Chapter 6

Non-well-founded set theory

Since its first appearance in the book by Joyal and Moerdijk [47], algebraic set theory has always claimed the virtue of being able to describe, in a single framework, various different set theories. However, despite the suggestion in [47] to construct sheaf models for the theory of non-well-founded set theory in the context of algebraic set theory, it appears that up until now no one ever tried to put small maps to use in order to model a set theory with the Anti-Foundation Axiom **AFA**.

This Chapter, which is joint work with Federico De Marchi, provides a first step in this direction. In particular, I build a categorical model of the weak constructive theory $\mathbf{CZF}_0$ of (possibly) non-well-founded sets, studied by Aczel and Rathjen in [7], extended by **AFA**. Classically, the universe of non-well-founded sets is known to be the final coalgebra of the powerclass functor [5]. Therefore, it should come as no surprise that one can build such a model from the final coalgebra for the functor $\mathcal{P}_s$ determined by a class of small maps.

Perhaps more surprising is the fact that such a coalgebra *always* exists. I prove this by means of a final coalgebra theorem, for a certain class of functors on a finitely complete and cocomplete category. The intuition that guides one along the argument is a standard proof of a final coalgebra theorem by Aczel [5] for set-based functors on the category of classes, that preserve inclusions and weak pullbacks. Given one such functor, he first considers the coproduct of all small coalgebras, and show that this is a weakly terminal coalgebra. Then, he quotients by the largest bisimulation on it, to obtain a final coalgebra. The argument works more generally for any functor of which one knows that there is a generating family of coalgebras, for in that case one can take the coproduct of that family, and perform the construction as above. The condition of a functor being set-based assures that one is in such a situation.

My argument is a recasting of the given one in the internal language of a category. Unfortunately, the technicalities that arise when externalising an argument which is given in the internal language can be off-putting at times. For instance, the externalisation of internal colimits forces one to work in the context of indexed categories and indexed functors. Within this context, I say that an indexed functor (which turns pullbacks into weak pullbacks) is small-based when there is a "generating family" of coalgebras. For such functors I prove an indexed final coalgebra theorem. I then apply this machinery to the case of a Heyting pretopos with a class of small maps, to show that the functor $\mathcal{P}_s$ is small-based and therefore has a final coalgebra. As a byproduct, I am able to build the M-type for any small map f (i.e. the final coalgebra for the polynomial functor P_f associated to f).

For sake of clarity, I have tried to collect as much indexed category theory as I could in a separate Section. This forms the content of Section 6.1. This should not affect readability of Section 6.2, where I prove the final coalgebra results. Finally, in Section 6.3 I prove that the final $\mathcal{P}_s$-coalgebra is a model of the theory $\mathbf{CZF}_0$+**AFA**.

The choice to focus on a weak set theory such as $\mathbf{CZF}_0$ is deliberate, since stronger theories can be modelled simply by adding extra requirements for the class of small maps. For example, one can model the theory **CST** of Myhill [62] (plus **AFA**), by adding the Exponentiation Axiom, or $\mathbf{IZF}^-$+**AFA** by adding the Powerset, Separation and Collection axioms. And one can force the theory to be classical by working in a Boolean pretopos. This gives a model of $\mathbf{ZF}^-$+**AFA**, the theory presented in Aczel's book [5], apart from the Axiom of Choice. And, finally, by adding appropriate axioms, it is possible to build a model of the theory $\mathbf{CZF}^-$+**AFA**, which was extensively studied by M. Rathjen in [71, 72].

The present results fit in the general picture described in the previous Chapter.[1] Recall that there I set myself the task of investigating a non-well-founded analogue to the established connection between Martin-Löf type theory, constructive set theory and the theory of ΠW-pretoposes. In the well-founded picture, W-types in ΠW-pretoposes can be used to obtain models for (well-founded) set theories, as explained in Chapter 4. The analogy suggests that M-types in ΠM-pretoposes provide the means for constructing models for non-well-founded set theories. But in this Chapter, it will turn out that the M-types in ΠM-pretoposes are not necessary for that purpose. This phenomenon resembles the situation in [52], where Lindström built a model of $\mathbf{CZF}^-$+**AFA** out of (intensional) Martin-Löf type theory with one universe, without making any use of M-types.

