}$$

for $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$.

Remark. The rational function q_1/q_2 is in fact determined explicitly in the following proof. It is given by an equation that involves Harish-Chandra's c-function for the Riemannian form G^d/K^d of G/H, cf. Lemma A.5 and (A.9).

Proof. We first recall how $\psi_\mu(Q : \xi)$ is defined. Let $\mathcal{H}_{\xi\lambda}$ denote the space $\mathcal{H}_\xi$ equipped with the representation $\xi \otimes \lambda \otimes 1$ of Q, and consider the G-equivariant map

$$T_\mu : C^{-\infty}(Q : \xi : \lambda) \otimes F \to C^{-\infty}\, \mathrm{Ind}_Q^G(\mathcal{H}_{\xi\lambda} \otimes F|_Q)$$

determined by

$$T_\mu(f \otimes v)(x) = f(x) \otimes \pi(x) v.$$

On the level of K-finite vectors, T_μ is an isomorphism (see [5, p. 359], where the map is denoted φ_λ). Let $p_\mu(Q : \xi : \lambda)$ denote the endomorphism of $C^{-\infty}(Q : \xi : \lambda) \otimes F$ given by projection along the infinitesimal character $\Lambda + \lambda + \mu$ (where Λ is the infinitesimal character of ξ), cf. [5, Prop. 8.3]. We refer to [5, p. 361] for the definitions of $b(Z, \lambda) \in \mathbb{C}$

and $D(Z,\lambda) \in \mathcal{Z}(\mathfrak{g})$ for $Z \in \mathcal{Z}(\mathfrak{g})$, $\lambda \in \mathfrak{a}^*_{\mathfrak{q}\mathbb{C}}$. Both objects depend polynomially on λ, and there exists $Z \in \mathcal{Z}(\mathfrak{g})$ such that $b(Z,\,\cdot\,)$ is not the zero polynomial. Then

$$p_\mu(Q\colon\xi\colon\lambda) = b(Z,\lambda)^{-1}[\mathrm{Ind}_Q^G(\xi\otimes\lambda\otimes 1)\otimes\pi](D(Z,\lambda)), \tag{A.2}$$

cf. [5, p. 362]. In particular we see that $p_\mu(Q\colon\xi\colon\lambda)$ acts as a differential operator. It follows from [5, p. 370, below (75)] that T_μ maps the image of $p_\mu(Q\colon\xi\colon\lambda)$ into the subspace $C^{-\infty}\,\mathrm{Ind}_Q^G(\mathcal{H}_{\xi\lambda}\otimes F_\mu)$ of $C^{-\infty}\,\mathrm{Ind}_Q^G(\mathcal{H}_{\xi\lambda}\otimes F|_Q)$. Here F_μ is the one-dimensional subspace in F of vectors of weight μ; it carries the representation $1\otimes\mu\otimes 1$ of Q (cf. [5, Prop. 5.5]).

Fix a non-zero vector e_μ in F_μ, then there is a natural G-equivariant isomorphism

$$S_\mu\colon C^{-\infty}(Q\colon\xi\colon\lambda+\mu) \xrightarrow{\sim} C^{-\infty}\,\mathrm{Ind}_Q^G(\mathcal{H}_{\xi\lambda}\otimes F_\mu);$$

the image of $f \in C^{-\infty}(Q\colon\xi\colon\lambda+\mu)$ is the generalized function

$$S_\mu f(x) = f(x)\otimes e_\mu$$

on G. Conversely, let $e^*_{-\mu} \in F^*$ be the (unique) vector of weight $-\mu$ such that $e^*_{-\mu}(e_\mu) = 1$, then testing with $e^*_{-\mu}$ on the second component of $\mathcal{H}_\xi\otimes F$ induces a G-equivariant linear map

$$\mathrm{t}_\mu\colon C^{-\infty}\,\mathrm{Ind}_Q^G(\mathcal{H}_{\xi\lambda}\otimes F|_Q) \to C^{-\infty}(Q\colon\xi\colon\lambda+\mu), \tag{A.3}$$

whose restriction to $C^{-\infty}\,\mathrm{Ind}_Q^G(\mathcal{H}_{\xi\lambda}\otimes F_\mu)$ is the inverse of S_μ. We define:

$$\tilde{T}_\mu := \mathrm{t}_\mu\circ T_\mu\colon C^{-\infty}(Q\colon\xi\colon\lambda)\otimes F \to C^{-\infty}(Q\colon\xi\colon\lambda+\mu).$$

Let $e_H \in F$ be a non-zero H-fixed vector, then

$$\tilde{T}_\mu[p_\mu(Q\colon\xi\colon\lambda)(j(Q\colon\xi\colon\lambda)\eta\otimes e_H)] \in C^{-\infty}(Q\colon\xi\colon\lambda+\mu)^H$$

for all $\eta \in V(\xi)$, by equivariance of $p_\mu(Q\colon\xi\colon\lambda)$ and $\tilde{T}_\mu$. We define the endomorphism $\tilde{\psi}_\mu(Q\colon\xi\colon\lambda)$ of $V(\xi)$ by

$$\tilde{\psi}_\mu(Q\colon\xi\colon\lambda)\eta = \mathrm{ev}(\tilde{T}_\mu[p_\mu(Q\colon\xi\colon\lambda)(j(Q\colon\xi\colon\lambda)\eta\otimes e_H)])$$

where $\mathrm{ev}\colon C^{-\infty}(Q\colon\xi\colon\lambda+\mu)^H \to V(\xi)$ is the evaluation map. Since by definition $j(Q\colon\xi\colon\lambda+\mu)$ is the inverse to ev, and since $T_\mu\circ p_\mu(Q\colon\xi\colon\lambda)$ maps into $C^{-\infty}\,\mathrm{Ind}_Q^G(\mathcal{H}_{\xi\lambda}\otimes F_\mu)$, on which $S_\mu\circ\mathrm{t}_\mu = I$, we have, equivalently, that $\tilde{\psi}_\mu(Q\colon\xi\colon\lambda)$ is determined by

$$T_\mu[p_\mu(Q\colon\xi\colon\lambda)(j(Q\colon\xi\colon\lambda)\eta\otimes e_H)] = S_\mu[j(Q\colon\xi\colon\lambda+\mu)\tilde{\psi}_\mu(Q\colon\xi\colon\lambda)\eta] \tag{A.4}$$

for $\eta \in V(\xi)$. Note that $\tilde{\psi}_\mu(Q\colon\xi\colon\lambda)$ maps each component $V(\xi,w)$ of $V(\xi)$ to itself, since $p_\mu(Q\colon\xi\colon\lambda)$ as well as $\tilde{T}_\mu$ are support preserving maps. The map $\psi_\mu(Q\colon\xi)$ in (A.1) is the restriction of $\tilde{\psi}_\mu(Q\colon\xi)$ to $V(\xi,1)$.

In the following it will be convenient to have some of the above mentioned notions from [5, §8] available in a slightly more general setting. In [5, §8] it is assumed that the finite dimensional irreducible representation ξ of M is unitary and has a non-trivial vector fixed by $w(M\cap H)w^{-1}$ for some $w \in \mathcal{W}$. Moreover, the linear form λ on $\mathfrak{a}_{\mathrm{q}}$ is extended to a linear form on $\mathfrak{a}$ with trivial restriction to $\mathfrak{a}_{\mathrm{h}} = \mathfrak{a}\cap\mathfrak{h}$. It is these assumptions on the representation $\xi\otimes\lambda\otimes 1$ of Q that we temporarily want to relax. It will also be convenient to deal with the σ-Langlands decomposition $Q = M_\sigma A_{\mathrm{q}} N$, instead of the ordinary Langlands decomposition $Q = MAN$. We recall that $M_\sigma = MA_{\mathrm{h}}$ where $A_{\mathrm{h}} = \exp \mathfrak{a}_{\mathrm{h}}$. We assume that $(\xi, \mathcal{H}_\xi)$ is a finite dimensional irreducible representation of M_σ, the infinitesimal character Λ of which is real with respect to the roots of $\mathfrak{j}$ in $\mathfrak{m}_\sigma$ (recall that $\mathfrak{j}$ is a Cartan subalgebra of $\mathfrak{g}$, defined as below [5, Cor. 5.3]). For $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ we then consider the representation $\xi\otimes\lambda\otimes 1$ of $Q = M_\sigma A_{\mathrm{q}} N$, and we use the notation $C(Q\colon\xi\colon\lambda)$ for the underlying space of the normally induced representation $\mathrm{Ind}_Q^G(\xi\otimes\lambda\otimes 1)$.

