Chapter 2

Induction in categories

Within a conception of mathematics that is constructive and predicative, it is very natural to assume the existence of a wide variety of inductively generated structures. To allow for inductively defined sets, Aczel extended his set theory **CZF** with the Regular Extension Axiom (**REA**), while for the same purpose, Martin-Löf has added well-founded types (or W-types) to his type theory. To study categorically constructive-predicative theories characterised by this liberal attitude towards inductively defined sets, it is necessary to isolate categories in which there exist objects that can be thought of as inductively generated.

In order to give a categorical formulation, the idea of an inductive definition is too informal, and also too broad, so one has to restrict oneself to a class of sets that result from an inductive definition of a determinate form. Following Moerdijk and Palmgren [60], I take Martin-Löf's W-types as the inductively defined sets to incorporate: they have a precise constructive justification in type theory, while they are also sufficient for obtaining models for set theory (see [61], generalising the work of Aczel in [2]; I will discuss this work in Chapter 4).

The categorical notion of a W-type will be discussed in the first Section of this Chapter. The main novelty of the discussion is a characterisation theorem which will be helpful in recognizing W-types in concrete cases.

Two classes of categories with W-types are then defined, ML-categories and ΠW-pretoposes. The latter, ΠW-pretoposes, were originally defined by Moerdijk and Palmgren, as predicative analogues of the notion of a topos. It is with these ΠW-pretoposes that Chapters 3 and 4 of this thesis will be concerned. ML-categories are almost ΠW-pretoposes, their only defect in this respect being that they are not exact. But this "defect" can be remedied by taking their exact completion, as will be discussed in Chapter 3.

The second Section of this Chapter discusses several examples of ΠW-pretoposes and ML-categories. More examples will be given in Chapter 3.

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2.1 W-types

Martin-Löf's W-types are sets generated by an inductive definition with a particular shape. The form of the inductive definition is determined by a signature: a set of term constructors, each with a small (set-sized) arity. The inductively generated set (the W-type) is then the free term algebra over this signature. There are sets which deserve to be called inductively generated, but are not of this form, but these are beyond the scope of this thesis (see [27] for a broader framework for inductive definitions).

To capture W-types categorically, the language of initial algebras (for an endofunctor) immediately suggests itself, for, in general, it is the appropriate categorical language to talk about inductively generated structures (see Appendix A for the basic facts and terminology on algebras and coalgebras on which this Section relies). The question then becomes which endofunctors one should require to possess initial algebras in order to have W-types. Moerdijk and Palmgren identified the following class, definable in any lccc (also for categorical terminology and notation, one should consult Appendix A).

Definition 2.1.1 In a cartesian category $\mathcal{C}$, there is a *polynomial functor*[1] P_f associated to every exponentiable map $f: B \longrightarrow A$. It is the endofunctor defined as the composite

$$\mathcal{C} \xrightarrow{B^*} \mathcal{C}/B \xrightarrow{\Pi_f} \mathcal{C}/A \xrightarrow{\Sigma_A} \mathcal{C}.$$

A more insightful way of writing $P_f(X)$ may be the following:

$$P_f(X) = \Sigma_A(X \times A \longrightarrow A)^{(f: B \longrightarrow A)},$$

or:

$$P_f(X) = \Sigma_{a \in A} X^{B_a},$$

where $B_a = f^{-1}(a)$ is the fibre of f over $a \in A$.

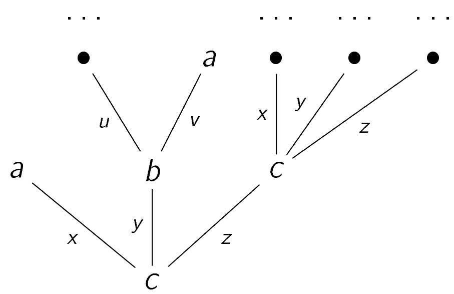
Definition 2.1.2 Let $f: B \longrightarrow A$ be an exponentiable morphism in a cartesian category $\mathcal{C}$. The *W-type* associated to f is the initial P_f-algebra, when it exists; the *M-type* associated to f is the final P_f-coalgebra, whenever it exists. If in a locally cartesian closed category $\mathcal{C}$, W-types (resp. M-types) exist for any map f, the category $\mathcal{C}$ is said to have W-types (resp. M-types).

[1] Polynomial functors have received different names in the literature: there is a tradition of calling them *partial product functors* (they are called like this in [44], for example), while recently a group of authors has emerged who use the name *containers* (for instance, see [1]).

To understand this definition better, it helps to compute W-types and M-types in the category of sets.