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6.1 Generating objects in indexed categories

As mentioned before, the aim is to prove a final coalgebra theorem for a special class of functors on finitely complete and cocomplete categories. The proof of such results will be carried out by repeating in the internal language of such a category $\mathcal{C}$ a classical

[1] Incidentally, I expect that, together with the results on sheaves therein, they should yield an adequate response to the suggestion by Joyal and Moerdijk.

set-theoretic argument. This forces one to consider $\mathcal{C}$ as an indexed category, via its canonical indexing $\mathbb{C}$, whose fibre over an object X is the slice category $\mathcal{C}/X$. I shall then focus on endofunctors on $\mathcal{C}$ which are components over 1 of indexed endofunctors on $\mathbb{C}$. For such functors, one can prove the existence of an indexed final coalgebra, under suitable assumptions. The component over 1 of this indexed final coalgebra will be the final coalgebra of the original $\mathcal{C}$-endofunctor.

Although I apply this result in a rather specific context, it turns out that all the basic machinery needed for the proof can be stated in a more general setting. This Section collects as much of the indexed category theoretic material as possible, hoping to make the other Sections easier to follow for a less experienced reader.

I will mostly be concerned with $\mathcal{S}$-cocomplete indexed categories for a cartesian base category $\mathcal{S}$. The reader should consult Appendix A for the relevant definitions. The notation follows closely that of Johnstone in Chapters B1 and B2 of [44].

The first step, in the set-theoretic argument to build the final coalgebra, is to identify a "generating family" of coalgebras, in the sense that any other coalgebra is the colimit of all coalgebras in that family mapping to it. When forming the internal diagram of those coalgebras that map into a given one, say (A, α), I need to select out of an object of maps to A those which are coalgebra morphisms. In order to consider such objects of arrows in the internal language, I need to introduce the following concept:

Definition 6.1.1 Let E and A be two objects, respectively in fibres $\mathcal{C}^U$ and $\mathcal{C}^I$ of an $\mathcal{S}$-indexed category $\mathbb{C}$. Whenever it exists, the object $\mathrm{Hom}(E, A)$ in $\mathcal{S}$ is called the *internal homset* from E to A (in $\mathcal{S}$), if it fits into a span

$$U \xleftarrow{\;s\;} \mathrm{Hom}(E, A) \xrightarrow{\;t\;} I \tag{6.1}$$

in $\mathcal{S}$ and there is a *generic arrow* $\varepsilon: s^*E \longrightarrow t^*A$ in $\mathcal{C}^{\mathrm{Hom}(E,A)}$, with the following universal property: for any other span in $\mathcal{S}$

$$U \xleftarrow{\;x\;} J \xrightarrow{\;y\;} I$$

and any arrow $\psi: x^*E \longrightarrow y^*A$ in $\mathcal{C}^J$, there is a unique arrow $\chi: J \longrightarrow \mathrm{Hom}(E, A)$ in $\mathcal{S}$ such that $s\xi = x$, $t\chi = y$ and $\chi^*\varepsilon \cong \psi$ (via the canonical isomorphisms arising from the two previous equalities). The object E is called *exponentiable*, if $\mathrm{Hom}(E, A)$ exists for all A in some fibre of $\mathbb{C}$.

Remark 6.1.2 It follows from the definition, via a standard diagram chasing, that the reindexing along an arrow $f: V \longrightarrow U$ in $\mathcal{S}$ of an exponentiable object E in $\mathcal{C}^U$ is again exponentiable.

Remark 6.1.3 The reader is advised to check that, in case $\mathcal{C}$ is a cartesian category and $\mathbb{C}$ is its canonical indexing over itself, the notion of exponentiable object agrees with the standard one of an exponentiable morphism (see Appendix A).

Given an exponentiable object E in $\mathcal{C}^U$ and an object A in $\mathcal{C}^I$, the *canonical cocone from E to A* is in the internal language the cocone of those morphisms from E to A. Formally, it is described as the internal diagram $(\mathbb{K}^A, L^A)$, where the internal category $\mathbb{K}^A$ and the diagram object L^A are defined as follows. K_0^A is the object $\mathsf{Hom}(E, A)$, with arrows s and t as in (6.1), and K_1^A is the pullback

$$\begin{array}{ccc} K_1^A & \xrightarrow{d_0} & K_0^A \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle s} \\ \mathsf{Hom}(E,E) & \xrightarrow[\bar{t}]{} & U, \end{array}$$

where

$$U \xleftarrow{\bar{s}} \mathsf{Hom}(E,E) \xrightarrow{\bar{t}} U$$

is the internal hom of E with itself. In the fibres over $\mathsf{Hom}(E, A)$ and $\mathsf{Hom}(E, E)$ one has generic maps $\varepsilon \colon s^*E \longrightarrow t^*A$ and $\bar{\varepsilon} \colon \bar{s}^*E \longrightarrow \bar{t}^*E$, respectively.

