Appendix A

Categorical background

This Appendix is meant to provide the prospective reader of this thesis with sufficient categorical background (or to refresh his, resp. her, memory). An excellent source on these matters is the first volume of Johnstone's Elephant [44].

Cartesian categories A.1 A category $\mathcal{C}$ is called *cartesian* if it possesses all finite limits. A functor between cartesian categories is called *cartesian* if it preserves finite limits.

Practically all categories in this thesis are cartesian. Slightly better categories are regular.

Regular categories A.2 There are several equivalent ways of defining regular categories. From a logical point of view, regular categories are cartesian categories in which one can interpret the existential quantifier. In any cartesian category, a morphism $f: Y \longrightarrow X$ induces a functor $f^*: \text{Sub } X \longrightarrow \text{Sub } Y$, by pullback. In regular categories, such functors f^* have left adjoints $\exists_f$. Applying $\exists_f$ to the maximal subobject $Y \subseteq Y$, one obtains the image of f: a subobject $X \subseteq A$ is called the *image* of a map $f: Y \longrightarrow X$ in a category $\mathcal{C}$, when it is the least subobject through which f factors. A morphism $f: Y \longrightarrow X$ having as image the maximal subobject $X \subseteq X$ is called a *cover*. As one can see, the notions of image and cover make sense in any category and will be used frequently in this thesis.

The notion of a regular category can now be defined as follows. A cartesian category $\mathcal{C}$ is called *regular* if every map in $\mathcal{C}$ factors, in a stable fashion, as a cover followed by a monomorphism. A cartesian functor between regular categories is called *regular* if it preserves covers.

For us, the most important fact about regular categories is the following result due to Joyal:

Lemma A.3 *In a regular category covers and regular epimorphism, i.e. epimorphisms that arise as coequalisers, coincide.*

To interpret full first-order intuitionistic logic, regular categories have to be equipped with more structure. In fact, a regular category $\mathcal{C}$ needs to satisfy the following two conditions to interpret disjunction and the universal quantifier (and implication) respectively:

- The subobject lattice Sub X of any object X in $\mathcal{C}$ has finite unions, preserved by the operation f^* for any $f: Y \longrightarrow X$.
- For any morphism $f: Y \longrightarrow X$, the functor $f^*: \text{Sub } X \longrightarrow \text{Sub } Y$ has a right adjoint $\forall_f$.

When these are satisfied, the category $\mathcal{C}$ is called a *Heyting category*.

Lemma A.4 *Let $R \subseteq A \times B$ be a relation from A to B in a Heyting category $\mathcal{C}$. R is the graph of a (necessarily unique) morphism $A \longrightarrow B$ in $\mathcal{C}$, if and only if the following two statements*

$$\forall a \in A \exists b \in B\, R(a, b)$$
$$\forall a \in A \forall b, b' \in B\, (R(a, b) \wedge R(a, b') \to b = b')$$

are valid in the internal logic of $\mathcal{C}$.

Relations R as in the lemma are called *functional*.

Even better than regular categories are exact categories, also called effective regular categories (in [44], for example).

Exact categories A.5 The idea behind exact categories is that equivalence relations have "good" quotients. I will say in an instant what I mean by a good quotient, but first I have to define what I mean by an equivalence relation in a categorical context.

Definition A.6 Two parallel arrows

$$R \underset{r_1}{\overset{r_0}{\rightrightarrows}} X$$

in category $\mathcal{C}$ form an *equivalence relation* when for any object A in $\mathcal{C}$ the induced function

$$\text{Hom}(A, R) \longrightarrow \text{Hom}(A, X)^2$$

is an injection defining an equivalence relation on the set $\text{Hom}(A, X)$. A morphism $q: X \longrightarrow Q$ is called the *quotient* of the equivalence relation, if the diagram

$$R \underset{r_1}{\overset{r_0}{\rightrightarrows}} X \overset{q}{\longrightarrow} Q$$

is both a pullback and a coequaliser. In this case, the diagram is called *exact*. The diagram is called *stably exact*, when for any $p\colon P \longrightarrow Q$ the diagram

$$p^*R \underset{p^*r_1}{\overset{p^*r_0}{\rightrightarrows}} p^*X \xrightarrow{p^*q} p^*Q$$

is also exact.