The maps T_μ, $\tilde T_\mu$ and S_μ make sense in this generality. It is seen as in [5, Prop. 8.1] that T_μ maps $p_\mu(Q\colon\xi\colon\lambda)(C(Q\colon\xi\colon\lambda)_K\otimes F)$ bijectively onto the space of K-finite vectors in $\mathrm{Ind}_Q^G(\mathcal{H}_{\xi\lambda}\otimes F_\mu)$, hence its composition $\tilde T_\mu$ with (A.3) restricts to an isomorphism

$$\tilde T_\mu\colon p_\mu(Q\colon\xi\colon\lambda)(C(Q\colon\xi\colon\lambda)_K\otimes F)\xrightarrow{\sim} C(Q\colon\xi\colon\lambda+\mu)_K$$

for $\mathbf{S}$-generic $\lambda\in\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$

The definitions of $b(Z,\lambda)$ and $D(Z,\lambda)$ immediately generalize to the present setting, and the analogue of [5, Lemma 8.4] holds; it states that (A.2) holds on the K-finite level, for $\mathbf{S}$-generic $\lambda\in\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$.

Before giving the proof of Lemma A.2 we establish an analogous result, in which H-fixed vectors are replaced by K-fixed vectors. The space $\mathcal{H}_\xi^{M\cap K}$ is either trivial or one-dimensional. At present we assume the latter and define

$$\epsilon(Q\colon\xi\colon\lambda)\colon \mathcal{H}_\xi^{M\cap K}\to C(Q\colon\xi\colon\lambda)$$

by

$$[\epsilon(Q\colon\xi\colon\lambda)\zeta](namk) = a^{\lambda+\rho_Q}\xi(m)\zeta$$

for $n\in N$, $a\in A_{\mathrm{q}}$, $m\in M_\sigma$, $k\in K$ and $\zeta\in\mathcal{H}_\xi^{M\cap K}$. Then $\epsilon(Q\colon\xi\colon\lambda)$ is a bijection of $\mathcal{H}_\xi^{M\cap K}$ onto $C(Q\colon\xi\colon\lambda)^K$; its inverse is given by the evaluation at the identity element.

Viewed as a function on G/K, $\epsilon(Q\colon\xi\colon\lambda)\zeta$ is a joint eigenfunction for $\mathbb{D}(G/K)$. This can be seen by factoring the Harish-Chandra homomorphism $\mathbb{D}(G/K)\to S(\mathfrak{a}_0)$ through $\mathbb{D}(M_\sigma/M\cap K)\otimes S(\mathfrak{a}_{\mathrm{q}})$, in analogy with [5, Lemma 4.4]; the function $m\mapsto\xi(m)\zeta$ on $M_\sigma/M\cap K$ is a joint eigenfunction for $\mathbb{D}(M_\sigma/M\cap K)$ (we recall that $\mathfrak{a}_0 = \mathfrak{j}\cap\mathfrak{p}$ is a Cartan subalgebra for the pair (G,K)). The eigenvalue homomorphism $\mathbb{D}(G/K)\to\mathbb{C}$ is obtained from the character $\Lambda_1+\lambda$ on $\mathfrak{a}_0$, where Λ_1 denotes the restriction to $\mathfrak{a}_0\cap\mathfrak{m}_\sigma$ of the infinitesimal character Λ of ξ.

Let $e_K\in F$ be a non-zero K-fixed vector and let $\zeta\in\mathcal{H}_\xi^{M\cap K}$. Then for generic $\lambda\in\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ the function $\tilde T_\mu[p_\mu(Q\colon\xi\colon\lambda)(\epsilon(Q\colon\xi\colon\lambda)\zeta\otimes e_K)]$ belongs to $C(Q\colon\xi\colon\lambda+\mu)^K$, hence its value at the identity is given by $\phi_\mu(Q\colon\xi\colon\lambda)\zeta$ for some complex scalar $\phi_\mu(Q\colon\xi\colon\lambda)$. In analogy with (A.4) we obtain

$$T_\mu[p_\mu(Q\colon\xi\colon\lambda)(\epsilon(Q\colon\xi\colon\lambda)\zeta\otimes e_K)] = \phi_\mu(Q\colon\xi\colon\lambda)S_\mu[\epsilon(Q\colon\xi\colon\lambda+\mu)\zeta]. \qquad \text{(A.5)}$$

Lemma A.3. *There exist polynomials $q_1, q_2 \in \Pi_{\Sigma,\mathbb{R}}(\mathfrak{a}_\mathrm{q})$ and a constant $c \neq 0$ such that*

$$\phi_\mu(Q\colon \xi\colon \lambda) = c\,\frac{q_1(\lambda)}{q_2(\lambda)}.$$

Proof. Fix $Z \in \mathcal{Z}(\mathfrak{g})$ such that $b(Z,\,\cdot\,) \neq 0$ and define an element $u_\lambda \in \mathcal{Z}(\mathfrak{g})$ depending rationally on $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$ by

$$u_\lambda = b(Z,\lambda)^{-1} D(Z,\lambda).$$

The elements of $\mathcal{Z}(\mathfrak{g})$ act on functions on G by the right regular representation R, and it follows from (A.2) and (A.5) that

$$R(u_\lambda)(T_\mu[\epsilon(Q\colon \xi\colon \lambda)\zeta \otimes e_K]) = \phi_\mu(Q\colon \xi\colon \lambda) S_\mu[\epsilon(Q\colon \xi\colon \lambda+\mu)\zeta]. \tag{A.6}$$

Let e_K^* be the K-fixed element of F^* determined by $e_K^*(e_K) = 1$. Then testing the expression on either side of (A.6) with e_K^* on the second component we obtain

$$R(u_\lambda)[\epsilon(Q\colon \xi\colon \lambda)\zeta(\,\cdot\,) \otimes e_K^*(\pi(\,\cdot\,)e_K)] = e_K^*(e_\mu)\phi_\mu(Q\colon \xi\colon \lambda)\epsilon(Q\colon \xi\colon \lambda+\mu)\zeta \tag{A.7}$$

on G. We now observe that $e_K^*(\pi(\,\cdot\,)e_K)$ equals $\varphi_{\mu+\rho_0}$, the elementary spherical function on G/K determined by the parameter $\mu+\rho_0$. Moreover, we have

$$\int_K [\epsilon(Q\colon \xi\colon \lambda)\zeta](kx)\,dk = \varphi_{\Lambda_1+\lambda}(x)\zeta. \tag{A.8}$$

Indeed, both sides of (A.8) are $\mathcal{H}_\xi^{M\cap K}$-valued joint eigenfunctions for $\mathbb{D}(G/K)$ with the same parameter $\Lambda_1+\lambda$, and they both take the value ζ when x is the identity. Integrating (A.7) over K we thus obtain

$$R(u_\lambda)[\varphi_{\Lambda_1+\lambda}\varphi_{\mu+\rho_0}] = e_K^*(e_\mu)\phi_\mu(Q\colon \xi\colon \lambda)\varphi_{\Lambda_1+\lambda+\mu}.$$

However, from the asymptotic expansions of the involved functions it follows (see [28, Thm. 4.5 and Lemma 4.6]) that

$$R(u_\lambda)[\varphi_{\Lambda_1+\lambda}\varphi_{\mu+\rho_0}] = \frac{\mathbf{c}(\mu+\rho_0)\mathbf{c}(\Lambda_1+\lambda)}{\mathbf{c}(\Lambda_1+\lambda+\mu)}\,\varphi_{\Lambda_1+\lambda+\mu},$$

where $\mathbf{c}\colon \mathfrak{a}^*_{0\mathbb{C}} \to \mathbb{C}$ denotes Harish-Chandra's c-function associated with the Riemannian symmetric space G/K. We conclude that

$$\phi_\mu(Q\colon \xi\colon \lambda) = \frac{\mathbf{c}(\mu+\rho_0)\mathbf{c}(\Lambda_1+\lambda)}{e_K^*(e_\mu)\mathbf{c}(\Lambda_1+\lambda+\mu)}. \tag{A.9}$$

The desired statement now follows from the Gindikin-Karpelevic formula for $\mathbf{c}$, cf. also [28, Cor. 4.7]. $\square$

We shall now translate (A.5) into an algebraic statement that will be used in the proof of Lemma A.2.