Fix a function $f: B \longrightarrow A$. One can think of f as specifying a signature: a term constructor for every element $a \in A$ of arity B_a. The W-type W_f is now the set of all terms over the signature specified by f. But it is even more suggestive to represent such terms as *well-founded trees* of a particular type. The W-type for f is the set of all well-founded trees in which nodes are labelled by elements $a \in A$ and edges are labelled by elements $b \in B$, in such a way that the edges into a certain node labelled by a are enumerated by $f^{-1}(a)$, as illustrated in the following picture.

$$\begin{array}{rcl} f^{-1}(a) & = & \emptyset \\ f^{-1}(b) & = & \{u, v\} \\ f^{-1}(c) & = & \{x, y, z\} \\ & \cdots & \end{array}$$



Let us first try to understand why this set has the structure of a P_f-algebra. Now $P_f(X)$ for a set X can be written as:

$$P_f(X) = \sum_{a \in A} X^{B_a}.$$

So it consists of an element $a \in A$ together with a function $t: B_a \longrightarrow X$. Therefore assume one is given an element $a \in A$ together with a function $t: B_a \longrightarrow W_f$. A new well-founded tree of the appropriate type can be constructed as follows: take a fresh node and label it with a. Draw edges into this node, one for every $b \in B_a$ and label these accordingly. Then stick to the edge labelled by $b \in B_a$ the well-founded tree tb. The new tree, which is easily seen to belong to W_f, is usually denoted by $\sup_a(t)$.

I have described an operation $\sup: P_f(W_f) \longrightarrow W_f$, giving W_f the structure of a P_f-algebra. The fact that the trees are well-founded means that one could actually generate all of them by (transfinitely) repeating this sup-operation. This construction terminates, because one has only a set of term constructors, and the arities are also small, so there is only a set of trees with the appropriate labelling, well-founded or not.

But this means that one can define functions by recursion on this generation process. And this is precisely what yields initiality of W_f. For if $m: P_f(X) \longrightarrow X$ is any P_f-algebra and one wishes to build a function $g: W_f \longrightarrow X$, one can do so by specifying the value of g on an element $\sup_a(t)$, assuming that one has already specified the value on elements of the form tb, where $b \in B_a$. Therefore one can put:

$$g(\sup_a(t)) = m(\lambda b \in f^{-1}(a).g(tb)),$$

expressing that g is a P_f-algebra morphism. The P_f-algebra morphism so defined is automatically unique, because functions defined by recursion always are. Although much of what I have said, applies only literally in the case of sets, thinking in terms of well-founded trees and recursion is the right intuition also in the abstract categorical case, to which I now turn.

Now let $f\colon B \longrightarrow A$ be an exponentiable morphism in a Heyting category $\mathcal{C}$. When the W-type W_f associated to f exists, one can say right away that it has two properties, both for abstract categorical reasons (see appendix A). First of all, the structure map $\sup\colon P_f(W_f) \longrightarrow W_f$ is an isomorphism, by Lambek's lemma (see Lemma A.14). Secondly, it will have the feature of possessing no proper P_f-subalgebras.

The second property is reflected as an induction principle in the internal logic. If R is a subobject of W that is *inductive* in the sense that

$$\forall b \in f^{-1}(a)(tb \in R) \Rightarrow \sup_a(t) \in R$$

holds in the internal logic, then $R = W$ as subobjects of W. Induction also holds when R depends on a parameter. This follows from the following fact (see [60, 33], and Remark 2.1.6 below):

Theorem 2.1.3 *Let $f\colon B \longrightarrow A$ be an exponentiable morphism in a cartesian category $\mathcal{C}$. The polynomial functor P_f is automatically indexed. If $\mathcal{C}$ is Heyting, a W-type for f is automatically an indexed well-founded fixpoint. If $\mathcal{C}$ is locally cartesian closed, a W-type for f is automatically an indexed initial algebra.*

As an illustration of this theme, there is this lemma:

Lemma 2.1.4 *Let $f\colon B \longrightarrow A$ be an exponentiable morphism in a Heyting category $\mathcal{C}$. When both the W-type and the M-type associated to f exist, the canonical map $i\colon W_f \longrightarrow M_f$ is monic.*

Proof. There is a canonical morphism i, because an M-type is a fixpoint for the polynomial functor and can therefore be regarded as a P_f-algebra (conversely, the W-type can be regarded as P_f-coalgebra). I will now actually prove something stronger: the P_f-algebra map i from W_f to any P_f-algebra with monic structure map $m\colon P_f X \longrightarrow X$, is a monomorphism.

For let

$$R = \{w \in W_f \mid \forall v \in W_f\colon i(w) = i(v) \Rightarrow w = v\}.$$

Then R is inductive: for suppose $w = \sup_a(t) \in W_f$ is such that $tb \in R$ for all $b \in B_a$. Let $v \in W_f$ be such that $i(w) = i(v)$. Since sup is an isomorphism, v is of the form $\sup_{a'}(t')$. Now

$$i(w) = m(a, \lambda b \in B_a.itb) = m(a', \lambda b' \in B_{a'}.it'b') = i(v).$$

Since m is monic, $a = a'$ and $itb = it'b$ for all $b \in B_a$. Using that $tb \in R$, one sees that $tb = t'b$ for all $b \in B_a$, i.e. $t = t'$. Therefore $w = v$ and the proof is finished. $\square$

In many categories, the two properties of being a fixpoint and being well-founded characterise W-types completely, as I will prove at the end of this Section. But before I do that, I will discuss P_f-coalgebras and M-types.

