ON TWO-PRIMARY ALGEBRAIC K-THEORY OF QUADRATIC NUMBER RINGS WITH FOCUS ON K_2

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ABSTRACT. We give explicit formulas for the 2-rank of the algebraic K-groups of quadratic number rings. A 4-rank formula for K_2 of quadratic number rings given in [1] provides further information about the actual group structure. The K_2 calculations are based on 2- and 4-rank formulas for Picard groups of quadratic number fields. These formulas are derived in a completely elementary way from the classical 2-rank formula for the narrow ideal class group of a quadratic number field. We also lift the K_2 results to higher algebraic K-groups.

1. INTRODUCTION

This paper deals with the two-primary algebraic K-theory of quadratic number rings. The problem is to determine the torsion part of these groups. In [15] Quillen proved that the algebraic K-groups of number rings are finitely generated Abelian groups, and Borel calculated their rank in [2].

To determine the torsion part of the algebraic K-groups of number rings is a very hard problem. One knows quite a lot about the K_2-functor, but still it is a highly nontrivial problem to determine the K_2-groups. A major early contribution to the theory is Tate's work in [17] that gives connections between K_2 and Galois cohomology. In fact p-rank formulas for K_2 of number rings can be quite easily deduced from results in [17]. These formulas involve classical number theory like ideal class groups and the number of primes lying above a rational prime. Unfortunately p is in general restricted to be a prime number. The situation is a bit better when it comes to quadratic number rings. In that case there are 4-rank formulas for K_2 given in terms of classical number theoretic invariants.

In this paper we describe a method to determine pieces of the two-primary part of K_2 of a quadratic number ring. There is a tight connection between the ordinary and the narrow K_2. This connection will be explained in the calculations. To the authors knowledge, the latest and most complete calculation of K_2 of quadratic number rings can be found in [13] and [14]. We would also like to mention the recent papers [18] and [19]. Further a large table of conjectural calculations is given in [3]. Our method can be used to capture the previous calculations as well. Furthermore we lift the K_2 calculations to higher even algebraic K-groups. This is done using the étale cohomology description of the two-primary algebraic K-theory of number rings from [16].

For the rest of the paper we fix the following notation:

-F is a number field with number ring $\mathcal{O}_F$

-R_F denotes the ring of two-integers of F

-$\triangle_F$ denotes the discriminant of F

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-s is the number of dyadic primes in F, i.e., the number of primes lying above 2 in F

-t is the number of distinct prime divisors of $\triangle_F$

-r_1 is the number of real embeddings of F

-d denotes a squarefree integer

-Pic denotes the Picard group and Pic_+ denotes the narrow Picard group

-$A\{n\}$ denotes the n-primary part of the Abelian group A and $rk_n(A)$ the n-rank of A, where n is a prime power

-C_n is the cyclic group of order n

The following result concerns the algebraic K-theory of R_F. However, the fibre sequence of spectra

$$\bigvee_{\wp|2} K(k_\wp) \longrightarrow K(\mathcal{O}_F) \longrightarrow K(R_F)$$

implies that the two-primary parts of the algebraic K-groups of $\mathcal{O}_F$ and R_F are isomorphic from degree 2 and on. This is an immediate consequence of the long exact sequence of two-completed algebraic K-groups induced by the fibre sequence above, and Quillen's calculation of algebraic K-theory of finite fields.

We recall Theorem 7.11 in [16]. We alert the reader that t denotes $rk_2Pic(R_F)$ and u denotes $rk_2Pic_+(R_F)$ in [16]. We make this result explicit for quadratic number fields in Theorem 3.1.

Theorem 1.1 (Rognes-Weibel). *Let $n \geq 1$. The number of even cyclic summands in the algebraic K-groups of R_F are given by*

a) If F is totally imaginary, then

$$rk_2K_n(R_F) = \begin{cases} s + rk_2Pic(R_F) - 1 & \text{for } n \text{ even,} \\ 1 & \text{for } n \text{ odd.} \end{cases}$$

b) If F is real, then

$$rk_2K_n(R_F) = \begin{cases} s + rk_2Pic(R_F) - 1 & \text{for } n = 8i, \\ 1 & \text{for } n = 8i+1, \\ r_1 + s + rk_2Pic(R_F) - 1 & \text{for } n = 8i+2, \\ r_1 & \text{for } n = 8i+3, \\ s + rk_2Pic_+(R_F) - 1 & \text{for } n = 8i+4, \\ 0 & \text{for } n = 8i+5, \\ s + rk_2Pic_+(R_F) - 1 & \text{for } n = 8i+6, \\ 1 & \text{for } n = 8i+7. \end{cases}$$

Now a quick outline of the paper. Fix $F = \mathbb{Q}(\sqrt{d})$ and $\overline{F} = \mathbb{Q}(\sqrt{-d})$.

In Section 2 we compute $rk_2Pic(R_F)$ and $rk_2Pic_+(R_F)$. Further we compare $rk_4Pic_+(\mathcal{O}_F)$ and $rk_4Pic_+(R_F)$. Here the number $rk_4Pic_+(\mathcal{O}_F)$ can be computed using a matrix M_F described in Subsection 3.2. The number $rk_4Pic_+(R_F)$ occurs in 4-rank formulas for both $K_2^+(R_F)$ and $K_2(R_F)$ given in [1], see Theorems 3.3 and 3.4 in this paper. From the comparison of $rk_4Pic_+(\mathcal{O}_F)$ and $rk_4Pic_+(R_F)$ we compute in Proposition 3.7 the 4-rank of $K_2^+(R_F)$ in terms of $rk_4Pic_+(\mathcal{O}_{\overline{F}})$. The 4-rank of $K_2(R_F)$ can then be found using Theorems 2.9, 3.3 and 3.4.

In Subsection 3.3 we give several examples of two-primary calculations of $K_2(R_F)$. In some cases we can completely determine the two-primary part of $K_2(R_F)$. To obtain more information on $K_2(R_F)\{2\}$ one would need a 2^ν-rank formula for $K_2(R_F)$ where $\nu \geq 3$.

In Subsection 3.4 we use the étale cohomology description of the two-primary algebraic K-theory of R_F given in [16] to lift the K_2 calculations to higher even algebraic K-groups of R_F. The main result is

Theorem 1.2. *Let F be any number field. Let e be the exponent of $(\mathbb{Z}/2^\nu)^*$ where $\nu \geq 2$ if F is a real number field. Then*

$$K_2(R_F)/2^\nu \cong H^2_{\acute{e}t}(R_F; \mathbb{Z}_2(e+2))/2^\nu.$$

We include an overview of the sections.

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2. Two and four rank formulas for quadratic Picard groups

In this section we will give 2-rank formulas for "quadratic Picard groups." These results will be vital when we turn to the two-primary algebraic K-theory of quadratic number rings. First we set the notation and recall some classical results.

Let S denote a finite set of primes in F including the infinite ones, and $\mathcal{O}_{F,S}$ the ring of S-integers in F. Let $Pic_+(\mathcal{O}_{F,S})$ denote the narrow Picard group of the S-integers and $Pic(\mathcal{O}_{F,S})$ the usual Picard group of $\mathcal{O}_{F,S}$. These groups come with a natural surjective map

$$\iota_S : Pic_+(\mathcal{O}_{F,S}) \longrightarrow Pic(\mathcal{O}_{F,S}).$$

For a fractional ideal $\mathfrak{a}$ in a number field F we let $[\mathfrak{a}]$ denote the class of $\mathfrak{a}$ in $Pic(\mathcal{O}_{F,S})$ while $[\mathfrak{a}]_+$ denotes the class of $\mathfrak{a}$ in $Pic_+(\mathcal{O}_{F,S})$. When S contains no finite primes then we obtain the ordinary and the narrow ideal class group. The notation will be $Pic(\mathcal{O}_F)$ and $Pic_+(\mathcal{O}_F)$, respectively. In this case we denote ι_S with ι. We mention two classical results.

Lemma 2.1. *Let F be a real quadratic number field with fundamental unit ϵ. Then*

$$ker\ \iota \cong \begin{cases} C_2 & \text{if } \boldsymbol{N}_{F/\mathbb{Q}}(\epsilon) = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. We refer to [6] p.181. ∎

For quadratic number fields, the 2-rank of $Pic_+(\mathcal{O}_F)$ was known to Gauss.

Theorem 2.2 (Gauss). *Let F be a quadratic number field. Then*

$$rk_2 Pic_+(\mathcal{O}_F) = t - 1.$$

Proof. A short and elegant proof can be found in [7]. □

2.1. **The 2-rank of the ideal class group of quadratic number fields.** As an application of the classical 2-rank formula for $rk_2 Pic_+(\mathcal{O}_F)$ we will now find the 2-rank of $Pic(\mathcal{O}_F)$. First we make some preparations.

