# Chapter 3

# Exact completion and glueing

This Chapter will be an exercise in "pure predicative topos theory". I prove two closure properties of $\Pi W$-pretoposes: closure under exact completion and under glueing. Closure under exact completion is especially noteworthy, because toposes are not closed under exact completion.

As an application of these two results, I give more examples of $\Pi W$-pretoposes and prove a result on the projectives in the free $\Pi W$-pretopos. The latter will imply that the free $\Pi W$-pretopos and the category of setoids are non-equivalent.

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## 3.1 Exact completion of a cartesian category

The examples of $\Pi W$-pretoposes that we have seen so far are toposes with nno and the category of setoids. The categorical construction called exact completion will provide us with a host of other examples. To show that they are examples, I need a set of conditions on a category $\mathcal{C}$ for its exact completion $\mathcal{C}_{ex}$ to be a $\Pi W$-pretopos. As always in the theory of exact completions, the category $\mathcal{C}$ has to satisfy the axioms for a $\Pi W$-pretopos in a weaker sense. I identify a set of conditions and I show that for the categories satisfying these conditions the exact completion is a $\Pi W$-pretopos. As it turns out, ML-categories are examples of such "weak $\Pi W$-pretoposes". This means that the exact completion of an ML-category is a $\Pi W$-pretoposes, so the ML-categories of the previous Chapter can be remedied in this way to become $\Pi W$-pretoposes. It also shows that the exact completion of a topos with nno is a $\Pi W$-pretopos. As exact completions of toposes are rarely toposes, this shows that there are many $\Pi W$-pretoposes that are not toposes.

Intuitively, the exact completion is the universal way of constructing an exact category out of a cartesian category. In more precise (2-categorical) terms it is the following. Write $\mathcal{C}art$ for the large category of (small) cartesian categories and $\mathcal{E}xact$ for the large category of (small) exact categories. The *exact completion* of a given a cartesian category $\mathcal{C}$ is an exact category $\mathcal{C}_{ex}$, together with a cartesian embedding $\mathbf{y}: \mathcal{C} \longrightarrow \mathcal{C}_{ex}$, such that for any exact category $\mathcal{D}$, composition with $\mathbf{y}$ induces an equivalence $\mathcal{E}xact(\mathcal{C}_{ex}, \mathcal{D}) \longrightarrow \mathcal{C}art(\mathcal{C}, \mathcal{D})$. As Joyal discovered, it is possible to explicitly describe $\mathcal{C}_{ex}$.

**Explicit description of an exact completion 3.1.1** Two parallel arrows

$$R \underset{r_1}{\overset{r_0}{\rightrightarrows}} X$$

in a cartesian category $\mathcal{C}$ form an *pseudo-equivalence relation* when for any object $A$ in $\mathcal{C}$ the image of the induced function

$$\text{Hom}(A, R) \longrightarrow \text{Hom}(A, X) \times \text{Hom}(A, X)$$

is an equivalence relation on the set $\text{Hom}(A, X)$. These pseudo-equivalence relations are the objects in the category $\mathcal{C}_{ex}$. A morphism from

$$R_X \underset{x_1}{\overset{x_0}{\rightrightarrows}} X$$

to

$$R_Y \underset{y_1}{\overset{y_0}{\rightrightarrows}} Y$$

in $\mathcal{C}_{ex}$ is an equivalence class of arrows $f: X \longrightarrow Y$ in $\mathcal{C}$ for which there exists a $g: R_X \longrightarrow R_Y$ such that $fx_i = y_i g$ for $i = 0, 1$. Two such arrows $f_0, f_1: X \longrightarrow Y$ are equivalent if there exists an $h: X \longrightarrow R_Y$ such that $f_i = y_i h$ for $i = 0, 1$.

The embedding $\mathbf{y}$ is given by the obvious functor $\mathbf{y}: \mathcal{C} \longrightarrow \mathcal{C}_{ex}$ that sends an object $A$ in $\mathcal{C}$ to

$$A \underset{1_A}{\overset{1_A}{\rightrightarrows}} A.$$

Besides being cartesian, the functor is evidently full and faithful. The proof that the category thus constructed is exact and actually the exact completion of $\mathcal{C}$ can be found in [21], [20].

For both the objects in the exact completion that are in the image of $\mathbf{y}$ and categories that arise as exact completions, there exist remarkable characterisation results. To state these, I need the following terminology.

**Projectives, external and internal 3.1.2** An object $P$ in a category $\mathcal{C}$ is *(externally) projective* if for any cover $g: X \longrightarrow Y$ and any morphism $f: P \longrightarrow Y$, there exists a morphism $h\colon P \longrightarrow X$ such that $gh = f$.[1] When $\mathcal{C}$ is cartesian, this is equivalent to: any cover $p\colon X \longrightarrow P$ has a section. An object $X$ is *covered by a projective*, if there exists a projective $P$ and a cover $f\colon P \longrightarrow X$. A category $\mathcal{C}$ has *enough projectives* if any object in $\mathcal{C}$ is covered by a projective.

These external projectives are to be distinguished from the following class of objects. In a cartesian category $\mathcal{C}$, an object $P$ is called *internally projective*, when for any cover $Y \longrightarrow X$ and any arrow $T \times P \longrightarrow X$, there exists a cover $T' \longrightarrow T$ and map $T' \times P \longrightarrow Y$ such that the square

$$\begin{array}{ccc} T' \times P & \longrightarrow & Y \\ \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow \\ T \times P & \longrightarrow & X \end{array}$$

commutes. A morphism $f\colon Y \longrightarrow X$ is called a *choice map*, when it is internally projective as an object of $\mathcal{C}/X$.

In case $P$ is exponentiable, this coincides with the more common definition: $P$ is internally projective iff the functor $(-)^P$ preserves covers. This means that in a Heyting category $\mathcal{C}$, for an exponentiable object $A$ that is also internally projective, the axiom of choice is valid "relative to $A$", in the sense that the following scheme is valid in the internal logic of $\mathcal{C}$:

$$\forall a \in A\, \exists x \in X\, \phi(a, x) \to \exists f \in X^A\, \forall a \in A\, \phi(a, f(a)).$$

The two characterisation results now are (see [21]):

**Lemma 3.1.3** *The objects in the image of* $\mathbf{y}\colon \mathcal{C} \longrightarrow \mathcal{C}_{ex}$ *are, up to isomorphism, the projectives of* $\mathcal{C}_{ex}$.

**Proposition 3.1.4** *An exact category $\mathcal{C}$ is an exact completion if and only if it has enough projectives and the projectives are closed under finite limits. In that case, $\mathcal{C}$ is the exact completion of the full subcategory of its projectives.*

An immediate consequence is (see [16]):

**Proposition 3.1.5** *If $\mathcal{C}$ is cartesian and $A$ an object in $\mathcal{C}$, then*

$$(\mathcal{C}/A)_{ex} \cong \mathcal{C}_{ex}/\mathbf{y}A.$$

One combines this with the following observation (which I am not the first to point out, see [38]) to show that morphisms of the form $\mathbf{y}f$ in $\mathcal{C}_{ex}$ are choice maps.

[1] Some mathematicians call such objects "regular projectives", but as this is to distinguish them from a class of objects that does not concern me, I do not follow their terminology.

**Lemma 3.1.6** *In an exact completion $\mathcal{C}_{ex}$ of a cartesian category $\mathcal{C}$, the external and internal projectives coincide.*

**Proof.** An internal projective is also externally projective, because in an exact completion the terminal object 1 is projective. An external projective is also internally projective, because in an exact completion, every object is covered by an external projective and external projectives are closed under products. □

## 3.2 Two existence results for W-types

For the main theorem of this Chapter, explaining which categories have a $\Pi W$-pretopos as exact completion, I need two auxiliary results on the existence of W-types, to be proved here. In both cases I rely essentially on the notion of path, introduced in the previous Chapter. Its main use is to help to define in a predicative fashion a certain predicate or relation, that would in an impredicative context (like that of toposes) be defined using transfinite induction.

To state the first theorem, I need the following definition.

**Definition 3.2.1** A square

$$\begin{array}{ccc} D & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & A \end{array}$$

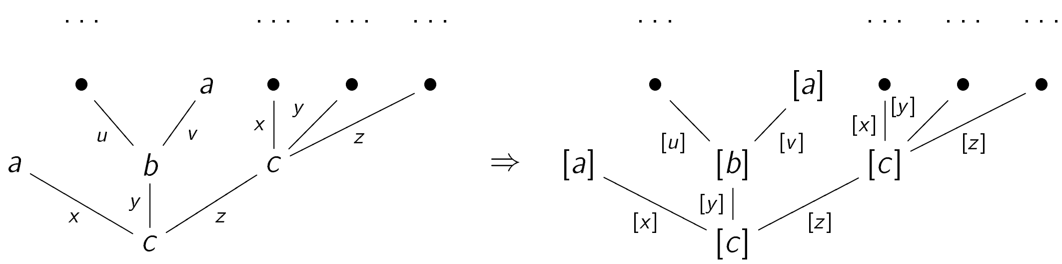
in a cartesian category $\mathcal{C}$ is called a *quasi-pullback*, when the induced map $D \longrightarrow B \times_A C$ is a cover.

**Theorem 3.2.2** *Suppose in a $\Pi$-pretopos $\mathcal{E}$ with a natural number object, one has a diagram of the following form:*

$$\begin{array}{ccc} D & \xrightarrow{[-]_B} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[{[-]_A}]{} & A \end{array} \tag{3.1}$$

*Suppose furthermore that this diagram is a quasi-pullback and that $g$ is a choice map for which there exists a W-type. Then there also exists a W-type for $f$.*

**Proof.** Write $W$ for the W-type for $g$ and sup for the structure map. The idea is to use the well-founded trees in $W$, whose branching type is determined by $g$, to represent well-founded trees whose branching type is determined by $f$. Intuitively this representation works as follows: a well-founded tree with branching type determined by $f$ is represented by an element $w \in W$ if it can be obtained by "bracketing" all labels in $w$.




While every tree with branching type determined by $f$ can be so represented, not every element in $W$ is suitable for representing such a tree. A tree $\sup_c(t)$ in $W$ is suitable for representing, or *representing* for short, whenever for any pair $d, d' \in g^{-1}(c)$ such that $[d]_B = [d']_B$, the trees $td$ and $td'$ are representing and represent the same tree. The trees $td$ and $td'$ are then identified in the bracketing process.

So the question is when two (representing) elements $\sup_c(t)$ and $\sup_{c'}(t')$ in $W$ represent the same tree (in which case I will write $\sup_c(t) \sim \sup_{c'}(t')$). They do, whenever $[c]_A = [c']_A$ and $td \sim t'd'$ for all pairs $d \in g^{-1}(c), d' \in g^{-1}(c')$. In an impredicative context, like the internal logic of a topos, one could define $\sim$ as the unique relation having this property. Here, with a predicative metatheory, one has to work a little harder and define $\sim$ explicitly in terms of paths. Then the property of being representing can be defined as being self-related via $\sim$.

The binary relation $\sim$ on $W$ is defined as follows: $w \sim w'$ if and only if

> all paths $\sigma$ in $\text{Paths}_w$ and $\sigma'$ in $\text{Paths}_{w'}$ having the same length ($2n+1$ say) and satisfying the equality
>
> $$[\sigma(2k+1)]_B = [\sigma'(2k+1)]_B$$
>
> for all $k < n$, also satisfy the equality
>
> $$[\rho(\sigma(2k))]_A = [\rho(\sigma'(2k))]_A$$
>
> for all $k \leq n$ ($\rho$ being the canonical map $W \cong \Sigma_C W^g \longrightarrow C$).

The reader should now verify that $\sim$ has the desired property:

$$\sup_c(t) \sim \sup_{c'}(t'),$$

if and only if $[c] = [c']$ and for all $d \in g^{-1}(c), d' \in g^{-1}(c')$: if $[d] = [d']$, then $td \sim t'd'$ (one proves this by induction).

Symmetry and transitivity of $\sim$ now follow. Symmetry is immediate, while transitivity

> $w \sim w'$ and $w' \sim w''$ imply $w \sim w''$

is proved by induction on $w'$, as follows. Suppose $\mathrm{sup}_c(t) \sim \mathrm{sup}_{c'}(t')$ and $\mathrm{sup}_{c'}(t') \sim \mathrm{sup}_{c''}(t'')$. Now clearly $[c] = [c'']$, because $[c] = [c']$ and $[c''] = [c']$. Suppose $d \in g^{-1}(c)$ and $d'' \in g^{-1}(c'')$ are such that $[d] = [d'']$. Since diagram (3.1) is a quasi-pullback, there exists a $d' \in g^{-1}(c')$ such that $[d'] = [d] = [d'']$. One has now that $td \sim t'd'$ and $t'd' \sim t''d''$, and so $td \sim t''d''$ by induction hypothesis. This shows that $\mathrm{sup}_c(t) \sim \mathrm{sup}_{c''}(t'')$.

