Chapter 1

Introduction

1.1 Background

Mathematical logic is the study of formal systems. In this way, the logician claims to understand something about mathematical reasoning. On the face of it, this is a surprising claim: as formal systems hardly play any rôle in the work of the ordinary mathematician, how can they illuminate her practices?

In everyday life, a mathematician who gives an argument never makes all her assumptions and reasoning steps fully explicit. She relies on her intuition: things she finds obvious and does not care to explain only remain implicit. This does not prevent her from communicating with her peers, because they have the same tacit understanding she has. If requested, she can explain herself more fully and fill in the gaps. In the end, she expects to be able to go back to some basic axioms, those of set theory, to make her argument absolutely rigorous, but this for her is an ideal possibility she would not care to pursue, except in some extreme cases.

Nevertheless, her basic assumptions, which are most probably made fully precise by the formal set theory **ZFC**, tell a lot about her conception of mathematics. From a philosophical point of view, **ZFC** expresses a belief in a static universe of mathematical objects, the properties of which she discovers rather than invents and which are generally independent of her cognitive activity. But different points of view are possible. Some mathematicians and philosophers have objected to what are called the non-constructive and impredicative aspects of **ZFC**.

Constructivism: An argument is constructive when the mathematical objects that are claimed to exist can actually be effectively found. Ordinary mathematics is full of arguments that are non-constructive (think of non-principal ultrafilters, bases for every vector space, maximal ideals in rings, etcetera). Such arguments express a strong belief in the mind-independent nature of mathematical objects, since it is hard to see how the objects that are claimed to exist can actually be built by the mathematician. A constructivist may for various philosophical reasons reject this picture, and insist that all arguments have to be constructive. Constructivist views go back to Kronecker at least, but have found a coherent and comprehensive expression in Brouwer's intuitionism. Not all constructivists have since then been intuitionists, but Brouwer's identification of the Law of Excluded Middle as the main culprit of the non-constructive nature of classical mathematics has been very influential.

Constructivism has its origin as a philosophy of mathematics, but is, ironically enough, more influential nowadays in computer science than in mathematics. A natural reading of the view that a constructive argument should always allow one to find the objects that are asserted to exist, is that these objects can be computed by an algorithm. In computer science, this has led to the paradigm that constructive proofs can in fact be regarded as programs, and vice versa. The same idea is behind what is called "realisability": realisability is an interpretation of the arguments of the constructive mathematician using the concepts of recursion theory, the mathematical theory of algorithms and computation.

Predicativism: Besides non-constructive arguments, **ZFC** also allows for the formalisation of impredicative arguments. These have been criticised by mathematicians like Russell, Poincaré and Weyl, also starting from a certain constructivist bias.

The view is basically that sets do not exist in themselves, but are the result of the mathematician collecting objects into a whole. Predicativists observe that certain definitions in ordinary mathematics define an element x in terms of a set A to which it might belong (think of a defining a real x as the supremum of a set of reals A bounded from above). If one thinks of the element x as being built by giving the definition, there is a clear problem here. The element x has to exist before the set was built (because it was collected in the set A), but at the same time it cannot exist before the set A was built (because it was defined in terms of it). Usually, the Powerset Axiom and the unrestricted Separation Axiom are considered to be the axioms that make the set theory **ZFC** intrinsically impredicative.

The discovery that made the foundations of mathematics a part of mathematics, was that **ZFC** and other foundational schemes, that are either intuitionistic or predicative (or both), can be studied *mathematically*. For that purpose, all the principles that the different foundational stances are committed to, have to be made completely explicit. Formalisation therefore not only makes fully precise what a conception of mathematics is committed to, so that one can see whether an argument is correct on such a conception, but it also allows the possibility of studying such conceptions mathematically. And that is what happens in this thesis.

But what kind of questions could the mathematician try to answer about these formal systems? She can compare different schemes in terms of their strength, for example, by providing a translation from one into the other. Or she can try to prove consistency of a formal system within another or try to prove that one can, without loosing consistency, add principles to a formal system. Another possibility is to prove certain principles independent: where one usually proves that certain principles can be proved by doing precisely that, one could also to try to prove that certain principles cannot be proved within a certain formal system. A statement is independent from a formal theory when it is both impossible to prove and to disprove it in the theory (the most famous examples of such independence results being the consistency of Peano Arithmetic within Peano Arithmetic, the Continuum Hypothesis in **ZFC** and the Axiom of Choice in **ZF**).