Definition 2.3. *Let k be an integer. We define*

$$M_k := \{d \in \mathbb{Z} \mid d \textit{ is squarefree and } k \textit{ is a norm from } F = \mathbb{Q}(\sqrt{d})\}.$$

Lemma 2.4. *We have*:

$M_{-1} = \{d \in \mathbb{Z} \mid d > 0, p \equiv 1 \bmod 4$ *for every odd prime* p *dividing* $d\}$

$M_{-2} = \{d \in \mathbb{Z} \mid d > 0, p \equiv 1$ *or* $3 \bmod 8$ *for every odd prime* p *dividing* $d\}$

$M_{2} = \{d \in \mathbb{Z} \mid p \equiv 1$ *or* $7 \bmod 8$ *for every odd prime* p *dividing* $d\}$

Proof. For basic properties of Hilbert symbols we refer to [12]. The product formula for Hilbert symbols tells us that

$$\prod_{\wp} (k,d)_{\wp} = 1$$

where $\wp$ runs over the finite and infinite primes of $\mathbb{Q}$, and k is an integer as in Definition 2.3. The celebrated Hasse norm theorem in [12] p.89 reads as

$$d \in M_k \Longleftrightarrow (k,d)_{\wp} = 1 \text{ for all primes } \wp \text{ of } \mathbb{Q}.$$

Hence from the product formula; $d \in M_k$ only if $(k,d)_{\wp} = 1$ for all finite and infinite primes $\wp$ of $\mathbb{Q}$ except 2. Let ν_∞ be the infinite prime in $\mathbb{Q}$. Then $(k,d)_{\nu_\infty}$ equals -1 if and only if both k and d are negative. Furthermore for an odd rational prime p

$$(k,d)_p = \begin{cases} (\frac{k}{p}) & \text{if } p \mid d \text{ and } p \nmid k, \\ (\frac{d}{p}) & \text{if } p \nmid d \text{ and } p \mid k, \\ (\frac{-kd/p^2}{p}) & \text{if } p \mid d \text{ and } p \mid k, \\ 1 & \text{if } p \nmid d \text{ and } p \nmid k. \end{cases}$$

Thus if k has no odd prime divisors (e.g., $k = -1, -2$ or 2), then

$$d \in M_k \Longleftrightarrow (k > 0 \text{ or } d > 0) \text{ and } ((\frac{k}{p}) = 1 \text{ for all odd primes } p \text{ dividing } d).$$

We easily read off the result from the well known formulas $(\frac{-1}{p}) = (-1)^{\frac{p-1}{2}}$ and $(\frac{2}{p}) = (-1)^{\frac{p^2-1}{8}}$. □

We will need the following elementary group theoretic result.

Lemma 2.5. *For a finite Abelian group G and $x \in G$*

$$rk_2(G/\langle x\rangle) = \begin{cases} rk_2(G) & \textit{if } x \in G^2, \\ rk_2(G) - 1 & \textit{otherwise.} \end{cases}$$

Proof. Say x generates a subgroup of order n, $\langle x\rangle = C_n$. Further let G' denote the quotient group, i.e., we have the exact sequence $0 \longrightarrow C_n \longrightarrow G \longrightarrow G' \longrightarrow 0$ inducing the exact sequence $C_n/2 \longrightarrow G/2 \longrightarrow G'/2 \longrightarrow 0$.

There are two cases to consider.

1) Suppose that $2 \nmid n$. Now $C_n^2 = C_n$, and hence $C_n/2 = 0$. Thus $x \in G^2$ and $rk_2(G) = rk_2(G')$.

2) Otherwise $2 \mid n$. Now $C_n/2 = C_2$ and the sequence $C_2 \xrightarrow{\theta} G/2 \longrightarrow G'/2 \longrightarrow 0$ is exact. If $x \in G^2$, then θ must be trivial and $rk_2(G) = rk_2(G')$. If $x \notin G^2$, then θ is injective. Hence $0 \longrightarrow C_2 \longrightarrow G/2 \longrightarrow G'/2 \longrightarrow 0$ is exact, and $\sharp(G/2) = 2\sharp(G'/2)$.

□

The next lemma will be used several times. To avoid possible confusion we adopt the notation that $\mathbf{N}$ denotes the ideal norm, while $\mathbf{N}_{F/\mathbb{Q}}$ denotes the element norm from F to $\mathbb{Q}$. For the connection between them, see p.57 in [6]. First we note the following easy observation; a norm 1 fractional ideal can be written on the form $\mathfrak{c}/\bar{\mathfrak{c}}$ where $\bar{\mathfrak{c}}$ denotes the Galois conjugate of $\mathfrak{c}$. Here $\mathfrak{c}$ equals a product of primes lying above split rational primes in $\mathbb{Q}$. It is easy to prove this. Start with a norm 1 fractional ideal, say $\mathfrak{a}$. Write $\mathfrak{a} = \frac{\mathfrak{b}}{\mathfrak{c}}$ where $\mathfrak{b}$ and $\mathfrak{c}$ are coprime integral $\mathcal{O}_F$-ideals. If $\wp$ is a prime ideal that divides $\mathfrak{b}$, then there must exist a prime ideal $\overline{\wp}$ that divides $\mathfrak{c}$ and with the same norm as $\wp$. Since $(\mathfrak{b},\mathfrak{c}) = 1$ we know that $\wp \neq \overline{\wp}$. Hence working in a quadratic number field we conclude that $\wp$ and $\overline{\wp}$ are conjugate.

Lemma 2.6. *For a fractional ideal* $\mathfrak{a}$ *of* $F = \mathbb{Q}(\sqrt{d})$ *the following hold*:

$$[\mathfrak{a}] \in Pic(\mathcal{O}_F)^2 \Longleftrightarrow \boldsymbol{N}(\mathfrak{a}) \text{ or } -\boldsymbol{N}(\mathfrak{a}) \text{ is a norm from } F$$

$$[\mathfrak{a}]_+ \in Pic_+(\mathcal{O}_F)^2 \Longleftrightarrow \boldsymbol{N}(\mathfrak{a}) \text{ is a norm from } F^+$$

Proof. $\Longrightarrow$: Suppose that $[\mathfrak{a}] \in Pic(\mathcal{O}_F)^2$, respectively $[\mathfrak{a}]_+ \in Pic_+(\mathcal{O}_F)^2$. Then we can find an element $\alpha \in F^*$, respectively $\alpha \in F^+$, and a fractional ideal $\mathfrak{b}$ such that $\mathfrak{a} = (\alpha)\mathfrak{b}^2$. We derive $|\mathbf{N}_{F/\mathbb{Q}}(\alpha\mathbf{N}(\mathfrak{b}))| = |\mathbf{N}_{F/\mathbb{Q}}(\alpha)|\mathbf{N}(\mathfrak{b}^2) = \mathbf{N}((\alpha)\mathfrak{b}^2) = \mathbf{N}(\mathfrak{a})$. For the narrow case we have $\mathbf{N}_{F/\mathbb{Q}}(\alpha\mathbf{N}(\mathfrak{b})) > 0$, so we cannot have a minus sign.

$\Longleftarrow$: Say $\beta \in F^*$, respectively $\beta \in F^+$, with $\mathbf{N}_{F/\mathbb{Q}}(\beta) = \pm\mathbf{N}(\mathfrak{a})$ (no minus sign in the narrow case). Consider the fractional ideal $\mathfrak{b} = (\beta)^{-1}\mathfrak{a}$. Clearly $\mathfrak{a}$ and $\mathfrak{b}$ represent the same class in both the narrow and the usual ideal class group. Furthermore

$$\mathbf{N}(\mathfrak{b}) = \frac{\mathbf{N}(\mathfrak{a})}{|\mathbf{N}_{F/\mathbb{Q}}(\beta)|} = 1.$$

We know that a norm one fractional ideal can be written on the form $\mathfrak{d}/\bar{\mathfrak{d}}$ where $\bar{\mathfrak{d}}$ denotes the Galois conjugate of $\mathfrak{d}$. Further $\mathfrak{d}\bar{\mathfrak{d}}$ is a principal fractional ideal representing the trivial class in both the narrow and the usual ideal class group. It follows that

$$[\mathfrak{a}] = [\mathfrak{b}] = \left[\frac{\mathfrak{d}^2}{\mathfrak{d}\bar{\mathfrak{d}}}\right] = [\mathfrak{d}]^2.$$

Of course the same holds for the narrow classes as well.

□

Now we have all the results we need in order to find $rk_2Pic(\mathcal{O}_F)$.