A $w = \mathrm{sup}_c(t) \in W$ such that $w \sim w$ will be called a *representing tree*. The point is that such a tree has the desired property that for any pair $d, d' \in g^{-1}(c)$, $td$ and $td'$ represent the same tree. Denote the set of all representing trees by $R$ and observe that $R$ is closed under subtrees.

Now, $\sim$ is an equivalence relation on $R$ and hence one can form the quotient $V$, together with a quotient map $q\colon R \longrightarrow V$. This map $q$ sends a representing tree to the tree it represents. Let me also define an object $R^*$ in $\mathcal{E}/C$ by setting for $c \in C$:

$$R^*_c \quad = \quad \{\, t \in W^{g^{-1}(c)} \mid \mathrm{sup}_c(t) \in R \,\}$$

Or, equivalently: $t \in R^*_c$ if and only if

> for any $d, d' \in g^{-1}(c)$ such that $[d]_B = [d']_B$, one has that $t(d) \sim t(d')$.

One clearly has a commuting diagram

$$\begin{array}{ccc} \Sigma_C R^* & \rightarrowtail & \Sigma_C W^g \\ {\scriptstyle \mathrm{sup}}\downarrow & & \downarrow{\scriptstyle \mathrm{sup}} \\ R & \rightarrowtail & W \end{array}$$

(in fact, this diagram is a pullback). I will now construct a commuting diagram of the following form:

$$\begin{array}{ccc} \Sigma_C R^* & \xrightarrow{q^*} & \Sigma_A V^f \\ {\scriptstyle \mathrm{sup}}\downarrow & & \downarrow{\scriptstyle s} \\ R & \xrightarrow[q]{} & V \end{array} \qquad (3.2)$$

To see that there is a morphism $q^*\colon \Sigma_C R^* \longrightarrow \Sigma_A V^f$ in $\mathcal{E}$, one needs to note that the subobject

$$Q^* \quad = \quad \{\, (t, h) \in \Sigma_C R^* \times \Sigma_A V^f \mid Q^*(t, h) \,\}$$

where $Q^*(t, h)$ is the statement:

> for the particular $c \in C$ and $a \in A$ such that $t \in R^*_c$ and $h \in V^{f^{-1}(a)}$, one has that $[c]_A = a$ and for all $d \in g^{-1}(c)$ that $q(t(d)) = h([d]_B)$.

is functional (for the definition of a functional relation, see Appendix A). The map $q^*$ so constructed is a cover: for let $h$ be an arbitrary element of $V^{f^{-1}(a)}$ for a certain $a \in A$. Pick a $c \in C$ such that $[c]_A = a$. One has

$$\forall d \in g^{-1}(c)\, \exists r \in R \colon q(r) = h([d]_B),$$

since $q$ is a cover. Since $g$ is a choice map, there is a map $t \colon g^{-1}(c) \longrightarrow R$ such that $q(t(d)) = h([d]_B)$ for all $d \in g^{-1}(c)$. If $d, d' \in g^{-1}(c)$ are such that $[d]_B = [d']_B$, then

$$q(t(d)) = h([d]_B) = h([d']_B) = q(t(d')),$$

so $t(d) \sim t(d')$. This means that $t \in R^*_c$ and hence that $(t, h) \in Q^*$. Since $h$ was arbitrary, this means that $q^*$ is a cover.

One now constructs $s \colon \Sigma_A V^f \longrightarrow V$ in $\mathcal{E}$ by using the fact that in a pretopos every epi is the coequaliser of its kernel pair. So suppose for certain $c, c' \in C$ elements $t \colon g^{-1}(c) \longrightarrow W \in R^*$ and $t' \colon g^{-1}(c') \longrightarrow W \in R^*$ are given such that $q^*(t) = q^*(t')$. This implies that $[c]_A = [c']_A$ and that

$$\forall d \in g^{-1}(c), d' \in g^{-1}(c') \colon [d]_B = [d']_B \Rightarrow td \sim t'd'.$$

This means that $\sup_c(t) \sim \sup_{c'}(t')$. Using the coequaliser property of $q^*$, this gives a morphism $s \colon \Sigma_A V^f \longrightarrow V$ making (3.2) commute.

This map $s$ is actually monic. For suppose $s_a(h) = s_{a'}(h')$ for some $h \colon f^{-1}(a) \longrightarrow V$ and $h' \colon f^{-1}(a') \longrightarrow V$. There are $t \colon g^{-1}(c) \longrightarrow W$ and $t' \colon g^{-1}(c') \longrightarrow W$, both in $\Sigma_C R^*$, such that $q^* t = h$ and $q^* g' = h'$. But now $q\sup_a(t) = q\sup_{a'}(t')$, i.e. $\sup_a(t) \sim \sup_{a'}(t')$. But this implies $[c] = [c']$, so $a = a'$, and also that for all $d \in g^{-1}(c)$ and $d' \in g^{-1}(c')$ such that $[d] = [d']$, $td \sim t'd'$. Hence $q^* t = q^* t'$ and $h = h'$. So $s$ is monic. But as $s$ is also clearly epic, $s$ is in fact an isomorphism.

I now claim that the $P_f$-algebra $(W, s \colon P_f(W) \longrightarrow W)$ is actually the W-type for $f$. I work towards applying Theorem 2.1.5.

Now, if $S$ is a subalgebra of $V$, i.e. a subobject of $V$ for which one has that

$$\forall a \in A\, \forall h \colon f^{-1}(a) \longrightarrow V\, (\forall b \in f^{-1}(a)\, (hb \in S) \Rightarrow s_a(h) \in S),$$

let $T$ be the following subobject of $W$:

$$\{\, w \in W \mid \text{if } w \text{ is representing, then } q(w) \in S \,\}.$$

I prove that $T = W$ as subobjects of $W$ by induction. This will immediately imply that $S = V$ as subobjects of $V$. Suppose $w = \sup_c(t) \in W$ is such that $td \in T$ for all $d \in g^{-1}(c)$. I assume that $w$ is representing and want to prove that $qw \in S$.

Because $w$ is representing, the trees $td$ $(d \in g^{-1}(c))$ are representing as well. Since they belong to $T$, $q(td)$ belongs to $S$. This means that for $h = q^* t$, $hb \in S$ for all $b \in f^{-1}(a)$, where $a = [c]$. So $s_a(h) \in S$, but $s_a(h) = s_a q^*(t) = q\sup_c(t)$.

So $V$ is the W-type for $f$ by Theorem 2.1.5 and the proof of Theorem 3.2.2 is completed. □

**Theorem 3.2.3** *Let $\mathcal{E}$ be a Π-pretopos with natural number object and $f: B \longrightarrow A$ be a choice map in $\mathcal{E}$. Assume that in a $P_f$-coalgebra $V$ with the following two properties exists: (1) its structure map $s$ is a cover; (2) the only subobject $R$ of $V$ for which*

$$v \in V, s(v) = (a, t) \text{ and } tb \in R \text{ for all } b \in f^{-1}(a) \text{ imply that } v \in R$$

*is the subobject $V$ itself. Then a W-type for $f$ exists.*

**Proof.** The idea is to turn $s$ into an isomorphism. This means identifying those $v$ and $v'$, with the property that for $(a, t) = s(v)$ and $(a', t') = s(v')$, one has that both $a = a'$ and $t$ and $t'$ are extensionally equal functions. In other words, I need a relation $\sim$ on $V$ such that:

$$v \sim v' \quad \Leftrightarrow \quad \text{if } (a, t) = s(v) \text{ and } (a', t') = s(v'), \text{ then } a = a' \text{ and } tb \sim t'b \text{ for all } b \in f^{-1}(a). \tag{3.3}$$

In other contexts, I might turn to a transfinite induction to construct such a relation, but here I again rely on paths.

First, I define an equivalence relation on the object of paths in $V$. I will call $\sigma$ and $\sigma'$ equivalent if they satisfy three conditions:

1. they have the same length, $2n + 1$ say.
2. they satisfy the equation
$$\sigma(2k + 1) = \sigma'(2k + 1)$$
for all $k < n$.
3. they satisfy the equation
$$\rho(\sigma(2k)) = \rho(\sigma'(2k))$$
for all $k \leq n$ ($\rho$ being the root map).

Then I define the following equivalence relation on $V$:

$$v \sim v' \quad \text{iff} \quad \text{for every } \sigma \text{ in } \text{Paths}_v \text{ there exists an equivalent } \sigma' \text{ in } \text{Paths}_{v'} \text{ and for every } \sigma' \text{ in } \text{Paths}_{v'} \text{ there exists an equivalent } \sigma \text{ in } \text{Paths}_v.$$

The reader should verify that $\sim$ now has the desired property 3.3.

Consider the quotient $W = V/\sim$ and the quotient map $q\colon V \longrightarrow W$. Note that $P_f q$ is also a cover, because $f$ is a choice map. I now want to complete the following diagram:

$$\begin{array}{ccc} V & \overset{q}{\twoheadrightarrow} & V \\ {\scriptstyle s}\downarrow & & \vdots\,{\scriptstyle m} \\ P_f V & \underset{P_f q}{\twoheadrightarrow} & P_f V. \end{array}$$

Using that in a pretopos every epi is the coequaliser of its kernel pair and the fact that $\sim$ satisfies 3.3, one can show that an isomorphism $m$ making the diagram commute, exists. Call its inverse $n$.

The proof will be completed once I show that $(W, n\colon P_f W \longrightarrow W)$ satisfies the conditions of Theorem 2.1.5. $n$ is certainly mono, so let $A$ be an arbitary $P_f$-subalgebra of $W$. Define

$$R = \{\, v \in V \mid q(v) \in A \,\}$$

It is easy to see that $R$ satisfies the hypothesis of condition (2): for assume $s(v) = (a, t)$ and $tb \in R$ for all $b \in f^{-1}(a)$. This means that $q(tb) \in A$ for all $b \in f^{-1}(a)$, and hence $n_a(qt) \in A$ because $A$ is subalgebra of $W$. But $n_a(qt) = (nP_f q)(a, t) = (nP_f qs)(v) = q(v)$. So $R = V$ and hence $A = W$. □

## 3.3 $\Pi W$-pretoposes as exact completions

This Section isolates a set of conditions on a cartesian category $\mathcal{C}$ sufficient for its exact completion to be a $\Pi W$-pretopos. Sufficient (and necessary) conditions for the exact completion to be a $\Pi$-pretopos can be extracted from the literature, but sufficient conditions for the exact completion to have W-types were unknown. I will recall the results available from the literature and then introduce the notion of a "weak W-type". In this way, I arrive at the notion of a "weak $\Pi W$-pretopos", and prove the main theorem of this Section:

**Theorem 3.3.1** *If $\mathcal{C}$ is a weak $\Pi W$-pretopos, then $\mathcal{C}_{ex}$ is a $\Pi W$-pretopos.*

How this can be used to give more examples of $\Pi W$-pretoposes will be the subject of the next Section.

The following terminology and results are taken from the literature, especially Menni's PhD thesis [59]. $\mathcal{C}$ is always a cartesian category.

**Proposition 3.3.2** *(See [59], proposition 4.4.1.) The exact completion of $\mathcal{C}$ is a pretopos if and only if $\mathcal{C}$ has finite, disjoint and stable sums. In this case, the embedding* $\mathbf{y}\colon \mathcal{C} \longrightarrow \mathcal{C}_{ex}$ *preserves the sums.*

For the exact completion to be locally cartesian closed, one weakens the requirement for dependent products, by dropping the uniqueness clause. So:

**Definition 3.3.3** For two morphisms $c: C \longrightarrow J$ and $t: J \longrightarrow I$ in a cartesian category $\mathcal{C}$, the *dependent product* of $c$ along $t$ is an object $w: W \longrightarrow I$ in $\mathcal{C}/I$ together with a morphism $\varepsilon: t^*w \longrightarrow c$ in $\mathcal{C}/J$ such that for any object $m: M \longrightarrow I$ in $\mathcal{C}/I$ together with a morphism $h: t^*m \longrightarrow c$ in $\mathcal{C}/J$ there exists a unique morphism $H: m \longrightarrow w$ in $\mathcal{C}/I$ such that $h = \varepsilon \circ t^*H$ in $\mathcal{C}/J$.