Let $\mathbb{C}_\rho$ denote $\mathbb{C}$ equipped with the structure of a $U(\mathfrak{m}+\mathfrak{a}+\mathfrak{n})$-module defined by $\rho = \rho_Q$ on $\mathfrak{a}$ and the trivial action on $\mathfrak{m}+\mathfrak{n}$. Note that since Q is $\sigma\theta$-stable, then so is ρ, that is, it vanishes on $\mathfrak{a}_{\mathrm{h}}$. Given a finite dimensional $U(\mathfrak{m}+\mathfrak{a}+\mathfrak{n})$-module V we shall write

$$\mathrm{Hom}_{\mathfrak{m}+\mathfrak{a}+\mathfrak{n}}(U(\mathfrak{g}), V)$$

for the space of $\mathfrak{m}+\mathfrak{a}+\mathfrak{n}$ homomorphisms $U(\mathfrak{g}) \to V$; here $U(\mathfrak{g})$ is viewed as a right $U(\mathfrak{m}+\mathfrak{a}+\mathfrak{n})$-module. Moreover, we define the $U(\mathfrak{g})$-module

$$\mathrm{I}_Q^G(V) := \mathrm{Hom}_{\mathfrak{m}+\mathfrak{a}+\mathfrak{n}}(U(\mathfrak{g}), V \otimes \mathbb{C}_\rho),$$

where the module structure is determined by

$$u \cdot F(v) = F(\check{u}v)$$

with $u \mapsto \check{u}$ the principal antiautomorphism of $U(\mathfrak{g})$.

We now consider a finite dimensional representation (δ, V_δ) of Q. We shall then also use the notation $\mathrm{I}_Q^G(V_\delta)$ for the $U(\mathfrak{g})$-module defined as above by means of the $U(\mathfrak{m}+\mathfrak{a}+\mathfrak{n})$-module structure on V_δ that arises from δ. On the other hand, we consider the normally induced representation $\mathrm{Ind}_Q^G(\delta)$. The underlying representation space consists of the space $C(Q\colon\delta)$ of continuous functions $G \to V_\delta$, transforming according to the rule

$$f(manx) = a^\rho \delta(man) f(x), \qquad (man \in Q, x \in G).$$

We define $C_Q^\omega(Q\colon\delta)$ to be the space of germs along Q of V_δ-valued real analytic functions, defined, and satisfying the above transformation rule, for x in some left Q-invariant neighborhood of e. Via differentiation from the right we equip this space with the structure of a $U(\mathfrak{g})$-module. Note that taking germs along Q induces a natural $U(\mathfrak{g})$-equivariant embedding

$$C(Q\colon\delta)_K \hookrightarrow C_Q^\omega(Q\colon\delta).$$

Given $f \in C_Q^\omega(Q\colon\delta)$, we define the map $\iota(f)\colon U(\mathfrak{g}) \to V_\delta \otimes \mathbb{C}_\rho$, by

$$\iota(f)(v) = [L(v)f](e) \otimes 1.$$

We view $\iota(f)$ as the power series of f at e. One readily verifies that ι is an equivariant embedding of the $U(\mathfrak{g})$-module $C_Q^\omega(Q\colon\delta)$ into $\mathrm{I}_Q^G(V_\delta)$.

It is readily seen that $\mathcal{Z}(\mathfrak{g})$ acts globally finitely on $\mathrm{I}_Q^G(V_\delta)$. Given $\Lambda_0 \in \mathfrak{j}_{\mathbb{C}}^*$ we denote by p_{Λ_0} the projection in $\mathrm{I}_Q^G(V_\delta)$ onto the generalized eigenspace for $\mathcal{Z}(\mathfrak{g})$ determined by the infinitesimal character Λ_0.

Fix $\zeta \in \mathcal{H}_\xi^{M\cap K}$ and denote by $\bar{\epsilon}_K(\lambda)$ the unique $\mathfrak{k}$-invariant element of $\mathrm{I}_Q^G(\mathcal{H}_{\xi\lambda} \otimes F|_Q)$ determined by $\bar{\epsilon}_K(\lambda)(1) = \zeta \otimes e_K \otimes 1$. Similarly, we denote by $\bar{\epsilon}_\mu(\lambda)$ the unique $\mathfrak{k}$-invariant element of $\mathrm{I}_Q^G(\mathcal{H}_{\xi\lambda} \otimes F_\mu)$ determined by $\bar{\epsilon}_\mu(\lambda)(1) = \zeta \otimes e_\mu \otimes 1$. Then one readily sees that

$$\bar{\epsilon}_K(\lambda) = \iota(T_\mu[\epsilon(Q\colon\xi\colon\lambda)\zeta \otimes e_K]), \quad \bar{\epsilon}_\mu(\lambda) = \iota(S_\mu[\epsilon(Q\colon\xi\colon\lambda+\mu)\zeta]). \tag{A.10}$$

Lemma A.4. *For generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ we have the following identity in $\mathrm{I}_Q^G(\mathcal{H}_{\xi\lambda} \otimes F|_Q)$:*

$$p_{\Lambda+\lambda+\mu}\,[\bar{\epsilon}_K(\lambda)] = \phi_\mu(Q\colon\xi\colon\lambda)\;\bar{\epsilon}_\mu(\lambda).$$

Proof. Since ι and T_μ are $U(\mathfrak{g})$-equivariant maps we obtain from (A.10) that

$$p_{\Lambda+\lambda+\mu}\,[\bar{\epsilon}_K(\lambda)] = \iota(T_\mu[p_\mu(Q\colon\xi\colon\lambda)(\epsilon(Q\colon\xi\colon\lambda)\zeta \otimes e_K)]).$$

Applying (A.5) and (A.10) we obtain the desired identity. $\square$

We now return to our original assumption on $(\xi, \mathcal{H}_\xi)$, that it belongs to $\hat{M}_H$. Our goal is to give the proof of Lemma A.2. For this we may as well assume that $V(\xi, 1) = \mathcal{H}_\xi^{M\cap H} \neq 0$, otherwise there is nothing to prove. Since M is of Harish-Chandra's class, the representation ξ has an infinitesimal character. This implies that the $\mathfrak{m}$-module $\mathcal{H}_\xi$ is a multiple of an irreducible representation, which we denote by $(\xi_0, \mathcal{H}_{\xi_0})$. It follows that we may assume that

$$\mathcal{H}_\xi = \mathcal{H}_{\xi_0} \otimes E,$$

with E a finite dimensional complex linear space, and such that

$$\xi(X) = \xi_0(X) \otimes I, \qquad (X \in \mathfrak{m}).$$

From the fact that $\mathcal{H}_\xi^{M\cap H} \neq 0$ it follows that ξ_0 possesses a non-trivial $\mathfrak{m}\cap\mathfrak{h}$ invariant vector. The space $\mathcal{H}_{\xi_0}^{\mathfrak{m}\cap\mathfrak{h}}$ is one-dimensional, and moreover

$$\mathcal{H}_\xi^{M\cap H} \subset \mathcal{H}_{\xi_0}^{\mathfrak{m}\cap\mathfrak{h}} \otimes E.$$

Let $\mathfrak{g}_+$ and $\mathfrak{g}_-$ denote the $+1$ and -1 eigenspaces for the involution $\sigma\theta$, respectively. Then $\mathfrak{g}^d := \mathfrak{g}_+ \oplus i\mathfrak{g}_-$ is a real form of the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$. It is called the dual real form of $\mathfrak{g}$. We denote the complex linear extensions of σ and θ to $\mathfrak{g}_\mathbb{C}$ by $\sigma_\mathbb{C}$ and $\theta_\mathbb{C}$, respectively. Let σ^d and θ^d denote the restrictions to $\mathfrak{g}^d$ of $\theta_\mathbb{C}$ and $\sigma_\mathbb{C}$, respectively. Then θ^d is a Cartan involution of $\mathfrak{g}^d$ and σ^d is an involution of $\mathfrak{g}^d$ that commutes with θ^d. We have associated eigenspace decompositions $\mathfrak{g}^d = \mathfrak{k}^d \oplus \mathfrak{p}^d = \mathfrak{h}^d \oplus \mathfrak{q}^d$. Note that $\mathfrak{p}^d \cap \mathfrak{q}^d = \mathfrak{p} \cap \mathfrak{q}$, hence $\mathfrak{a}_\mathrm{q}^d := \mathfrak{a}_\mathrm{q}$ is maximal abelian in $\mathfrak{p}^d \cap \mathfrak{q}^d$. Note that the root space decomposition of $\mathfrak{g}_\mathbb{C}$ relative to $\mathfrak{a}_\mathrm{q}$ is stable under the conjugations determining the real forms $\mathfrak{g}$ and $\mathfrak{g}^d$. Hence Σ^d, the collection of roots of $\mathfrak{a}_\mathrm{q}^d$ in $\mathfrak{g}^d$ equals Σ.