In the category of sets, the M-type associated to a function $f\colon B \longrightarrow A$ is the set of all trees (well-founded or otherwise) labelled in the familiar way: nodes are labelled by elements $a \in A$, while edges into a node labelled by a are enumerated by $b \in B_a$. If one wishes for a more concrete description, one could regard M_f as the set of all sets S of sequences of the form

$$\langle a_0, b_0, a_1, b_1, \ldots, a_n \rangle$$

where $a_i \in A$ and $b_i \in B$ satisfy:

1. $f(b_i) = a_i$;
2. $\langle a \rangle \in S$ for some $a \in A$;
3. if $\langle a_0, b_0, a_1, b_1, \ldots, a_n, b_n, a_{n+1} \rangle \in S$, then also $\langle a_0, b_0, a_1, b_1, \ldots, a_n \rangle \in S$;
4. if $\langle a_0, b_0, a_1, b_1, \ldots, a_n \rangle \in S$ and $b \in B_{a_n}$, then $\langle a_0, b_0, a_1, b_1, \ldots, a_n, b, a \rangle \in S$ for some $a \in A$.

That this yields the final P_f-coalgebra, one sees as follows.

There is the projection morphism $P_f(X) \longrightarrow A$ for any X, so to any P_f-coalgebra $m\colon X \longrightarrow P_f X$ one can associate a *root map* $\rho\colon X \longrightarrow A$. The name is suggested by the case of M-types and W-types, where ρ assigns to a tree the label of its root (a W-type is also naturally a P_f-coalgebra, because it is a fixpoint for P_f). Another notion that makes sense in any P_f-coalgebra $m\colon X \longrightarrow P_f X$ is that of a *path*, the name again being suggested by the case of W- and M-types. A sequence of the form

$$\langle x_0, b_0, x_1, b_1, \ldots, x_n \rangle$$

is called a *path from x_0 to x_n*, if $x_i \in X$ and $b_i \in B$ are such that they satisfy the following compatibility condition: if for an $i < n$, $m(x_i)$ is of the form (a_i, t_i), then $f(b_i) = a_i$ and $x_{i+1} = t_i b_i$.

A P_f-coalgebra morphism $g\colon X \longrightarrow M_f$ can now be obtained as follows: thinking of trees as sets of paths, one sends an element $x \in X$ to the set of all sequences of the form

$$\langle \rho(x_0), b_0, \rho(x_1), b_1, \ldots, \rho(x_n) \rangle,$$

where $\langle x_0, b_0, x_1, b_1, \ldots, x_n \rangle$ is a path starting from x. One readily sees that g is well-defined, a P_f-coalgebra morphism and the unique such.

An important observation is that the notions of a root and a path are readily formalized in the internal logic of locally cartesian closed regular categories $\mathcal{C}$ with finite disjoint sums and natural number object: an object Paths = Paths(m) can be defined in any such $\mathcal{C}$ as the subobject of $(X + B + 1)^{\mathbb{N}}$ consisting of those $\sigma \in (X + B + 1)^{\mathbb{N}}$ for which one has:

1. $\sigma(0) \in X$.

2. If $\sigma(n) \in X$ for an even natural number n, then either $\sigma(n+1) \in B$ or $\sigma(n+1) = *$ (with $*$ denoting the unique element of 1).

3. If $\sigma(n) = x$ and $\sigma(n+1) = b$ for an even number n, then $fb = a$ and $\sigma(n+2) = tb$, where $(a, t) = r(x)$.

4. $n < k$ and $\sigma(n) = *$ imply $\sigma(k) = *$.

5. There is a natural number n such that $\sigma(n) = *$.

Since paths will play a prominent rôle in this thesis, I establish some terminology and notation, which will also make sense in the internal logic of $\mathcal{C}$. Because there is a natural number n such that $\sigma(n) = *$, there is also a least such (this is a consequence of the constructively valid principle that every inhabited decidable subset of the natural numbers has a least element). That number will be called the length of σ. Then $\sigma(0)$ will be called the beginning, while σ(the length of $\sigma - 1$) will be called the end of the path. One has a map Paths $\to X$ defined by assigning to a path the beginning of the path. The fibre above x for this map will be denoted by Paths$_x$. Also in the internal logic, I will continue to use the notation $\langle \ldots, \ldots, \ldots \rangle$ for writing down sequences and $*$ as a symbol for concatenation.

One can then use the notion of path to define the notion of *subtree*. Once again, this notion has a clear meaning in the case of W- and M-types, but makes sense in any P_f-coalgebra. I will call y a *subtree* of x if there is a path σ in Paths$_x$ such that $\sigma(n) = y$ for some even natural number n. If n can be chosen to be bigger than 1, y is called a *proper subtree*. Observe that the subtree relation is reflexive and transitive. For example, in a W-type, the tree $t(b)$ is a subtree of $w = \sup_a(t)$ (the tbs are really the immediate proper subtrees of $\sup_a(t)$).

As an application of the notion of path, one has the following characterisation or recursion theorem for W-types, which will be very helpful in recognizing W-types.

Theorem 2.1.5 (Characterisation Theorem) *Let $\mathcal{C}$ be a locally cartesian closed regular category with finite disjoint sums and a natural number object, and $f : B \longrightarrow A$ a morphism in $\mathcal{C}$. The following are equivalent for a P_f-algebra $(W, s : P_f(W) \longrightarrow W)$:*

1. *It is the W-type for f.*

2. *It is an indexed well-founded fixpoint.*

3. *It is a well-founded fixpoint.*

The idea of the proof of the characterisation theorem is essentially that of Cantor's general recursion theorem (see, for example, [49], Theorem 5.6): one builds a map by pasting together "attempts" (partial approximations). Some care has to be taken, because the argument has to be predicative. In the impredicative context of elementary toposes, the characterisation would follow from a general result (see Theorem 2.2.3), but that argument does not obviously carry over to a predicative one.