Theorem 2.7. *For* $F = \mathbb{Q}(\sqrt{d})$ *we have*

$$rk_2Pic(\mathcal{O}_F) = \begin{cases} t-1 & \textit{if } d<0 \textit{ or} \\ & (d>0 \textit{ and } p \equiv 1 \bmod 4 \textit{ for all odd primes } p \mid d), \\ t-2 & \textit{if } d>0 \textit{ and } p\equiv 3 \bmod 4 \textit{ for some odd prime } p \mid d. \end{cases}$$

Proof. We may assume $d > 0$, and due to Lemma 2.4 it suffices to show that

$$rk_2Pic(\mathcal{O}_F) = \begin{cases} t-1 & \text{if } -1 \text{ is a norm from } F, \\ t-2 & \text{otherwise.} \end{cases}$$

The sequence $0 \longrightarrow ker\, \iota \longrightarrow Pic_+(\mathcal{O}_F) \longrightarrow Pic(\mathcal{O}_F) \longrightarrow 0$ is exact, where $ker\, \iota$ is known from Lemma 2.1. This gives us the exact sequence $ker\, \iota/2 \longrightarrow Pic_+(\mathcal{O}_F)/2 \longrightarrow Pic(\mathcal{O}_F)/2 \longrightarrow 0$. Clearly $ker\, \iota$ is generated by some $[(a)]_+$ where a has negative norm. From Lemma 2.5 we are reduced to prove that

$$[(a)]_+ \in Pic_+(\mathcal{O}_F)^2 \Longleftrightarrow d \in M_{-1}$$

for any element $a \in F^*$ with negative norm.

$\Longrightarrow$: Say $[(a)]_+ = [\mathfrak{a}]_+^2$, i.e., $(a) = (\alpha)\mathfrak{a}^2$ for some $\alpha \in F^+$. We get

$$\mathbf{N}((a)) = \mathbf{N}((\alpha))\mathbf{N}(\mathfrak{a})^2, \text{ i.e., } -\mathbf{N}_{F/\mathbb{Q}}(a) = \mathbf{N}_{F/\mathbb{Q}}(\alpha)\mathbf{N}(\mathfrak{a})^2$$

and hence

$$-1 = \mathbf{N}_{F/\mathbb{Q}}(\frac{\alpha}{a}\mathbf{N}(\mathfrak{a})) \in \mathbf{N}_{F/\mathbb{Q}}(F).$$

$\Longleftarrow$: Choose $\alpha \in F^*$ where $\mathbf{N}_{F/\mathbb{Q}}(\alpha) = -1$, i.e., $\mathbf{N}((\alpha)) = 1$. The rest of the argument can be found in the proof of Lemma 2.6.

□

Remark 2.8. *The short exact sequence*

$$0 \longrightarrow ker\, \iota \longrightarrow Pic_+(\mathcal{O}_F) \xrightarrow{\iota} Pic(\mathcal{O}_F) \longrightarrow 0$$

is in general not a split exact sequence. One example is $F = \mathbb{Q}(\sqrt{34})$. *In this case* $\mathbf{N}_{F/\mathbb{Q}}(\epsilon) = 1$ *where* $\epsilon = 35+6\sqrt{34}$ *denotes the fundamental unit. Hence we have that* $ker\, \iota \cong C_2$. *From Theorem 2.7 we see that* $rk_2Pic(\mathcal{O}_{\mathbb{Q}(\sqrt{34})}) = rk_2Pic_+(\mathcal{O}_{\mathbb{Q}(\sqrt{34})}) = 1$. *In fact the ordinary ideal class group is cyclic of order* 2, *so the narrow ideal class group is cyclic of order* 4.

2.2. **The 2-rank of Pic and Pic_+ for the two-integers in quadratic number fields.** We will now see what happens with the 2-rank of the narrow and the usual Picard group of the two-integers of $F = \mathbb{Q}(\sqrt{d})$.

Theorem 2.9. *For* $F = \mathbb{Q}(\sqrt{d})$ *we have*

$$rk_2Pic(R_F) = \begin{cases} rk_2Pic(\mathcal{O}_F) & \textit{if } d \equiv 5 \bmod 8 \textit{ or } d \in M_{-2} \cup M_2, \\ rk_2Pic(\mathcal{O}_F) - 1 & \textit{otherwise,} \end{cases}$$

$$rk_2Pic_+(R_F) = \begin{cases} rk_2Pic_+(\mathcal{O}_F) & \textit{if } d \equiv 5 \bmod 8 \textit{ or } d \in M_2, \\ rk_2Pic_+(\mathcal{O}_F) - 1 & \textit{otherwise.} \end{cases}$$

Proof. The crucial thing here is of course the factorization of 2 in F. There are three possibilities:

1) If $d \equiv 1 \bmod 8$, then 2 splits

2) If $d \equiv 3 \bmod 4$ or $2 \mid d$, then 2 ramifies

3) If $d \equiv 5 \bmod 8$, then 2 is inert

In the cases where 2 is inert, i.e., when $d \equiv 5 \bmod 8$ there will be no effect on neither the narrow nor the ordinary ideal class group if we invert 2. Hence we may assume that the residue field index of 2 is not two, so $\mathbf{N}(\wp) = 2$ for $\wp$ a dyadic prime. Clearly $Pic(R_F) = Pic(\mathcal{O}_F)/\langle[\wp]\rangle$ for $\wp$ a dyadic prime of F. The situation for the narrow ideal class group is identical to this result, i.e., $Pic_+(R_F) = Pic_+(\mathcal{O}_F)/\langle[\wp]_+\rangle$. We are left to prove that

$$[\wp] \in Pic(\mathcal{O}_F)^2 \iff \pm 2 \in \mathbf{N}_{F/\mathbb{Q}}(F) \text{ (only the plus sign in the narrow case).}$$

Now mimic the proof of Lemma 2.6 and Theorem 2.7. □

Later we will need the following explicit description of $rk_2Pic(R_F)$.

Corollary 2.10. *For $F = \mathbb{Q}(\sqrt{d})$ we have the following calculation.*

1) $rk_2Pic(R_F) = t - 1$ *in the cases*

$d < 0$, $d \equiv 5 \bmod 8$.

$d < 0$, $p \equiv 1$ *or* $7 \bmod 8$ *for every odd prime p dividing d.*

$d > 0$, $d \equiv 5 \bmod 8$ *and* $p \equiv 1 \bmod 4$ *for every odd prime p dividing d.*

$d > 0$, $p \equiv 1 \bmod 8$ *for every odd prime p dividing d.*

2) $rk_2Pic(R_F) = t - 3$ *in the cases*

$d > 0$, $d \equiv 1, 2, 3, 6$ *or* $7 \bmod 8$ *and there exists prime numbers p_1 and p_2 dividing d where*

$$p_1 \equiv 5 \bmod 8 \text{ and } p_2 \equiv 3 \bmod 4.$$

3) $rk_2Pic(R_F) = t - 2$ *in all other cases.*

Proof. We compare Theorems 2.7 and 2.9. There are two cases to consider.

1) The case $rk_2Pic(R_F) = rk_2Pic(\mathcal{O}_F) = t - 1$. If F is imaginary, then $d \equiv 5 \bmod 8$ or $d \in M_2$. Lemma 2.4 gives the claimed restriction on d. For the real case we have $d \equiv 5 \bmod 8$ or $d \in M_{-2} \cup M_2$, while $p \equiv 1 \bmod 4$ for every odd prime p dividing d. If $d \in M_{-2}$, then $p \equiv 1$ or $3 \bmod 8$ for every odd prime p dividing d. If $d \in M_2$, then $p \equiv 1$ or $7 \bmod 8$ for every odd prime p dividing d.

2) The case $rk_2Pic(R_F) = t-3$. From the table of $rk_2Pic(\mathcal{O}_F)$; d is positive and $d \notin M_{-1}$, i.e., there exists an odd prime number congruent 3 modulo 4 dividing d. The "otherwise" condition in the table for $rk_2Pic(R_F)$ is that $d \equiv 1, 2, 3, 6$ or $7 \bmod 8$ and that there exists an odd prime number dividing d being congruent 5 modulo 8.

□

A straightforward comparison of Theorem 2.2 and Theorem 2.9 yields

Corollary 2.11. *We have the following formula for the 2-rank of the narrow Picard group of the 2-integers in $F = \mathbb{Q}(\sqrt{d})$*

$$rk_2 Pic_+(R_F) = \begin{cases} t-1 & \textit{if } d \equiv 5 \bmod 8 \textit{ or } d \in M_2, \\ t-2 & \textit{otherwise.} \end{cases}$$

We also note the following two results.