Let G^d be a connected group of Harish-Chandra's class with Lie algebra $\mathfrak{g}^d$ to which both involutions θ^d and σ^d lift. Standard notations introduced in the context of G will also be used for G^d; a superscript d will indicate that an object originally defined for G, H, K is defined in exactly the same way, but with (G^d, H^d, K^d) in place of (G, H, K).

In this spirit, let A_q^d be the image of $\mathfrak{a}_\mathrm{q}^d$ under $\exp\colon \mathfrak{g}^d \to G^d$. Moreover, let Q^d be the minimal σ^d-parabolic subgroup containing A_q^d, determined by the system Σ^+ of positive roots for $\Sigma^d = \Sigma$, and let $Q^d = M_\sigma^d A_\mathrm{q}^d N^d$ be its σ^d-Langlands decomposition. This is compatible with the σ-Langlands decomposition of Q, in the sense that $\mathfrak{m}_\sigma^d = \mathfrak{m}_{\sigma\mathbb{C}} \cap \mathfrak{g}^d$ and $\mathfrak{n}^d = \mathfrak{n}_\mathbb{C} \cap \mathfrak{g}^d$.

We extend the representation ξ_0 of $\mathfrak{m}$ in $\mathcal{H}_{\xi_0}$ to a representation ξ_1 of $\mathfrak{m}_\sigma = \mathfrak{m} \oplus \mathfrak{a}_{\mathrm{h}}$ by triviality on $\mathfrak{a}_{\mathrm{h}}$. The restriction to $\mathfrak{m}_\sigma^d$ of the complexification of ξ_1 is denoted by ξ_1^d. The representation ξ_1^d is irreducible, and possesses a one-dimensional subspace of vectors that are annihilated by $[\mathfrak{m}_{\sigma\mathbb{C}} \cap \mathfrak{h}_{\mathbb{C}}] \cap \mathfrak{g}^d = \mathfrak{m}_\sigma^d \cap \mathfrak{k}^d$. We fix such a vector (non-trivial) and denote it ζ^d. Since M_σ^d is a group of Harish-Chandra's class, it follows that ξ_1^d lifts to a unique $M_\sigma^d \cap K^d$-spherical representation, also denoted ξ_1^d.

The finite dimensional irreducible representation (π, F) of G is K-spherical, hence the associated infinitesimal representation of $\mathfrak{g}_{\mathbb{C}}$ in F is irreducible. Let π^d denote the restriction to $\mathfrak{g}^d$ of this infinitesimal representation. Then, since π is also H-spherical, π^d has a non-trivial $\mathfrak{k}^d$-fixed vector. Since G^d is of Harish-Chandra's class, the representation π^d lifts to a unique K^d-spherical representation of G^d in $F^d := F$, which is again denoted by π^d. Note that μ is an extremal $\mathfrak{a}_{\mathrm{q}}^d$-weight of π^d. We assume that μ is Q-dominant; then μ is also Q^d-dominant. As before we select a non-trivial vector $e_\mu = e_\mu^d$ in the weight space F_μ of F. Moreover, we select a non-trivial H-fixed vector $e_H \in F$; then e_H is K^d-fixed as well, and we put $e_K^d := e_H$.

According to Lemma A.4, applied to G^d, we now have for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* = \mathfrak{a}_{\mathrm{q}\mathbb{C}}^{d*}$, that

$$p_{\Lambda+\lambda+\mu}(\bar{\epsilon}_K^d(\lambda)) = \phi_\mu^d(Q^d \colon \xi_1^d \colon \lambda)\, \bar{\epsilon}_\mu^d(\lambda), \tag{A.11}$$

in the representation space

$$\mathrm{I}_{Q^d}^{G^d}(\mathcal{H}_{\xi_1^d\lambda} \otimes F^d|_{Q^d}) = \mathrm{Hom}_{\mathfrak{m}_\sigma^d \oplus \mathfrak{a}_{\mathrm{q}}^d \oplus \mathfrak{n}^d}(U(\mathfrak{g}^d), \mathcal{H}_{\xi_1^d\lambda} \otimes F \otimes \mathbb{C}_{\rho^d}).$$

Here $\mathcal{H}_{\xi_1^d\lambda}$ denotes $\mathcal{H}_{\xi_0}$, equipped with the $U(\mathfrak{m}_\sigma^d + \mathfrak{a}_{\mathrm{q}}^d + \mathfrak{n}^d)$-module structure $\xi_1^d \otimes \lambda \otimes 1$. Note that $U(\mathfrak{g}^d) = U(\mathfrak{g})$ and $(\mathfrak{m}_\sigma^d + \mathfrak{a}_{\mathrm{q}}^d + \mathfrak{n}^d)_{\mathbb{C}} = (\mathfrak{m}_\sigma + \mathfrak{a}_{\mathrm{q}} + \mathfrak{n})_{\mathbb{C}}$; hence, the space in the above equation equals

$$\begin{aligned}\mathrm{I}_Q^G(\mathcal{H}_{\xi_1\lambda} \otimes F|_Q) &= \mathrm{Hom}_{\mathfrak{m}_\sigma + \mathfrak{a}_{\mathrm{q}} + \mathfrak{n}}(U(\mathfrak{g}), \mathcal{H}_{\xi_1\lambda} \otimes F \otimes \mathbb{C}_\rho)\\ &= \mathrm{Hom}_{\mathfrak{m}+\mathfrak{a}+\mathfrak{n}}(U(\mathfrak{g}), \mathcal{H}_{\xi_0\lambda} \otimes F \otimes \mathbb{C}_\rho).\end{aligned}$$

It follows that we have a natural isomorphism of $U(\mathfrak{g})$-modules:

$$\mathrm{I}_Q^G(\mathcal{H}_{\xi\lambda} \otimes F|_Q) \simeq \mathrm{I}_{Q^d}^{G^d}(\mathcal{H}_{\xi_1^d\lambda} \otimes F^d|_{Q^d}) \otimes E; \tag{A.12}$$

here $U(\mathfrak{g})$ acts on the first component of the tensor product on the right-hand side.

Lemma A.5. *The function $\psi_\mu(Q \colon \xi) \colon \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \to \mathrm{End}(V(\xi, 1))$ is given by*

$$\psi_\mu(Q \colon \xi \colon \lambda) = \phi_\mu^d(Q^d \colon \xi_1^d \colon \lambda)\, I_{V(\xi,1)}. \tag{A.13}$$

Proof. We recall that

$$V(\xi, 1) \simeq \mathcal{H}_\xi^{M\cap H} \subset \mathcal{H}_{\xi_0}^{\mathfrak{m}\cap\mathfrak{h}} \otimes E = \mathcal{H}_{\xi_1^d}^{\mathfrak{m}_\sigma^d \cap \mathfrak{k}^d} \otimes E = \mathbb{C}\zeta^d \otimes E,$$

where ζ^d is the fixed $\mathfrak{m}^d \cap \mathfrak{k}^d$ invariant vector for ξ_1^d. For each $\eta \in V(\xi, 1)$ we have accordingly $\eta = \zeta^d \otimes \eta_E$ for a uniquely determined $\eta_E \in E$.