There is immediately one complication: what mathematical principles does one employ in studying formal systems? If this is to be a mathematical investigation, the arguments have to be conducted (ideally) within a certain formal system. Unfortunately, there is no Archimedean point from which one can judge any formal system in absolute terms. Also the mathematical logician is working within a mathematical theory, her "metatheory", and the best she can do is to make this as weak as possible (preferably, no stronger than the weakest theory she is studying). This is just a fact of life.

The methods to establish metamathematical results can roughly be classified as either "syntactic" (proof-theoretic) or "semantic" (model-theoretic). As I understand it, proof theorists assign ordinals to formal theories in order to measure their strength. The starting point is Gödel's Incompleteness Theorem: a theory S that proves the consistency of a theory T is stronger than T. So when Gentzen proves that adding induction up to ϵ_0 to Peano Arithmetic (**PA**) allows you to prove the consistency of the original system, **PA** plus induction up to ϵ_0 is a stronger system that **PA** itself. Induction up to lower ordinals is provable in **PA**, so in a sense ϵ_0 measures the strength of Peano Arithmetic. Theories that prove induction up to ϵ_0 are stronger than **PA** and principles that imply this statement cannot be provable within Peano Arithmetic.

This thesis takes a model-theoretic approach. A model-theorist proves the consistency of a collection of statements by exhibiting a model, a mathematical structure in which they are all correct. Suppose one takes the famous translation of 2-dimensional elliptic geometry into 3-dimensional Euclidean geometry in which the plane is a 2-sphere, points are diametrically opposed points, and lines are concentric circles. If you believe in the existence of the 2-sphere, diametrically opposed points and concentric circles, you have to believe in the consistency of elliptic geometry. The work of Cohen proceeds along the same lines: starting with a model of **ZFC**, one can manipulate it in such a way that it becomes a model of **ZFC** including the negation of the Continuum Hypothesis. This shows that one can consistently add the negation of Continuum Hypothesis to **ZFC**. Assuming **ZFC** has a model, of course, which is (in view of Gödel's Completeness theorem that guarantees the existence of models of consistent (first-order) theories) the same as assuming the consistency of **ZFC**.

The thesis will be about formal systems that are different from **ZFC**, in that they are constructive as well as predicative. Predicative systems with classical logic descend from Weyl's *Das Kontinuum* and have mainly been studied from a proof-theoretic point of view, while constructive systems that are impredicative, like **IZF** (which is basically **ZF** with intuitionistic logic) and higher-order type theory, have also been studied from a more semantic point of view, in the form of topos theory (I will come back to this).

The historical origin of the interest in formal systems that are both constructive and predicative lies in Bishop's book on constructive analysis [17]. The book did a lot to rekindle the interest in constructive analysis (as can be seen from the difference in the lengths of the bibliographies in the first and second edition). There was of course an intuitionistic school in constructive analysis, but many became disheartened by the less attractive features of Brouwer's thought.[1] Bishop skillfully managed to avoid those and showed that it is possible to develop constructive analysis in an elegant and coherent fashion. Among other things, he managed not to unnerve the classical mathematician by only using methods she also believed to be valid, and not making analysis dependent on strange entities called choice sequences. He did manage to include variants of the well-known results, possibly with stronger assumptions, and was able to develop some "higher" mathematics (more, say, than the mathematical theory covered in any first-year analysis course). The impression the book left behind on many people was that Bishop made the constructive program look much more attractive than it had ever done.

The relative success of the book made it a worthwhile task to understand the conception of mathematics that was expressed in the book. Although the book talked about sets and functions in a way familiar to any mathematician, it is clear that these terms cannot have the meaning they are usually taken to have (as is clear from Bishop's insistence that all his mathematical statements have "numerical meaning"). But how the notions of sets and functions were now supposed to be understood, Bishop did not make very explicit. It may be considered an advantage that Bishop did not spend page after page to explain his basic notions and instead went straight on to develop his mathematics, but, as I explained, a formal framework is essential for studying such notions mathematically.

As Bishop did not make his commitments fully precise, the task of formalising his approach to mathematics fell to other people. The first attempt was made by Myhill in [62]: he formulated a set theory **CST** that, he claimed, allows for a formalisation of Bishop's book. **CST** was a theory much like ordinary **ZFC**, which has the advantage

[1] Think of the (in)famous obscurity of Brouwers philosophical views, which included a degree of metaphysical solipsism, a considerable amount of mysticism and a distaste for formalisation. In addition his views seemed to make doing mathematics very cumbersome as certain methods were considered taboo, allowed for certain mathematical objects, like choice sequences, that were very much unlike anything introduced into mathematics before, included mathematical principles that are false from the classical point of view, and finally seemed to be destructive of certain portions of mathematics treasured by anyone who has ever studied it. One can understand why they were not considered very inviting.

of looking familiar to the general mathematician. But it had the features of being both constructive and predicative. That its underlying logic is intuitionistic, so as not to include the Law of Excluded Middle, was to be expected, but Myhill also argued that the framework had to be formulated within predicativist limits. More precisely, he argued for a restricted version of the Separation Axiom and the exclusion of the Powerset Axiom. In general, one might think (as I do) that a consistent constructivist has to be a predicativist as well, but that being as it may, in his book, Bishop abided by predicativist strictures.