**Definition 3.3.4** For two morphisms $c: C \longrightarrow J$ and $t: J \longrightarrow I$ in a cartesian category $\mathcal{C}$, a *weak dependent product* of $c$ along $t$ is an object $w: W \longrightarrow I$ in $\mathcal{C}/I$ together with a morphism $\varepsilon: t^*w \longrightarrow c$ in $\mathcal{C}/J$ such that for any object $m: M \longrightarrow I$ in $\mathcal{C}/I$ together with a morphism $h: t^*m \longrightarrow c$ in $\mathcal{C}/J$ there exists a (not necessarily unique) morphism $H: m \longrightarrow w$ in $\mathcal{C}/I$ such that $h = \varepsilon \circ t^*H$ in $\mathcal{C}/J$. One says that a cartesian category $\mathcal{C}$ has weak dependent products if it has all possible weak dependent products.

The following proposition is contained in [22] (see also [16]):

**Proposition 3.3.5** *The exact completion $\mathcal{C}_{ex}$ of a cartesian category $\mathcal{C}$ is locally cartesian closed if and only if $\mathcal{C}$ has weak dependent products.*

**Remark 3.3.6** Unfortunately, the authors do not point out, although it follows from their proofs, that in case $\mathcal{C}$ has genuine dependent products, the embedding $\mathbf{y}: \mathcal{C} \longrightarrow \mathcal{C}_{ex}$ preserves them. Hence the following argument.

By Proposition 3.1.5, it suffices to show that $\mathbf{y}$ preserves exponentials. How are exponentials of projectives $A$ and $B$ computed in $\mathcal{C}_{ex}$? It is not hard to see that you can compute $B^A$ in $\mathcal{C}$ and obtain the exponential in $\mathcal{C}_{ex}$ by taking the quotient of the following equivalence relation:

$$R = \{(f, g) \,|\, \forall b \in B. f(b) = g(b)\} \rightrightarrows \mathbf{y}(B^A).$$

For the purpose of computing the universal quantifier $\forall b \in B$, let me introduce the notion of a *proof*.

For any object $X$ in $\mathcal{C}$, pre-order the slice category $\mathcal{C}/X$ by declaring that

$$A \longrightarrow X \leq B \longrightarrow X,$$

whenever there is a morphism $A \longrightarrow B$ making the obvious triangle commute. The set of *proofs* (or *weak subobjects*) Prf $X$ is then the poset obtained by identifying $A \longrightarrow X$ and $B \longrightarrow X$ in case both $A \longrightarrow X \leq B \longrightarrow X$ and $B \longrightarrow X \leq A \longrightarrow X$. Clearly, any morphism $f: Y \longrightarrow X$ in $\mathcal{C}$ induces an order-preserving map $f^*: \text{Prf } X \longrightarrow \text{Prf } Y$ by pullback. The fact that $\mathcal{C}$ has weak dependent products means precisely that $f^*$ always has a right adjoint. When $\mathcal{C}$ has genuine dependent products, these right adjoints can be computed by taking these real dependent products.

The functor $\mathbf{y}$ now induces an order-preserving bijection $\mathrm{Prf}\, X \longrightarrow \mathrm{Prf}\, \mathbf{y}X$, basically by taking images, commuting with $f^*$ for any morphism $f: Y \longrightarrow X$ in $\mathcal{C}$. This means that in $\mathcal{C}_{ex}$, the operation of pulling back subobjects along a morphism $f: Y \longrightarrow X$ between projectives, has a right adjoint, and for this reason universal quantifiers along such $f$ exist. The universal quantifier that concerns me is precisely of this form, so, in a way, it can be computed "type-theoretically" (by taking $\Pi_f$) in the original category $\mathcal{C}$.

Therefore, to compute $R$ in $\mathcal{C}_{ex}$, I should take the following object in $\mathcal{C}$:

$$\Sigma_{f,g \in A^B} \Pi_{b \in B} \{ * \mid f(b) = g(b) \}.$$

But this is just $A^B$, because the principle of extensionality holds in $\mathcal{C}$. So the equivalence relation in question takes the following form:

$$\mathbf{y}(B^A) \rightrightarrows \mathbf{y}(B^A).$$

Hence its quotient is simply $\mathbf{y}(B^A)$, and therefore $\mathbf{y}$ preserves exponentials.

There are a number of special cases of the notion of a weak dependent product that will be important later on. There is the *weak exponential*, which is a weakening of the familiar notion of an exponential. A weak version of the exponential $Y^X$ can be defined as a weak dependent product of the projection $X \times Y \longrightarrow X$ along $X \longrightarrow 1$. More concretely this means that it is an object $Z$ together with a "weak evaluation" $\varepsilon: Z \times X \longrightarrow Y$ such that for every map $h: X \times A \longrightarrow Y$ there is a (not necessarily unique) morphism $H: A \longrightarrow Z$ such that $h = \varepsilon \circ (X \times H)$.

Furthermore, there is the notion of a weak simple product. Not surprisingly, this is the weakening of the notion of a simple product, which may not be so familiar. One calls

$$\begin{array}{ccc} W \times K & \xrightarrow{\epsilon} & C \\ & \searrow^{w \times K} & \downarrow c \\ & & I \times K \end{array} \tag{3.4}$$

a *simple product diagram*, if for any other such diagram

$$\begin{array}{ccc} X \times K & \xrightarrow{f} & C \\ & \searrow^{x \times K} & \downarrow c \\ & & I \times K \end{array}$$

there exists a unique $f': X \longrightarrow W$ over $I$ such that $f = \epsilon \circ (f' \times K)$. In this case $w: W \longrightarrow I$ together with $\epsilon$ will be the *simple product* of $c: C \longrightarrow I \times K$ with respect to $K$. (Observe that this is equivalent to being the dependent product of $c\colon C \longrightarrow I \times K$ along the projection $I \times K \longrightarrow I$.)

If one drops the uniqueness condition for $f'$, then diagram (3.4) is called a *weak simple product diagram*. $w$ together with $\epsilon$ will be called a *weak simple product* and this is equivalent to being a weak dependent product of $c$ along the projection $I \times K \to I$.

In addition, one can weaken the notion of a natural number object.

**Definition 3.3.7** Let $\mathcal{C}$ be a cartesian category. A diagram

$$1 \longrightarrow A \longrightarrow A$$

is called an *inductive structure*. $t\colon A \longrightarrow B$ is a morphism of inductive structures with domain $1 \longrightarrow A \to A$ and codomain $1 \longrightarrow B \longrightarrow B$, if the following diagram commutes:

$$\begin{array}{ccccc} & \nearrow & A & \longrightarrow & A \\ 1 & & \downarrow t & & \downarrow t \\ & \searrow & B & \longrightarrow & B \end{array}$$

A *natural number object* is an inductive structure

$$1 \xrightarrow{0} N \xrightarrow{s} N$$

that is initial in the category of inductive structures. It is a *weak natural number object* if it is weakly initial in the category of inductive structures (meaning that for any inductive structure $1 \longrightarrow A \longrightarrow A$ there exists a morphism of inductive structures $t\colon N \longrightarrow A$).

The following result follows from Proposition 5.1 in [16]:

**Proposition 3.3.8** *If $\mathcal{C}$ is cartesian category with weak dependent products and a weak natural number object, then $\mathcal{C}_{ex}$ has a natural number object.*

The results contained in the literature can therefore be summarized as follows:

**Corollary 3.3.9** *When $\mathcal{C}$ is a cartesian category with finite, disjoint sums, weak dependent products and a weak natural number object, $\mathcal{C}_{ex}$ is a $\Pi$-pretopos with a natural number object.*

**Proof.** Combine Proposition 3.3.2, Proposition 3.3.5 and Proposition 3.3.8. □

What is missing is a sufficient condition for the exact completion to have W-types. To fill this gap, I will introduce the notion of a "weak W-type", inspired by Theorem 2.1.5, and subsequently prove that it has the desired property. Unfortunately, it is also rather involved. To make things easier it will be good to set some notation and terminology.

Fix a morphism $f\colon B \longrightarrow A$ in a cartesian category $\mathcal{C}$. As in any cartesian category, one has for any object $X$ in $\mathcal{C}$ two functors: $X^*\colon \mathcal{C} \longrightarrow \mathcal{C}/X$ (the pullback along $X \longrightarrow 1$) and $\Sigma_X\colon \mathcal{C}/X \longrightarrow \mathcal{C}$ (its left adjoint, given by composition).

**Definition 3.3.10** A *$P_f$-structure* is a quadruple (4-tuple) $\mathbf{x} = (X, X^*, \sigma_X, \varepsilon_X)$ with $X$ an object in $\mathcal{C}$, $X^*$ an object in $\mathcal{C}/A$, $\sigma_X$ a map $\Sigma_A(X^*) \to X$ in $\mathcal{C}$ and $\varepsilon_X$ a map $X^* \times f \longrightarrow A^*X$ in $\mathcal{C}/A$. A homomorphism of $P_f$-structures from $\mathbf{x} = (X, X^*, \sigma_X, \varepsilon_X)$ to $\mathbf{z} = (Z, Z^*, \sigma_Z, \varepsilon_Z)$ is a pair $\mathbf{t} = (t, t^*)$, where $t$ is a map in $\mathcal{C}$ from $X$ to $Z$, and $t^*$ is a map from $X^*$ to $Z^*$ in $\mathcal{C}/A$. Furthermore, the following diagrams should commute:

$$\begin{array}{ccc} \Sigma_A X^* & \xrightarrow{\Sigma_A t^*} & \Sigma_A Z^* \\ {\scriptstyle \sigma_X}\downarrow & & \downarrow{\scriptstyle \sigma_Z} \\ X & \xrightarrow[t]{} & Z \end{array} \qquad \begin{array}{ccc} X^* \times f & \xrightarrow{t^* \times f} & Z^* \times f \\ {\scriptstyle \varepsilon_X}\downarrow & & \downarrow{\scriptstyle \varepsilon_Z} \\ A^*X & \xrightarrow[A^*t]{} & A^*Z \end{array}$$

It is easy to see that this defines a category, one I shall denote by $P_f(\mathcal{C})$.

**Definition 3.3.11** A map $\mathbf{t}\colon \mathbf{x} \longrightarrow \mathbf{z}$ in $P_f(\mathcal{C})$ is said to be a *weak $P_f$-substructure map*, if for the pullback $L$ in this diagram in $\mathcal{C}/A$:

$$\begin{array}{ccc} L & \xrightarrow{p_0} & Z^* \times f \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \varepsilon_Z} \\ A^*X & \xrightarrow[A^*t]{} & A^*Z \end{array}$$

the following is a weak simple product diagram:

$$\begin{array}{ccc} X^* \times f & \xrightarrow{\alpha_X} & L \\ & {\scriptstyle t^* \times f}\searrow & \downarrow{\scriptstyle p_0} \\ & & Z^* \times f \end{array}$$

where $\alpha_X = \langle (t^* \times f), \varepsilon_X \rangle$.

Before I can define weak W-types, I first have to define the notion of a weak $P_f$-algebra.

**Definition 3.3.12** A *weak $P_f$-algebra* is a $P_f$-structure

$$\mathbf{x} = (X, X^*, \sigma_X, \varepsilon_X)$$

such that $X^*$ is a weak version of the exponential $(A^*X)^f$ in $\mathcal{C}/A$ with $\varepsilon_X$ as weak evaluation map. The morphisms of weak $P_f$-algebras are simply the morphisms of $P_f$-structures. (So the category of weak $P_f$-algebras is a full subcategory of the category of $P_f$-structures.)

**Definition 3.3.13** A morphism $\mathbf{t} = (t, t^*: \mathbf{x} \longrightarrow \mathbf{z})$ is a *weak $P_f$-subalgebra*, if it is a weak $P_f$-substructure and both $\mathbf{x}$ and $\mathbf{z}$ are weak $P_f$-algebras.

It would have been enough to require that $\mathbf{z}$ is weak $P_f$-algebra, in view of the following easy lemma.

**Lemma 3.3.14** *If $\mathbf{t}: \mathbf{x} \longrightarrow \mathbf{z}$ is a weak $P_f$-substructure map and $\mathbf{z}$ is a weak $P_f$-algebra, then so is $\mathbf{x}$.*

And finally:

**Definition 3.3.15** A *weak W-type for $f$* is a weak $P_f$-algebra $\mathbf{v}$ with two properties: (i) its structure map $\sigma_V$ is an isomorphism; (ii) every weak $P_f$-subalgebra $\mathbf{i}: \mathbf{x} \longrightarrow \mathbf{v}$ has a section.

The second property (ii) is supposed the be a weakening of the property of having no proper subalgebras. Although very technical, I would like to stress that the property is precisely what one would expect, in that it is the strong property with uniqueness clauses dropped and subobjects replaced by "weak subobjects" or "proofs" (see above).