Corollary 2.12. *Let $F = \mathbb{Q}(\sqrt{d})$. We have $rk_2 Pic(R_F) = 0$ in precisely the following cases:*

d	*with the condition*
$-1, \pm 2$	
$-p$	$p \equiv 3 \bmod 4$ *or* $p \equiv 5 \bmod 8$
$-2p$	$p \equiv 3 \bmod 8$ *or* $p \equiv 5 \bmod 8$
$-p_1p_2$	$p_1 \equiv 3 \bmod 8$ *and* $p_2 \equiv 5 \bmod 8$
p	p *is odd*
$2p$	$p \equiv 3 \bmod 4$ *or* $p \equiv 5 \bmod 8$
p_1p_2	$p_1 \equiv 5 \bmod 8$ *and* $p_2 \equiv 3 \bmod 4$
$2p_1p_2$	$p_1 \equiv 5 \bmod 8$ *and* $p_2 \equiv 3 \bmod 4$
p_1p_2	$p_1 \equiv 3 \bmod 4$ *and* $p_2 \equiv 3 \bmod 4$
p_1p_2	$p_1 \equiv 5 \bmod 8$ *and* $p_2 \equiv 5 \bmod 8$
$p_1p_2p_3$	$p_1 \equiv 5 \bmod 8$ *and* $p_2, p_3 \equiv 3 \bmod 4$

Proof. Consult Corollary 2.10 where we calculated $rk_2 Pic(R_F)$. We omit the details. □

Corollary 2.13. *Let $F = \mathbb{Q}(\sqrt{d})$ with d positive. We have $rk_2 Pic_+(R_F) = 0$ in precisely the following cases:*

d	*with the condition*
2	
p	$p \equiv 1 \bmod 4$
$2p$	$p \equiv 3 \bmod 8$ *or* $p \equiv 5 \bmod 8$
p_1p_2	$p_1 \equiv 3 \bmod 8$ *and* $p_2 \equiv 3 \bmod 8$
p_1p_2	$p_1 \equiv 5 \bmod 8$ *and* $p_2 \equiv 5 \bmod 8$

Proof. Use Corollary 2.11. □

2.3. **The 4-rank of $Pic_+(\mathcal{O}_F)$ and $Pic_+(R_F)$ for F a quadratic number field.** Next we would like to compare the 4-rank of $Pic_+(\mathcal{O}_F)$ with the 4-rank of $Pic_+(R_F)$. This turns out to be harder than for the 2-rank. We start with a result about the 4-rank of a finite Abelian group. For an element x in a group G, let $ord(x)$ denote the order of the subgroup generated by x.

Lemma 2.14. *For a finite Abelian group G and $x \in G$*

$$rk_4(G/\langle x\rangle) = \begin{cases} rk_4(G) & \textit{if } x \in G^4, \textit{ or } (x \notin G^2 \textit{ and } ord(x) \equiv 2 \bmod 4) \\ & \textit{or } (x \notin G^2 \textit{ and } x^2 \in G^4 \textit{ and } ord(x) \equiv 0 \bmod 4), \\ rk_4(G) - 1 & \textit{otherwise.} \end{cases}$$

Proof. This goes like the analogous lemma we had for the 2-rank. We use the same notion, namely that x generates a subgroup of order n, $\langle x\rangle = C_n$. Further let G' denote the quotient group. The exact sequence $0 \longrightarrow C_n \longrightarrow G \longrightarrow G' \longrightarrow 0$ induces the exact sequences

$$C_n/2 \xrightarrow{\theta} G/2 \longrightarrow G'/2 \longrightarrow 0 \quad \text{and} \quad C_n/4 \xrightarrow{\lambda} G/4 \longrightarrow G'/4 \longrightarrow 0.$$

We divide the proof into the following cases.

1) Suppose $x \in G^4$. Then $\mathrm{im}(\lambda) = 0$, i.e., $G/4 \cong G^{'}/4$. In particular their 4-ranks agree.

2) Clearly $2 \nmid n$ gives $C_n^4 = C_n$ and $x \in C_n^4$. Hence we are in the situation above.

3) The case $n \equiv 2 \bmod 4$ gives $C_n^2 = C_n^4 = C_{n/2}$, so we get the exact sequences

$$C_2 \xrightarrow{\theta} G/2 \longrightarrow G^{'}/2 \longrightarrow 0 \quad \text{and} \quad C_2 \xrightarrow{\lambda} G/4 \longrightarrow G^{'}/4 \longrightarrow 0.$$

$*$) If $x \in G^2 \backslash G^4$, then θ is trivial and λ is injective. Thus $\sharp(G/2) = \sharp(G^{'}/2)$ and $\sharp(G/4) = 2\sharp(G^{'}/4)$. Therefore $rk_2(G) = rk_2(G^{'})$ and $rk_4(G) = rk_4(G^{'}) + 1$.

$*$) Now for the possibility $x \notin G^2$. Both θ and λ are now injective, so $\sharp(G/2) = 2\sharp(G^{'}/2)$ and $\sharp(G/4) = 2\sharp(G^{'}/4)$. Consequently $rk_2(G) = rk_2(G^{'}) + 1$ and $rk_4(G) = rk_4(G^{'})$.

4) The case $n \equiv 0 \bmod 4$ gives $C_n^2 = C_{n/2}$ and $C_n^4 = C_{n/4}$. Hence we get the exact sequences

$$C_2 \xrightarrow{\theta} G/2 \longrightarrow G^{'}/2 \longrightarrow 0 \quad \text{and} \quad C_4 \xrightarrow{\lambda} G/4 \longrightarrow G^{'}/4 \longrightarrow 0.$$

$*$) If $x \in G^2 \setminus G^4$, the map θ must be trivial while $\ker \lambda \cong C_2$. As before $\sharp(G/2) = \sharp(G^{'}/2)$ and $\sharp(G/4) = 2\sharp(G^{'}/4)$. We get $rk_2(G) = rk_2(G^{'})$ and $rk_4(G) = rk_4(G^{'}) + 1$.

$*$) If $x \notin G^2$ and $x^2 \in G^4$ then θ is injective while $\ker \lambda \cong C_2$. Hence $\sharp(G/2) = 2\sharp(G^{'}/2)$ and $\sharp(G/4) = 2\sharp(G^{'}/4)$. Thus $rk_2(G) = rk_2(G^{'}) + 1$ and $rk_4(G) = rk_4(G^{'})$.

$*$) The only possibility left is $x \notin G^2$ and $x \notin G^4$. Now both θ and λ are injective. We get $\sharp(G/2) = 2\sharp(G^{'}/2)$ and $\sharp(G/4) = 4\sharp(G^{'}/4)$. This implies $rk_2(G) = rk_2(G^{'}) + 1$ and $rk_4(G) = rk_4(G^{'}) + 1$.

$\square$

Let $\wp$ be a dyadic prime in F. From Lemma 2.14 we know when equality occurs between $rk_4 Pic_+(\mathcal{O}_F)$ and $rk_4 Pic_+(R_F)$, namely if and only if we are in one of the following cases:

1) $[\wp]_+ \in Pic_+(\mathcal{O}_F)^4$

2) $[\wp]_+ \notin Pic_+(\mathcal{O}_F)^2$ and $ord([\wp]_+) \equiv 2 \bmod 4$

3) $[\wp]_+ \notin Pic_+(\mathcal{O}_F)^2$, $[\wp]_+^2 \in Pic_+(\mathcal{O}_F)^4$ and $ord([\wp]_+) \equiv 0 \bmod 4$

Otherwise we have $rk_4 Pic_+(R_F) = rk_4 Pic_+(\mathcal{O}_F) - 1$. It is not easy to tell if a class is a fourth power in $Pic_+(\mathcal{O}_F)$. However, we can find answers in some cases.