We consider the germ $\gamma_1(\lambda)$ along Q of the function $T_\mu[j(Q\colon \xi\colon \lambda)\eta \otimes e_H]$, which restricts to a real analytic map $QH \to \mathcal{H}_\xi \otimes F$. Then $\gamma_1(\lambda) \in C_Q^\omega \operatorname{Ind}_Q^G(\mathcal{H}_{\xi\lambda} \otimes F|_Q)$. Via the identification (A.12) we may view the associated formal power series $\iota(\gamma_1(\lambda))$ as an element of $\mathrm{I}_{Q^d}^{G^d}(\mathcal{H}_{\xi_1^d\lambda} \otimes F^d|_{Q^d}) \otimes E$; it is $\mathfrak{k}^d \otimes I$-invariant, and its value at $1 \in U(\mathfrak{g}^d)$ equals $\zeta^d \otimes e_K^d \otimes 1 \otimes \eta_E$. From this we obtain that

$$\iota(\gamma_1(\lambda)) = \bar{\epsilon}_K^d(\lambda) \otimes \eta_E. \tag{A.14}$$

We consider similarly the germ $\gamma_2(\lambda) \in C_Q^\omega \operatorname{Ind}_Q^G(\mathcal{H}_{\xi\lambda} \otimes F_\mu)$ along Q of the function $S_\mu[j(Q\colon \xi\colon \lambda + \mu)\eta]$. Its formal power series, viewed as an element of $\mathrm{I}_{Q^d}^{G^d}(\mathcal{H}_{\xi_1^d\lambda} \otimes F_\mu^d) \otimes E$ is $\mathfrak{k}^d \otimes I$-invariant, and its value at $1 \in U(\mathfrak{g}^d)$ equals $\zeta^d \otimes e_\mu^d \otimes 1 \otimes \eta_E$. It follows that

$$\iota(\gamma_2(\lambda)) = \bar{\epsilon}_\mu^d(\lambda) \otimes \eta_E. \tag{A.15}$$

It follows from (A.14) that

$$\iota \circ p_\mu(Q\colon \xi\colon \lambda)\,[\gamma_1(\lambda)] = p_{\Lambda+\lambda+\mu} \circ \iota\,[\gamma_1(\lambda)] = p_{\Lambda+\lambda+\mu}\,[\bar{\epsilon}_K^d(\lambda)] \otimes \eta_E,$$

which by (A.11) and (A.15) equals $\phi_\mu^d(Q^d\colon \xi_1^d\colon \lambda)$ times $\iota\,[\gamma_2(\lambda)]$. Since ι is an embedding, it follows that

$$p_\mu(Q\colon \xi\colon \lambda)[\gamma_1(\lambda)] = \phi_\mu^d(Q^d\colon \xi_1^d\colon \lambda)\gamma_2(\lambda)$$

on a neighborhood of e, for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. We conclude that

$$T_\mu[p_\mu(Q\colon \xi\colon \lambda)(j(Q\colon \xi\colon \lambda)\eta \otimes e_H)] = \phi_\mu^d(Q^d\colon \xi_1^d\colon \lambda)\, S_\mu[j(Q\colon \xi\colon \lambda + \mu\colon x)\eta]$$

on HQ, and comparing this with (A.4) we obtain (A.13). □

Finally, Lemma A.2 follows from Lemmas A.5 and A.3. □

Appendix B. Induction of relations

In this appendix we recall a result from [12] that is used in Sections 8 and 10. We first introduce the notion of a Laurent functional and discuss its relation to the previously defined notion of a Laurent operator. Let V be a real linear space, equipped with a positive definite inner product $\langle\cdot,\cdot\rangle$, and let $V_\mathbb{C}$ denote its complexification, equipped with the complex linear extension of the inner product $\langle\cdot,\cdot\rangle$.

Let X be a (possibly empty) finite set of non-zero elements of V, such that $\mathbb{R}\xi_1 \neq \mathbb{R}\xi_2$ for all distinct $\xi_1, \xi_2 \in X$. By an X-hyperplane in $V_\mathbb{C}$ we mean an affine hyperplane of the form $H = a + \alpha_{H\mathbb{C}}^\perp$, with $a \in V_\mathbb{C}$, $\alpha_H \in X$. Note that α_H is uniquely determined in view of our assumption on X; hence the polynomial function $l_H\colon V_\mathbb{C} \to \mathbb{C},\ z \mapsto \langle \alpha_H, z - a\rangle$ is also uniquely determined, and we have $H = l_H^{-1}(0)$. A locally finite collection of X-hyperplanes in $V_\mathbb{C}$ is called an X-configuration in $V_\mathbb{C}$.

If $a \in V_{\mathbb{C}}$, then we denote the (finite) collection of all X-hyperplanes containing a by $\mathcal{H}(a, X)$. Moreover, we denote by $\mathcal{M}(a, X)$ the ring of germs of meromorphic functions at a whose singular locus at a is contained in the germ of $\cup\mathcal{H}(a, X)$ at a. By $\mathbb{N}^X$ we denote the space of functions $X \to \mathbb{N}$. For $d \in \mathbb{N}^X$ we define the polynomial function $\pi_{a,d} = \pi_{a,X,d} \colon V_{\mathbb{C}} \to \mathbb{C}$ by

$$\pi_{a,d}(z) = \prod_{\xi \in X} \langle \xi, z - a\rangle^{d(\xi)}, \qquad (z \in V_{\mathbb{C}}).$$

By $\mathcal{O}_a = \mathcal{O}_a(V_{\mathbb{C}})$ we denote the ring of germs of holomorphic functions at a. Then

$$\mathcal{M}(a, X) = \cup_{d \in \mathbb{N}^X} \ \pi_{a,d}^{-1}\mathcal{O}_a.$$

We define the space $\mathcal{M}(a, X)^*_{\mathrm{laur}}$ of X-Laurent functionals at a to be the space of linear functionals $\mathcal{L} \colon \mathcal{M}(a, X) \to \mathbb{C}$ such that for every $d \in \mathbb{N}^X$ there exists an element $u_d \in S(V)$ such that

$$\mathcal{L}\varphi = u_d[\pi_{a,d}\varphi](a),$$

for all $\varphi \in \pi_{a,d}^{-1}\,\mathcal{O}_a$. It is immediate from this definition that the string $(u_d)_{d\in\mathbb{N}^X}$ is uniquely determined by $\mathcal{L}$; we denote it by $u_{\mathcal{L}}$.

Remark B.1. Let $T_a \colon z \mapsto z + a$ denote translation by a in $V_{\mathbb{C}}$. Pull-back under T_a induces an isomorphism of rings $T_a^* \colon \mathcal{O}_a \to \mathcal{O}_0$, $\varphi \mapsto \varphi \circ T_a$. Moreover, $T_a^*(\pi_{a,d}) = \pi_{0,d}$ for every $d \in \mathbb{N}^X$, and we see that pull-back under T_a also induces an isomorphism of rings $T_a^* \colon \mathcal{M}(a, X) \to \mathcal{M}(0, X)$. From the definition of an X-Laurent functional one sees that transposition induces a linear map $T_{a*} \colon \mathcal{M}(0, X)^*_{\mathrm{laur}} \to \mathcal{M}(a, X)^*_{\mathrm{laur}}$. Obviously, T_{a*} is a linear isomorphism; moreover, one readily checks that $u_{T_{a*}\mathcal{L}} = u_{\mathcal{L}}$ for every $\mathcal{L} \in \mathcal{M}(0, X)^*_{\mathrm{laur}}$.

We shall now investigate which strings $(u_d)_{d\in\mathbb{N}^X}$ arise from Laurent functionals, following the method of [11, Sect. 1.3]. We write $\varpi_d = \pi_{0,d}$ and equip the space $\mathbb{N}^X$ with the partial ordering $\preceq$ defined by $d' \preceq d$ if and only if $d'(\xi) \leq d(\xi)$ for every $\xi \in X$. If $d' \preceq d$ then we define $d - d'$ componentwise as suggested by the notation.

In [11, Sect. 1.3] we defined $S_{\leftarrow}(V, X)$ as the linear space of strings $(u_d)_{d\in\mathbb{N}^X}$ satisfying

$$u_d(\varpi_{d-d'}\varphi)(0) = u_{d'}(\varphi)(0), \tag{B.1}$$

for all $d', d \in \mathbb{N}^X$ with $d' \preceq d$ and for every germ $\varphi \in \mathcal{O}_0$. This space is a projective limit space in a natural way, see [11] for details.