The set theory **CST** was extended by Aczel to a stronger set theory he called **CZF**. Besides formulating a framework for doing Bishop-style constructive mathematics, he also provided a clear constructive justification, by interpreting **CZF** into Martin-Löf's type theory. Martin-Löf type theory is another attempt to elucidate the activity of the constructive mathematician. Its strength is that it provides a direct analysis of the basic concept of constructivism: that of a mathematical construction. Thus Aczel's interpretation of **CZF** into type theory makes explicit how the constructive nature of his theory is to be understood.

Martin-Löf type theory is a remarkable theory, which has made an impact on computer science as well. While the system was originally formulated by Martin-Löf as a formal analysis of constructivist mathematics, he also pointed out that it could be regarded as a programming language. As such it has two noteworthy features: first, programs written within this system are always correct, in the following sense. Typically, programs are written to calculate the solution to a problem which has been specified in advance. While writing a program, one simultaneously (mathematically) proves that the program computes what is the solution to the problem (and not something else). Furthermore, it is impossible to write programs that do not terminate: so-called "loops" are guaranteed not to occur.

Also in the mechanical verification of mathematical arguments, type theory has been influential. The most impressive feat in this respect might be the complete formal verification of the Four Colour Theorem in the system **COQ** by Gonthier. **COQ** is based on the Calculus of Constructions, an extension of Martin-Löf type theory.

Both **CZF** and Martin-Löf type theory are still predicative formal theories. An early formulation of Martin-Löf type theory allowed for impredicative definitions, but as that turned out to be inconsistent (Girard's paradox), its successors have been formulated within predicative limits. And, like **CST**, **CZF** is not committed to the existence of all powersets, and contains the Separation Axiom only in a restricted form. Now that both **CZF** and Martin-Löf type theory have emerged as formal systems for doing constructive-predicative mathematics, and books are currently being written on how to do that (on **CZF** by Aczel and Rathjen, on type theory by Dybjer, Coquand, Setzer and Palmgren), the time seems ripe for a mathematical investigation of these systems. This thesis hopes to contribute to that.

But before misunderstandings arise, several remarks should perhaps be made. As a first remark, I should say that I will focus on the set theory **CZF**. Implications of my work for type theory will not be pursued and are left for the future. (I will, however, say something about the connection in Appendix B.)

Secondly, I should say that I am not the first and only person to work on **CZF**. The first results concerning **CZF** are, of course, due to Peter Aczel [2, 3, 4], but recently the set theory has been investigated mainly from a proof-theoretic point of view by Rathjen and Lubarsky [70, 74, 73, 75, 54, 53] (some of which is as yet unpublished). Sheaf models (in particular, forcing) for **CZF** have been investigated by Grayson [34] and, more recently, Gambino [30, 31, 32]. The best introduction to **CZF** is [7]. My work distuingishes itself from the approach of these authors in its heavy reliance on categorical language and methods and in emphasising inductively generated sets.

My source of inspiration for the categorical approach to the subject is topos theory. While topos theory was initially developed in the Grothendieck school for the purposes of algebraic geometry, the theory became interesting for logicians when Lawvere emphasised the importance of the "subobject classifier" and gave the definition of an elementary topos. For the logician, elementary toposes are models for a higher-order intuitionistic type theory. Topos theory turned out to be a rich subject (as witnessed by the two thick volumes of Johnstone's Elephant [44] and [45], with a third to come), with plenty of implications for higher-order intuitionistic type theory.

Not only type theory, but also (intuitionistic) set theory profited from the development of topos theory. In the eighties of the last century, the study of the constructive, but impredicative set theory **IZF**, was conducted mainly within a topos-theoretic framework (as can be seen from sources like [51], Part III and [81], Chapter 15).[2]

This thesis studies constructive-predicative formal theories in a similar spirit. Some adaptations are in order, as a topos is a structure that is far too rich for my purposes. Due to the subobject classifier, it models impredicative structures as well. Therefore one of the themes of this thesis is the quest of a predicative analogue to the notion of a topos, a kind of "predicative topos". Such predicative toposes should have the same properties as toposes, especially having the same closure properties that have proved important for logical applications, while simultaneously providing models for constructive and predicative formal theories.