The codomain map d_0 of $\mathbb{K}^A$ is the top row of the pullback above, whereas d_1 is induced by the composite

$$(\bar{s}x)^*E \xrightarrow{x^*\bar{\varepsilon}} (\bar{t}x)^*E \cong (sd_0)^*E \xrightarrow{d_0^*\varepsilon} (td_0)^*A$$

via the universal property of $\mathsf{Hom}(E, A)$ and ε.

The internal diagram L^A is now the object s^*E in $\mathcal{C}^{K_0^A}$, and the arrow from $d_1^*L^A$ to $d_0^*L^A$ is (modulo the coherence isomorphisms) $x^*\bar{\varepsilon}$.

When the colimit of the canonical cocone from E to A is A itself, one should think of A as being generated by the maps from E to it. Therefore, it is natural to introduce the following terminology.

Definition 6.1.4 The object E is called a *generating object* if, for any A in $\mathcal{C} = \mathcal{C}^1$, $A = \mathrm{colim}_{\mathbb{K}^A} L^A$.

Later, we shall see how F-coalgebras form an indexed category. Then, a generating object for this category will provide, in the internal language, a "generating family" of coalgebras. The set-theoretic argument then goes on by taking the coproduct of all coalgebras in that family. This provides a weakly terminal coalgebra. Categorically, the argument translates to the following result.

Proposition 6.1.5 *Let $\mathbb{C}$ be an $\mathcal{S}$-cocomplete $\mathcal{S}$-indexed category with a generating object E in $\mathcal{C}^U$. Then, $\mathcal{C} = \mathcal{C}^1$ has a weakly terminal object (an object is* weakly terminal *if it is satisfies the existence but not necessarily the uniqueness requirement for a terminal object).*

Proof. One builds a weakly terminal object in $\mathcal{C}$ by taking the internal colimit Q of the diagram $(\mathbb{K}, L)$ in $\mathbb{C}$, where $K_0 = U$, $K_1 = \mathrm{Hom}(E, E)$ (with domain and codomain maps $\overline{s}$ and $\overline{t}$, respectively), $L = E$ and the map from $d_0^* L$ to $d_1^* L$ is precisely $\overline{\varepsilon}$.

Given an object $A = \mathrm{colim}_{\mathbb{K}^A} L^A$ in $\mathcal{C}$, notice that the serially commuting diagram

$$\begin{array}{ccc} K_1^A & \overset{d_1}{\underset{d_0}{\rightrightarrows}} & K_0^A \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle s} \\ \mathrm{Hom}(E, E) & \overset{\overline{s}}{\underset{\overline{t}}{\rightrightarrows}} & U \end{array}$$

defines an internal functor $J \colon \mathbb{K}^A \longrightarrow \mathbb{K}$. One has a commuting triangle of internal $\mathcal{S}$-categories

$$\begin{array}{ccc} \mathbb{K}^A & \xrightarrow{J} & \mathbb{K} \\ & \searrow \quad \swarrow & \\ & 1. & \end{array}$$

Taking left adjoint along the reindexing functors which this induces on categories of internal diagrams, one gets that $\mathrm{colim}_{\mathbb{K}^A} \cong \mathrm{colim}_{\mathbb{K}} \circ \mathrm{Lan}_J$. Hence, to give a map from $A = \mathrm{colim}_{\mathbb{K}^A} L^A$ to $Q = \mathrm{colim}_{\mathbb{K}} L$ it is sufficient to give a morphism of internal diagrams from $(\mathbb{K}, \mathrm{Lan}_J L^A)$ to $(\mathbb{K}, L)$, or, equivalently, from $(\mathbb{K}^A, L^A)$ to $(\mathbb{K}^A, J^* L)$, but the reader can easily check that these two diagrams are in fact the same. □

Once the coproduct of coalgebras in the "generating family" is formed, the set-theoretic argument is concluded by quotienting it by its largest bisimulation. One way to build such a bisimulation constructively is to identify a generating family of bisimulations and then taking their coproduct.