Lemma 2.15. *Let $F = \mathbb{Q}(\sqrt{d})$ where $d \equiv 3$ or $5 \bmod 8$. Then $rk_4 Pic_+(\mathcal{O}_F) = rk_4 Pic_+(R_F)$.*

Proof. We know that 2 is inert if $d \equiv 5 \bmod 8$, so we can focus on $d \equiv 3 \bmod 8$. In this case 2 ramifies and hence $[\wp]_+$ is trivial or represents a class of order two. Recall that $\mathbf{N}(\wp) = 2$ in this case. If the order of $[\wp]_+$ is 1, then we are in case 1). If the order is two, we are in case 2) since

$$[\wp]_+ \in Pic_+(\mathcal{O}_F)^2 \Longleftrightarrow d \in M_2 \Longrightarrow d \equiv 1, 2, 6 \text{ or } 7 \bmod 8.$$

We have already seen proofs of both the equivalence and the implication above, so we will not repeat the argument once more. $\square$

Lemma 2.16. *For $F = \mathbb{Q}(\sqrt{d})$ where $d \equiv 2, 6$ or $7 \bmod 8$ we have $rk_4 Pic_+(\mathcal{O}_F) = rk_4 Pic_+(R_F)$ if and only if:*

$$i)\ d \notin M_2 \quad or \quad ii)\ [\wp]_+ \text{ is a fourth power in } Pic_+(\mathcal{O}_F)$$

Proof. If $ii)$ holds, then we are in situation 1). If $i)$ holds, i.e., $[\wp]_+ \notin Pic_+(\mathcal{O}_F)^2$, then $ord([\wp]_+) = 1$ or $ord([\wp]_+) = 2$. These possibilities are covered by 1), respectively 2). Conversely suppose that equality holds. We can be in case 1), which gives us that $ii)$ is true. Case 2) implies that $i)$ is true. We can never be in situation 3) since 2 ramifies, and hence the order of $[\wp]_+$ equals 1 or 2. $\square$

Lemma 2.17. *For $F = \mathbb{Q}(\sqrt{d})$ where $d \equiv 1 \bmod 8$ we have $rk_4 Pic_+(\mathcal{O}_F) = rk_4 Pic_+(R_F)$ if and only if:*

1) *$[\wp]_+$ is a fourth power in $Pic_+(\mathcal{O}_F)$, or*

2) *$ord([\wp]_+) \equiv 2 \bmod 4$ and $d \notin M_2$, or*

3) *$ord([\wp]_+) \equiv 0 \bmod 4$, $d \notin M_2$ and $([\wp]_+^2) \in Pic_+(\mathcal{O}_F)^4$*

Proof. In this case the norm of $\wp$ equals 2. This is where the condition $d \notin M_2$ comes in and replaces $[\wp]_+^2 \notin Pic_+(\mathcal{O}_F)^2$. □

3. Calculations of two-primary algebraic K-theory for quadratic number rings

In this section we will do the announced algebraic K-theory calculations. From the results on the quadratic Picard groups in Section 2 we can give a complete description of the 2-rank of $K_n(R_F)$ for F a quadratic number field. This will be carried out in Subsection 3.1. Further Subsection 3.2 describes a method to determine parts of the two-primary algebraic K-theory of quadratic number rings. Several examples will be treated in Subsection 3.3. The final effort is to lift the K_2 calculations to higher even algebraic K-groups. This will be done in Subsection 3.4.

3.1. **Calculation of $rk_2 K_n(R_F)$ for F a quadratic number field.** We will give a version of Theorem 1.1 for quadratic number fields. Our 2-rank calculation of $Pic(R_F)$ and $Pic_+(R_F)$ for $F = \mathbb{Q}(\sqrt{d})$ gives an answer given in terms of the number of distinct prime divisors t of $\triangle_F$ and the number of dyadic primes s in F. The integer s is known to be 2 for $d \equiv 1 \bmod 8$ and 1 otherwise.

Theorem 3.1. *Let $F = \mathbb{Q}(\sqrt{d})$ with d squarefree. Let s be the number of dyadic primes and t the number of distinct prime divisors of $\triangle_F$. We have the following calculation of $rk_2 K_n(R_F)$.*

1) *Let F be imaginary. If ($d \equiv 5 \bmod 8$) or ($p \equiv 1$ or $7 \bmod 8$ for all odd primes p dividing d), then:*

$n \geq 1 \bmod 2$	$rk_2 K_n(R_F)$
1	1
2	$s+t-2$

In all other cases we have:

$n \geq 1 \bmod 2$	$rk_2 K_n(R_F)$
1	1
2	$s+t-3$

2) *Let F be real. If ($d \equiv 5 \bmod 8$ and $p \equiv 1 \bmod 4$ for all odd primes p dividing d) or ($p \equiv 1 \bmod 8$ for all odd primes p dividing d), then:*

$n \geq 1 \bmod 8$	$rk_2 K_n(R_F)$
1	1
2	$s+t$
3	2
8	$s+t-2$

If $d \equiv 1, 2, 3, 6$ or $7 \bmod 8$ and there exists odd prime numbers p_1 and p_2 dividing d such that $p_1 \equiv 5 \bmod 8$ and $p_2 \equiv 3 \bmod 4$, then:

$n \geq 1 \bmod 8$	$rk_2 K_n(R_F)$
1	1
2	$s+t-2$
3	2
8	$s+t-4$

In all other cases we have:

$n \geq 1 \bmod 8$	$rk_2 K_n(R_F)$
1	1
2	$s+t-1$
3	2
8	$s+t-3$

For the other values of n modulo 8 we have the following.

If ($d \equiv 5 \bmod 8$) *or* ($p \equiv 1$ *or* $7 \bmod 8$) *for all odd primes p dividing d, then*:

$n \geq 1 \bmod 8$	$rk_2 K_n(R_F)$
4	$s+t-2$
5	0
6	$s+t-2$
7	1

In all other cases:

$n \geq 1 \bmod 8$	$rk_2 K_n(R_F)$
4	$s+t-3$
5	0
6	$s+t-3$
7	1

Proof. Compare Theorem 1.1, Corollaries 2.10 and 2.11. □

3.2. **A method for two-primary calculations of K_2 for quadratic number rings.** In this subsection we describe a method that in some favorable cases calculates the two-primary part of K_2 of quadratic number rings.

The narrow version of $K_2(F)$ is defined by a split short exact sequence

$$0 \longrightarrow K_2^+(F) \longrightarrow K_2(F) \longrightarrow \bigoplus_{\wp \text{ real}} \mu_2 \longrightarrow 0$$

induced by Hilbert symbols. Next one defines $K_2^+(\mathcal{O}_{F,S}) := K_2(\mathcal{O}_{F,S}) \cap K_2^+(F)$.

The following result is part of Corollary 3.9 in [8].

Theorem 3.2 (Keune). *For $F = \mathbb{Q}(\sqrt{d})$ we have*

$$rk_2 K_2^+(R_F) = rk_2 Pic_+(R_F) + s - 1.$$

One main reason for focusing on K_2 is that we have 4-rank formulas for K_2 of quadratic number fields. We continue to work with the quadratic number field $F = \mathbb{Q}(\sqrt{d})$ where d is a squarefree integer. Let $\overline{F} = \mathbb{Q}(\sqrt{-d})$ with two-integers $R_{\overline{F}}$ and let $\delta = rk_2 Pic_+(R_F) - rk_2 Pic(R_F)$. It follows from Theorem 2.9 that δ equals 0 if $d \in M_{-1} \cup M_{-2}$ and 1 in all all other cases. Various 4-rank formulas for K_2 are known, see [5] and [9]. We will use the following results from [1].

Theorem 3.3 (Boldy). *For $F = \mathbb{Q}(\sqrt{d})$ we have*

$$rk_4 K_2^+(R_F) = rk_4 Pic_+(R_{\overline{F}}) + \overline{s} - 1.$$

Theorem 3.4 (Boldy). *For $F=\mathbb{Q}(\sqrt{d})$ we have*

$$rk_4K_2(R_F)=rk_4Pic_+(R_{\overline{F}})+\overline{s}+\delta-1.$$

Definition 3.5. *Let rk_n denote the n-rank of $K_2(R_F)$ and rk_n^+ denote the n-rank of $K_2^+(R_F)$. Here n is a prime power. We will only consider $n=2$ and $n=4$.*

Let $p_1,...,p_{\overline{t}}$ be the prime divisors of $\triangle_{\overline{F}}$ with the modification that $p_{\overline{t}}=2$ in case $2\mid\triangle_{\overline{F}}$. Likewise we let $\overline{s}$ denote the number of dyadic primes in $\overline{F}$. The matrix

$$M_F=\begin{bmatrix} (\frac{d/p_1}{p_1}) & (\frac{p_1}{p_2}) & \cdots & (\frac{p_1}{p_{\overline{t}-1}})\\ (\frac{p_2}{p_1}) & (\frac{d/p_2}{p_2}) & \cdots & (\frac{p_2}{p_{\overline{t}-1}})\\ \vdots & \vdots & \ddots & \vdots\\ (\frac{p_{\overline{t}-1}}{p_1}) & (\frac{p_{\overline{t}-1}}{p_2}) & \cdots & (\frac{d/p_{\overline{t}-1}}{p_{\overline{t}-1}})\\ (\frac{p_{\overline{t}}}{p_1}) & (\frac{p_{\overline{t}}}{p_2}) & \cdots & (\frac{p_{\overline{t}}}{p_{\overline{t}-1}}) \end{bmatrix}$$

is described in [11], using ideas of Redei. Here $(\frac{p}{q})$ denotes the quadratic residue symbol of p and q. We adopt the following conventions. The value in M_F of $(\frac{p}{q})=1$ is 0, while the M_F value of $(\frac{p}{q})=-1$ is 1. Let $r_2=rk_2Pic_+(\mathcal{O}_{\overline{F}})=\overline{t}-1$ and $r_4=rk_4Pic_+(\mathcal{O}_{\overline{F}})$. Then $r_4=r_2-rank_{\mathbb{F}_2}(M_F)$. This fact motivates the definition of M_F.