Lemma B.2. *The map $\mathcal{L} \mapsto u_{\mathcal{L}}$ is a linear isomorphism from $\mathcal{M}(a, X)^*_{\mathrm{laur}}$ onto $S_{\leftarrow}(V, X)$.*

Proof. In view of Remark B.1 we may as well assume that $a = 0$. Let $\mathcal{L} \in \mathcal{M}(0, X)^*_{\mathrm{laur}}$, and let $u_{\mathcal{L}} = (u_d)_{d\in\mathbb{N}^X}$ be the associated string in $S(V)$. Then for all d', d with $d' \preceq d$ we have $\varpi_{d-d'} = \pi_{0,d}\pi_{0,d'}^{-1}$. Hence, for every $\varphi \in \mathcal{O}_0$,

$$u_d(\varpi_{d-d'}\varphi)(0) = \mathcal{L}(\pi_{0,d'}^{-1}\varphi) = u_{d'}(\varphi)(0),$$

so that (B.1) holds. It follows that $u_{\mathcal{L}} \in S_{\leftarrow}(V, X)$. Obviously the map $\mathcal{L} \to u_{\mathcal{L}}$ is a linear injection. We will finish the proof by establishing its surjectivity.

Let $u \in S_{\leftarrow}(V, X)$. For $d \in \mathbb{N}^X$ we define $\mathcal{L}_d \colon \pi_{0,d}^{-1}\mathcal{O}_0 \to \mathbb{C}$ by $\mathcal{L}_d(\psi) = u_d(\pi_{0,d}\psi)(0)$. If $d, d' \in \mathbb{N}^X$, $d' \preceq d$, then from (B.1) it follows that $\mathcal{L}_d = \mathcal{L}_{d'}$ on $\pi_{0,d'}^{-1}\mathcal{O}_0$. Therefore, there exists a unique $\mathcal{L} \in \mathcal{M}(0, X)^*$ such that $\mathcal{L} = \mathcal{L}_d$ on $\pi_{0,d}^{-1}\mathcal{O}_0$ for every $d \in \mathbb{N}^X$. By definition we have $\mathcal{L} \in \mathcal{M}(0, X)^*_{\mathrm{laur}}$ and $u_{\mathcal{L}} = u$. □

In the following we shall see that the notion of a Laurent functional is closely related to the notion of a Laurent operator introduced in [11], see also Section 5. For this, we need some notation as well as a slight generalization of the concept of a Laurent operator from the setting of a real X-configuration to that of an arbitrary one.

By an X-subspace in $V_{\mathbb{C}}$ we mean any non-empty intersection of X-hyperplanes in $V_{\mathbb{C}}$. We denote the set of such affine subspaces by $\mathcal{A} = \mathcal{A}(V_{\mathbb{C}}, X)$. For $L \in \mathcal{A}$ there exists a unique real linear subspace $V_L \subset V$ such that $L = a + V_{L\mathbb{C}}$ for some $a \in V_{\mathbb{C}}$. The intersection $V_{L\mathbb{C}}^{\perp} \cap L$ consists of a single point, called the central point of L; we denote it by $c(L)$. The space L is said to be real if $c(L) \in V$; this means precisely that L is the complexification of an affine subspace of V.

For an X-configuration $\mathcal{H}$ we define $\mathcal{M}(V_{\mathbb{C}}, \mathcal{H})$ to be the space of meromorphic functions on $V_{\mathbb{C}}$ whose singular locus is contained in $\cup\mathcal{H}$. If $\mathcal{H}$ consists of real hyperplanes, we put $\mathcal{H}_V = \{H \cap V \mid H \in \mathcal{H}\}$; then $\mathcal{M}(V_{\mathbb{C}}, \mathcal{H})$ equals the space $\mathcal{M}(V, \mathcal{H}_V)$ defined in Section 5.

If $L \in \mathcal{A}$ we write $\mathcal{H}(L, X)$ for the collection of X-hyperplanes containing L, and $X(L) = \{\alpha_H \mid H \in \mathcal{H}(L, X)\}$. From the assumptions on X it follows that $X(L) = X \cap V_L^{\perp}$. Let X_r be the orthogonal projection of $X \setminus X(L)$ onto V_L. Let X_r^0 be a subset of X_r such that for every $\xi \in X_r$ there exists a unique $\xi^0 \in X_r^0$ with $\xi \in \mathbb{R}\xi^0$. Translation by $c(L)$ induces an affine isomorphism from $V_{L\mathbb{C}}$ onto L. Via this isomorphism we equip L with the structure of a complex linear space together with a real form with inner product; moreover, we write X_L for the image of X_r^0 in L. If $\mathcal{H}$ is an X-configuration in $V_{\mathbb{C}}$, then $\mathcal{H}_L = \{H \cap L \mid H \in \mathcal{H},\ \emptyset \subsetneq H \cap L \subsetneq H\}$ is an X_L-configuration in L.

We can now define the space $\mathrm{Laur}(V_{\mathbb{C}}, L, \mathcal{H})$ of Laurent operators from $\mathcal{M}(V_{\mathbb{C}}, \mathcal{H})$ to $\mathcal{M}(L, \mathcal{H}_L)$ as in [11, Sect. 1.3], see also Section 5. Lemma 1.5 of [11] is now readily seen to generalize to the present setting. It provides us with an isomorphism

$$\mathrm{Laur}(V_{\mathbb{C}}, L, \mathcal{H}) \xrightarrow{\simeq} S_{\leftarrow}(V_L^{\perp}, X(L)), \quad R \mapsto u_R. \tag{B.2}$$

Lemma B.3. *Assume that $L \in \mathcal{A}$, and let $\mathcal{H}$ be an X-configuration in $V_{\mathbb{C}}$ containing $\mathcal{H}(L, X)$.*

(a) *If $\varphi \in \mathcal{M}(V_{\mathbb{C}}, \mathcal{H})$, then for $w \in L \setminus \cup\mathcal{H}_L$ the function $z \mapsto \varphi(w + z)$ is meromorphic on $V_{L\mathbb{C}}^{\perp}$, with a germ at 0 that belongs to $\mathcal{M}(0, X(L))$.*

(b) *If $\mathcal{L} \in \mathcal{M}(0, X(L))^*_{\mathrm{laur}}$ is a Laurent functional in $V_{L\mathbb{C}}^{\perp}$, then for $\varphi \in \mathcal{M}(V_{\mathbb{C}}, \mathcal{H})$ the function*

$$\mathcal{L}_*\varphi \colon w \mapsto \mathcal{L}(\varphi(w + \cdot\,))$$

belongs to the space $\mathcal{M}(L, \mathcal{H}_L)$. The operator $\mathcal{L}_* \colon \mathcal{M}(V_{\mathbb{C}}, \mathcal{H}) \to \mathcal{M}(L, \mathcal{H}_L)$ is a Laurent operator.

(c) *The map* $\mathcal{L} \mapsto \mathcal{L}_*$ *is an isomorphism from* $\mathcal{M}(0, X(L))^*_{\rm laur}$ *onto* $\mathrm{Laur}(V_\mathbb{C}, L, \mathcal{H})$. *This isomorphism corresponds with the identity on* $S_\leftarrow(V_L^\perp, X(L))$, *via the isomorphisms of Lemma B.2 and Eqn.* (B.2).

Proof. (a) Let $w \in L \setminus \cup\mathcal{H}_L$. Assume that $H \in \mathcal{H}$ is a hyperplane containing w. Then $H \cap L \neq \emptyset$ and from $w \notin \cup\mathcal{H}_L$ it follows that $H \in \mathcal{H}(L, X)$. Thus, any hyperplane $H \in \mathcal{H}$ containing w satisfies $V_L^\perp \not\subset V_H$, hence $w + V_{L\mathbb{C}}^\perp \not\subset H$. It follows that $\cup\mathcal{H}$ has a non-empty complement in $w + V_{L\mathbb{C}}^\perp$. Hence if $\varphi \in \mathcal{M}(V_\mathbb{C}, \mathcal{H})$, then $\varphi^w \colon z \mapsto \varphi(w + z)$ is a meromorphic function on $V_{L\mathbb{C}}^\perp$. The germ $(\varphi^w)_0$ has its singular locus contained in the union of the hyperplanes $H_w := -w + (w + V_{L\mathbb{C}}^\perp) \cap H$, with $H \in \mathcal{H}, H \ni w$, hence with $H \in \mathcal{H}(L, X)$. We note that $w \in H$ implies $H_w = V_{L\mathbb{C}}^\perp \cap H$; the latter is an $X(L)$-hyperplane in $V_{L\mathbb{C}}^\perp$, containing 0. This proves (a).