This suggests that to apply Proposition 6.1.5 twice; first in the indexed category of coalgebras, in order to obtain a weakly terminal coalgebra (G, γ), and then in the (indexed) category of bisimulations over (G, γ). To this end, one needs to prove co-completeness and existence of a generating object for these categories. The language of inserters allows one to do that in a uniform way.

Instead of giving the general definition of an inserter in a 2-category, I will only describe an inserter explicitly for the 2-category of $\mathcal{S}$-indexed categories.

Definition 6.1.6 Given two $\mathcal{S}$-indexed categories $\mathbb{C}$ and $\mathbb{D}$ and two parallel $\mathcal{S}$-indexed functors $F, G \colon \mathbb{C} \longrightarrow \mathbb{D}$, the *inserter* $\mathbb{I} = \mathbb{I}\mathrm{ns}(F, G)$ *of* F *and* G has as fibre $\mathcal{I}^X$ the category whose objects are pairs (A, α) consisting of an object A in $\mathcal{C}^X$ and an arrow in $\mathcal{D}^X$ from $F^X A$ to $G^X A$, an arrow $\phi \colon (A, \alpha) \longrightarrow (B, \beta)$ being a map $\phi \colon A \longrightarrow B$ in $\mathcal{C}^X$ such that $G^X(\phi)\alpha = \beta F^X(\phi)$.

The reindexing functor for a map $f \colon Y \longrightarrow X$ in $\mathcal{S}$ takes an object (A, α) in $\mathcal{I}^X$ to the object $(f^* A, f^* \alpha)$, where $f^* \alpha$ has to be read modulo the coherence isomorphisms of $\mathbb{D}$, but I shall ignore these thoroughly.

There is an indexed *forgetful* functor $U\colon \mathbb{I}\mathrm{ns}(F,G)\longrightarrow\mathbb{C}$ which takes a pair (A,α) to its carrier A; the maps α determine an indexed natural transformation $FU\longrightarrow GU$. The triple $(\mathbb{I}\mathrm{ns}(F,G), U, FU\longrightarrow GU)$ has a universal property, like any good categorical construction, but it will not be used. The situation is depicted as below:

$$\mathbb{I}\mathrm{ns}(F,G)\xrightarrow{U}\mathbb{C}\underset{G}{\overset{F}{\rightrightarrows}}\mathbb{D}. \tag{6.2}$$

A tedious but otherwise straightforward computation, yields the proof of the following:

Lemma 6.1.7 *Given an inserter as in (6.2), if $\mathbb{C}$ and $\mathbb{D}$ are $\mathcal{S}$-cocomplete and F preserves indexed colimits, then $\mathbb{I}\mathrm{ns}(F,G)$ is $\mathcal{S}$-cocomplete and U preserves colimits (in other words, U creates colimits). In particular, $\mathbb{I}\mathrm{ns}(F,G)$ has all internal colimits, and U preserves them.*

Example 6.1.8 Here, we shall be concerned with two particular examples of inserters. One is the indexed category $F-\mathbb{C}\mathrm{oalg}$ of coalgebras for an indexed endofunctor F on $\mathbb{C}$, which can be presented as the inserter

$$\mathbb{I}\mathrm{ns}(\mathrm{Id},F)\xrightarrow{U}\mathbb{C}\underset{F}{\overset{\mathrm{Id}}{\rightrightarrows}}\mathbb{C}. \tag{6.3}$$

More concretely, $(F\text{–}\mathbb{C}\mathrm{oalg})^I = F^I-\mathrm{coalg}$ consists of pairs (A,α) where A is an object and $\alpha\colon A\longrightarrow F^I A$ a map in $\mathcal{C}^I$, and morphisms from such an (A,α) to a pair (B,β) are morphisms $\phi\colon A\longrightarrow B$ in $\mathcal{C}^I$ such that $F^I(\phi)\alpha=\beta\phi$. The reindexing functors are the obvious ones.

The other inserter we shall encounter is the indexed category $\mathbb{S}\mathrm{pan}(M,N)$ of spans over two objects M and N in $\mathcal{C}^1$ of an indexed category. This is the inserter

$$\mathbb{I}\mathrm{ns}(\Delta,\langle M,N\rangle)\xrightarrow{U}\mathbb{C}\underset{\langle M,N\rangle}{\overset{\Delta}{\rightrightarrows}}\mathbb{C}\times\mathbb{C} \tag{6.4}$$

Where $\mathbb{C}\times\mathbb{C}$ is the product of $\mathbb{C}$ with itself (which is defined fibrewise), Δ is the diagonal functor (also defined fibrewise), and $\langle M,N\rangle$ is the pairing of the two constant indexed functors determined by M and N. By this I mean that an object in $\mathcal{C}$ is mapped to the pair (M,N) and an object in $\mathcal{C}^X$ is mapped to the pair (X^*M, X^*N).