Proof. The difficult part will be to show that (3) implies (1). Before Theorem 2.1.3 I pointed out that (1) implies (2), while the implication from (2) to (3) is trivial.

So assume W is a fixpoint having no proper P_f-subalgebras. In particular, it is a P_f-coalgebra and hence it makes sense to talk about paths in W. This can therefore be used to define a notion of attempt.

Now let $(X, m \colon P_f(X) \longrightarrow X)$ be an arbitrary P_f-algebra. An attempt for an element $w \in W$ is a morphism $g \colon \mathrm{Paths}_w \longrightarrow X$ with the additional property that for any path $\sigma \in \mathrm{Paths}_w$ ending with an element $w' = s_{a'}(t')$ the following equality holds:

$$g(\sigma) = m(\lambda b' \in f^{-1}(a').g(\sigma * \langle b', t'b' \rangle))$$

Later it will become apparent that $g(\sigma)$ is $p(w')$ where w' is the last element of σ and p is the unique P_f-algebra morphism $p \colon W \longrightarrow X$. So an attempt will turn out to be the restriction to the subtrees of w of the unique P_f-algebra morphism $p \colon W \longrightarrow X$, which I still have to construct.

Let S be the collection of all those $w \in W$ for which there exists a unique attempt. Then S is a subalgebra of W. For let $w = s_a(t)$ be such that $tb \in S$ for all $b \in f^{-1}(a)$. This means that there are for every $b \in f^{-1}(a)$ unique attempts g_b for tb. Write $x_b = g_b(\langle tb \rangle)$. Now define a morphism $g \colon \mathrm{Paths}_w \longrightarrow X$ for w as follows:

$$\begin{aligned} g(\langle w \rangle) &= m(\lambda b \in f^{-1}(a).x_b) \text{ and} \\ g(\langle w, b \rangle * \sigma) &= g_b(\sigma) \end{aligned}$$

The fact that this is an attempt is easily verified.

Uniqueness relies on the following observation: if σ is a path starting with w and ending w' and g is an attempt for w, then $h(\langle w' \rangle * \tau) = g(\sigma * \tau)$ defines an attempt for w'. So if h is any attempt for w, then $h_b(\langle tb \rangle * \tau) = g(\langle w, b, tb \rangle * \tau)$ defines an attempt for tb for every $b \in f^{-1}(a)$. So $h_b = g_b$ and what remains is the proof of $h(\langle w \rangle) = g(\langle w \rangle)$. But that is established by an easy calculation:

$$\begin{aligned} h(\langle w \rangle) &= m(\lambda b \in f^{-1}(a).h(\langle w, b, tb \rangle)) \\ &= m(\lambda b \in f^{-1}(a).h_b(\langle tb \rangle)) \\ &= m(\lambda b \in f^{-1}(a).g_b(\langle tb \rangle)) \\ &= g(\langle w \rangle) \end{aligned}$$

So $h = g$ and $w \in S$. Therefore S is a subalgebra of W.

As W has no proper subalgebras, $S = W$ as subobjects of W. But that allows one to define $p: W \longrightarrow X$ by:

$$p(w) = g(\langle w \rangle)$$

for the unique attempt g for w. This defines a P_f-algebra morphism, which is actually the unique such: for if q is another P_f-algebra morphism the equaliser of p and q is a subalgebra of W. □

Remark 2.1.6 It would not be difficult to give a direct proof that (3) implies (2). The crucial fact is that in case T is a P_{X^*f}-subalgebra of X^*W in $\mathcal{C}/X$, then

$$S = \{ w \in W \mid \forall x \in X : (x, w) \in T \}$$

defines a P_f-subalgebra of W in $\mathcal{C}$.

I can now give the following two, important, definitions:

Definition 2.1.7 A locally cartesian closed category $\mathcal{C}$ with finite disjoint sums and W-types is called an *ML-category*.

Observe that ML-categories possess a natural number object, because the functor $X \mapsto 1 + X$ is polynomial, and an initial algebra for this functor is a nno (see Appendix A). In fact, $X \mapsto 1 + X$ is $X \mapsto P_f(X)$, where f is the right inclusion $1 \longrightarrow 1 + 1$.

Definition 2.1.8 A locally cartesian closed pretopos $\mathcal{C}$ with W-types is called a *ΠW-pretopos*. (So a ΠW-pretopos is an exact ML-category.)

It is with these kinds of categories that the first part of this thesis will mostly be concerned. In Chapters 3 and 4, I will show that, for logical purposes, ΠW-pretoposes can be regarded as a kind of predicative toposes, in that a theory analogous to topos theory, with similar closure conditions, can be developed for them, while they at the same time provide a natural habitat for models of set theory.

In the next Section I will give a number of examples of ML-categories and ΠW-pretoposes. Once I have developed more theory, especially concerning exact completions, I will be able to give more.

Generalised polynomial functors 2.1.9 Which functors, aside from polynomial ones, automatically have initial algebras in a category, when that category has W-types? The question is interesting, but I will not attempt to give an answer.