The idea of developing a predicative theory of toposes, helpful in studying set theories like **CZF** and Martin-Löf type theory, goes back to two papers by Moerdijk and Palmgren ([60] and [61]). In the first, they put forward a categorical definition of W-type and the notion of a ΠW-pretopos, which I believe to be a very suitable candidate for the title of "predicative topos". They prove stability of ΠW-pretoposes under a number of topos-theoretic constructions, including slicing, taking sheaves and glueing (over $\mathcal{S}ets$).

[2] The work by Cohen on the independence of the Continuum Hypothesis and the Axiom of Choice from classical set theory can also be reformulated topos-theoretically (see [18] and [56], Chapter 6).

In their second paper, Moerdijk and Palmgren use "algebraic set theory" to connect Π*W*-pretoposes with set theory: with this machinery, they show how a model of **CZF** can be built inside a Π*W*-pretopos. Essentially, this is a generalisation of Aczel's interpretation of **CZF** inside Martin-Löf type theory (this will be recapitulated in Chapter 4 of this thesis).

The idea of categories as a place ("topos") where models of set theory can live, is the starting point of algebraic set theory. This subject dates back to a book by Joyal and Moerdijk [47] with the same name and provides a uniform categorical tool for studying formal set theories. In the meantime the approach has been taken up by several categorical logicians, including Steve Awodey, Alex Simpson, Carsten Butz, Thomas Streicher, Henrik Forsell, Michael Warren, my former colleague Claire Kouwenhoven-Gentil and present colleague Jaap van Oosten.[3] Steve Awodey and his PhD-student Michael Warren have also studied constructive-predicative set theories within the setting of algebraic set theory [10].

Besides the usage of categorical methods, another respect in which the approach taken here differs from that of other logicians working on **CZF**, is in emphasising inductively defined structures. It is very natural to allow for a wide class of inductively generated sets within a constructive-predicative viewpoint. For example, Poincaré took in his philosophical papers the principle of induction as the only truly mathematical (as opposed to logical) principle. Some predicative theories have been proposed that do not allow for particular inductively defined sets, but that looks unnecessarily restrictive, and theories like **CZF** and Martin-Löf type theory have in one form or another incorporated features to build inductively defined sets. To be more precise, Aczel extended **CZF** with a Regular Extension Axiom (**REA**) to allow for inductively generated sets and Martin-Löf type theory contains a class of inductive types called W-types.[4] W-types are thought of as sets of well-founded trees, and the categories I work with, will contain inductively defined structures of that form.

The emphasis on inductively generated structures, is complemented by a discussion of coinduction. I do not mean to give an introduction here to the notions of coinduction and bisimulation (for that, see [42] and [83]), but I do want to provide some historical background.

The idea of a "non-well-founded" analogue to set theory was popularised by Peter Aczel in his book [5]. He forcefully argued that it would be worthwhile to investigate alternatives to the Axiom of Foundation (or Regularity Axiom), which (classically) says that there are no infinitely descending ϵ-chains. The axiom is not necessary to prevent set-theoretic paradoxes, and has no relevance for the work of "ordinary" mathematicians, but is mainly of use to metamathematicians, in that it provides them with a convenient picture of the universe of sets (the so-called cumulative hierarchy).

[3] References can be found at the webpage devoted to the subject: `http://www.phil.cmu.edu/projects/ast/`.

[4] As explained in Appendix B, all the types in Martin-Löf type theory are in a sense inductively generated sets.

In set-theoretic texts, it is usually pointed out that it does no harm to accept the axiom (a result by Von Neumann), but that does not exclude the possibility of interesting alternatives.

In his book, Peter Aczel made a case for one such alternative: the Anti-Foundation Axiom. The name is a bit misleading in that it suggests it might be the only possible alternative, but it is more colourful than "X_1-axiom", as it was originally called by Forti and Honsell [28]. Aczel's starting point was the old idea that sets can be presented as trees: the representation of a set x consists of a node, with one edge into this node for every element of y of x, on which one sticks the representation of that element, which consists of a node, etcetera. The Foundation Axiom implies that only well-founded trees will now represent sets, and the sets they represent are automatically unique (basically by the Extensionality Axiom). The idea of the Anti-Foundation Axiom is to have *all* trees (well-founded or not) represent *unique* sets. (Hence "non-well-founded" set theory.)