Remark 6.1.9 Notice that, in both cases, the forgetful functors preserve $\mathcal{S}$-indexed colimits in $\mathbb{C}$, hence both $F-\mathbb{C}\mathrm{oalg}$ and $\mathbb{S}\mathrm{pan}(M,N)$ are $\mathcal{S}$-cocomplete, and also internally cocomplete, if $\mathbb{C}$ is.

In order to apply Proposition 6.1.5 to these indexed categories, one needs to find generating objects for them. This will be achieved by means of the following two lemmas.

First of all, consider an $\mathcal{S}$-indexed inserter $\mathbb{I} = \mathbb{I}\mathrm{ns}(F, G)$ as in (6.2), such that F preserves exponentiable objects. Then, given an exponentiable object E in $\mathcal{C}^U$, define an arrow $\overline{U} \stackrel{r}{\to} U$ in $\mathcal{S}$ and an object $(\overline{E}, \overline{\varepsilon})$ in $\mathcal{I}^{\overline{U}}$, as follows.

Then form the generic map $\varepsilon: s^*F^U E \longrightarrow t^*G^U E$ associated to the internal hom of $F^U E$ and $G^U E$ (which exists because F preserves exponentiable objects), and then define $\overline{U}$ as the equaliser of the following diagram

$$\overline{U} \xrightarrow{e} \mathrm{Hom}(F^U E, G^U E) \underset{t}{\overset{s}{\rightrightarrows}} U, \tag{6.5}$$

the arrow $r: \overline{U} \longrightarrow U$ being one of the two equal composites $se = te$.

Put $\overline{E} = r^*E$ and

$$\overline{\varepsilon} \;=\; F^{\overline{U}}(r^*E) \xrightarrow{\cong} e^*s^*F^U E \xrightarrow{e^*\varepsilon} e^*t^*G^U E \xrightarrow{\cong} G^{\overline{U}}(r^*E).$$

The pair $(\overline{E}, \overline{\varepsilon})$ defines an object in $\mathcal{I}^{\overline{U}}$.

Lemma 6.1.10 *The object* $(\overline{E}, \overline{\varepsilon})$ *is exponentiable in* $\mathbb{I}\mathrm{ns}(F, G)$.

Proof. Consider an object (A, α) in a fibre $\mathcal{I}^X$. Then, I define the internal hom $\mathrm{Hom}((\overline{E}, \overline{\varepsilon}), (A, \alpha))$ as follows.

First, I build the internal homset

$$\overline{U} \xleftarrow{s} L = \mathrm{Hom}(\overline{E}, A) \xrightarrow{t} X$$

of A and $\overline{E}$ in $\mathbb{C}$, with generic map $\chi: s^*\overline{E} \longrightarrow t^*A$. Because F preserves exponentiable objects, it is also possible to form the internal hom in $\mathbb{D}$

$$\overline{U} \xleftarrow{\overline{s}} \mathrm{Hom}(F^{\overline{U}}\overline{E}, G^X A) \xrightarrow{\overline{t}} X$$

with generic map $\overline{\chi}: \overline{s}^* F^{\overline{U}}\overline{E} \longrightarrow \overline{t}^* G^X A$. By the universal property of $\overline{\chi}$, the two composites in $\mathcal{D}^L$

$$s^*F^{\overline{U}}\overline{E} \xrightarrow{\cong} F^L s^*\overline{E} \xrightarrow{F^L\chi} F^L(t^*A) \xrightarrow{\cong} t^*F^X A \xrightarrow{t^*\alpha} t^*G^X A$$

and

$$s^*F^{\overline{U}}\overline{E} \xrightarrow{s^*\overline{\varepsilon}} s^*G^{\overline{U}}\overline{E} \xrightarrow{\cong} G^L s^*\overline{E} \xrightarrow{G^L\chi} G^L t^*A \xrightarrow{\cong} t^*G^X A$$

give rise to two maps $p_1, p_2: L \longrightarrow \mathrm{Hom}(F^{\overline{U}}\overline{E}, G^X A)$ in $\mathcal{S}$, whose equaliser $i: M \longrightarrow L$ has as domain the internal hom $\mathrm{Hom}((\overline{E}, \overline{\varepsilon}), (A, \alpha))$.