A regular category $\mathcal{C}$ is now called *exact*, when any equivalence relation fits into a stably exact diagram. A functor between exact categories is called *exact*, if it is regular.

Among exact categories, pretoposes are of special interest. To identify these, one needs the following definition.

Definition A.7 A cartesian category $\mathcal{C}$ has *finite disjoint, stable sums*, when it has an initial object 0 (the empty sum) and for any two objects A and B a binary sum $A+B$ that is disjoint in the sense that

$$\begin{array}{ccc} 0 & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & A+B \end{array}$$

is a pullback, and stable in the sense that for all maps $A \longrightarrow X$, $B \longrightarrow X$ and $Y \longrightarrow X$, the canonical map $Y \times_X A + Y \times_X B \longrightarrow Y \times_X (A+B)$ is an isomorphism.

A *pretopos* is an exact category with finite disjoint, stable sums. On pretoposes there is the following important result, that will frequently be used.

Lemma A.8 *In a pretopos, every epimorphism fits into a stably exact diagram. Put differently, every epimorphism is the coequaliser of its kernel pair. In particular, epimorphisms, regular epimorphisms and covers coincide.*

Because of the sums, disjunction can be interpreted in a pretopos, but it does not necessarily have the structure to interpret universal quantification. A pretopos $\mathcal{C}$ is therefore called *Heyting*, if for any morphism $f\colon Y \longrightarrow X$ in $\mathcal{C}$ the functor $f^*\colon \mathrm{Sub}\ X \longrightarrow \mathrm{Sub}\ Y$ induced by pullback, has a right adjoint Π_f. So Heyting pretoposes are exact Heyting categories.

Lcccs A.9 Lccc is an abbreviation for "locally cartesian closed category". The quickest way to define an lccc is by first observing that for any morphism $f\colon Y \longrightarrow X$ in a cartesian category $\mathcal{C}$, pulling back along f determines a functor $f^*\colon \mathcal{C}/X \longrightarrow \mathcal{C}/Y$. Such functors always have a left adjoint Σ_f given by composition with f, but when they also have right adjoints, the category $\mathcal{C}$ is an *lccc*. One also sometimes says that $\mathcal{C}$ has *dependent products*.

This definition is the quickest, but it is not the one I will use most often. However, to state the other equivalent definitions, I first need to recall the definitions of an exponential and a cartesian closed category.

Definition A.10 In a category $\mathcal{C}$ with products, an object Z is the *exponential* of two objects A and B, if it is equipped with an evaluation morphism $\epsilon: Z \times A \longrightarrow B$ such that for any morphism $f: X \times A \longrightarrow B$ there is a unique morphism $\overline{f}: X \longrightarrow Z$ such that

$$\begin{array}{ccc} X \times A & \xrightarrow{\overline{f} \times 1} & Z \times A \\ & \searrow^{f} & \downarrow{\scriptstyle \epsilon} \\ & & B \end{array}$$

commutes. In this case Z is often written as B^A. An object A in $\mathcal{C}$ is called *exponentiable*, if B^A exists for any object B. A map $f: X \longrightarrow Y$ is called *exponentiable*, if it is exponentiable as an object of $\mathcal{C}/Y$. A category $\mathcal{C}$ with products in which every object is exponentiable, is called *cartesian closed*.

When an object A in a category $\mathcal{C}$ with products is exponentiable, the association $(-)^A: B \mapsto B^A$ is functorial. In fact, it is right adjoint to the functor $(-) \times A: B \mapsto B \times A$. Therefore, lcccs are certainly cartesian closed.

A cartesian category $\mathcal{C}$ is now *locally cartesian closed*, when it satisfies any of the following equivalent conditions:

1. All pullback functors $f^*: \mathcal{C}/X \longrightarrow \mathcal{C}/Y$ for a map $f: Y \longrightarrow X$ have a right adjoint Π_f.
2. Any morphism $f: Y \longrightarrow X$ is exponentiable.
3. Any slice category of $\mathcal{C}$ is cartesian closed.

The existence of the right adjoints Π_f has a number of consequences. For example, since pullback functors are now also left adjoints, they preserve all colimits. This means in particular that in an lccc, sums are always stable.