As Peter Aczel discovered, this means that one can solve systems of equations like:

$$\begin{aligned} x &= \{x, y\} \\ y &= \{x, \{x, \{y\}\}\} \end{aligned}$$

uniquely. This proved a very fruitful idea in computer science, where people lacked a conceptual language to describe circular (and, more generally, non-terminating) phenomena. The other concepts that Aczel isolated (coinduction, bisimulation), frequently put in a category-theoretic framework, now belong to the standard arsenal of tools in the theory of concurrency and program specification, as well as in the study of semantics for programming languages with coinductive types [23, 24, 42, 83, 12].

1.2 Contents and results

One of the main aims of this thesis is to convince the reader that the notion of a ΠW-pretopos is a sensible predicative analogue of the notion of a topos. In order to make a persuasive case, I need to show two things: first, I have to make clear that the class of ΠW-pretoposes shares many of the properties with the class of toposes. This applies in particular to the closure properties that have been exploited in the logical applications of topos theory. Secondly, I should explain how ΠW-pretoposes provide models for constructive and predicative formal theories, like Aczel's **CZF**.

The contents of this thesis are therefore as follows. Chapter 2 introduces W-types in a categorical context and ΠW-pretoposes. It also proves a (new) characterisation theorem that helps one to recognise W-types in categories. This is then used to identify and concretely describe W-types in various categories.

In Chapter 3, I prove two closure properties of ΠW-pretoposes, both of them new.

The first of these is the closure under exact completion. This result is surprising, as the corresponding result for toposes is false. As will be discussed, less than the structure of ΠW-pretopos is needed to get an exact completion that is a ΠW-pretopos. I identify weaker categorical structures ("weak ΠW-pretoposes"), whose exact completions are ΠW-pretoposes. This is then used to give more examples of ΠW-pretoposes, one of which is a kind of predicative realisability topos. It shows that ΠW-pretoposes are closed under a notion of realisability, which promises to be important for logical purposes. Finally, I prove closure of ΠW-pretoposes under (general) glueing. Combined with closure under exact completion, one obtains a result concerning the projectives in the free ΠW-pretopos.

Chapter 4 leaves the area of pure "predicative topos theory" and studies an application to the set theory **CZF**. Using the framework of algebraic set theory along the lines of [61], I prove that the models of **CZF** of Streicher in [80] and Lubarsky in [53] exist as objects in the effective topos, and are in fact the same. The model is then further scrutinised and shown to validate a host of constructivist principles. The result that these are therefore collectively consistent with **CZF** is new.

The two final Chapters are joint work with Federico De Marchi and are concerned with categories with coinductively generated structures. In the same way as a W-type is an inductively generated set of a particular type, which is to be thought of as a set of well-founded trees, the dual notion of M-type is to be thought of as a coinductively generated set of non-well-founded trees.

Chapter 5 studies M-types in categories. I prove some existence results concerning M-types: the main result in this direction is that the existence of a fixpoint for a polynomial functor implies the existence of an M-type. This is also used to strengthen a result by Santocanale on the existence of M-types in locally cartesian closed pretoposes with natural number object. The Chapter also introduces the notion of a ΠM-pretopos and continues to investigate the possibility of developing a theory of ΠM-pretoposes analogous to the theory of ΠW-pretoposes. More particularly, it studies the stability of ΠM-pretoposes under various topos-theoretic constructions, like slicing, coalgebras for a cartesian comonad and sheaves. In topos theory, these closure properties have proved useful for logical applications and the hope is that these results will have applications to models of non-well-founded set theory and type theories with coinductive types.

An interesting question is whether coinductively defined structures are essentially impredicative. This question is also discussed by Rathjen in [72]: his conclusion is that Aczel's theory of non-well-founded sets can be developed without using such impredicative objects as powersets and the like. For that reason, he feels that the circularity that the predicativist discerns in impredicative definitions is of a different kind than the circularity in coinductively defined sets. Such views are of course highly philosophical, but are at least confirmed in that there are models of predicative formal theories where the Anti-Foundation Axiom is valid. A general method for constructing such models (classical or constructive, predicative or impredicative) is obtained in

Chapter 6.

Following Aczel [5], I use a final coalgebra theorem to construct such models. Therefore I first prove an abstract categorical final coalgebra theorem applicable in the setting of algebraic set theory. This is then applied to prove the existence of M-types and models of non-well-founded set theory in settings very much like that of the original book on algebraic set theory by Joyal and Moerdijk [47]. This is also joint work with Federico De Marchi.

The thesis concludes with three appendices that are meant to provide some background for this thesis. The first introduces the category theory and categorical terminology that is needed to understand this thesis, while the second gives an introduction to Martin-Löf type theory. The third is on partial combinatory algebras and realisability toposes.