The generic map $(si)^*(\overline{E},\overline{\varepsilon})\longrightarrow(ti)^*(A,\alpha)$ in $\mathcal{I}^M$ associated to this internal hom forms the central square of the following diagram, and this commutes because its outer sides are the reindexing along the maps $p_1 i = p_2 i$ of the generic map $\overline{\chi}$ above:

$$\begin{array}{ccc}
(si)^* F^{\overline{U}}\overline{E} & \xrightarrow{(si)^*\overline{\varepsilon}} & (si)^* G^{\overline{U}}\overline{E} \\
{\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\
F^M(si)^*\overline{E} & \xrightarrow{(si)^*(\overline{E},\overline{\varepsilon})} & G^M(si)^*\overline{E} \\
{\scriptstyle F^M i^*\chi}\downarrow & & \downarrow{\scriptstyle G^M i^*\chi} \\
F^M(ti)^*A & \xrightarrow{(ti)^*(A,\alpha)} & G^M(ti)^*A \\
{\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\
(ti)^*F^X A & \xrightarrow[(ti)^*\alpha]{} & (ti)^*G^X A.
\end{array}$$

The verification of its universal property is a lengthy but straightforward exercise. □

Next, I find a criterion for the exponentiable object $(\overline{E},\overline{\varepsilon})$ to be generating.

Lemma 6.1.11 *Consider an inserter of $\mathcal{S}$-indexed categories as in (6.2), where $\mathbb{C}$ and $\mathbb{D}$ are $\mathcal{S}$-cocomplete, and F preserves $\mathcal{S}$-indexed colimits. If $(\overline{E},\overline{\varepsilon})$ is an exponentiable object in $\mathcal{I}^{\overline{U}}$ and for any (A,α) in $\mathcal{I}^1$ the equation*

$$\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} UL^{(A,\alpha)} \cong U(A,\alpha) = A$$

holds, where $(\mathbb{K}^{(A,\alpha)}, L^{(A,\alpha)})$ is the canonical cocone from $(\overline{E},\overline{\varepsilon})$ to (A,α), then $(\overline{E},\overline{\varepsilon})$ is generating in $\mathbb{I}\mathrm{ns}(F,G)$.

Proof. Recall from Lemma 6.1.7 that $\mathbb{I}\mathrm{ns}(F,G)$ is internally cocomplete and the forgetful functor $U\colon \mathbb{I}\mathrm{ns}(F,G)\longrightarrow\mathbb{C}$ preserves internal colimits. Therefore, given an arbitrary object (A,α) in $\mathcal{I}^1$, one can always form the colimit $(B,\beta)=\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} L^{(A,\alpha)}$. All I need to show is that $(B,\beta)\cong(A,\alpha)$. The isomorphism between B and A exists because, by the assumption,

$$B = U(B,\beta) = U\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} L^{(A,\alpha)} \cong \mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} UL^{(A,\alpha)} \cong A.$$

Now, it is not too hard to show that the transpose of the composite

$$\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} F^{\overline{U}}U^{\overline{U}}L^{(A,\alpha)} \cong FU\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} L^{(A,\alpha)} \xrightarrow{\beta} GU\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} L^{(A,\alpha)}$$

is (modulo isomorphisms preserved through the adjunction $\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} \dashv \mathbb{K}^{(A,\alpha)^*}$) the transpose of α. Hence, $\beta\cong\alpha$ and I am done. □

As an example, I show the following result about the indexed category of spans:

Proposition 6.1.12 *Given an $\mathcal{S}$-cocomplete indexed category $\mathbb{C}$ and two objects M and N in $\mathcal{C}^1$, if $\mathbb{C}$ has a generating object, then so does the indexed category of spans $\mathbb{P} = \mathbb{S}\mathrm{pan}(M, N)$.*

Proof. Recall from Example 6.1.8 that the functor $U: \mathbb{S}\mathrm{pan}(M, N) \longrightarrow \mathbb{C}$ creates indexed and internal colimits. If E in $\mathcal{C}^U$ is a generating object for $\mathbb{C}$, then, by Lemma 6.1.10 one can build an exponentiable object

$$(\overline{E}, \overline{\varepsilon}) = M \xleftarrow{\overline{\varepsilon}_1} \overline{E} \xrightarrow{\overline{\varepsilon}_2} N$$

in $\mathcal{P}^{\overline{U}}$. I am now going to prove that $\mathbb{S}\mathrm{pan}(M, N)$ meets the requirements of Lemma 6.1.11 to show that $\overline{E}$ is a generating object.