From Theorems 3.3 and 3.4 we see that if we know the value of rk_4^+, then we also know the value of rk_4 since their difference is given by δ. What we will do is to express rk_2^+ and rk_4^+ in terms of r_2 and r_4. The first problem is to find $rk_4Pic_+(R_{\overline{F}})$. As we saw in Subsection 2.3 there are cases where it is hard to find this number. We will run into trouble in the cases:

$Sit.1)$ $[\wp]_+\in Pic_+(\mathcal{O}_{\overline{F}})^4$

$Sit.2)$ $(ord([\wp]_+)\equiv 0 \bmod 4$ and $[\wp]_+\in Pic_+(\mathcal{O}_{\overline{F}})^4)$ or $(ord([\wp]_+)\equiv 2 \bmod 4)$

Translating the Lemmas 2.15, 2.16 and 2.17 from F to $\overline{F}$ yields

Corollary 3.6. *We have $rk_4Pic_+(R_{\overline{F}})=rk_4Pic_+(\mathcal{O}_{\overline{F}})$ in the three cases:*

1) $d\equiv 3,5 \bmod 8$

2) $d\equiv 1,2,6 \bmod 8$ *with the restriction* $(d\notin M_2)$ *or* $Sit.1$

3) $d\equiv 7 \bmod 8$ *with the restriction*

$$Sit.1 \text{ or } (d\notin M_2, ord([\wp]_+)\equiv 2 \bmod 4) \text{ or } (d\notin M_2, [\wp]_+^2\in Pic_+(\mathcal{O}_{\overline{F}})^4, ord([\wp]_+)\equiv 0 \bmod 4)$$

In all other cases we have $rk_4Pic_+(\mathcal{O}_{\overline{F}})-rk_4Pic_+(R_{\overline{F}})=1$.

Proposition 3.7. *We have the following calculation of rk_2^+ and rk_4^+*

$d \equiv \bmod 8$	*Case*	*with the condition*	rk_2^+	rk_4^+
1	1.1	$d \notin M_2$	$r_2 - 1$	r_4
1	1.2	$d \in M_2$ *and* $Sit.1$	r_2	r_4
1	1.2	$d \in M_2$ *and not* $Sit.1$	r_2	$r_4 - 1$
3	2		r_2	r_4
5	3		$r_2 - 1$	r_4
$2, 6$	4.1	$d \notin M_2$	$r_2 - 1$	r_4
$2, 6$	4.2	$d \in M_2$ *and* $Sit.1$	r_2	r_4
$2, 6$	4.2	$d \in M_2$ *and not* $Sit.1$	r_2	$r_4 - 1$
7	5.1	$d \notin M_2$ *and* $Sit.1$	r_2	$r_4 + 1$
7	5.1	$d \notin M_2$ *and not* $Sit.1$	r_2	r_4
7	5.2	$d \in M_2$ *and* $Sit.1$	$r_2 + 1$	$r_4 + 1$
7	5.2	$d \in M_2$ *and not* $Sit.1$	$r_2 + 1$	r_4

with the additional remark:

In case $d \equiv 7 \bmod 8$ *and* $d \in M_2$, *then* $r_4 = 0$ *implies* $rk_4^+ = 1$

Proof. Case 1.1: $d \equiv 1 \bmod 8$ and $d \notin M_2$. Then $rk_2 Pic_+(\mathcal{O}_F) = r_2 - 1$, $rk_2 Pic_+(R_F) = r_2 - 2$ and $rk_4 Pic_+(R_{\overline{F}}) = r_4$. Further $s = 2$ and $\overline{s} = 1$. Hence $rk_2^+ = r_2 - 1$ and $rk_4^+ = r_4$.

Case 1.2: $d \equiv 1 \bmod 8$, $d \in M_2$. We get that $rk_2 Pic_+(\mathcal{O}_F) = r_2 - 1$, $rk_2 Pic_+(R_F) = r_2 - 1$, $rk_4 Pic_+(R_{\overline{F}}) = r_4$ in $Sit.1$ and $rk_4 Pic_+(R_{\overline{F}}) = r_4 - 1$ in all other cases. Further $s = 2$ and $\overline{s} = 1$. Hence $rk_2^+ = r_2, rk_4^+ = r_4$ if $Sit.1$ holds and $rk_4^+ = r_4 - 1$ otherwise.

Case 2: $d \equiv 3 \bmod 8$. We derive that $rk_2 Pic_+(\mathcal{O}_F) = r_2 + 1$, $rk_2 Pic_+(R_F) = r_2$ since $d \notin M_2$, and also $rk_4 Pic_+(R_{\overline{F}}) = r_4$. Further $s = \overline{s} = 1$. We find that $rk_2^+ = r_2$ and $rk_4^+ = r_4$.

Case 3: $d \equiv 5 \bmod 8$. We get that $rk_2 Pic_+(\mathcal{O}_F) = r_2 - 1$, $rk_2 Pic_+(R_F) = r_2 - 1$ and $rk_4 Pic_+(R_{\overline{F}}) = r_4$. Further $s = \overline{s} = 1$. This time we find $rk_2^+ = r_2$ and $rk_4^+ = r_4$.

Case 4.1: $d \equiv 2, 6 \bmod 8$, $d \notin M_2$. We get that $rk_2 Pic_+(\mathcal{O}_F) = r_2 - 1$, $rk_2 Pic_+(R_F) = r_2 - 1$, $rk_4 Pic_+(R_{\overline{F}}) = r_4$. Further $s = \overline{s} = 1$. Hence $rk_2^+ = r_2 - 1$ and $rk_4^+ = r_4$.

Case 4.2: $d \equiv 2, 6 \bmod 8$, $d \in M_2$. Now $rk_2 Pic_+(\mathcal{O}_F) = rk_2 Pic_+(R_F) = r_2$, $rk_4 Pic_+(R_{\overline{F}}) = r_4$ in case $Sit.1$ holds and $rk_4 Pic_+(R_{\overline{F}}) = r_4 - 1$ otherwise. Again $s = \overline{s} = 1$, so $rk_2^+ = r_2, rk_4^+ = r_4$ if $Sit.1$ holds and $rk_4^+ = r_4 - 1$ otherwise.

Case 5.1: $d \equiv 7 \bmod 8$, $d \notin M_2$. We get that $rk_2 Pic_+(\mathcal{O}_F) = r_2 + 1$, $rk_2 Pic_+(R_F) = r_2$, $rk_4 Pic_+(R_F) = r_4$ in case $Sit.1$ holds and $rk_4 Pic_+(R_{\overline{F}}) = r_4 - 1$ otherwise. Again $s = 1$ and $\overline{s} = 2$. Hence $rk_2^+ = r_2, rk_4^+ = r_4 + 1$ if $Sit.1$ holds and $rk_4^+ = r_4$ otherwise.

Case 5.2: $d \equiv 7 \bmod 8$, $d \in M_2$. We get $rk_2 Pic_+(\mathcal{O}_F) = rk_2 Pic_+(R_F) = r_2 + 1$, $rk_4 Pic_+(R_{\overline{F}}) = r_4$ in $Sit.1$ and $rk_4 Pic_+(R_{\overline{F}}) = r_4 - 1$ otherwise. Trivially $s = 1$ and $\overline{s} = 2$. So $rk_2^+ = r_2 + 1, rk_4^+ = r_4 + 1$ if $Sit.1$ holds and $rk_4^+ = r_4$ otherwise.

Now for the additional remark. When $r_4 = 0$ we claim that $[\wp]_+ \in Pic_+(\mathcal{O}_{\overline{F}})^4$, i.e., we are in $Sit.1$. The condition on d implies $[\wp]_+ \in Pic_+(\mathcal{O}_{\overline{F}})^2$ from Lemmas 2.5 and 2.7. The assumption $r_4 = 0$ implies $Pic_+(\mathcal{O}_{\overline{F}})^2 = Pic_+(\mathcal{O}_{\overline{F}})^4$, i.e., $[\wp]_+$ is a fourth power in $Pic_+(\mathcal{O}_{\overline{F}})$.