(b) Let $\mathcal{L} \in \mathcal{M}(0, X(L))^*_{\rm laur}$ and put $u_\mathcal{L} = (u_d)_{d \in \mathbb{N}^{X(L)}}$. If d' is a map $\mathcal{H} \to \mathbb{N}$, then via the bijection $\mathcal{H}(L, X) \simeq X(L)$, we may identify $d'|\mathcal{H}(L, X)$ with an element $d \in \mathbb{N}^{X(L)}$. For $\varphi \in \mathcal{H}(V_\mathbb{C}, \mathcal{H}, d')$ we then have $\mathcal{L}_*\varphi(w) = u_d(\pi_{0,d}\varphi^w)(0)$. We now observe that $\pi_{0,d} = \pi_{0,X(L),d}$ equals the polynomial $q_{L,d'}$ defined in [11, Eqn. (1.5)]. Hence $\mathcal{L}_*$ is a Laurent operator from $\mathcal{M}(V_\mathbb{C}, \mathcal{H})$ to $\mathcal{M}(L, \mathcal{H}_L)$.

(c) From the reasoning in (b) we see that the element $u_\mathcal{L}$ of $S_\leftarrow(V_L^\perp, X(L))$ equals the element $u_{\mathcal{L}_*}$ corresponding to $\mathcal{L}_*$ under the isomorphism of (B.2). It follows that the map $\mathcal{L} \mapsto \mathcal{L}_*$ corresponds to the identity on $S_\leftarrow(V_L^\perp, X(L))$. In particular, it is an isomorphism from $\mathcal{M}(0, X(L)^*_{\rm laur}$ onto $\mathrm{Laur}(V_\mathbb{C}, L, \mathcal{H})$. $\square$

Remark B.4. In particular, we may apply the above lemma with $L = \{a\}$. Then $V_L^\perp = V$ and $X(L) = X$; hence for $\mathcal{H}$ an X-configuration containing $\mathcal{H}(a, X)$ we have $\mathcal{M}(0, X)^*_{\rm laur} \simeq \mathrm{Laur}(V_\mathbb{C}, \{a\}, \mathcal{H})$. Composing with the isomorphism T_{a*} discussed in Remark B.1 we obtain an isomorphism

$$\mathcal{M}(a, X)^*_{\rm laur} \simeq \mathrm{Laur}(V_\mathbb{C}, \{a\}, \mathcal{H}).$$

We have $\mathcal{H}_L = \emptyset$; hence $\mathcal{M}(L, \mathcal{H}_L) \simeq \mathbb{C}$ naturally via evaluation at a and we may identify $\mathrm{Laur}(V_\mathbb{C}, \{a\}, \mathcal{H})$ with a subspace of $\mathcal{M}(V_\mathbb{C}, \mathcal{H})^*$. If $\mathcal{L} \in \mathcal{M}(a, X)^*_{\rm laur}$, then the associated Laurent operator $\mathcal{L}_* \in \mathcal{M}(V_\mathbb{C}, \mathcal{H})^*$ is given by $\mathcal{L}_*(\varphi) = \mathcal{L}(\varphi_a)$.

Let $\mathcal{M}(*, X)^*_{\rm laur}$ denote the disjoint union of the spaces $\mathcal{M}(a, X)^*_{\rm laur}$, $a \in V_\mathbb{C}$. A map $\mathcal{L} \colon V_\mathbb{C} \to \mathcal{M}(*, X)^*_{\rm laur}$ with $\mathcal{L}_a := \mathcal{L}(a) \in \mathcal{M}(a, X)^*_{\rm laur}$ for all $a \in V_\mathbb{C}$ is called a section of $\mathcal{M}(*, X)^*_{\rm laur}$. The closure of the set $\{a \in V_\mathbb{C} \mid \mathcal{L}_a \neq 0\}$ is called the support of $\mathcal{L}$, and denoted by $\mathrm{supp}(\mathcal{L})$. A finitely supported section of $\mathcal{M}(*, X)^*_{\rm laur}$ is called an X-Laurent functional on $V_\mathbb{C}$. The space of such Laurent functionals is denoted by $\mathcal{M}(V_\mathbb{C}, X)^*_{\rm laur}$. If S is a subset of $V_\mathbb{C}$, we put

$$\mathcal{M}(S, X)^*_{\rm laur} = \{\mathcal{L} \in \mathcal{M}(V_\mathbb{C}, X)^*_{\rm laur} \mid \mathrm{supp}\,\mathcal{L} \subset S\},$$

and call this the space of X-Laurent functionals supported on S. If Ω is an open subset of $V_\mathbb{C}$, then by $\mathcal{M}(\Omega)$ we denote the ring of meromorphic functions on Ω. Moreover, if $a \in \Omega$, then by $\mathcal{M}(\Omega, a, X)$ we denote the subring of those $\varphi \in \mathcal{M}(\Omega)$ whose germ φ_a at a belongs to $\mathcal{M}(a, X)$. If $S \subset \Omega$, we define

$$\mathcal{M}(\Omega, S, X) := \cap_{a \in S}\mathcal{M}(\Omega, a, X).$$

Finally, we write $\mathcal{M}(\Omega, X)$ for $\mathcal{M}(\Omega, \Omega, X)$. In particular, $\mathcal{M}(V_{\mathbb{C}}, X)$ is the ring of meromorphic functions whose singular locus is contained in the union of an X-configuration.

There is a natural pairing $\mathcal{M}(S, X)^*_{\mathrm{laur}} \times \mathcal{M}(\Omega, S, X) \to \mathbb{C}$, given by

$$\mathcal{L}\varphi = \sum_{a \in \operatorname{supp} \mathcal{L}} \mathcal{L}_a \varphi_a.$$

The pairing naturally induces a linear map $\mathcal{M}(S, X)^*_{\mathrm{laur}} \to \mathcal{M}(\Omega, S, X)^*$ which is injective; however, we will not need this injectivity here.

If E is a finite dimensional complex linear space, $\psi \in \mathcal{M}(\Omega, S, X) \otimes E$ and $\mathcal{L} \in \mathcal{M}(S, X)^*_{\mathrm{laur}}$, then we shall write $\mathcal{L}\psi$ for $(\mathcal{L} \otimes I_E)\psi$.

Now assume that $L \in \mathcal{A}$, and let the sets $X(L) \subset V_L^\perp$ and $X_L \subset L$ be as defined in Remark B.1.

Lemma B.5. *Let $\mathcal{L}$ be an $X(L)$-Laurent functional on $V_{L\mathbb{C}}^\perp$, and let $\varphi \in \mathcal{M}(V_{\mathbb{C}}, X)$. Then for w in the complement of an X_L-configuration in L, the function $z \mapsto \varphi(w + z)$ belongs to $\mathcal{M}(V_{L\mathbb{C}}^\perp, \operatorname{supp} \mathcal{L}, X(L))$. Moreover, the function*

$$\mathcal{L}_*\varphi \colon w \mapsto \mathcal{L}(\varphi(w + \cdot\,))$$

belongs to the space $\mathcal{M}(L, X_L)$.

Proof. It suffices to prove the assertions for a Laurent functional $\mathcal{L}$ whose support consists of a single point $a \in V_{L\mathbb{C}}^\perp$. Composing $\mathcal{L}$ with a translation if necessary, we may as well assume that $a = 0$ (use Remark B.1).