Furthermore, because Π_f as a right adjoint preserves monos, right adjoints to the operation of pulling back subobjects along an arbitrary map exist in an lccc. Therefore universal quantifiers can be interpreted. This means that a locally cartesian closed regular category with disjoint sums is a Heyting category. In particular, locally cartesian closed pretoposes, or Π-pretoposes as I will frequently call them, are Heyting pretoposes.

Algebras and coalgebras A.11 The setting is that of a category $\mathcal{C}$ equipped with an endofunctor $T\colon \mathcal{C} \longrightarrow \mathcal{C}$. A category of *$T$-algebras* can then be defined as follows. Objects are pairs consisting of an object X together with a morphism $x\colon TX \longrightarrow X$ in $\mathcal{C}$. A morphism from $(X, x\colon TX \longrightarrow X)$ to $(Y, y\colon TY \longrightarrow Y)$ is a morphism $p\colon X \longrightarrow Y$ in $\mathcal{C}$ such that

$$\begin{array}{ccc} TX & \xrightarrow{Tp} & TY \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X & \xrightarrow[p]{} & Y \end{array}$$

commutes.

I will frequently be interested in the initial object in this category, whenever it exists. This initial object (I, i) is then called the *initial* or *free T-algebra*. As the name free T-algebra suggests, the idea is that the structure of I has been freely generated so as to make it a T-structure. Very often, at least in the cases I am interested in, I has been generated by an inductive definition. Its initiality is then a consequence of the recursive property such an inductively defined object automatically possesses. In fact, the language of initial algebras is the right categorical language for studying inductively defined structures.

For example, in case $\mathcal{C}$ is a Π-pretopos, consider the endofunctor T on $\mathcal{C}$ sending an object X to $1 + X$. Then T-algebras are morphisms $x\colon 1 + X \longrightarrow X$, or equivalently pairs of morphisms $(x_0\colon 1 \longrightarrow X, x_1\colon X \longrightarrow X)$, usually depicted as:

$$1 \xrightarrow{x_0} X \xrightarrow{x_1} X.$$

Morphisms of T-algebras are then commuting diagrams like:

$$\begin{array}{ccccc} 1 & \xrightarrow{x_0} & X & \xrightarrow{x_1} & X \\ {\scriptstyle =}\downarrow & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ 1 & \xrightarrow[y_0]{} & Y & \xrightarrow[y_1]{} & Y. \end{array}$$

The initial T-algebra is called the *natural number object (nno)* in X and is usually depicted as:

$$1 \xrightarrow{0} N \xrightarrow{s} N.$$

It is easy to see that in the category of sets, this is precisely the set of natural numbers with zero and successor, and to verify this fact one uses precisely the fact that functions can be (uniquely) defined by recursion on the natural numbers. One sees that the language of initial algebras allows us to make sense of the notion of the natural numbers in more general categories.

In case $\mathcal{C}$ is just a cartesian category, an indexed version of the above is more useful. An *indexed natural number object* in a cartesian category $\mathcal{C}$ is an object N equipped with the following structure

$$1 \xrightarrow{0} N \xrightarrow{s} N,$$

such that for any (parameter) object P and any arrows $f\colon P \longrightarrow Y$ and $t\colon P \times Y \longrightarrow Y$, there is a unique $\overline{f}\colon P \times N \longrightarrow Y$ for which the diagram

$$\begin{array}{ccccc} P \times 1 & \xrightarrow{1\times 0} & P \times N & \xrightarrow{1\times s} & P \times N \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle(\pi_1,\overline{f})} & & \downarrow{\scriptstyle\overline{f}} \\ P & \xrightarrow[(1,f)]{} & P \times Y & \xrightarrow[t]{} & Y \end{array}$$

commutes. When $\mathcal{C}$ is cartesian closed, it is sufficient to check this for $P = 1$ and the difference between the two definitions disappears.

The following lemma is a result that illustrates the usefulness of an (indexed) nno.

Lemma A.12 *A pretopos $\mathcal{C}$ with an indexed nno is cocartesian, i.e. it has all finite colimits and these are stable.*

Initial algebras have special properties: they are well-founded fixpoints. In some cases, this characterises them completely, but that is not always the case.