In the definition, it would have been sufficient to require that the structure map $\sigma_V$ is monic, because of the following lemma:

**Lemma 3.3.16** *If $\mathbf{w} = (W, W^*, \sigma_W, \varepsilon_W)$ is a weak $P_f$-algebra for some $f$ in a cartesian category $\mathcal{C}$ with weak dependent products with the property that every weak $P_f$-subalgebra $\mathbf{t}: \mathbf{x} \longrightarrow \mathbf{w}$ has a section, then the structure map $\sigma_W$ has a section.*

(For those who know how to derive Lambek's result concerning initial algebras, proving this lemma should be easy.)

**Lemma 3.3.17** *Let $\mathcal{C}$ be a locally cartesian closed category. A W-type $W_f$ for a morphism $f: B \longrightarrow A$ is also a weak W-type for $f$.*

**Proof.** It is easy to see that $W_f$ can be considered as a weak $P_f$-algebra $\mathbf{w}$. Then the first condition for being a weak W-type is certainly satisfied, because $\sup: P_f W_f \longrightarrow W_f$ is an isomorphism. To verify the second condition, let $\mathbf{x} = (X, X^*, \sigma_X, \varepsilon_X)$ be any weak $P_f$-algebra and $\mathbf{t} = (t, t^*): \mathbf{x} \longrightarrow \mathbf{w}$ be a weak $P_f$-subalgebra morphism in $\mathcal{C}$. Because

**t** is a weak $P_f$-subalgebra, there is a morphism $r\colon (A^*X)^f \longrightarrow X^*$ in $\mathcal{C}/A$ such that $t^*r = (A^*t)^f$. Now $(X, \sigma_X \Sigma_A r\colon P_f X \longrightarrow X)$ is a $P_f$-algebra and $t$ is a morphism of $P_f$-algebras from this algebra to $W_f$. Hence $t$ has a section $u$ in the category of $P_f$-algebras. Then $\mathbf{s} = (u, r(A^*u)^f)$ is a section of **t**. □

**Definition 3.3.18** A cartesian category $\mathcal{C}$ is called a *weak* $\Pi W$*-pretopos*, if it has finite disjoint and stable sums, weak dependent products, a weak natural number object and weak W-types for all morphisms.

Now the main theorem of this Section has a precise meaning.

**Theorem 3.3.19** *(= Theorem 3.3.1.) If $\mathcal{C}$ is a weak $\Pi W$-pretopos, then $\mathcal{C}_{ex}$ is a $\Pi W$-pretopos.*

To prove this theorem, it suffices to show that $\mathcal{C}_{ex}$ has W-types for all maps lying in the image of **y** (proof: use the remark before Lemma 3.1.6 to see that these are choice maps and then apply Theorem 3.2.2). So the main theorem will follow from:

**Proposition 3.3.20** *Let $\mathcal{C}$ be a cartesian category with finite disjoint and stable sums, weak dependent products and a weak natural number object. If $\mathcal{C}$ has a weak W-type for a map $f$ in $\mathcal{C}$, then $\mathcal{C}_{ex}$ has a genuine W-type for the map* $\mathbf{y}f$.

To prove Proposition 3.3.20, I will make use of Theorem 3.2.3. What I show is that if $\mathbf{w} = (W, W^*, \sigma_W, \varepsilon_W)$ is a weak W-type in $\mathcal{C}$ for a map $f\colon B \longrightarrow A$, then $\mathbf{y}W$ has the structure of a $P_{\mathbf{y}f}$-coalgebra in $\mathcal{C}_{ex}$, with the special properties formulated in Theorem 3.2.3. This is established by the following sequence of lemmas.

**Warning 3.3.21** In the remainder of this Section, I will drop the occurences of **y**; I trust that the reader will not get confused.

From now on, suppose $\mathcal{C}$ is a cartesian category with finite disjoint and stable sums, weak dependent products and a weak natural number object, and suppose that $\mathbf{w} = (W, W^*, \sigma_W, \varepsilon_W)$ is a weak W-type for a map $f\colon B \longrightarrow A$ in $\mathcal{C}$.

**Lemma 3.3.22** *The unique map $q\colon W^* \longrightarrow (A^*W)^f$ in $\mathcal{C}_{ex}/A$ such that*

$$\begin{array}{ccc} W^* \times f & \xrightarrow{\quad q \times f \quad} & (A^*W)^f \times f \\ & {\scriptstyle \varepsilon_W} \searrow \quad \swarrow {\scriptstyle ev} & \\ & A^*W & \end{array}$$

*commutes is a cover.*

**Proof.** Since **w** is a weak $P_f$-algebra, one knows that $W^*$ is a weak version of $(A^*W)^f$ in $\mathcal{C}/A$. One can now define the equivalence relation

$$R_a = \{\, (g, h) \in W^*_a \times W^*_a \,|\, \forall b \in f^{-1}(a)\colon \varepsilon_W(g, b) = \varepsilon_W(h, b) \,\}$$

on $W^*_a$ $(a \in A)$ in $\mathcal{C}_{ex}/A$. It is not difficult to see that the quotient $W^*/R$ in $\mathcal{C}_{ex}/A$ is a strong version of the exponential $(A^*W)^f$. So $q$ is (up to iso) the quotient map and hence a cover. ☐

This establishes that $W$ has the structure of a $P_f$-coalgebra in $\mathcal{C}_{ex}$, with an epic structure map

$$n\colon W \overset{\sigma_W^{-1}}{\twoheadrightarrow} \Sigma_A W^* \overset{\Sigma_A q}{\twoheadrightarrow} P_f W = \Sigma_A (A^*W)^f.$$

Notice that **w** is also a $P_f$-structure in $\mathcal{C}_{ex}$, via **y**.

**Lemma 3.3.23** *If* $\mathbf{r} = (R, R^*, \sigma_R, \varepsilon_R)$ *is a* $P_f$*-structure in* $\mathcal{C}_{ex}$ *and* $\mathbf{t}\colon \mathbf{r} \longrightarrow \mathbf{w}$ *is a weak* $P_f$*-substructure map, then* $\mathbf{t}$ *has a section in* $P_f(\mathcal{C}_{ex})$*.*

**Proof.** Consider the pullback $L$ in $\mathcal{C}_{ex}/A$ in the diagram

$$\begin{array}{ccc} L & \xrightarrow{p_0} & W^* \times f \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon_W} \\ A^*R & \xrightarrow[A^*t]{} & A^*W \end{array}$$

Since $\mathbf{t}$ is a weak $P_f$-substructure map, the following is a weak simple product diagram:

$$\begin{array}{ccc} R^* \times f & \xrightarrow{\alpha_R} & L \\ & {\scriptstyle t^* \times f}\searrow & \big\downarrow{\scriptstyle p_0} \\ & & W^* \times f \end{array}$$

where $\alpha_R = \langle (t^* \times f), \varepsilon_R \rangle$.

Let $\xi\colon K \longrightarrow R$ be a cover by an object in the image of **y**. Now consider the following two pasted pullback diagrams:

$$\begin{array}{ccccc} L' & \xrightarrow{l_0} & L & \xrightarrow{p_0} & W^* \times f \\ {\scriptstyle l_1}\big\downarrow & & {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon_W} \\ A^*K & \xrightarrow[A^*\xi]{} & A^*R & \xrightarrow[A^*t]{} & A^*W \end{array}$$

Since the objects $K$, $W$ and $W^* \times f$ lie in the image of **y**, and since this functor preserves pullbacks, I may assume that $L'$ also lies in the image of **y**.

Construct the following pullback:

$$\begin{array}{ccc} L'' & \xrightarrow{j_0} & L' \\ {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle l_0} \\ R^* \times f & \xrightarrow[\alpha_R]{} & L \end{array}$$

And construct the strong version of $\Pi_{\pi_0}(j_1)$ in $\mathcal{C}_{ex}/A$ (where $\pi_0$ is the projection $R^* \times f \longrightarrow R^*$). This means that one has an object $K_0^*$ with maps $\xi_2^* \colon K_0^* \longrightarrow R^*$ and $\alpha_{K_0} \colon K_0^* \times f \longrightarrow L''$ such that

$$\begin{array}{ccc} K_0^* \times f & \xrightarrow{\alpha_{K_0}} & L'' \\ & {\scriptstyle \xi_2^* \times f}\searrow & \downarrow{\scriptstyle j_1} \\ & & R^* \times f \end{array}$$

is a simple product diagram.

It is not hard to verify that

$$\begin{array}{ccc} K_0^* \times f & \xrightarrow{j_0 \alpha_{K_0}} & L' \\ & {\scriptstyle (t^* \xi_2^*) \times f}\searrow & \downarrow{\scriptstyle p_0 l_0} \\ & & W^* \times f \end{array}$$

is a weak simple product diagram. Now let $\xi_1^* \colon K^* \longrightarrow K_0^*$ be a cover by an element in the image of $\mathbf{y}$. This implies that

$$\begin{array}{ccc} K^* \times f & \xrightarrow{\alpha_K} & L' \\ & {\scriptstyle (t^* \xi^*) \times f}\searrow & \downarrow{\scriptstyle p_0 l_0} \\ & & W^* \times f \end{array} \tag{3.5}$$

with $\alpha_K = j_0 \alpha_{K_0}(\xi_1^* \times f)$ and $\xi^* = \xi_2^* \xi_1^*$, can be seen as a weak simple product diagram in $\mathcal{C}/A$.

Using the fact that $K^*$ is projective, one constructs a map $\sigma_K$ making

$$\begin{array}{ccc} \Sigma_A K^* & \xrightarrow{\Sigma_A \xi^*} & \Sigma_A R^* \\ {\scriptstyle \sigma_K}\downarrow & & \downarrow{\scriptstyle \sigma_R} \\ K & \xrightarrow[\xi]{} & R \end{array}$$

commutative. This means that one has a $P_f$-structure $\mathbf{k} = (K, K^*, \sigma_K, \varepsilon_K = l_1 \alpha_K)$ in $\mathcal{C}_{ex}$, that can also be seen as a $P_f$-structure in $\mathcal{C}$, and a $P_f$-structure map ¸ $= (\xi, \xi^*)$ in $\mathcal{C}_{ex}$. Now $\mathbf{t}$¸ can be seen as a $P_f$-structure map in $\mathcal{C}$, and it is actually a weak $P_f$-substructure map in $\mathcal{C}$ (since (3.5) is a weak simple product diagram). Therefore

**k** can be seen as a weak $P_f$-algebra in $\mathcal{C}$, and since **w** is a weak W-type in $\mathcal{C}$, one has a $P_f$-structure map $\mathbf{s}'$ such that $(\mathbf{t}\,¸)\mathbf{s}' = \mathbf{1}_{\mathrm{W}}$ in $P_f(\mathcal{C})$ and $P_f(\mathcal{C}_{ex})$. So $\mathbf{s} = ¸\mathbf{s}'$ is a $P_f$-structure map in $\mathcal{C}_{ex}$ that is a section of **t**. □

**Corollary 3.3.24** *Let $R \subseteq W$ be a subobject in $\mathcal{C}_{ex}$ and assume that the following statement holds in the internal logic of $\mathcal{C}_{ex}$:*

$$\forall w \in W \,(\, \text{If } n(w) = (a, t) \text{ and } \forall b \in f^{-1}(a)\colon tb \in R\,,\ \text{then } w \in R\,). \qquad (3.6)$$

*Then $R = W$ as subobjects of $W$.*

**Proof.** Define the following object in $\mathcal{C}_{ex}/A$: for any $a \in A$

$$R^*_a = \{\, \tau \in W^*_a \mid \forall b \in f^{-1}(a)\colon \mathrm{proj}_W(\varepsilon_W(\tau, b)) \in R\,\}.$$

Or, equivalently:

$$R^*_a = \{\, \tau \in W^*_a \mid \forall b \in f^{-1}(a)\colon q_a(\tau)(b) \in R\,\}.$$

The validity of statement (3.6) implies that for the inclusion map $j^*\colon R^* \longrightarrow W^*$, $\sigma_W \Sigma_A j^*$ factors through $R$. For if $\tau \in R^*_a$, write $w = (\sigma_W \Sigma j^*)(\tau)$. Since $n(w) = (q\Sigma j^*)(\tau) = (a, q(\tau))$, $q(\tau)(b) \in R$ for all $b \in f^{-1}(a)$, and so $w \in R$. Hence there is a map $\sigma_R$ making

$$\begin{array}{ccc} \Sigma R^* & \xrightarrow{\Sigma j^*} & \Sigma W^* \\ {\scriptstyle \sigma_R}\big\downarrow & & \big\downarrow{\scriptstyle \sigma_W} \\ R & \xrightarrow[j]{} & W \end{array}$$

commute. By the first definition of $R^*$, the map $\varepsilon_W(j^* \times f)$ factors through $A^*R$, so one has a map $\varepsilon_R$ making

$$\begin{array}{ccc} R^* \times f & \xrightarrow{j^* \times f} & W^* \times f \\ {\scriptstyle \varepsilon_R}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon_W} \\ A^*R & \xrightarrow[A^*j]{} & A^*W \end{array}$$

commute. So $\mathbf{r} = (R, R^*, \sigma_R, \varepsilon_R)$ is a $P_f$-structure in $\mathcal{C}_{ex}$ and $\mathbf{j} = (j, j^*)$ is a $P_f$-structure map. It is actually a weak $P_f$-substructure map, so **j** has a section $\mathbf{s}\colon \mathbf{w} \longrightarrow \mathbf{r}$. This implies that $j$ is iso, and $R = W$ as subobjects. □

This completes the proof of the main result, Theorem 3.3.1.