To this end, consider a span

$$(A, \alpha) = M \xleftarrow{\alpha_1} A \xrightarrow{\alpha_2} N$$

in $\mathcal{P}^1$. Then, I can form the canonical cocone $(\mathbb{K}^{(A,\alpha)}, L^{(A,\alpha)})$ from $(\overline{E}, \overline{\varepsilon})$ to (A, α) in $\mathbb{S}\mathrm{pan}(M, N)$, and the canonical cocone $(\mathbb{K}^A, L^A)$ from E to A in $\mathbb{C}$. The map $r: \overline{U} \longrightarrow U$ of (6.5) induces an internal functor $u: \mathbb{K}^{(A,\alpha)} \longrightarrow \mathbb{K}^A$, which is an isomorphism. Therefore, the induced reindexing functor $u^*: \mathbb{C}^{\mathbb{K}^A} \longrightarrow \mathbb{C}^{\mathbb{K}^{(A,\alpha)}}$ between the categories of internal diagrams in $\mathbb{C}$ is also an isomorphism, and hence $\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} u^* \cong \mathrm{colim}_{\mathbb{K}^A}$. Moreover, it is easily checked that $u^* L^A = U L^{(A,\alpha)}$. Therefore, one has

$$\mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} U L^{(A,\alpha)} \cong \mathrm{colim}_{\mathbb{K}^{(A,\alpha)}} u^* L^A \cong \mathrm{colim}_{\mathbb{K}^A} L^A \cong A$$

and this finishes the proof. □

6.2 Final coalgebra theorems

In this Section, I am going to use the machinery of Section 6.1 in order to prove an indexed final coalgebra theorem. I then give an axiomatisation for class of small maps, which is a bit different from the one studied in Chapter 4, for a Heyting pretopos with an (indexed) natural number object, and apply the theorem in order to derive existence of final coalgebras for various functors in this context. In more detail, I shall show that every small map has an M-type, and that the functor $\mathcal{P}_s$ has a final coalgebra.

6.2.1 An indexed final coalgebra theorem

In this Section, $\mathcal{C}$ is a category with finite limits and stable finite colimits (that is, its canonical indexing $\mathbb{C}$ is a $\mathcal{C}$-cocomplete $\mathcal{C}$-indexed category), and F is an indexed endofunctor over it (I shall write F for F^1). Recall from Remark 6.1.9 that the indexed category $F - \mathbb{C}\text{oalg}$ is $\mathcal{C}$-cocomplete (and the indexed forgetful functor U preserves indexed colimits).

I say that F is *small-based* whenever there is an exponentiable object (E, ε) in F^U–coalg such that, for any other F-coalgebra (A, α), the canonical cocone $(\mathbb{K}^{(A,\alpha)}, L^{(A,\alpha)})$ from (E, ε) to (A, α) has the property that

$$\text{colim}_{\mathbb{K}^{(A,\alpha)}} UL^{(A,\alpha)} \cong U(A, \alpha) = A. \tag{6.6}$$

It is immediate from Example 6.1.8 and Lemma 6.1.11 that, whenever there is a pair (E, ε) making F small-based, this is automatically a generating object in F–$\mathbb{C}$oalg. I shall make an implicit use of this generating object in the proof of:

Theorem 6.2.1 *Let F be a small-based indexed endofunctor on a category $\mathcal{C}$ as above. If F^1 takes pullbacks to weak pullbacks, then F has an indexed final coalgebra.*

Before giving a proof, I need to introduce a little technical lemma. A *weak pullback* is a square that is satisfies the existence requirement for pullbacks (but not necessarily the uniqueness requirement).

Lemma 6.2.2 *If $F = F^1$ turns pullbacks into weak pullbacks, then every pair of arrows*

$$(A, \alpha) \xrightarrow{\phi} (C, \gamma) \xleftarrow{\psi} (B, \beta)$$

can be completed to a commutative square by the arrows

$$(A, \alpha) \xleftarrow{\mu} (P, \chi) \xrightarrow{\nu} (B, \beta)$$

in such a way that the underlying square in $\mathcal{C}$ is a pullback. Moreover, if ψ is a coequaliser in $\mathcal{C}$, then so is μ.