Definition A.13 Let $\mathcal{C}$ be a category equipped with an endofunctor $T\colon \mathcal{C} \longrightarrow \mathcal{C}$. A *fixpoint* is an object X together with an isomorphism $TX \cong X$.

Fixpoints can always be regarded as T-algebras, and on the other hand one has the following elementary, but very useful, result by Lambek (see [50]):

Lemma A.14 *An initial T-algebra is a fixpoint.*

Definition A.15 Let $\mathcal{C}$ be a category equipped with an endofunctor $T\colon \mathcal{C} \longrightarrow \mathcal{C}$. A T-algebra X together with a morphism f to a T-algebra Y is called a *T-subalgebra* of Y, when the underlying map of f in $\mathcal{C}$ is a monomorphism. A T-algebra Y is called *well-founded*, when in all its T-subalgebras $f\colon X \longrightarrow Y$, f is an isomorphism.

Instead of saying "X is well-founded", one also says that "X has no proper subalgebras". It is a trivial observation that initial algebras are always well-founded.

Where algebras form the right categorical language to study inductively defined structures, coalgebras are the right categorical language to study phenomena like coinduction and bisimulation, with which I will also be concerned. The setting is again that of a category $\mathcal{C}$ equipped with an endofunctor $T\colon \mathcal{C} \longrightarrow \mathcal{C}$ and the category of *T-coalgebras* is defined dually to that of the category of T-algebras. So objects are pairs consisting of an object X together with a morphism $x\colon X \longrightarrow TX$ in $\mathcal{C}$, and a morphism $p\colon X \longrightarrow Y$ in $\mathcal{C}$ is a morphism of T-coalgebras from $(X, x\colon X \longrightarrow TX)$ to $(Y, y\colon Y \longrightarrow TY)$, when $(Tp)x = yp$. The terminal object in this category, when it exists, is called the *final* or *cofree T-coalgebra*. Some results on initial algebras simply carry over by duality to final coalgebras, in particular Lambek's result that they are fixpoints.

Indexed categories A.16 Algebras and coalgebras can also be defined in an indexed setting. For more on indexed categories, see again the Elephant [44], whose notational conventions I will follow.

An indexed category $\mathbb{C}$ is defined by giving for every object I in a fixed category $\mathcal{S}$, the base of the indexed category, a category $\mathcal{C}^I$. Furthermore, there should be so-called reindexing functors $x^*: \mathcal{C}^I \longrightarrow \mathcal{C}^J$ for every $x: J \longrightarrow I$ in $\mathcal{S}$. Finally, for any two composable arrows $x: J \longrightarrow I$ and $y: K \longrightarrow J$ in $\mathcal{S}$, the functors $(xy)^*$ and y^*x^* are required to be naturally isomorphic, and $(\mathrm{id}_I)^*$ is supposed to be naturally isomorphic to the identity on $\mathcal{C}^I$. The natural isomorphisms, which are part of the data of an indexed category, are in turn demanded to satisfy a number of coherence conditions, which I shall not state here.

An indexed terminal object is given by a family of objects T_I, one for every I in $\mathcal{S}$, such that T_I is final in every category $\mathcal{C}^I$, and for every $x: J \longrightarrow I$, $x^*T_I \cong T_J$. The definition of an indexed initial object is similar. In case the base category $\mathcal{S}$ has a terminal object 1, an indexed terminal object is given by the following data: a terminal object T in $\mathcal{C}^1$, whose reindexings I^*T are still final for every $I = I \longrightarrow 1$ in $\mathcal{S}$.

An indexed functor $F: \mathbb{C} \longrightarrow \mathbb{D}$ for two categories indexed over the same base category $\mathcal{S}$ is given by a family of functors $F^I: \mathcal{C}^I \longrightarrow \mathcal{D}^I$, one for every object I in $\mathcal{S}$. These functors are given together with natural isomorphisms for every $x: J \longrightarrow I$ that fill the squares

$$\begin{array}{ccc} \mathcal{C}^I & \xrightarrow{x^*} & \mathcal{C}^J \\ {\scriptstyle F^I}\downarrow & & \downarrow{\scriptstyle F^J} \\ \mathcal{D}^I & \xrightarrow[x^*]{} & \mathcal{D}^J. \end{array}$$

I will again omit the coherence conditions that these natural isomorphisms need to satisfy.