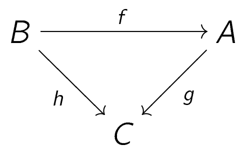
However, in [33], Gambino and Hyland identify a class of functors that have initial algebras, when all W-types exist. Suppose $\mathcal{C}$ is an ML-category, and recall that, since $\mathcal{C}$ is an lccc, pullback functors

$$f^*: \mathcal{C}/X \longrightarrow \mathcal{C}/Y$$

have left and right adjoints for all $f: Y \longrightarrow X$, called Σ_f and Π_f, respectively. Consider all possible compositions of such functors f^*, Σ_f and Π_f, possibly for different f. When such a composition has the same slice of $\mathcal{C}$ as domain and codomain, the functor is called *generalised polynomial*.

The class of generalised polynomial functors is less unwieldy than one might initially think, because of the following "normal form lemma".

Lemma 2.1.10 *For any generalised polynomial functor Δ there is a (not necessarily commutative) triangle*



such that Δ is naturally isomorphic to $\Sigma_g \Pi_f h^*$.

In their paper, Gambino and Hyland prove:

Theorem 2.1.11 *(See [33], Theorem 12.) All generalised polynomial functors on a ML-category $\mathcal{C}$ have initial algebras in the appropriate slice.*

One could prove an extension of the characterisation result Theorem 2.1.5 for initial algebras for generalised polynomial functors: when an algebra is a well-founded fixpoint for a generalised polynomial functor, it is the initial algebra.

2.2 Categories with W-types

In this Section, I will introduce various examples of the notions of ML-category and ΠW-pretopos, introduced above. It is well known that these categories have this structure, W-types excepted, and for proofs, the reader is therefore referred to the literature. Frequently, a concrete description of W-types or M-types was not available in the literature, and in some cases their presence was unknown. Therefore concrete descriptions will be provided.

Toposes 2.2.1 Elementary toposes with a natural number object form an important class of examples of ΠW-pretoposes. Recall that a morphism $\top: 1 \longrightarrow \Omega$ in a cartesian category $\mathcal{C}$ is the *subobject classifier* of $\mathcal{C}$, when for any monomorphism $m: A \longrightarrow X$ there is a unique map $c_A: X \longrightarrow \Omega$ such that

$$\begin{array}{ccc} A & \longrightarrow & 1 \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle \top} \\ X & \xrightarrow[c_A]{} & \Omega \end{array}$$

is a pullback. And recall that *(elementary) topos* is a cartesian closed category with pullbacks and a subobject classifier.

Topos theory is a rich subject with plenty of examples, among which are Grothendieck toposes, in particular categories of sheaves over a topological space (see [56]), the free topos (with nno) (see [51]) and the effective topos (see [39]). Any (good) book on topos theory will prove that a topos is a Π-pretopos (for example, [44], Corollaries A2.3.4 and A2.4.5). The fact that toposes with nno have W- and M-types is folklore. The description that I gave of an M-type for a map $f: B \longrightarrow A$ as consisting of sets of sequences

$$\langle a_0, b_0, a_1, b_1, \ldots, a_n \rangle$$

having four properties, makes sense in the internal logic of any topos with nno. It is a routine exercise in the use of internal logic to show that the object so defined is the M-type for the map f. Therefore:

Proposition 2.2.2 *(See Lemma 2.4 in [46].) Toposes with nno have M-types.*

The fact that toposes have W-types now follows from another folklore result:

Theorem 2.2.3 *(See Proposition 1 in [43].) Let F be an indexed functor on a topos $\mathcal{E}$ preserving pullbacks. If F has a fixpoint, it also has a well-founded fixpoint (see Appendix A), which necessarily is the initial F-algebra.*

Since M-types are fixpoints and polynomial functors are indexed and preserve pullbacks, one immediately obtains:

Proposition 2.2.4 *(See Proposition 2.3.5 in [68].) Toposes with nno have W-types.*

Alternatively, one can simply select in the internal logic the trees in the M-type that are well-founded (the internal logic of toposes is impredicative, so this is expressible), and obtain the W-type in that way (see Proposition 3.6 in [60]).

The next couple of examples concern full subcategories of the realisability topos $\mathrm{RT}(\mathcal{Q})$ on a pca $\mathcal{Q}$. For a (brief) discussion of pcas and realisability toposes, see Appendix C.

Convention 2.2.5 When in the name of a category depending on a pca $\mathcal{Q}$, this index is omitted, it is to be understood that the relativization is to K_1, Kleene's first pca. So RT is RT(K_1), i.e. the effective topos.

Assemblies 2.2.6 Fix a pca $\mathcal{Q}$. An *assembly on* $\mathcal{Q}$ consists of set X together with a function $[-]_X \colon X \longrightarrow \mathcal{P}_i Q$ (by $\mathcal{P}_i Q$, I mean the set of non-empty subsets of Q). Instead of $q \in [x]_X$, I will frequently write $q \vdash_X x$, or simply $q \vdash x$, when X is understood. In this case q is called a *realiser* of x. A function $f \colon X \longrightarrow Y$ is a morphism of assemblies from $(X, [-]_X)$ to $(Y, [-]_Y)$, when there is an element $t \in Q$ such that for all $x \in X, q \in Q$:

$$q \vdash_X x \Rightarrow t \cdot q \downarrow \text{ and } q \cdot p \vdash_Y f(x).$$

Such a t is said to *track* f, or to *realise* f. In this way, one obtains a category $\mathcal{A}sm(\mathcal{Q})$ of assemblies on $\mathcal{Q}$.

