Appendix C

Pcas and realisability

This Appendix briefly discusses the definitions of pcas and realisability toposes. References for pcas are [11] and [35], while on realisability toposes the reader should consult [40], [39] and [69].

Pcas C.1 Before being able to define pcas, I need the notion of a *partial applicative structure*. A partial applicative structure $\mathcal{Q} = (Q, \cdot)$ is a set Q equipped with a partial binary operation $(a, b) \mapsto a \cdot b$. The partial application $\cdot$ is frequently not written down: one very often writes ab instead of $a \cdot b$. The usual conventions for working with partial operation are assumed to be in place. For two expressions ϕ and ψ involving elements of Q and the binary operation $\cdot$, one writes $\phi \downarrow$ to mean "ϕ is defined", $\phi = \psi$ to mean "ϕ and ψ are defined and equal" and $\phi \simeq \psi$ to mean "when ϕ or ψ is defined, so is the other and they are equal". Another convention is that of "bracketing to the left": abc should be read as $(ab)c$.

Given a pca $\mathcal{Q}$ and a countable set of fresh variables $x_0, x_1, x_2, \ldots$, the set of terms $T(\mathcal{Q})$ is the smallest set closed under:

1. $a \in T(\mathcal{Q})$ for all $a \in Q$,
2. $x_i \in T(\mathcal{Q})$ for all $i \in \mathbb{N}$,
3. whenever $a, b \in T(\mathcal{Q})$, then $(ab) \in T(\mathcal{Q})$.

One should think of the elements of $T(\mathcal{Q})$ as the set of polynomials with coefficients in $\mathcal{Q}$.

A *partial combinatory algebra (pca)* $\mathcal{Q} = (Q, \cdot)$ is a partial applicative structure that is *combinatory complete*, in the sense that for every term $t(x_0, \ldots, x_n) \in T(\mathcal{Q})$ there is an element $q \in Q$ such that for all $a_0, \ldots, a_n \in Q$:

(i) $qa_0 \ldots a_{n-1} \downarrow$ and

(ii) $qa_0 \ldots a_n \simeq t(a_0, \ldots, a_n)$.

As is well-known, to get combinatory completeness it is necessary and sufficient to require the existence of two elements k and s in Q satisfying the following laws:

1. $kab = a$,
2. $sab\downarrow$,
3. $sabc \simeq ac(bc)$.

Actually, pcas are usually defined in terms of k and s, but since there are in any pca an indefinite number of ks and ss having these properties, the definition in this form is less canonical. And combinatory completeness is where pcas are all about.

The important facts about pcas, from my point of view, are the following. Due to combinatory completeness, elements in a pca may be denoted by lambda terms, like $\lambda x_0, \ldots, x_n.t(x_0, \ldots, x_n)$. This is a bit tricky, since pcas are not models of the lambda calculus, as there may be no good interpretation of lambda terms containing free variables (see [11]). But, like in the lambda calculus, one can solve fixpoint equations, there is a choice of Church numerals in any pca (which will usually be denoted by the ordinary numerals), and there are pairing operations with associated projections. By the latter I mean that there are always elements j, j_0, j_1 in a pca $\mathcal{Q}$ such $jab\downarrow$, $j_0(jab) = a$ and $j_1(jab) = b$ for all $a, b \in Q$. Instead of jab I will also frequently write $\langle a, b\rangle$. Results of repeated pairings will often be denoted by terms of the type $\langle a_1, a_2, \ldots, a_n\rangle$, with the associated projections denoted by j_i $(1 \leq i \leq n)$.

The prime example of a pca is that of the natural numbers equipped with Kleene application: one fixes a particular coding $\{-\}$ of the partial recursive functions as natural numbers, so that $\{m\}$ is the partial recursive function encoded by the natural number m. Then defines $m \cdot n \simeq \{m\}n$ to obtain K_1, Kleenes pca. Models of the lambda calculus provide other examples, like Scott's graph model $\mathcal{P}\omega$.

Heyting pre-algebras C.2 A *Heyting pre-algebra* is a pre-order, that has finite limits and colimits and is cartesian closed as a category. As for partial orders, products and coproducts are denoted by $\wedge$ and $\vee$, respectively, while the exponentials a^b are denoted $b \to a$. The order is usually denoted by $\vdash$.

For any pca $\mathcal{Q}$, write $\Sigma = \mathcal{P}Q$ for the powerset of Q. Σ carries the structure of pre-order as follows: $A \vdash B$, when there is a $q \in Q$ such that $q \cdot a\downarrow$ for all $a \in A$, and $q \cdot a \in B$. It has, in fact, the structure of a Heyting pre-algebra in which:

$$
\begin{aligned}
A \wedge B &= \{\langle a, b\rangle \mid a \in A, b \in B\} \\
A \vee B &= \{\langle a, 0\rangle \mid a \in A\} \cup \{\langle b, 1\rangle \mid b \in B\} \\
A \to B &= \{q \in Q \mid q \cdot a\downarrow \text{ and } q \cdot a \in B \text{ for all } a \in A\}.
\end{aligned}
$$

For any set X, one could give Σ^X the structure of a Heyting pre-algebra, by defining the ordering pointwise. But there is another possibility, which is more important for our purposes. Say $F \vdash G$ for $F, G \in \Sigma^X$, when there is a $q \in Q$ such that for all

$x \in X, a \in F(x)$, $q \cdot a$ is defined and in $G(x)$. The point is commonly expressed by saying that there should be a realiser q that works *uniformly* for all $x \in X$. It can easily be shown by extending the definitions above that also when the order of Σ^X is defined in this way, it has the structure of a Heyting pre-algebra.