For an indexed endofunctor F on an indexed category $\mathbb{C}$, one can define a new indexed category: the indexed category F–$\mathbb{A}\mathrm{lg}$ of F-algebras. For any I in $\mathcal{S}$, its fibre $(F\text{–}\mathbb{A}\mathrm{lg})^I$ is the category of F^I-algebras in the category $\mathcal{C}^I$, and the reindexing functors are defined in the obvious way. By an indexed initial algebra, one means an indexed initial object in this indexed category. These are automatically indexed well-founded fixpoints: by this, I mean a family of algebras A_I, one for every I in $\mathcal{S}$, such that each A_I is a well-founded fixpoint for F^I in $\mathcal{C}^I$.

For any cartesian category $\mathcal{C}$, there is the canonical indexing of $\mathcal{C}$ over itself. The fibre $\mathcal{C}^I$ for any I in $\mathcal{C}$ is the slice $\mathcal{C}/I$, while x^* is defined by pullback. Remark that $\mathcal{C}^1$ is really just $\mathcal{C}$. By an indexed endofunctor on a cartesian category $\mathcal{C}$, one means an endofunctor that is indexed with respect to the canonical indexing of $\mathcal{C}$ over itself. In this case, $(F - \mathbb{A}\mathrm{lg})^1$ is also just the ordinary category of F-algebras on $\mathcal{C}$.

When F has an indexed initial algebra, this means that F has an ordinary initial algebra A, with the additional property that for every object I in $\mathcal{S}$, I^*A is initial in $(F - \mathbb{A}\mathrm{lg})^I$. These are also indexed well-founded fixpoints, that is, A is a well-founded fixpoint for

F, and so are all its reindexings. Indexed natural number objects are examples of such indexed initial algebras.

The definitions of an indexed category of coalgebras for an indexed endofunctor on an indexed category, and an indexed final coalgebra, should now be obvious.

Internal categories and colimits A.17 Suppose $\mathcal{S}$ is a cartesian category. An *internal category* $\mathbb{K}$ in $\mathcal{S}$ consists of a diagram

$$K_1 \underset{d_0}{\overset{d_1}{\rightrightarrows}} K_0,$$

where d_1 is the *domain* map, d_0 is the *codomain* one, and they have a common left inverse i. Furthermore there is a composition. If K_2 is the object of composable arrows, i.e. the object

$$\begin{array}{ccc} K_2 & \xrightarrow{p_1} & K_1 \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle d_1} \\ K_1 & \xrightarrow[d_0]{} & K_0, \end{array}$$

there is a morphism $c\colon K_2 \longrightarrow K_1$ such that $d_1 c = d_1 p_0$ and $d_0 c = d_0 p_1$. Composition behaves well with respect to identities: $c\langle \mathrm{id}, i d_0\rangle = \mathrm{id}$ and $c\langle i d_1, \mathrm{id}\rangle = \mathrm{id}$. Finally, composition is associative: if one forms the limit of

$$\begin{array}{ccccccc} K_1 & & & K_1 & & & K_1 \\ & {\scriptstyle d_0}\searrow & \swarrow{\scriptstyle d_1} & & {\scriptstyle d_0}\searrow & \swarrow & \\ & & K_0 & & K_0, & & \end{array}$$

then two possible composites $K_3 \longrightarrow K_2 \longrightarrow K_1$ are equal. There is also a notion of internal functor between internal categories, and this gives rise to the category of internal categories in $\mathcal{S}$ (see Section B2.3 of [44] for the details).

An *internal diagram L of shape* $\mathbb{K}$ in an $\mathcal{S}$-indexed category $\mathbb{C}$ consists of an internal $\mathcal{S}$-category $\mathbb{K}$, an object L in $\mathcal{C}^{K_0}$, and a map $d_1^* L \longrightarrow d_0^* L$ in $\mathcal{C}^{K_1}$ which interacts properly with the categorical structure of $\mathbb{K}$. Moreover, one can consider the notion of a *morphism of internal diagrams*, and these data define the category $\mathbb{C}^{\mathbb{K}}$ of *internal diagrams of shape* $\mathbb{K}$ *in* $\mathbb{C}$.