$\square$

Now an outline of the method that we will use for the calculations. Proposition 3.7 calculates rk_2^+ and rk_4^+ in terms of r_2 and r_4, respectively. It is trivial to find r_2, while the matrix M_F takes care of r_4. In this way we automatically determine rk_4 from δ. The matrix M_F will play the vital part in the calculations. Using this method with ideas going all the way back to Redei we have a quite powerful machine in hand that provides lots of examples. This will be discussed in the next subsection.

3.3. **Examples of K_2 of quadratic number rings.** This subsection consists of examples. We will heavily use the formulas obtained in the previous subsection and the calculations will usually be based on the matrix M_F. Whenever we write the word matrix we refer to M_F. First we make an elementary observation.

Example 3.8. *Let F be a real quadratic number field and suppose that $rk_4^+ = 0$. Then we have*
$$K_2(R_F)\{2\} \cong (\mathbb{Z}/2)^{rk_2 - rk_4} \oplus (\mathbb{Z}/4)^{rk_4}.$$

Proof. Recall that $rk_2 + rk_4 = rk_2^+ + rk_4^+ + 2$ from the 2 and 4-rank formulas. From the exact sequence $0 \longrightarrow K_2^+(R_F) \longrightarrow K_2(R_F) \longrightarrow \mu_2 \oplus \mu_2 \longrightarrow 0$ we get that $rk_2 + rk_4 + ... = 2 + rk_2^+ + rk_4^+ + ...$, i.e., $rk_8 + rk_{16} + ... = rk_8^+ + rk_{16}^+ + ...$. But the assumption is that $rk_4^+ = 0$. □

Before we begin with the examples we ought to remind ourself of the following. In case $d < 0$ we clearly have $rk_2 = rk_2^+$ and $rk_4 = rk_4^+$ since the narrow and the usual K_2 are equal. For d positive we have $rk_2 = rk_2^+ + 2 - \delta$ and $rk_4 = rk_4^+ + \delta$. Again we emphasize that $\delta = 0$ if $d \in M_{-1} \cup M_{-2}$ and 1 in all other cases. All the p, q and r's occurring in the examples are prime numbers. In the calculations we will refer to the various cases occurring in Proposition 3.7. Let us begin with an easy example.

Example 3.9. *Let $F = \mathbb{Q}(\sqrt{-p})$. Then*

$$K_2(R_F)\{2\} \cong \begin{cases} \mathbb{Z}/2^n & \textit{for } p \equiv 1 \bmod 8, \\ \mathbb{Z}/2 & \textit{for } p \equiv 7 \bmod 8, \\ 0 & \textit{otherwise,} \end{cases}$$

for some $n \geq 2$.

Proof. We divide the calculation into cases according to the congruence class of p modulo 8.

1) $p \equiv 1 \bmod 8$. Now $d \equiv 7 \bmod 8$, $d \in M_2$, so we are in Case 5.2. From $d \equiv 3 \bmod 4$ we find that $\triangle = p$, i.e., $\bar{t} = 1$, $r_2 = r_4 = 0$. Thus we are in $Sit.1$ and $rk_2 = rk_4 = 1$.

2) $p \equiv 3 \bmod 8$. We have $d \equiv 5 \bmod 8$, i.e., Case 3. From $d \equiv 1 \bmod 4$ we find that $\triangle = 4p$, i.e., $\bar{t} = 2$, $r_2 = 1$ and hence $rk_2 = 0$.

3) $p \equiv 5 \bmod 8$. We have $d \equiv 3 \bmod 8$, $d \notin M_2$, i.e., Case 2. From $d \equiv 3 \bmod 4$ we find that $\triangle = p$, i.e., $\bar{t} = 1$, $r_2 = 0$ and hence $rk_2 = 0$.

4) $p \equiv 7 \bmod 8$. We have $d \equiv 1 \bmod 8$, $d \in M_2$, i.e., Case 1.2. From $d \equiv 1 \bmod 4$ we find that $\triangle = 4p$, i.e., $\bar{t} = 2$, $r_2 = 1$. The matrix is

$$\begin{bmatrix} (\frac{-d/p}{p}) \\ (\frac{2}{p}) \end{bmatrix} = \begin{bmatrix} (\frac{-1}{p}) \\ (\frac{2}{p}) \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and hence $r_4 = r_2 - 1 = 0$, i.e., $rk_2 - 1 = rk_4 = 0$.

□

Remark 3.10. *One can prove that n equals 2 in the case $p \equiv 9 \bmod 16$. The situation is far more complicated if $p \equiv 1 \bmod 16$, because in this case $n \geq 3$. As an example we mention that $n = 3$ for $p = 17$. All these claims follow from unpublished work of Qin, see also* [4] *for similar computations.*

Next we consider $F = \mathbb{Q}(\sqrt{-p_1 ... p_{2m+1} q_1 ... q_{2n+1}})$ with $p_i \equiv 7 \bmod 8$ and $q_i \equiv 5 \bmod 8$. Due to the residue classes of the p_i's and the q_i's modulo eight we have that $(\frac{p_i}{p_j}) = (\frac{p_j}{p_i})$. We are in Case 3, and we easily find $rk_2 = 2(m+n)+1$ from Theorem 3.1. Let $\widehat{p_i}$ denote the product $p_1 \ldots p_{i-1}p_{i+1}\ldots q_{2n-1}$ and likewise for $\hat{q}_i$. The matrix is given as:

$$M_F = \begin{bmatrix} -(\frac{\widehat{p_1}}{p_1}) & (\frac{p_1}{p_2}) & \cdots & (\frac{p_1}{q_{2n-1}}) \\ (\frac{p_2}{p_1}) & -(\frac{\widehat{p_2}}{p_2}) & \cdots & (\frac{p_2}{q_{2n-1}}) \\ \vdots & \vdots & \ddots & \vdots \\ (\frac{q_{2n+1}}{p_1}) & (\frac{q_{2n+1}}{p_2}) & \cdots & (\frac{\widehat{q_{2n+1}}}{q_{2n+1}}) \\ (\frac{2}{p_1}) & (\frac{2}{p_2}) & \cdots & (\frac{2}{q_{2n+1}}) \end{bmatrix}$$

We make some observations on M_F. First assume that all the quadratic residue symbols equals 1. Then the entries in the first $2m+1$ columns are all zero except the entries $e_{ii} = 1$ for $1 \leq i \leq 2m+1$. Next consider the remaining $2n+1$ columns. All the entries are zero, except for the last row where all the entries are equal to 1. Thus the $\mathbb{F}_2$-rank of M_F is $2m+2$ and $rk_4 = 2n$. Hence we record

Example 3.11. *Let $F = \mathbb{Q}(\sqrt{-p_1 \ldots p_{2m+1}q_1 \ldots q_{2n+1}})$ with $p_i \equiv 7 \bmod 8$ and $q_i \equiv 5 \bmod 8$. If all the quadratic symbols we can make from the p_i's and the q_i's equals 1, then*

$$K_2(R_F)\{2\} \cong (\mathbb{Z}/2)^{2m+1} \oplus (\mathbb{Z}/2^{a_1}) \oplus \ldots \oplus (\mathbb{Z}/2^{a_{2n}})$$

for some $a_i \geq 2$.

Next consider the following slightly modification of the last example. We adjoint one more p_i, say p_{2m+2}, such that there is an even number of p_i's. This change of input brings us over to Case 2. The 2-rank rk_2 equals $2(m+n+1)$, and:

$$M_F = \begin{bmatrix} -(\frac{\widehat{p_1}}{p_1}) & (\frac{p_1}{p_2}) & \cdots & (\frac{p_1}{q_{2n-1}}) \\ (\frac{p_2}{p_1}) & -(\frac{\widehat{p_2}}{p_2}) & \cdots & (\frac{p_2}{q_{2n-1}}) \\ \vdots & \vdots & \ddots & \vdots \\ (\frac{q_{2n}}{p_1}) & (\frac{q_{2n}}{p_2}) & \cdots & (\frac{\widehat{q_{2n}}}{q_{2n}}) \\ (\frac{q_{2n+1}}{p_1}) & (\frac{q_{2n+1}}{p_2}) & \cdots & (\frac{q_{2n+1}}{q_{2n}}) \end{bmatrix}$$

If all the quadratic residue symbols are 1, then $e_{ii} = 1$ for $1 \leq i \leq 2m+2$ while all the other entries are zero. Consequently

$$K_2(R_F)\{2\} \cong (\mathbb{Z}/2)^{2m+2} \oplus (\mathbb{Z}/2^{a_1}) \oplus \ldots \oplus (\mathbb{Z}/2^{a_{2n}})$$

for some $a_i \geq 2$. Suppose we have the situation that the only nontrivial quadratic symbols are $(\frac{p_1}{p_2}) = \ldots = (\frac{p_i}{p_{i+1}}) = -1$. The number rk_4 equals $2n+1$ if i is odd, as one easily checks by looking at M_F. However, rk_4 will be $2n+1$ if i is even.