Triposes C.3 A tripos (over $\mathcal{S}ets$) is an indexed category $\mathbb{P}$ over $\mathcal{S}ets$ whose fibres $\mathcal{P}^I$ are Heyting pre-algebras, with some more properties. In particular, the reindexing functors along functions $f \colon J \longrightarrow I$ are required to preserve the structure of a Heyting pre-algebra, and the reindexing functors have left and right adjoints $\exists_f$ and $\forall_f$, satisfying the Beck-Chevalley condition. This means that triposes have the structure to model many-sorted, first-order intuitionistic logic. I will skip the formal details, but the idea is that the elements of $\mathcal{P}^I$ are predicates on the set I, and formulas $\phi(i)$ in first-order logic with a free variable of sort I can be interpreted in the tripos as such predicates (formulas with more free variables, maybe of different sorts, are interpreted using the products in $\mathcal{S}ets$). Then such formulas $\phi(i)$ are valid, when their corresponding element in $\mathcal{P}^I$ is isomorphic to the terminal object in that fibre. One writes:

$$\mathbb{P} \vdash \phi(i),$$

or simply $\vdash \phi(i)$, when $\mathbb{P}$ is understood.

Any pca $\mathcal{Q}$ gives rise to a tripos $\mathbb{P}$. The fibre $\mathcal{P}^I$ is Σ^I, and reindexing is defined by precomposition. For a predicate $F \in \Sigma^J$ and a function $f \colon J \longrightarrow I$, the quantifiers are defined by:

$$\begin{aligned} \exists_f(F)(i) &= \{q \in Q \mid \exists j \in f^{-1}(i).\, q \in F(j)\} \\ \forall_f(F)(i) &= \{q \in Q \mid \forall j \in f^{-1}(i)\, \forall a \in Q.\, q \cdot a \downarrow \text{ and } q \cdot a \in F(j)\}. \end{aligned}$$

Realisability toposes C.4 Given a tripos $\mathbb{P}$, consider the following category. Objects are pairs $(X, =)$, where X is a set, and $=$ is an element of $\mathcal{P}^{X \times X}$, which the tripos believes to be a partial equivalence relation (i.e. a symmetric and transitive relation), in the sense that

$$\begin{aligned} &\mathbb{P} \vdash x = x' \to x' = x \\ &\mathbb{P} \vdash x = x' \wedge x' = x'' \to x = x''. \end{aligned}$$

The statement that $x = x$ is sometimes abbreviated as Ex (and one thinks of this as saying that "x exists").

Morphisms from $(X, =)$ to $(Y, =)$ are equivalence classes of functional relations. A functional relation is an element $F \in \mathcal{P}^{X \times Y}$, such that the following are valid:

$$\begin{aligned} &Fxy \wedge x = x' \wedge y = y' \to Fx'y' \\ &Fxy \to Ex \wedge Ey \\ &Fxy \wedge Fxy' \to x = x' \\ &Ex \to \exists y\, Fxy. \end{aligned}$$

Two such functional relations F, G are equivalent, when they are extensionally equal in the sense that

$$\mathbb{P} \vdash Fxy \leftrightarrow Gxy$$

(for this to obtain, the validity of one implication is sufficient).

This defines a category (not quite, but identities and compositions can be constructed), which is actually a topos: for this, one uses some of the structure of a tripos that I have not explained, but also does not concern me. The important thing is the following theorem.

Theorem C.5 *The category defined out of a tripos in the way explained above, is a topos.*

When the tripos derives from a pca $\mathcal{Q}$, the topos built in this fashion is called the *realisability topos* over $\mathcal{Q}$, and denoted by $\mathrm{RT}(\mathcal{Q})$.

Theorem C.6 *The category $\mathrm{RT}(\mathcal{Q})$ is a topos with nno.*

In case $\mathcal{Q}$ is Kleene's pca K_1, $\mathrm{RT}(\mathcal{Q})$ is what is called the *effective topos $\mathcal{E}ff$*, which is therefore also a topos with nno.

In the thesis, I use many results on the effective topos, but I do not think it is worthwhile to summarise them here. However, I do want to record the following two facts, which are useful to know. They both concern canonical representations of categorical notions in a realisability topos.

Subobjects of an object $(X, =)$ in $\mathrm{RT}(\mathcal{Q})$ are in one-to-one correspondence to equivalence classes of strict relations, i.e. elements $R \in \mathcal{P}^X$ such that the following are valid:

$$\begin{array}{l} Rx \wedge x = x' \rightarrow Rx' \\ Rx \rightarrow Ex. \end{array}$$

Two such strict relations R, S are equivalent, when $Rx \leftrightarrow Sx$ is valid.

Quotients of an object $(X, =)$ in $\mathrm{RT}(\mathcal{Q})$ are in one-to-one correspondence to equivalence classes of elements $R \in \mathcal{P}^{X \times X}$ satisfying the following:

$$\begin{array}{l} \mathbb{P} \vdash Ex \rightarrow Rxx \\ \mathbb{P} \vdash Ex \wedge Ey \wedge Rxy \rightarrow Ryx \\ \mathbb{P} \vdash Ex \wedge Ey \wedge Ez \wedge Rxy \wedge Ryz \rightarrow Rxz. \end{array}$$

Again, two such elements $R, S \in \mathcal{P}^{X \times X}$ are equivalent, when $Rxy \leftrightarrow Sxy$ is valid.