## 3.4 More examples of $\Pi W$-pretoposes

I have identified a categorical structure, that of a weak $\Pi W$-pretopos, whose exact completion always is a $\Pi W$-pretopos. Using this, I give several more examples of $\Pi W$-pretoposes. One of them is what one might call a "predicative realisability topos", which is analogous to the realisability toposes in topos theory.

**Exact completion of an ML-category 3.4.1** Any ML-category $\mathcal{C}$ is a weak $\Pi W$-pretopos. It is clear that a category that has genuine dependent products, also has weak dependent products, and in Lemma 3.3.17, I also showed that it has all weak W-types. Therefore:

**Theorem 3.4.2** *The exact completion $\mathcal{C}_{ex}$ of an ML-category $\mathcal{C}$ is again an ML-category. Moreover, the embedding $\mathbf{y}: \mathcal{C} \longrightarrow \mathcal{C}_{ex}$ is a morphism of ML-categories.*

**Proof.** That $\mathcal{C}_{ex}$ is an ML-category is a direct application of Theorem 3.3.1. Of course, $\mathbf{y}$ is cartesian (it always is), but it also preserves the sums and the dependent products by Proposition 3.3.2 and Remark 3.3.6. It remains to check to $\mathbf{y}$ preserves W-types.

Because $\mathbf{y}$ preserves $\Pi$, it is clear that whenever $W$ is the W-type for a morphism $f: B \longrightarrow A$ in $\mathcal{C}$, $\mathbf{y}W$ is also an algebra for $P_{\mathbf{y}f}$ in $\mathcal{C}_{ex}$. It is weakly initial for the following reason: when $X$ is an object with a $P_{\mathbf{y}f}$-algebra structure $t: P_{\mathbf{y}f}X \longrightarrow X$, cover $X$ with a projective $Y$ via some map $q: Y \longrightarrow X$. Since $P_{\mathbf{y}f}Y$ is again projective (because it can be computed in $\mathcal{C}$), the following diagram can be filled:

$$\begin{array}{ccc} P_{\mathbf{y}f}Y & \xrightarrow{P_{\mathbf{y}f}q} & P_{\mathbf{y}f}X \\ \vdots & & \downarrow t \\ Y & \xrightarrow[q]{} & X. \end{array}$$

Therefore $Y$ has the structure of a $P_f$-algebra in $\mathcal{C}$ and there exists a $P_f$-algebra morphism $p: W \longrightarrow Y$. Then $qp$ is a $P_{\mathbf{y}f}$-algebra morphism in $\mathcal{C}_{ex}$.

But then $\mathbf{y}W$ is initial, because it possesses no non-trivial $P_f$-algebra endomorphisms. For if $m: \mathbf{y}W \longrightarrow \mathbf{y}W$ is a $P_{\mathbf{y}f}$-algebra morphism in $\mathcal{C}_{ex}$, $m$ is also a $P_f$-algebra morphism in $\mathcal{C}$, since $\mathbf{y}$ is full and faithful. Therefore $m$ is the identity on $\mathbf{y}W$. This is sufficient to prove that $\mathbf{y}W$ is initial, because whenever $s, t: \mathbf{y}W \longrightarrow X$ are two $P_f$-algebra morphisms, their equaliser $i: E \longrightarrow \mathbf{y}W$ is also a $P_f$-algebra, with $i$ preserving this structure. Because $\mathbf{y}W$ is weakly initial, there is a $P_f$-algebra morphism $k: \mathbf{y}W \longrightarrow E$. So $ik$ is a $P_f$-algebra endomorphism on $\mathbf{y}W$, hence the identity. Therefore $E = \mathbf{y}W$ as subobjects of $\mathbf{y}W$ and $s = t$. □

This has several consequences. First of all, the exact completions of all the ML-categories discussed in Chapter 1 are $\Pi W$-pretoposes. In general, it shows that it is not a serious loss of generality to require predicative toposes to be exact. The reason for including it is that exactness is very useful in obtaining models of set theory. *Prima facie* this might look unnecessarily restrictive, but I believe that this result shows that this is not so. In particular, it shows that there is no reason not to develop a predicative theory analogous to topos theory only for $\Pi W$-pretoposes and not for general ML-categories, as I am doing in these Chapters.

Secondly, this theorem also shows that $\Pi W$-pretoposes are closed under exact completion. But beware, the inclusion **y** will rarely be a morphism of $\Pi W$-pretoposes. In that case $\mathbf{y}: \mathcal{C} \longrightarrow \mathcal{C}_{ex}$ would be exact, which can only happen when all objects in $\mathcal{C}$ are projective and $\mathcal{C}$ is its own exact completion.

Finally, since the exact completion of a topos (with nno) is seldom again a topos, but it is a $\Pi W$-pretopos, there are many examples of $\Pi W$-pretoposes that are not toposes. It also shows that there is a closure property of "predicative toposes" that has no analogue in the topos-theoretic case. This will be exploited in the next Section.

**Realisability toposes 3.4.3** This example is basically a warm-up exercise for the following one. I am going to prove that the realisability topos $\mathrm{RT}(\mathcal{Q})$ is a $\Pi W$-pretopos. The point is that I try not to rely on the fact that $\mathrm{RT}(\mathcal{Q})$ is a topos with nno, but instead try to give a predicative proof that admits relativisation to a $\Pi W$-pretopos. But that is the next example.

So let $\mathcal{Q}$ be a pca with underlying set $Q$. A *partitioned assembly* (over a $\mathcal{Q}$ in $\mathcal{S}ets$) consists of a set $X$ together with a morphism $X \longrightarrow Q$. A morphism of partitioned assemblies from $[-]_X: X \longrightarrow Q$ to $[-]_Y: Y \longrightarrow Q$ is a function $f: X \longrightarrow Y$ for which there exists an element $r \in Q$ such that:

$$\forall x \in X: r \cdot [x]_X \downarrow \text{ and } r \cdot [x]_X = [f(x)]_Y.$$

This defines a category $\mathcal{P}asm(\mathcal{Q})$, which, I claim, is a weak $\Pi W$-pretopos.

It is readily seen to be a full subcategory of the category $\mathcal{A}sm(\mathcal{Q})$. The finite limits and sums are computed as in this category. To be more explicit, assume that the conventions for pcas as explained in Appendix C are in place. In particular, assume one has chosen a pairing operator $j$ with projections $j_0$ and $j_1$ and a set $C$ of Church numerals, which will simple be denoted by the standard natural numbers. A product $(X, [-]_X) \times (Y, [-]_Y)$ would then be constructed by taking $X \times Y$ as underlying set, where a pair $(x, y)$ is realised by $\langle [x]_X, [y]_Y \rangle$. A sum $(X, [-]_X) + (Y, [-]_Y)$ has as underlying set $X + Y$, where $x$ is realised by $\langle 0, x \rangle$ and $y$ by $\langle 1, y \rangle$.

If $(X, [-]_X)$ and $(Y, [-]_Y)$ are partitioned assemblies, a weak version of $X^Y$ is the following: $F = \{(f, t) \in X^Y \times Q \mid t \text{ tracks } f\}$. So $F$ consists of pairs $(f, t)$ such that for all $x \in X$, the expression $t \cdot [x]_X$ is defined and its value equals $[f(x)]_Y$. The map $[-]_F: F \longrightarrow Q$ is given by the second projection and the evident evaluation morphism $F \times Y \longrightarrow X$ is tracked by the element in $Q$ coding application. To see that this is indeed the weak exponential, let $r$ be the realiser of some $H: Z \times Y \longrightarrow X$ in

$\mathcal{P}asm(\mathcal{Q})$. The transpose of $H$ in $\mathcal{S}ets$, $l: Z \to X^Y$, extends to a morphism $h: Z \to F$ in $\mathcal{P}asm(\mathcal{Q})$ by sending $z$ to $(l(z), \lambda n \in Q.r \cdot \langle [z], n \rangle)$. (I will refer to the weak exponentials constructed in this fashion as the "canonical" weak exponentials.)

More or less the same argument will establish that $\mathcal{P}asm(\mathcal{Q})$ has weak dependent products. Let me just give the construction. To construct the weak dependent product of $c: C \longrightarrow J$ along $t: J \longrightarrow I$ one sets $W_i$, for every $i \in I$, to be as follows:

$$\{ (f, a_f, a_i) \in C^{J_i} \times Q \times Q \mid f \text{ is a section of } c, a_f \text{ realises } f \text{ and } a_i \text{ realises } i \}$$

The morphism $W \longrightarrow Q$ is given by projection onto the last two coordinates (suitably coded). The morphism $\varepsilon$ is defined on a $j \in J$ by sending $(f, a_f, a_i) \in t^*W_i$ (where $i = t(j)$) to $f(j)$. (I will refer to the weak dependent products constructed in this fashion as the "canonical" weak dependent products.)

Finally, the construction of the weak natural number object in $\mathcal{P}asm(\mathcal{Q})$ is easy: it is simply the set $C$ of all Church numerals together with the inclusion of $C$ in $Q$.

Weak W-types are constructed as follows. Recall the construction of real W-types in $\mathcal{A}sm(\mathcal{Q})$ via the notion of a *decoration*. As I pointed out, a morphism $f: B \longrightarrow A$ of partitioned assemblies can also be regarded as a morphism of assemblies, and therefore one can associate the set of decorations, a particular set of elements in $Q$, to every well-founded tree $w$ in the W-type associated to the underlying map of $f$ in set. The weak W-type associated to $f$ is now the set of *decorated trees*, pairs $(w, a)$ where $a$ is a decoration of the tree $w$, together with the projection on the second coordinate. A proof of this claim will follow later.

An immediate corollary is that the exact completion of $\mathcal{P}asm(\mathcal{Q})$ is a $\Pi W$-pretopos. Assuming the axiom of choice, one can prove this is a topos, in fact it is the realisability topos on $\mathcal{Q}$ (see [77]), so in that case this is something that is well-known. In case one is unwilling to assume the axiom of choice, that it is a $\Pi W$-pretopos seems to be the best one can say.

**Intermezzo: W-types in realisability toposes 3.4.4** Since, under the assumption of the axiom of choice, the realisability topos on a pca is the exact completion of its full subcategory of partitioned assemblies, one can use the theory developed in this Chapter to give a concrete description of W-types in realisability toposes. This has been worked out in a small note by Claire Kouwenhoven-Gentil and me.

Since $\mathrm{RT}(\mathcal{Q})$ is the exact completion of $\mathcal{P}asm(\mathcal{Q})$, every object in $\mathrm{RT}(\mathcal{Q})$ is covered by a partitioned assembly (in fact, $(X, =)$ is covered by $\{(x, n) \in X \times Q \mid n \in E(x)\}$ with second projection). The partitioned assemblies are also internally projective and maps between partitioned assemblies are choice maps in $\mathrm{RT}(\mathcal{Q})$. This implies that for any morphism $f: B \longrightarrow A$, there exists a choice map $\phi: B' \longrightarrow A'$ between partitioned assemblies such that

$$\begin{array}{ccc} B' & \xrightarrow{\{-\}} & B \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle f} \\ A' & \xrightarrow[\{-\}]{} & A \end{array}$$

is a quasi-pullback.

By Theorem 3.2.2, given such a square, $W{:}= W_f$ can be constructed as a subquotient of $W'{:}= W_\phi$. More precisely, consider the following relation on $W'$, defined inductively in the internal logic by: $\sup_\alpha \tau \sim \sup_{\alpha'} \tau'$ iff

$$\{\alpha\} = \{\alpha'\} \wedge \forall \beta \in \phi^{-1}(\alpha), \forall \beta' \in \phi^{-1}(\alpha') \colon \{\beta\} = \{\beta'\} \rightarrow \tau\beta \sim \tau'\beta'.$$

$\sim$ is symmetric and transitive. One now constructs $W$ by considering the reflexive elements and dividing out by the equivalence relation $\sim$.