Example 3.12. *Let $F = \mathbb{Q}(\sqrt{-p_1 \ldots p_{2m+2}q_1 \ldots q_{2n+1}})$ with $p_i \equiv 7 \bmod 8$ and $q_i \equiv 5 \bmod 8$. If the only nontrivial quadratic symbols are $(\frac{p_1}{p_2}) = \ldots = (\frac{p_i}{p_{i+1}}) = -1$ where i is even, then*

$$K_2(R_F)\{2\} \cong (\mathbb{Z}/2)^{2m+1} \oplus (\mathbb{Z}/2^{a_1}) \oplus \ldots \oplus (\mathbb{Z}/2^{a_{2n+1}})$$

for some $a_i \geq 2$.

Next consider the real quadratic number field $F = \mathbb{Q}(\sqrt{p_1 \ldots p_{2n+1}})$ for some $p_i \equiv 3 \bmod 8$. One checks that $rk_2 = 2n+2$, $rk_2^+ = 2n$ and $(\frac{p_i}{p_j}) = -(\frac{p_j}{p_i})$. The matrix is given as:

$$M_F = \begin{bmatrix} \widehat{(\frac{p_1}{p_1})} & \cdots & (\frac{p_1}{p_{2n}}) \\ (\frac{p_2}{p_1}) & \cdots & (\frac{p_2}{p_{2n}}) \\ \vdots & \ddots & \vdots \\ (\frac{p_{2n-1}}{p_1}) & \cdots & \widehat{(\frac{p_{2n-1}}{p_{2n-1}})} \\ (\frac{p_{2n}}{p_1}) & \cdots & (\frac{p_{2n}}{p_{2n-1}}) \end{bmatrix}$$

If all the possible quadratic residue symbols we can make out of the p_i's are trivial, then $rk_4 = rk_2$. By looking at the matrix one finds that if $(\frac{p_1}{p_2}), ..., (\frac{p_{i-1}}{p_i})$ are the only quadratic residue symbols equal to -1, then $rk_4 = rk_4^+ = 2n - i + 1$. Hence

Example 3.13. *Let $F = \mathbb{Q}(\sqrt{p_1 ... p_{2n+1}})$ with $p_i \equiv 3 \bmod 8$. If $(\frac{p_1}{p_2}) = ... = (\frac{p_{i-1}}{p_i})$ are the only quadratic residue symbols equal to -1, then*

$$K_2^+(R_F)\{2\} \cong (\mathbb{Z}/2)^{i+1} \oplus (\mathbb{Z}/2^{a_1}) \oplus ... \oplus (\mathbb{Z}/2^{a_{2n-i-1}})$$

and

$$K_2(R_F)\{2\} \cong (\mathbb{Z}/2)^{i+3} \oplus (\mathbb{Z}/2^{b_1}) \oplus ... \oplus (\mathbb{Z}/2^{b_{2n-i-1}})$$

for some $a_j, b_j \geq 2$.

We end the section with two more concrete examples.

Example 3.14. *For $F = \mathbb{Q}(\sqrt{-pqr})$ with $p, q \equiv 7 \bmod 8$, $r \equiv 5 \bmod 8$ and $(\frac{p}{r}) = -1$*

$$K_2(R_F)\{2\} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

Proof. We are in Case 2. One finds that $rk_2 = 2$ and $rk_4 = r_4$. The matrix is:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Hence $r_4 = 2 - rk_{\mathbb{F}_2}(M_F) = 0$. □

Example 3.15. *Let $F = \mathbb{Q}(\sqrt{2pqr})$ with $p \equiv 1 \bmod 8$, $q \equiv 3 \bmod 8$ and $r \equiv 5 \bmod 8$. Assume that $-(\frac{q}{p}) = (\frac{r}{p}) = 1$. Then*

$$K_2^+(R_F)\{2\} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

and

$$K_2(R_F)\{2\} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2^n$$

for some $n \geq 2$.

Proof. We are in Case 4.1, $rk_2 = 2$ and $rk_4 = r_4$. The matrix is

$$\begin{bmatrix} (\frac{2qr}{p}) & (\frac{p}{q}) & (\frac{p}{r}) \\ (\frac{q}{p}) & (\frac{2pr}{q}) & (\frac{q}{r}) \\ (\frac{r}{p}) & (\frac{r}{q}) & (\frac{2pq}{r}) \\ (\frac{2}{p}) & (\frac{2}{q}) & (\frac{2}{r}) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & (\frac{r}{q}) & (\frac{r}{q}) \\ 0 & (\frac{r}{q}) & -(\frac{r}{q}) \\ 0 & 1 & 1 \end{bmatrix} \sim ... \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and hence $rk_4^+ = 0$. Since $d \notin M_{-1} \cup M_{-2}$ we find that $\delta = 1$, $rk_2 = 3$ and $rk_4 = 1$.

□

3.4. **Lifting the K_2 calculations to higher K-groups.** The basis for lifting the previous K_2 calculations to higher algebraic K-groups is

Theorem 3.16 (Rognes-Weibel). *Let F be a totally imaginary number field. Then*

$$K_{2n}(R_F)\{2\} \cong H^2_{\acute{e}t}(R;\mathbb{Z}_2(n+1)).$$

For a real number field F

$$K_{8n+2}(R_F)\{2\} \cong H^2_{\acute{e}t}(R;\mathbb{Z}_2(4n+2)).$$

Let e denote the exponent of $(\mathbb{Z}/2^\nu)^*$. The motivation for introducing e is the isomorphism $\mathbb{Z}/2^\nu(i) \cong \mathbb{Z}/2^\nu(e+i)$ of sheaves for the étale topology. The exact sequence

$$0 \longrightarrow \mathbb{Z}_2(i) \xrightarrow{2^\nu} \mathbb{Z}_2(i) \longrightarrow \mathbb{Z}/2^\nu(i) \longrightarrow 0$$

of coefficient sheaves induces the horizontal exact sequence diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^2_{\acute{e}t}(R_F;\mathbb{Z}_2(2))/2^\nu \arrow[r] & H^2_{\acute{e}t}(R_F;\mathbb{Z}/2^\nu(2)) \arrow[r] \arrow[d, "\cong"] & {}_{2^\nu}H^3_{\acute{e}t}(R_F;\mathbb{Z}_2(2)) \arrow[r] & 0 \\
0 \arrow[r] & H^2_{\acute{e}t}(R_F;\mathbb{Z}_2(e+2))/2^\nu \arrow[r] & H^2_{\acute{e}t}(R_F;\mathbb{Z}/2^\nu(e+2)) \arrow[r] & {}_{2^\nu}H^3_{\acute{e}t}(R_F;\mathbb{Z}_2(e+2)) \arrow[r] & 0
\end{tikzcd}$$

Lemma 3.17. *Let $i \neq 0, 1$. Then*

$$H^3_{\acute{e}t}(R_F;\mathbb{Z}_2(i)) \cong \begin{cases} (\mathbb{Z}/2)^{r_1} & \text{for } i \text{ odd}, \\ 0 & \text{for } i \text{ even}. \end{cases}$$

Proof. Combine Corollary 4.4, Propositions 2.4 and 4.6 in [16]. □

Combining the diagram, Theorem 3.16 and Lemma 3.17 yields

Theorem 3.18. *Let e be the exponent of $(\mathbb{Z}/2^\nu)^*$ where $\nu \geq 2$ if F is a real number field. Recall that $e = 2^{\nu-1}$ for $1 \leq \nu \leq 2$, and $e = 2^{\nu-2}$ for $\nu \geq 3$. We have the isomorphism*

$$K_2(R_F)/2^\nu \cong H^2_{\acute{e}t}(R_F;\mathbb{Z}_2(e+2))/2^\nu.$$

In particular if F is totally imaginary, or if F is real and $e \equiv 0 \bmod 4$, then

$$K_2(R_F)/2^\nu \cong K_{2e+2}(R_F)/2^\nu.$$

Remark 3.19. *This result lifts the K_2 calculations to higher K-groups. This illustrates in yet another way how important and interesting K_2 is!*

We illustrate this result using the field F from Example 3.14. Recall that $K_2(R_F)\{2\} \cong (\mathbb{Z}/2)^2$. Thus $K_2(R_F)\{2\} \cong K_2(R_F)/2$ and $K_2(R_F)\{2\} \cong K_{2e+2}(R_F)/2^\nu$.

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