An indexed functor $F\colon \mathbb{C} \longrightarrow \mathbb{D}$ induces an ordinary functor $F^{\mathbb{K}}\colon \mathbb{C}^{\mathbb{K}} \longrightarrow \mathbb{D}^{\mathbb{K}}$ between the corresponding categories of internal diagrams of shape $\mathbb{K}$. Dually, given an internal functor $F\colon \mathbb{K} \longrightarrow \mathbb{J}$, this (contravariantly) determines by reindexing of $\mathbb{C}$ an ordinary functor on the corresponding categories of internal diagrams: $F^*\colon \mathbb{C}^{\mathbb{J}} \longrightarrow \mathbb{C}^{\mathbb{K}}$. One says that $\mathbb{C}$ has *internal left Kan extensions* if these reindexing functors have left adjoints, denoted by Lan_F. In the particular case where $\mathbb{J} = 1$, the trivial internal category with one object, I write $\mathbb{K}^*\colon \mathbb{C} \longrightarrow \mathbb{C}^{\mathbb{K}}$ for the functor, and $\mathrm{colim}_{\mathbb{K}}$ for its left adjoint $\mathrm{Lan}_{\mathbb{K}}$, and $\mathrm{colim}_{\mathbb{K}} L$ is called the *internal colimit* of L.

Furthermore, suppose $\mathbb{C}$ and $\mathbb{D}$ are $\mathcal{S}$-indexed categories with internal colimits of shape $\mathbb{K}$. Then, one says that an $\mathcal{S}$-indexed functor $F\colon \mathbb{C} \longrightarrow \mathbb{D}$ *preserves colimits* if the canonical natural transformation filling the square

$$\begin{array}{ccc} \mathbb{C}^{\mathbb{K}} & \xrightarrow{F^{\mathbb{K}}} & \mathbb{D}^{\mathbb{K}} \\ {\scriptstyle \mathrm{colim}_{\mathbb{K}}}\downarrow & & \downarrow{\scriptstyle \mathrm{colim}_{\mathbb{K}}} \\ \mathbb{C} & \xrightarrow[F]{} & \mathbb{D} \end{array}$$

is an isomorphism.

Indexed cocomplete categories A.18 A $\mathcal{S}$-indexed category $\mathbb{C}$ is called *$\mathcal{S}$-cocomplete*, in case every fibre is finitely cocomplete, finite colimits are preserved by reindexing functors, and these functors have left adjoints satisfying the Beck-Chevalley condition. If $\mathcal{S}$ has a terminal object, there is the following easy lemma:

Lemma A.19 *If the fibre $\mathcal{C} = \mathcal{C}^1$ of an $\mathcal{S}$-cocomplete $\mathcal{S}$-indexed category $\mathbb{C}$ has a terminal object T, then this is an indexed terminal object, i.e. X^*T is terminal in $\mathcal{C}^X$ for all X in $\mathcal{S}$.*

From Proposition B2.3.20 in [44] it follows that:

Lemma A.20 *If $\mathbb{C}$ is an $\mathcal{S}$-cocomplete $\mathcal{S}$-indexed category, then it has colimits of internal diagrams and left Kan extensions along internal functors in $\mathcal{S}$. Moreover, if an indexed functor $F\colon \mathbb{C} \longrightarrow \mathbb{D}$ between $\mathcal{S}$-cocomplete categories preserves $\mathcal{S}$-indexed colimits, then it also preserves internal colimits.*

The next result is pointed out in Chapter 6 (Remark 6.1.9):

Lemma A.21 *If F is an indexed functor on a $\mathcal{S}$-cocomplete indexed category $\mathbb{C}$, the indexed category $F - \mathbb{C}\mathrm{oalg}$ of F-coalgebras is again $\mathcal{S}$-cocomplete, and the indexed forgetful functor $U\colon F - \mathbb{C}\mathrm{oalg} \longrightarrow \mathbb{C}$ preserves colimits (in other words, U creates colimits).*