Besides, the structure map $s \colon P_f(W) \longrightarrow W$ is the unique arrow making the following diagram commute:

$$\begin{array}{ccccc} P_f(W) & \xleftarrow{q^*} & R^* & \rightarrowtail & P_\phi(W') \\ {\scriptstyle s}\downarrow & & \downarrow \sup & & \downarrow \sup \\ W & \xleftarrow[q]{} & R & \rightarrowtail & W'. \end{array}$$

Here $R$ is the object of reflexive elements, $q$ the quotient map and $q^*$ is defined on a pair $(\alpha, \tau \colon B'_\alpha \rightarrow W')$ with $\sup_\alpha \tau \in R$ as the pair $(a, t)$, with $a = \{\alpha\}$ and $t \colon B_a \rightarrow W$ defined by $t(\{\beta\}) = [\tau(\beta)]$ (which is well-defined, as $\sup_\alpha \tau \in R$).

Consider the following object in that category in $\mathrm{RT}(\mathcal{Q})$:

$$(W_{\mathcal{S}ets}(\phi), \sim),$$

where $\phi$ is as above, and $r \vdash w \sim w'$ for $w = \sup_\alpha \tau$ and $w' = \sup_{\alpha'} \tau'$, if and only if $r = \langle r_0, r_1, r_2 \rangle$ is such that the following hold:

- $r_0 \vdash Ew \wedge Ew'$.
- $r_1 \vdash a = a'$.
- for all $\beta, \beta', m$ such that $m \vdash \beta \in \phi^{-1}(\alpha) \wedge \beta' \in \phi^{-1}(\alpha') \wedge b = b'$, $r_2 \cdot m$ is defined and $r_2 \cdot m \vdash \tau\beta \sim \tau'\beta'$.

In these conditions, $a = \{\alpha\}$, $a' = \{\alpha'\}$, $b = \{\beta\}$, $b' = \{\beta'\}$ and $Ew$ is the set of decorations of $w$.

**Corollary 3.4.5** *The object under consideration is the W-type for $f$ in $\mathrm{RT}(\mathcal{Q})$.*

**Proof.** From Chapter 2, one knows how to compute W-types for $\phi$ in the categories of assemblies or in the realisability topos. Then the proof consists in rewriting in terms of realisers the description given above in terms of the internal logic of $\mathrm{RT}(\mathcal{Q})$. □

**Predicative realisability toposes 3.4.6** One can relativise the preceeding example to a fixed $\Pi W$-pretopos $\mathcal{E}$, which will then act as a kind of predicative metatheory. But first, one has to agree on a notion of an internal pca in $\mathcal{E}$. The notion will have to be more stringent than might be expected at first, in order to circumvent problems related to choice. What I will need is that the elements of the pca that are required to exist in the condition of combinatory completeness are given as a function of the initial data (by a morphism in $\mathcal{E}$). For this it suffices to assume that the combinators $k$ and $s$ are given as global elements (morphisms $1 \longrightarrow Q$).

Then the definition of a partitioned assembly can go through as follows: a *partitioned assembly* over an internal pca $\mathcal{Q}$ in a $\Pi W$-pretopos $\mathcal{E}$ consists of an object $X$ in $\mathcal{E}$ together with a morphism $[-]_X \colon X \longrightarrow Q$. A morphism of partitioned assemblies $f \colon (X, [-]_X) \longrightarrow (Y, [-]_Y)$ is a morphism $f \colon X \longrightarrow Y$ for which there exists a global element[2] $r \colon 1 \longrightarrow Q$ such that:

$$\forall x \in X \colon r \cdot [x]_X \downarrow \text{ and } r \cdot [x]_X = [f(x)]_Y$$

holds in the internal logic of $\mathcal{E}$. As usual, $r$ is said to *track* or *realise* $f$.

The construction of the finite limits, finite sums and weak dependent products is the same as in the more specific case of the previous example. That it has weak W-types is far from obvious. One somehow needs to be able to define the notion of a decoration predicatively, which is possible by giving a key rôle to the notion of path. Defining decorations will thereby inevitably become a rather technical exercise, but it can done, as I will now show.

Suppose $f$ is a morphism in $\mathcal{P}asm(\mathcal{Q})$. Now fix a tree $w \in W(f)$. A function $\kappa \colon \mathrm{Paths}_w \longrightarrow Q$ is called a *decoration* of $w$, if for any path $\sigma$ ending with the subtree $w' = \sup_a(t)$, one has that $\kappa(\sigma)$ codes a pair $\langle n_0, n_1 \rangle$ where $n_0$ equals $[a]$ and $n_1$ has the property that

$$\forall b \in f^{-1}(a) \colon n_1 \cdot [b] \text{ is defined and is equal to } \kappa(\sigma * \langle b, tb \rangle).$$

Observe that there is a lot of redundancy in this definition. In fact, all the information is already contained in the element $\kappa(\langle w \rangle) \in Q$. One might call the element $\kappa \in Q$ a decoration of $w$ if for every path $\sigma$ of length $l$, say, there exists a function $c \colon \{0, 2, \ldots, l-1\} \longrightarrow Q$ such that (1) $c(0) = \kappa$; (2) for any even $m < l - 1$, $c(m)$ codes a pair $\langle n_0, n_1 \rangle$ such that (a) $n_0 = [\rho\sigma(m)]$ and (b) $n_1 \cdot [\sigma(m+1)]$ is defined and equals $c(m+2)$. Notice that for fixed $\kappa$ and $\sigma$, a function $c$ having these properties, if it exists, is necessarily unique: $\kappa$ determines $c(0)$ by (1) and $c(m)$ determines $c(m+2)$ by (2b). For this reason, I may write $c_\sigma$, whenever $\kappa$ is understood.

So one has a notion of decoration in the "functional" and the "elementary" sense. The numerical definition of a decoration may contain less redundancy, but is, I feel,

[2] It is necessary to require the existence of a global element, rather than the existence of such an $r \in Q$ in the internal logic of $\mathcal{E}$, for otherwise the resulting category would not have weak exponentials.

somewhat opaque. It is convenient to have both perspectives available and I will make use of both of them. (That they are indeed equivalent, as I am suggesting, is something one may see as follows: every decoration $\kappa$ in the functional sense induces one in the elementary sense by taking $\kappa(\langle w \rangle)$. Then the function $c_\sigma$ for a path $\sigma$ is given by $c_\sigma(m) = \kappa\langle \sigma(0), \ldots \sigma(m) \rangle$. Conversely, because $c$ is a function of $\sigma$, one can put $\kappa(\sigma) = c_\sigma(l-1)$.)

A pair $v = (w, \kappa) \in W(f) \times Q$ such that $\kappa$ is a decoration of $w$ is called a *decorated tree*. Furthermore, $v' = (w', \kappa')$ will be called a *decorated subtree* of $v = (w, \kappa)$ if there is a path $\sigma$ in $\mathrm{Paths}_w$, of length $n$ say, such that $\sigma(n-1) = w'$ and $\kappa' = \kappa(\sigma)$. (In the equation $\kappa' = \kappa(\sigma)$, $\kappa'$ is a decoration in the elementary sense and $\kappa$ is a decoration in the functional sense. Here one clearly sees it pays off to have both perspectives available.) One might call $v'$ a proper decorated subtree, if the length $n$ can be chosen to be bigger than 1. I will denote the collection of decorated subtrees of $v$ by $\mathrm{DSubTr}_v$. One again sees that the notion of a decorated subtree is reflexive and transitive, and that there are immediate decorated subtrees of $(\mathrm{sup}_a(t), \kappa)$, namely the $(tb, \kappa\langle \mathrm{sup}_a(t), b, tb \rangle)$'s $(b \in f^{-1}(a))$. These are obviously proper.

After these preliminaries, the weak W-types in $\mathcal{P}asm(\mathcal{Q})$ can quickly be constructed. Set

$$V = \{\, v = (w, \kappa) \in W(f) \times Q \mid v \text{ is a decorated tree} \,\}$$

This is an object in $\mathcal{P}asm(\mathcal{Q})$ by defining $[-]_V \colon V \longrightarrow Q$ to be the second projection. Let $V^*$ be the "canonical" weak version of $V^f$ in the slice over $A$, so:

$$V^*_a = \{\, (t, (n_0, n_1)) \in V^{f^{-1}(a)} \times P \mid n_1 \text{ tracks } t \text{ and } n_0 = [a] \,\}.$$

In more detail: $(t, (n, m))$ is in $V^*_a$ if $m = [a]$ and $n \cdot [b]$ is defined and equal to the "decoration-component" of $t(b)$ for every $b \in f^{-1}(a)$. (Now $\varepsilon_W$ is, of course, the corresponding weak evaluation.)

The morphism $\sigma_V \colon \Sigma_A V^* \longrightarrow V$ is defined by sending $(t, (n, m)) \in V^*_a$ to the pair $(\mathrm{sup}_a(t), (n, m))$, where the pair $(n, m)$ is suitably coded. (The reader should verify that this pair consists of a tree together with a decoration for this tree, and that $\sigma_V$ is tracked by the identity, basically.)

Observe that $\sigma_V$ is actually an isomorphism. The unique element $\tau$ such that $\sigma(\tau) = v = (w = \mathrm{sup}_a(t), \kappa)$ is $((a, \lambda b \in f^{-1}(a).(tb, \kappa(\langle w, b, tb \rangle))), \kappa)$.

This completes the construction of the quadruple $\mathbf{v} = (V, V^*, \sigma_V, \varepsilon_V)$. That it is a weak $P_f$-algebra is immediate by the construction. That it is the weak W-type is not easy to show, but it follows from the following sequence of lemmas.

I have to show that every weak $P_f$-subalgebra morphism $\mathbf{i} \colon \mathbf{x} \to \mathbf{v}$ has a section. So suppose one has a weak $P_f$-algebra $\mathbf{x} = (X, X^*, \sigma_X, \varepsilon_X)$, together with a weak $P_f$-subalgebra map $\mathbf{i} \colon \mathbf{x} \longrightarrow \mathbf{v}$. If $L = (V^* \times f) \times_V X$ and if $p_0$ is the map $L \longrightarrow V^* \times f$, one may assume that $i^* \colon X^* \longrightarrow V^*$ is the "canonical" weak dependent product of $p_0$ along the projection $V^* \times f \longrightarrow V^*$ constructed above with $\varepsilon_X$ the "canonical" weak evaluation map, in view of the following lemma:

**Lemma 3.4.7** *Let a weak $P_f$-algebra $\mathbf{x} = (X, X^*, \sigma_X, \varepsilon_X)$ together with a weak $P_f$-subalgebra morphism $\mathbf{i}: \mathbf{x} \longrightarrow \mathbf{v}$ in $\mathcal{PASL}(\mathcal{P})$ be given. Now there exists a weak $P_f$-algebra $\mathbf{z} = (Z, Z^*, \sigma_Z, \varepsilon_Z)$ with a weak $P_f$-subalgebra morphism $\mathbf{j}: \mathbf{z} \longrightarrow \mathbf{v}$ where $j^*: Z^* \longrightarrow V^*$ is the canonical weak dependent product of $p_0$ along the projection $Z^* \times f \longrightarrow Z^*$ and $\varepsilon_Z$ the canonical weak evaluation map, together with a weak $P_f$-algebra morphism $\mathbf{k}: \mathbf{z} \longrightarrow \mathbf{x}$.*

**Proof.** Suppose a weak $P_f$-algebra $\mathbf{x} = (X, X^*, \sigma_X, \varepsilon_X)$ is given together with a weak $P_f$-subalgebra morphism $\mathbf{i}: \mathbf{x} \longrightarrow \mathbf{v}$. Now put $Z = X$ and $j = i$. Now let $j^*: Z^* \longrightarrow V^*$ be the canonical weak dependent product of $p_0$ along the projection $V^* \times f \longrightarrow V^*$ and let $\varepsilon_Z$ be the canonical weak evaluation map. Let $k: Z \longrightarrow X$ be the identity.