The category $\mathcal{A}sm(\mathcal{Q})$ is a regular ML-category for any pca $\mathcal{Q}$. It is not a ΠW-pretopos, as it is not exact. $\mathcal{A}sm(\mathcal{Q})$ occurs as the full subcategory of $\mathrm{RT}(\mathcal{Q})$ consisting of the $\neg\neg$-separated objects, and the inclusion of $\mathcal{A}sm(\mathcal{Q})$ in $\mathrm{RT}(\mathcal{Q})$ preserves the regular ML-category structure. These facts are well-known. Even the fact that the category of assemblies has W-types and M-types, and that they are preserved by the inclusion, seems to be familiar to many people, although it is seldom pointed out explicitly (it is implicit in [67], for example). Also for the sake of completeness, I explain the construction here, based on some unpublished notes by Ieke Moerdijk (another description is contained in [37]).

But before I can give a concrete description of the W-types and M-types in the category of assemblies for a pca $\mathcal{Q}$, I first need to investigate the behaviour of the functor P_f for a morphism $f \colon B \longrightarrow A$ of assemblies. The underlying set of $P_f(X)$ consists of those pairs (a, t) where a is an element of Q and t is a function from B_a to X that has a realiser. An element $n \in \mathcal{A}$ is a realiser for (a, t), if $n = \langle n_0, n_1 \rangle$ is such that n_0 realises a and n_1 tracks t (the latter meaning, of course, that for every $b \in B_a$ and every realiser m of b, $n_1 \cdot m$ is defined and equal to a realiser of tb).

The key concept behind the construction of the W-types is the notion of *decoration*. This notion will recur a number of times, so I will take some time explaining it. The idea behind this notion is as follows. Elements w of the W-type W_f belonging to a function $f \colon B \longrightarrow A$ are thought of as being constructed by repeated application of the sup operation to maps of the form $f^{-1}(a) \longrightarrow W_f$. When constructing the W-types in $\mathcal{A}sm(\mathcal{Q})$, I only want those elements w of $W(f)$ that can be constructed by applying the operation sup to "trackable" maps of the form $f^{-1}(a) \longrightarrow W_f$. A decoration of an element in the W-type specifies for each application of the sup operation that has been used to generate the element, a realiser for the "applicant", i.e. both an element in Q that tracks the map $t \colon f^{-1}(a) \longrightarrow W_f$ to which it has been applied, as well as a realiser for a.

Therefore the W-type for f is constructed as follows. First construct the W-type W for the underlying function f in the category of sets. Next define a function $E \colon W \longrightarrow \mathcal{P}Q$

by transfinite induction: $E(\sup_a t)$ consists of those elements $n = \langle n_0, n_1 \rangle \in Q$ such that (i) n_0 realises a; and (ii) n_1 tracks t, that is, for every $b \in B_a$ and every realiser m of b, $n_1 \cdot m$ is defined and a member of $E(tb)$. I call a member n of $E(w)$ a *decoration* or a *realiser* of the tree $w \in W$. The trees w that have a decoration are called *decorable* and V will be the name of the set of all decorable trees.

The set V is the underlying set of an assembly whose realisability relation is determined by restriction of E to V. This assembly, also to be called V, is, I claim, the W-type for f in the category of assemblies. It is not hard to see that it has the structure of a P_f-algebra. Let a be an element of A and t be a function $B_a \to V$. The element $\sup_a(t) \in W$ is actually an element of V, because if n is a realiser of (a, t) in $P_f(V)$, then n is a decoration of $\sup_a(t)$ (this is immediate from the definition of E and the description of $P_f(V)$). So there is a map of assemblies $s\colon P_f(V) \to V$, which is tracked by the identity.

To verify that the constructed object is the W-type, I appeal to the characterisation theorem, Theorem 2.1.5. The map s is iso, basically because the underlying map sup is, therefore I only need to show that V has no proper subalgebras.

Let $(X, m\colon P_f(X) \longrightarrow X)$ be a subalgebra of V. One may assume that the underlying set X is actually a subset of V and that m is the restriction of s to $P_f(X)$ on the level of underlying functions. First of all, one sees that $X = V$ on the level of sets. Let P be set of trees $w \in W$ for which one has that

$$w \in V \Rightarrow w \in X.$$

That $P = W$ can be proved by transfinite induction, which immediately shows that $X = V$ as sets. For suppose $\sup_a t \in W$ and $tb \in P$ for all $b \in B_a$ (here $a \in A, t\colon B_a \longrightarrow W$, of course). One needs to show that $\sup_a t \in P$, so assume that $\sup_a t \in V$. Because s is iso, one has that $tb \in V$ and hence, by induction hypothesis, $tb \in X$. Since on the level of sets, m is the restriction of s, which is a restriction of sup, one has that $\sup_a t$ is in X. This completes the proof.

To show, finally, that the X and V are isomorphic as assemblies, I have to show that the identity map $i\colon V \to X$ is tracked by some element $r \in Q$. For this, let q be the element in Q tracking m and let H be the element computing the composition of two elements in $\mathcal{Q}$ (that is, $H(x, y) \cdot n = x \cdot (y \cdot n)$). Now use the fact that one can solve recursion equations, to obtain an r satisfying the following equation:

$$r \cdot j(n_0, n_1) = q \cdot j(n_0, H(r, n_1)).$$

It is easy to see that r tracks i, by proving by a transfinite induction that for any tree $w \in W$ and any decoration n of w, $r \cdot n$ is defined and a realiser of $i(w)$.