Let $\varphi \in \mathcal{M}(V_{\mathbb{C}}, X)$. Then there exists an X-configuration $\mathcal{H}$ in $V_{\mathbb{C}}$ containing $\mathcal{H}(L, X)$, such that $\varphi \in \mathcal{M}(V_{\mathbb{C}}, \mathcal{H})$. All assertions now follow from Lemma B.3. $\square$

We now specialize to the setting of a reductive symmetric space. We take $V = \mathfrak{a}_{\mathrm{q}}^*$ and $X = \bar{\Sigma}^+$, the set of indivisible roots in Σ^+. The space $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \bar{\Sigma}^+)$ is denoted by $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \Sigma)$. Moreover, with notation as in Section 8, let $F \subset \Delta$ and let $\Sigma_F := \Sigma \cap \mathfrak{a}_{F\mathrm{q}}^{*\perp}$ denote the set of roots of $\mathfrak{a}_{F\mathrm{q}}^\perp$ in $\mathfrak{m}_F$. Note that if $\lambda \in \mathfrak{a}_{F\mathrm{q}}^{*\perp}$ and $L = \lambda + \mathfrak{a}_{F\mathrm{q}\mathbb{C}}^*$, then $V_L^\perp = \mathfrak{a}_{F\mathrm{q}}^{*\perp}$ and $X(L)$ equals the set $\bar{\Sigma}_F^+$ of indivisible roots in Σ_F^+. By a Σ_F-Laurent functional on $\mathfrak{a}_{F\mathrm{q}\mathbb{C}}^{*\perp}$ we mean a $\bar{\Sigma}_F^+$-Laurent functional on $\mathfrak{a}_{F\mathrm{q}\mathbb{C}}^{*\perp}$.

The following theorem is proved in [12]. Its displayed equations concern equalities between meromorphic functions, in view of Lemma B.5.

Theorem B.6. *Let $v \in {}^F\mathcal{W}$. Let $\mathcal{L}_1, \mathcal{L}_2$ be Σ_F-Laurent functionals on $\mathfrak{a}_{F\mathrm{q}\mathbb{C}}^{*\perp}$ and let $\phi_1, \phi_2 \in \mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \Sigma) \otimes {}^\circ\mathcal{C}_{F,v}$. Assume that*

$$\mathcal{L}_1(E^\circ(X_{F,v}\colon \cdot : m)\phi_1(\nu + \cdot\,)) = \mathcal{L}_2(E_+(X_{F,v}\colon \cdot : m)\phi_2(\nu + \cdot\,))$$

for all $m \in X_{F,v,+}$ and generic $\nu \in \mathfrak{a}_{F\mathrm{q}\mathbb{C}}^*$. Define $\psi_i = (I \otimes \mathrm{i}_{F,v})\phi_i \in \mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \Sigma) \otimes {}^\circ\mathcal{C}$ for $i = 1, 2$. Then, for every $x \in X_+$,

$$\mathcal{L}_1(E^\circ(\nu + \cdot : x)\psi_1(\nu + \cdot\,)) = \mathcal{L}_2\Big(\sum_{s \in W^F} E_{+,s}(\nu + \cdot : x)\psi_2(\nu + \cdot\,)\Big), \tag{B.3}$$

as an identity of V_τ-valued meromorphic functions in the variable $\nu \in \mathfrak{a}_{F\mathrm{q}\mathbb{C}}^*$.

The following result is a dual version of the above theorem.

Corollary B.7. *Let $v \in {}^F\mathcal{W}$. Let $\mathcal{L}_1, \mathcal{L}_2$ be Σ_F-Laurent functionals on $\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}$, and let $\phi_1, \phi_2 \in \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \Sigma) \otimes ({}^\circ\mathcal{C}_{F,v})^*$. Assume that*

$$\mathcal{L}_1(\phi_1(\nu + \,\cdot\,)E^*(X_{F,v}\colon \cdot : m)) = \mathcal{L}_2(\phi_2(\nu + \,\cdot\,)E^*_+(X_{F,v}\colon \cdot : m)) \tag{B.4}$$

for all $m \in X_{F,v,+}$ and generic $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$. Define $\psi_i = (I \otimes \mathrm{pr}^*_{F,v})\phi_i \in \mathcal{M}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \Sigma) \otimes {}^\circ\mathcal{C}^*$ for $i = 1, 2$. Then, for every $x \in X_+$,

$$\mathcal{L}_1(\psi_1(\nu + \,\cdot\,)E^*(\nu + \,\cdot : x)) = \mathcal{L}_2(\psi_2(\nu + \,\cdot\,) \sum_{s \in W^F} E^*_{+,s}(\nu + \,\cdot : x)), \tag{B.5}$$

as an identity of V^*_τ-valued meromorphic functions in the variable $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$.

Proof. We prove this corollary by dualization of Theorem B.6.

If $\psi \in \mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F)$, then the function $\psi^\vee\colon \lambda \mapsto \overline{\psi(-\bar{\lambda})}$ is readily seen to belong to $\mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F)$ as well. If $\mathcal{L}$ is a Σ_F-Laurent functional on $\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}$, then there is a unique Σ_F-Laurent functional $\mathcal{L}^\vee$ on $\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}$ such that

$$\mathcal{L}^\vee(\psi^\vee) = (\mathcal{L}(\psi))^*, \tag{B.6}$$

where the star denotes conjugation of a complex number. If H is a finite dimensional complex Hilbert space, then we shall use the following notation. If $v \in \mathrm{H}$, then by v^* we denote the element of the dual Hilbert space H^* determined by $v^*(w) = \langle w | v \rangle$, for $w \in \mathrm{H}$. If $\psi \in \mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F) \otimes \mathrm{H}$, then we define the function $\psi^\vee \in \mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F) \otimes \mathrm{H}^*$ by

$$\psi^\vee(\lambda) = \psi(-\bar{\lambda})^*.$$

With this notation, equation (B.6) still holds if $\mathcal{L}$ is a Σ_F-Laurent functional on $\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}$ and if $\psi \in \mathcal{M}(\mathfrak{a}^{*\perp}_{F\mathrm{q}\mathbb{C}}, \Sigma_F) \otimes \mathrm{H}$.

Let now $\mathcal{L}_1, \mathcal{L}_2, \phi_1, \phi_2$ be as in the corollary. Then replacing ν by $-\bar{\nu}$ in (B.4) and applying a star to both sides of the resulting equation, we obtain that

$$\mathcal{L}_1^\vee(E^\circ(X_{F,v}\colon \cdot : m)\phi_1^\vee(\nu + \,\cdot\,)) = \mathcal{L}_2^\vee(E_+(X_{F,v}\colon \cdot : m)\phi_2^\vee(\nu + \,\cdot\,)),$$

for all $m \in X_{F,v,+}$ and generic $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$. Applying Theorem B.6 with $\mathcal{L}_i^\vee, \phi_i^\vee$ in place of $\mathcal{L}_i, \phi_i$, respectively, we then obtain, for all $x \in X_+$, that

$$\mathcal{L}_1^\vee(E^\circ(\nu + \cdot : x)\, \mathrm{i}_{F,v}\, \phi_1^\vee(\nu + \,\cdot)) = \mathcal{L}_2^\vee(\sum_{s \in W^F} E_{s,+}(\nu + \cdot : x)\, \mathrm{i}_{F,v}\, \phi_2^\vee(\nu + \,\cdot)) \tag{B.7}$$

as a meromorphic identity in ν. We now observe that

$$[\mathrm{i}_{F,v}(\phi_i^\vee(\mu))]^* = \mathrm{pr}^*_{F,v}(\phi_i(-\bar{\mu})) = \psi_i(-\bar{\mu}).$$

Thus, applying a star to both sides of (B.7) and inserting $-\bar{\nu}$ for ν we obtain (B.5), for all $x \in X_+$ and for generic ν. Since both members of (B.5) are meromorphic functions of ν, by Lemma B.5, equation (B.5) holds as an identity of meromorphic functions. $\square$

Remark B.8. The above results have two features that are worthwhile noting explicitly. First of all, the results enable us to extend certain sums of 'partial' Eisenstein integrals to smooth functions on all of X. Indeed, a priori the expression on the right-hand side of equation (B.3) is only defined for $x \in X_+$. However, the expression on the left-hand side of the equation is a smooth function of $x \in X$.

Secondly the above results are also of interest if $\mathcal{L}_2 = 0$. In that case the statements amount to asserting that the Eisenstein integrals satisfy relations of a particular type, if certain leading coefficients in their expansions along the wall $A^+_{F\mathrm{q}}v$ satisfy these relations. The title of this section is motivated by the well known fact that taking such leading coefficients essentially inverts the procedure of parabolic induction.

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