Because $X^*$ is a weak dependent product of $p_0$ along the projection $V^* \times f \longrightarrow V^*$ there exists a morphism $k^*: Z^* \longrightarrow X^*$ such that $\varepsilon_X \circ (k^* \times f) = k \circ \varepsilon_Z$. Now set $\sigma_Z = \sigma_X \circ k^*$. Now $\mathbf{z} = (Z, Z^*, \sigma_Z, \varepsilon_Z)$ is a canonical weak $P_f$-subalgebra, with $\mathbf{j} = (j, j^*)$ as weak $P_f$-subalgebra morphism. Furthermore, $\mathbf{k} = (k, k^*)$ is a weak $P_f$-algebra morphism. □

So for a given $\tau$ in $V^*$, one may assume that $X^*_\tau$ is defined as

$$\{(h \in L^f, n_h \in Q, n_\tau \in Q) \mid (p_0 h)(-) = (\tau, -), n_h \text{ realises } h \text{ and } n_\tau \text{ realises } \tau\}.$$

Or, equivalently, defined as

$$\{(h \in X^f, n_h \in Q, n_\tau \in Q) \mid ih = \varepsilon_W(\tau, -), n_h \text{ realises } h \text{ and } n_\tau \text{ realises } \tau\}.$$

The latter will be my working definition.

After making this simplifying assumption, one chooses an $s: 1 \longrightarrow Q$ such that $s$ tracks $\sigma_X$ and constructs a solution $r$ of the recursion equation:[3]

$$r \cdot j(n_0, n_1) = s \cdot j(n_0, H(r, n_1))$$

(here $H$ is the realiser of the function yielding the code of the composition of two elements).

The idea behind the construction of the $P_f$-algebra morphism $\mathbf{d}: \mathbf{v} \longrightarrow \mathbf{x}$ that is going to be a section of $\mathbf{i}$ is essentially the same as that behind the construction of the $P_f$-algebra morphism in Theorem 2.1.5, although technical details will make this construction more complex. Again, the crux is an appropriate notion of an attempt. Here I define an *attempt* for some element $v$ of $V$ as a function $g: \mathrm{DSubTr}_v \longrightarrow X$ such that:

1. $r \cdot [v']_V = [g(v')]_X$ for all decorated subtrees $v'$ of $v$.
2. If $v' = \sigma_X(\tau)$ is some decorated subtree of $v$, then the function $h = g \circ \tau$ is tracked by $m = H(r, j_1[v'])$ and satisfies the equation $(\sigma_X)_\tau(h, (m, [\tau])) = g(v')$.

[3] It is here that one needs the strict requirements on the pca $\mathcal{Q}$.

3. $ig(v') = v'$ for all decorated subtrees $v'$ of $v$.

One should think of an attempt as a partial approximation of a section **d** of **i**. Once the construction of **d** is completed, a attempt will turn out to be a restriction of **d** to the subtrees of a particular element $v$ of $V$.

Concerning attempts one proves the following two lemmas.

**Lemma 3.4.8** *Attempts are unique, so if $g$ and $h$ are two maps $DSubTr_v \longrightarrow X$ both satisfying the defining condition for attempts for an element $v$, then $g = h$.*

**Proof.** Let

$$Q = \{ w \in W \mid \text{For all decorations } \kappa \text{ of } w, \text{ attempts for } (w, \kappa) \text{ are unique.} \}$$

I use induction to show that $Q = W$: that will immediately imply the desired result. Assume that $w \in W$ is such that for all proper subtrees $w'$ and decorations $\kappa'$ of $w'$ attempts are unique for $(w', \kappa')$. Let $\kappa$ be a decoration of $w$ and notice that attempts are unique for proper decorated subtrees of $v = (w, \kappa)$, in particular for the immediate subtrees $v_b = (tb, \kappa(\langle w, b, tb \rangle))$.

Suppose $g$ is a attempt on $v$. The values of $g$ on proper decorated subtrees of $v$ are uniquely determined by the fact that the restriction of a attempt to the decorated subtrees of a particular decorated subtree is again a attempt for that decorated subtree. In particular, the value of $g$ on the immediate subtrees $v_b$ is fixed. Then the second element in the definition of a attempt determines the value of $g$ on $v$ itself. This completes the induction step and the proof. □

**Lemma 3.4.9** *Attempts exist for every $v$.*

**Proof.** Let

$$Q = \{ w \in W \mid \text{For all decorations } \kappa \text{ of } w, \text{ attempts for } (w, \kappa) \text{ exist.} \}$$

Again, by induction I show that $Q = W$, which will prove the lemma. Now, assume that $w \in W$ is such that for all proper subtrees $w'$ and decorations $\kappa'$ of $w'$ attempts exist for $(w', \kappa')$. Let $\kappa$ be a decoration of $w$ and observe that (necessarily unique) attempts $g_b$ exist for the immediate subtrees $v_b = (tb, \kappa(\langle w, b, tb \rangle))$.

If one wants to define a attempt $g\colon \mathrm{DSubTr}_v \longrightarrow X$ on $v$, one is forced to put $g(v') = g_b(v')$ if $v'$ is some decorated subtree of some $v_b$ with $b \in f^{-1}(a)$ (this is independent of the particular $b$ involved in view of the previous lemma). It remains to define $g(v)$. In the previous lemma, I already observed that I have no choice in how to define $g(v)$. Let me now be more detailed. Let

$$h = \lambda b \in f^{-1}(a).g_b(v_b)$$

and $m = H(r, j_1\kappa)$. Write $\tau = (t, \kappa) \in V_a^*$, so $[\tau] = \kappa$. First I claim that $m$ tracks $h$. Let $b \in f^{-1}(a)$ be arbitrary and calculate:

$$\begin{aligned} m \cdot [b] &= H(r, j_1\kappa) \cdot [b] \\ &= r \cdot (j_1\kappa \cdot [b]) \\ &= r \cdot \kappa(\langle w, b, tb \rangle) \\ &= r \cdot [v_b] \\ &= [g(v_b)]. \end{aligned}$$

This means that $(h, (m, [\tau]))$ is actually a member of $X_\tau^*$ and one puts (is even forced to put) $g(v) = \sigma_X(h, (m, [\tau]))$.

The map $g \colon \mathrm{DSubTr}_v \longrightarrow X$ satisfies the second condition for being a attempt by construction. What about the first?

$$\begin{aligned} r \cdot [v] &= r \cdot \kappa \\ &= s \cdot (j_0\kappa, H(r, j_1\kappa)) \\ &= s \cdot [(h, ([a], m))] \\ &= [\sigma_X(h, ([a], m))] \\ &= [g(v)] \end{aligned}$$

This being satisfied: what about the third?

$$\begin{aligned} ig(v) &= i(\sigma_X)_\tau(h, (m, [\tau])) \\ &= \sigma_V i_\tau^*(h, (m, [\tau])) \\ &= \sigma_V(\tau) \\ &= v \end{aligned}$$

So this one is also satisfied. This means that $g$ has the required properties and hence the induction step is completed. This also completes the proof. □

Using these two lemmas, one can define the map $d \colon V \longrightarrow X$ by setting $d(v) = g(v)$, where $g$ is the unique attempt $g \colon \mathrm{SubTr}_v \longrightarrow X$. It is immediate from the proof of the second lemma, where the attempts were actually built, that the natural number $r$ tracks $s$ and that $s$ extends to a weak $P_f$-algebra map $\mathbf{d}$ that is a section of $\mathbf{i}$. So $\mathbf{v}$ is a weak W-type for $f$ in $\mathcal{P}asm(\mathcal{Q})$.

In this way, within a predicative metatheory, one shows that the exact completion of $\mathcal{P}asm(\mathcal{Q})$ is a $\Pi W$-pretopos. I would argue that this deserves to be called "the predicative realisability topos on $\mathcal{Q}$ relative to $\mathcal{E}$", as it would yield $\mathrm{RT}(\mathcal{Q})$ in case $\mathcal{E} = \mathcal{S}ets$. Then the argument shows that $\Pi W$-pretoposes are closed under a notion of realisability, like toposes.

**Subcountables in the effective topos 3.4.10** Again fix a pca $\mathcal{Q}$. A *base* on $\mathcal{Q}$ is a subobject $X \subseteq Q$. A morphism $f: X \longrightarrow Y$ of bases is a function $f: X \longrightarrow Y$ that is tracked by an element $r \in Q$ in the sense that

$$\forall x \in X : r \cdot x \downarrow \text{ and } r \cdot x = f(x).$$

This yields a category, which will be denoted by $\mathcal{B}ase(\mathcal{Q})$. Bases can be identified by partitioned assemblies $(X, [-]_X)$ where $[-]_X$ is injective, so where realisers are unique. It is then rather easy to see that the category of bases inherits the weak $\Pi W$-pretopos structure of $\mathcal{P}asm(\mathcal{Q})$.

In case $\mathcal{Q} = K_1$, and assuming the axiom of choice, the exact completion of the category of bases is a subcategory of the effective topos. Actually, it is the full subcategory of subcountables, which is therefore a $\Pi W$-pretopos (an object is *subcountable*, when it is covered by a subobject of the natural number object).

**Corollary 3.4.11** *The subcountable objects in the effective topos form a* $\Pi W$*-pretopos.*

**Proof.** To prove that the category of subcountables is the exact completion of the category of bases, it suffices to show that both contain the same objects, as the exact completion of the category of bases is also a full subcategory of the effective topos, since $\mathcal{E}ff = \mathcal{P}asm_{ex}$ and $\mathcal{B}ase$ is a full subcategory of $\mathcal{P}asm$.

The natural number object $N$ in $\mathcal{E}ff$ is the same as in assemblies: the underlying set is that of the natural numbers, and $n$ is realised solely by $n$, so $En = \{n\}$. As the bases are precisely the $\neg\neg$-closed subobjects of $N$, and objects in $\mathcal{B}ase_{ex}$ are covered by bases, they are certainly subcountable. Conversely, a subobject of $N$ in $\mathcal{E}ff$ can be represented by a predicate $P: \mathbb{N} \longrightarrow \mathcal{P}\mathbb{N}$ such that $\vdash P(x) \to Ex$. It is in $\mathcal{B}ase_{ex}$, because it can be obtained as the quotient:

$$\{(x, m, m') \mid m, m' \vdash P(x)\} \rightrightarrows \{(x, m) \mid m \vdash P(x)\}.$$

A subcountable in $\mathcal{E}ff$ is represented by a symmetric, transitive relation on $\mathbb{N}$ in $\mathcal{E}ff$, more precisely, a function $R: \mathbb{N} \times \mathbb{N} \longrightarrow \mathcal{P}\mathbb{N}$ such that

$$\begin{aligned} &\vdash R(x, x) \to Ex, \\ &\vdash R(x, y) \to R(y, x), \\ &\vdash R(x, y) \wedge R(y, z) \to R(x, z). \end{aligned}$$

Therefore it can be obtained as the following quotient of subobjects of $N$:

$$\{(x, y) \mid R(x, y) \neq \emptyset\} \rightrightarrows \{x \mid R(x, x) \neq \emptyset\},$$

and hence it is in $\mathcal{B}ase_{ex}$. □

The subcountables in the effective topos will in the next Chapter be exploited to give a model of constructive-predicative set theory, that validates a principle incompatible with the existence of the powerset of the natural numbers.

It would be interesting to see to what extent the subcountables in the effective topos can be regarded as a kind of "modified PERs". The point is that they are modified so that the category will be exact, and it may therefore model quotient types in addition to what is modelled by the category of ordinary PERs.

## 3.5 Glueing and the free $\Pi W$-pretopos

This Section discusses another closure property of $\Pi W$-pretoposes, one that they share with toposes: closure under glueing. When combined with the theory of exact completions, it yields a (to me) surprising fact concerning the free $\Pi W$-pretopos. Among other things, it shows that the free $\Pi W$-pretopos cannot be the same as the category of setoids.

Consider any cartesian functor $F: \mathcal{E} \longrightarrow \mathcal{F}$ between $\Pi W$-pretoposes. Out of these data, one builds a new category $Gl(F)$ as follows. Objects are triples $(A, X, \alpha)$, where $A$ and $B$ are objects of $\mathcal{E}$ and $\mathcal{F}$ respectively and $\alpha: B \longrightarrow FA$ in $\mathcal{F}$. Such triples are also sometimes denoted by $\alpha: B \longrightarrow FA$. Morphisms $(A, X, \alpha) \longrightarrow (B, Y, \beta)$ are pairs $(f: A \longrightarrow B, g: X \longrightarrow Y)$ such that

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & FA \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle Ff} \\ Y & \xrightarrow[\beta]{} & FB \end{array}$$

commutes.

I will prove in an instant that the category $Gl(F)$ so defined is actually a $\Pi W$-pretopos. But more is true. There is an adjoint pair of functors

$$\mathcal{E} \underset{\widehat{F}}{\overset{P}{\rightleftarrows}} Gl(F), \quad (\perp)$$

where $P$ is a forgetful functor, sending a triple $(A, X, \alpha)$ to $A$, and $\widehat{F}$ sends an object $A$ to the triple $(A, FA, 1_{FA})$. $P$ will be a morphism of $\Pi W$-pretoposes, while $\widehat{F}$ will typically preserve whatever $F$ preserves (so it will be at least cartesian). Clearly, $P\widehat{F} \cong 1$.