This completes the proof of the fact that V is the W-type of f in the category of assemblies. The fact that the inclusion of assemblies on $\mathcal{Q}$ into the realisability topos on $\mathcal{Q}$ preserves W-types, is easily seen as follows. Since the inclusion preserves the lccc structure, the inclusion also preserves the functor P_f in the sense that

$$\begin{array}{ccc} \mathcal{A}sm(\mathcal{Q}) & \xrightarrow{i} & \mathrm{RT}(\mathcal{Q}) \\ {\scriptstyle P_f}\downarrow & & \downarrow{\scriptstyle P_{if}} \\ \mathcal{A}sm(\mathcal{Q}) & \xrightarrow[i]{} & \mathrm{RT}(\mathcal{Q}) \end{array}$$

commutes (up to natural isomorphism). Therefore the W-type in $\mathcal{A}sm(\mathcal{Q})$ is a fixpoint in $\mathrm{RT}(\mathcal{Q})$. It is also well-founded, because subobjects of $\neg\neg$-separated objects are again $\neg\neg$-separated. Therefore it is also the W-type in $\mathrm{RT}(\mathcal{Q})$ by either Theorem 2.1.5 or Theorem 2.2.3.

M-types are constructed in a similar fashion. When $f: B \longrightarrow A$ is a morphism of assemblies, write M for the M-type of sets associated to the underlying function of f in sets. Write sup for the inverse of the structure map of the coalgebra M. A decoration for an element $m = \mathrm{sup}_a(t) \in M$ is an element $d \in Q$ such that d codes a pair $\langle d_0, d_1 \rangle$, where d_0 realises $a = \rho(m)$ and $d_1 \cdot y$ is defined for every realiser y of some $b \in B_a$ and is a decoration of tb, in the sense that it codes a pair $\langle e_0, e_1 \rangle$, where …etcetera (somewhat pedantically, one can say that the notion of a decoration is coinductively defined). More formally, say that a sequence of elements

$$\langle d_0, e_0, d_1, e_1, \ldots, d_n \rangle$$

in Q *tracks a path*

$$\langle m_0, b_0, m_1, b_1, \ldots, m_n \rangle$$

in M, when $j_0(d_i)$ realises $\rho(m_i)$, e_i realises b_i and $j_1(d_i) \cdot e_i = e_{i+1}$. An element $d \in Q$ is a decoration for m, when every path starting from m is tracked by a sequence beginning with d. The advantage of defining the M-type in this manner, is that it makes clear that any reliance on impredicative methods turns out be only apparent. Since this definition in the case of well-founded trees coincides with the definition given above, this means that both the construction of the W-type as that of the M-type can also be performed within a predicative metatheory, as long as W-types, respectively M-types are available in that metatheory.

Modest sets 2.2.7 Again, fix a pca $\mathcal{Q}$. A *modest set on $\mathcal{Q}$* consists of a set X together with a function $[-]_X: X \longrightarrow \mathcal{P}_i Q$ mapping distinct elements of X to disjoint subsets of Q. Again, one writes $a \vdash_X x$ or simply $a \vdash x$ to mean $a \in [x]_X$, and one says that *a realises x*. Equivalently, a modest set as a set X together with a relation $\vdash_X \subseteq Q \times X$ satisfying

$$x = y \Leftrightarrow \exists a \in Q \colon a \vdash x \text{ and } a \vdash y.$$

A morphism of modest sets from $(X, \vdash_X)$ to $(Y, \vdash_Y)$ is a function of sets $f: X \longrightarrow Y$ having the property that there is a $t \in Q$ *tracking* f in the sense that for all $a \in Q, x \in X$

$$a \vdash x \Rightarrow t \cdot a \downarrow \text{ and } t \cdot a \vdash f(x).$$

In this way one obtains a category $\mathcal{M}od(\mathcal{Q})$, another regular ML-category that is a full subcategory of the realisability topos on $\mathcal{Q}$, where the inclusion is a morphism of regular ML-categories.

$\mathcal{M}od(\mathcal{Q})$ is equivalent to the category $\mathrm{PER}(\mathcal{Q})$ of *partial equivalence relations on* $\mathcal{Q}$. A partial equivalence relation (or simply a PER) on $\mathcal{Q}$ is a symmetric, transitive relation on Q. If R and S are partial equivalence relations on Q, one calls $f \in Q$ *equivalence preserving*, when for all $a, b \in Q$

$$aRb \Rightarrow f \cdot a\downarrow, f \cdot b\downarrow \text{ and } (f \cdot a)S(f \cdot b).$$

Two equivalence preserving elements $f, g \in P$ are considered equivalent when $(f \cdot a)S(g \cdot a)$ for all $a \in P$. A morphism of PERs is an equivalence class of equivalence preserving elements. A good reference on modest sets and PERs is [13].

Usually, when people talk about PERs, they mean partial equivalence relation on the pca K_1, but the definition makes good sense for any pca. For example, as proved in [13], when $\mathcal{Q}$ is Scott's graph model $\mathcal{P}\rightarrow$, $\mathrm{PER}(\mathcal{Q})$ is equivalent to the category of countably based equilogical spaces, which, for that reason, is also a regular ML-category.

That $\mathcal{M}od(\mathcal{Q})$ for any pca $\mathcal{Q}$ has both W-types and M-types is pointed out in [13]. But the description there is not very concrete, and one can easily give a concrete description along the lines of the previous example, so that is what I will do here.

Fix a morphism $f: B \longrightarrow A$ of modest sets. Let W be the W-type of the underlying map of f in sets. Again, define a function $E: W \longrightarrow \mathcal{P}Q$ by transfinite induction as follows: $E(\sup_a t)$ consists of those elements $n = j(n_0, n_1) \in Q$ such that (i) n_0 realises a; and (ii) n_1 tracks t, that is, for every $b \in B_a$ and every realiser m of b, $n_1 \cdot m$ is defined and a member of $E(tb)$. I call a member n of $E(w)$ a *decoration* or a *realiser* of the tree $w \in W$. By a straightforward proof by transfinite induction, one shows that E maps distinct well-founded trees to disjoint subsets. So if V is a the set of all decorated trees ($w \in W$ such that $Ew \neq \emptyset$), then (V, E) is a modest set, which is actually the W-type for f in modest sets. The proof of this fact is completely similar to the one given above and therefore omitted. The inclusion of modest sets over a pca into the realisability topos over that pca again preserves W-types, because it preserves the lccc structure and modest sets are closed under subobjects in $\mathrm{RT}(\mathcal{Q})$. Also the construction of M-types contains no surprises.

Heyting-valued sets 2.2.8 The category of sets valued on a frame forms another regular ML-category. Let H be a frame (a.k.a. a complete Heyting algebra). An object of the category $\mathcal{H}_+$ of H-valued sets is a set X together with a function $[-]_X: X \longrightarrow H$. A function $f: X \longrightarrow Y$ is a morphism of H-valued sets, whenever for all $x \in X$, $[x]_X \leq [f(x)]_Y$. It is not so hard to see that it is a regular ML-category, but references seem to be scarce (some facts are collected in [59] and [66]). The presence of W-types and M-types appears to be new.

For a morphism $f\colon B \longrightarrow A$ of H-valued sets, $P_f(X)$ for an H-valued set X is constructed as follows. Writing $|\ldots|$ for the obvious forgetful functor $|\ldots|\colon \mathcal{H}_+ \longrightarrow \mathrm{Sets}$, $|P_f(X)| = P_{|f|}(|X|)$. An element $(a, t) \in |P_f(X)|$ is then mapped to

$$\bigwedge_{b \in B_a} ([b] \Rightarrow [tb]) \wedge [a].$$

The W-type W_f is then also computed as in sets on the level of the underlying sets. An element $\sup_a(t)$ has the above expression as its value (this inductively defines an element in H for the well-founded tree). M-types are a bit more involved, but not harder, to describe, and will be omitted.

Heyting algebras 2.2.9 Heyting algebras can be considered as categories, like any poset. Considered in this way, they are cartesian closed, because of the implication $\rightarrow$: $a^b = b \rightarrow a$. They are in fact locally cartesian closed, because for any a in a Heyting algebra H, $H/a = \{x \in H \,|\, x \leq a\}$ is again a Heyting algebra, with implication given by $b \rightarrow b'$ (in H/a) $= (b \rightarrow b') \wedge a$. So it makes sense to ask whether Heyting algebras possess W-types and/or M-types. The curious answer is that they have both: for a map $f\colon b \longrightarrow a$ (which simply means $b \leq a$),

$$P_f(x) = (b \rightarrow (x \wedge a)) \wedge a.$$

So the W-type is the least fixpoint for this, which is $\neg b \wedge a$, and the M-type is the greatest fixpoint, which is a.

Setoids 2.2.10 The last example, for now, is built from the syntax of (intensional) Martin-Löf type theory (see Appendix B for an introduction to type theory). A *setoid* is a type X together with an equivalence relation, meaning a type $R(x, y)$ in the context $x \in X, y \in X$ with proof terms for reflexivity, symmetry and transitivity. A morphism of setoids from (X, R) to (Y, S) is an equivalence class of terms t of type $X \rightarrow Y$ preserving the equivalence relation (meaning that there is a term of type $\Pi x, y \in X.\, R(x, y) \rightarrow S(tx, ty)$). Such terms s and t are considered equivalent, when there is a term of type $\Pi x\colon \in X.\, S(sx, tx)$. The category obtained in this fashion, will be denoted by $\mathcal{S}etoids$.

Theorem 2.2.11 *(See [60], Section 7.) $\mathcal{S}etoids$ is a ΠW-pretopos.*

This theorem has "ideological" importance, in that it shows that a ΠW-pretopos is a predicative structure (I consider Martin-Löf type theory to be the paradigmatic constructive-predicative theory). In the next Chapter, I will show that $\mathcal{S}etoids$ is *not* the free ΠW-pretopos. This is unfortunate, because if it were, there would have been a way in which the theory of ΠW-pretoposes would have been useful also for studying intensional type theory. Now it seems that all light that it will shed on type theory, will fall on the extensional version (see the Appendix for the difference between intensional and extensional type theory).