**Theorem 3.5.1** *If $F: \mathcal{E} \longrightarrow \mathcal{F}$ is a cartesian functor between $\Pi W$-pretoposes, then $Gl(F)$ is a $\Pi W$-pretopos. Furthermore, there is a pair of adjoint functors*

$$\mathcal{E} \underset{\widehat{F}}{\overset{P}{\rightleftarrows}} Gl(F), \quad (\perp)$$

*where $P$ is a morphism of $\Pi W$-pretoposes, $\widehat{F}$ is cartesian, and $P\widehat{F} \cong 1$. In case $F$ is a morphism of ML-categories, so is $\widehat{F}$.*

**Proof.** All the claims will follow from the concrete description of the $\Pi W$-pretopos structure of $Gl(F)$.

That $Gl(F)$ is cartesian is obvious, as finite limits can be computed componentwise and they are preserved by $F$. There is no difficulty in seeing that $Gl(F)$ has finite, disjoint sums, because the sum of $X \longrightarrow FA$ and $Y \longrightarrow FB$ is $X + Y \longrightarrow FA + FB \longrightarrow F(A + B)$. To see that $Gl(F)$ is regular, observe the following facts, where $(f, g)$ is a morphism in $Gl(F)$.

1. When $f$ and $g$ are both monic, so is $(f, g)$.
2. When $f$ and $g$ are both covers, so is $(f, g)$.
3. When $(f, g)$ is a cover (monic), so are both $f$ and $g$.
4. $Gl(F)$ is regular.

1 is obvious, while 2 follows from Joyal's result that covers in a regular category are the coequalisers of their kernel pair (see Lemma A.3). Now one can see that any morphism $(f, g)$ can be factored as a cover followed by a mono, by doing this componentwise. Since such factorisations are unique up to isomorphism, 3 follows. 4 is then immediate.

That $Gl(F)$ is a pretopos follows from the fact that coequalisers of equivalence relations can be computed componentwise, and that it has a natural number object is also trivial (it is $\mathbb{N} \longrightarrow F\mathbb{N}$). To see that $Gl(F)$ is a $\Pi$-pretopos, it is sufficient to show that is a cartesian closed, because for any $(A, X, \alpha)$ in $Gl(F)$, the slice category $Gl(F)/(A, X, \alpha)$ is again a glueing category: it is $Gl(G)$, where $G$ is the composite:

$$\mathcal{E}/A \xrightarrow{F_A} \mathcal{F}/FA \xrightarrow{\alpha^*} \mathcal{F}/X.$$

More explicitly, $t \colon B \longrightarrow A$ is sent by $G$ to the upper side of the pullback square:

$$\begin{array}{ccc} GB & \xrightarrow{Gt} & X \\ {\scriptstyle \sigma_X}\downarrow & & \downarrow{\scriptstyle \alpha} \\ FB & \xrightarrow[Ft]{} & FA. \end{array} \tag{3.7}$$

As the composite of two cartesian functors, $G$ is cartesian as well.

$Gl(F)$ is cartesian closed, because the exponential $(A, X, \alpha)^{(B, Y, \beta)}$ is computed by first forming the pullback ($\theta$ is the obvious comparison map):

$$\begin{array}{ccccc} Z & \longrightarrow & & \longrightarrow & X^Y \\ {\scriptstyle \gamma}\downarrow & & & & \downarrow{\scriptstyle \alpha^Y} \\ F(A^B) & \xrightarrow[\theta]{} & FA^{FB} & \xrightarrow[FA^{\beta}]{} & FA^Y, \end{array}$$

when it will be $(A^B, Z, \gamma)$.

It is more complicated to see that $GI(F)$ inherits W-types. First one should describe polynomial functors for morphisms $\phi = (f, g) \colon (B, Y, \beta) \longrightarrow (A, X, \alpha)$ in $GI(F)$. Let $G \colon \mathcal{E}/A \longrightarrow \mathcal{F}/X$ be as above, and observe that there is a natural transformation

$$\tau_C \colon G(P_f C) \longrightarrow P_g(FC),$$

which is the composite of

$$\begin{aligned} & G(P_f C) = G\Sigma_A(C \times A \longrightarrow A)^{(B \longrightarrow A)} \longrightarrow \Sigma_X G((C \times A \longrightarrow A)^{(B \longrightarrow A)}) \longrightarrow \\ & \Sigma_X G(C \times A \longrightarrow A)^{G(B \longrightarrow A)} = P_{Gf}(FC), \end{aligned}$$

and the natural transformation $P_{Gf} \longrightarrow P_g$ induced by the commuting triangle:

$$\begin{array}{ccc} Y & & \\ \downarrow & \searrow^{g} & \\ GB & \xrightarrow[Gf]{} & X, \end{array}$$

obtained from (3.7) (see [60], Section 4.2). For any triple $(C, Z, \gamma)$ in $GI(F)$, let $P_g^C(Z, \gamma)$ be defined by taking the pullback:

$$\begin{array}{ccc} P_g^C(Z) & \longrightarrow & P_g(Z) \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle P_g(\gamma)} \\ G(P_f C) & \longrightarrow & P_g(FC). \end{array}$$

$P_g^C(Z, \gamma)$ can be regarded as an object in $\mathcal{F}/(FP_f C)$, by composing $\delta$ with $\sigma_{P_f C}$. $P_\phi$ computed on the triple $(C, Z, \gamma)$ is now $(P_f C, P_g^C Z, \sigma_{P_f C}\delta)$.

When $W$ is the initial $P_f$-algebra in $\mathcal{E}$, $W \cong P_f W$, so $P_g^W$ is an endofunctor on $\mathcal{F}/FW$. In the terminology of Gambino and Hyland [33], $P_g^W$ is a generalised polynomial functor, hence has an initial algebra $(V, \psi)$. I claim that $(W, V, \psi)$ is the W-type for $\phi$ in $GI(F)$. It is a fixpoint by construction, and it is not hard to see that an extension of Theorem 2.1.5 will prove that it is initial. □

The promised application to the free $\Pi W$-pretopos is the following theorem. In the remainder of the Section, write $\mathcal{D}$ for the free ML-category and $\mathcal{E}$ for the free $\Pi W$-pretopos.

**Theorem 3.5.2** *If $B \colon \mathcal{D} \longrightarrow \mathcal{E}$ is the unique morphism of ML-categories from the free ML-category to the free $\Pi W$-pretopos, all the objects in the image of $B$ are projective.*

**Proof.** The proof relies on the combination of Theorem 3.4.2 with the previous theorem. Let $\mathcal{E}$ be the free $\Pi W$-pretopos, and take its exact completion $\mathcal{E}_{ex}$. From

Theorem 3.4.2, one knows that $\mathbf{y}: \mathcal{E} \longrightarrow \mathcal{E}_{ex}$ is a morphism of ML-categories. If one writes $\mathcal{F}$ for the $\Pi W$-pretopos obtained by glueing along $\mathbf{y}$, one obtains by the previous result a pair of adjoint functors

$$\mathcal{E} \underset{P}{\overset{\widehat{y}}{\rightleftarrows}} \mathcal{F}, \quad (\top)$$

where $P$ is a morphism of $\Pi W$-pretoposes, $\widehat{y}$ is a morphism of ML-categories, and $P\widehat{y} \cong 1$. Since $\mathcal{F}$ is a $\Pi W$-pretopos and $\mathcal{E}$ is initial among $\Pi W$-pretoposes, there is a morphism $S: \mathcal{E} \longrightarrow \mathcal{F}$ of $\Pi W$-pretoposes, such that also $PS \cong 1$. If $B: \mathcal{D} \longrightarrow \mathcal{E}$ is the unique morphism of ML-categories from the free ML-category to the free $\Pi W$-pretopos, one also has $\widehat{y}B \cong SB$.

It is easy to see that objects of the form $\widehat{y}X$ are projective in $\mathcal{F}$, because objects of the form $\mathbf{y}X$ are, and $\mathbf{y}$ is full and faithful (also use the characterisation of covers in $Gl(F)$ given in the proof of the previous theorem). It is also not hard to that in case $SX$ is projective for an object $X$ in $\mathcal{E}$, $X$ is itself projective, because $S$, as a morphism of $\Pi W$-pretoposes, preserves covers. Since objects in the image of $B$ are such objects, the statement of the theorem is proved. □

What is most surprising (to me, at least) about this result is that it shows that all higher types, like $\mathbb{N}^{\mathbb{N}}$, are projective in the free $\Pi W$-pretopos. What is *not* true, however, is that $\mathbb{N}^{\mathbb{N}}$ is *internally* projective in the free $\Pi W$-pretopos, as the following result shows.

**Proposition 3.5.3** *1. If $\mathcal{F}$ is a $\Pi$-pretopos in which $\mathbb{N}^{\mathbb{N}}$ is internally projective, then Church's Thesis is false in the internal logic of $\mathcal{F}$.*

*2. $\mathbb{N}^{\mathbb{N}}$ is not internally projective in the free $\Pi W$-pretopos.*

**Proof.** If $\mathbb{N}^{\mathbb{N}}$ is internally projective in a $\Pi W$-pretopos $\mathcal{F}$, its internal logic will model $\mathbf{HA}^{\omega} + AC_{1,0} + EXT$. It is a well-known result by Troelstra [82] (see also [81]) that this theory refutes Church's Thesis.

Because the validity of statements in the internal logic is preserved by morphisms of $\Pi W$-pretoposes, validity of the negation of Church's Thesis in the free $\Pi W$-pretopos would imply validity of the negation of Church's Thesis in *all* $\Pi W$-pretoposes. But since Church's Thesis is valid in the effective topos, for instance, this is impossible. Therefore $\mathbb{N}^{\mathbb{N}}$ is not internally projective in the free $\Pi W$-pretopos. □

**Corollary 3.5.4** *The following three $\Pi W$-pretoposes are all different:*

- *The free $\Pi W$-pretopos $\mathcal{E}$.*

- *The exact completion $\mathcal{D}_{ex}$ of the free ML-category $\mathcal{D}$.*
- *The category $\mathcal{S}etoids$.*

**Proof.** This corollary is an immediate consequence of the following table:

| Category | $\mathbb{N}^{\mathbb{N}}$ externally projective | $\mathbb{N}^{\mathbb{N}}$ internally projective |
|---|---|---|
| $\mathcal{S}etoids$ | No | No |
| $\mathcal{E}$ | Yes | No |
| $\mathcal{D}_{ex}$ | Yes | Yes |

The two bottom rows are consequences of the results obtained in this Chapter. The entries for $\mathcal{D}_{ex}$ follow immediately from Theorem 3.4.2 and Lemma 3.1.6, while the previous two results give the entries for $\mathcal{E}$.

The entries for the category of setoids are consequences of the following sequence of facts. Among the setoids, there are the "pure types", consisting of a type with its intensional equality as equivalence relation. These pure types are projective. This includes the pure type $1 = N_1$, which is the terminal object in the category of setoids. So the terminal object in $\mathcal{S}etoids$ is projective, and hence the internal projectives are also externally projective.

The object $\mathbb{N}^{\mathbb{N}}$ in $\mathcal{S}etoids$ is the type $N \to N$ together with the "extensional" equality relation

$$\Pi n \in N. \, \mathrm{Id}(N, fn, gn).$$

This object is covered by the pure type $N \to N$, so if it were projective, this cover would have a section. This would imply that there is a definable operation $s \in (N \to N) \to (N \to N)$ such that the following types are provably inhabited:

$$\Pi f \in N \to N. \, \mathrm{EXTEQ}(f, sf)$$
$$\Pi f, g \in N \to N. \, \mathrm{EXTEQ}(f, g) \to \mathrm{INTEQ}(sf, sg),$$

where

$$\mathrm{INTEQ}(f, g) \; := \; \mathrm{Id}(N \to N, f, g)$$
$$\mathrm{EXTEQ}(f, g) \; := \; \Pi n \in N. \, \mathrm{Id}(N, fn, gn).$$

Such an $s$ cannot exist, because if it would, one could decide extensional equality of terms of type $N \to N$, which is known to be impossible: for any two closed terms $p, r$ of type $N \to N$, the type $\mathrm{EXTEQ}(p, r)$ is inhabited, iff $\mathrm{INTEQ}(sp, sr)$ is inhabited, iff $sp$ and $sr$ are convertible, which is decidable (many thanks to Thomas Streicher for helping me out on this). Therefore $\mathbb{N}^{\mathbb{N}}$ is not projective in $\mathcal{S}etoids$, and, a fortiori, not internally projective either. □