Proof. Build P as the pullback of ψ and ϕ in $\mathcal{C} = \mathcal{C}^1$. Then, since F turns pullbacks into weak pullbacks, there is a map $\chi: P \longrightarrow FP$, making both μ and ν into coalgebra morphisms. The second statement follows at once by the assumption that finite colimits in $\mathcal{C}$ are stable. □

Proof of Theorem 6.2.1. Because F–$\mathbb{C}$oalg is $\mathcal{C}$-cocomplete, it is enough, by Lemma A.19, to show that the fibre over 1 of this indexed category admits a terminal object.

Given that (E, ε) is a generating object in $F - \mathbb{C}\text{oalg}$, Proposition 6.1.5 implies the existence of a weakly terminal F-coalgebra (G, γ). The classical argument now goes on taking the quotient of (G, γ) by the maximal bisimulation on it, in order to obtain a terminal coalgebra. I do that as follows. Let $\mathbb{B} = \mathbb{S}\text{pan}((G, \gamma), (G, \gamma))$ be the indexed category of spans over (G, γ), i.e. bisimulations. Then, by Remark 6.1.9, $\mathbb{B}$ is a $\mathcal{C}$-cocomplete $\mathcal{C}$-indexed category, and by Proposition 6.1.12 it has a generating object. Applying again Proposition 6.1.5, I get a weakly terminal span (i.e. a weakly terminal bisimulation)

$$(G, \gamma) \xleftarrow{\lambda} (B, \beta) \xrightarrow{\rho} (G, \gamma).$$

I now want to prove that the coequaliser

$$(B, \beta) \underset{\rho}{\overset{\lambda}{\rightrightarrows}} (G, \gamma) \xrightarrow{q} (T, \tau)$$

is a terminal F-coalgebra.

It is obvious that (T, τ) is weakly terminal, since (G, γ) is. On the other hand, suppose (A, α) is an F-coalgebra and $f, g \colon (A, \alpha) \longrightarrow (T, \tau)$ are two coalgebra morphisms; then, by Lemma 6.2.2, the pullback s (resp. t) in $\mathcal{C}$ of q along f (resp. g) is a coequaliser in $\mathcal{C}$, which carries the structure of a coalgebra morphism into (A, α). One further application of Lemma 6.2.2 to s and t yields a commutative square in $F-\text{coalg}$

$$\begin{array}{ccc} (P, \pi) & \xrightarrow{s'} & \bullet \\ {\scriptstyle t'}\downarrow & & \downarrow{\scriptstyle t} \\ \bullet & \xrightarrow[s]{} & (A, \alpha) \end{array}$$

whose underlying square in $\mathcal{C}$ is a pullback. Furthermore, the composite $d = ts' = st'$ is a regular epi in $\mathcal{C}$, hence an epimorphism in $F-\text{coalg}$.

Write $\widetilde{s}$ (resp. $\widetilde{t}$) for the composite of t' (resp. s') with the projection of the pullback of f (resp. g) and q to G. Then, the triple $((P, \pi), \widetilde{s}, \widetilde{t})$ is a span over (G, γ); hence, there is a morphism of spans

$$\chi \colon ((P, \pi), \widetilde{s}, \widetilde{t}) \longrightarrow ((B, \beta), \lambda, \rho).$$

It is now easy to compute that $fd = q\lambda\chi = q\rho\chi = gd$, hence $f = g$, and the proof is complete. □

As a particular instance of Theorem 6.2.1, one can recover the classical result from Aczel [5, p. 87].

Corollary 6.2.3 (Final Coalgebra Theorem) *Any standard functor (on the category of classes) that preserves weak pullbacks has a final coalgebra.*

Proof. First of all, notice that preservation of weak pullbacks is equivalent to our requirement that pullbacks are mapped to weak pullbacks. Moreover, the category of classes has finite limits and stable finite colimits. As an exponentiable object, take the class U of all small coalgebras.

Now, consider a standard functor F on classes (in Aczel's terminology). This can easily be seen as an indexed endofunctor, since for any two classes X and I, one has $X/I \cong X^I$ (so, the action of F can be defined componentwise). It is now sufficient to observe that every F-coalgebra is the union of its small subcoalgebras, therefore the functor is small-based in our sense. □

Remark 6.2.4 With a bit of effort, the reader can see in the present proof of Theorem 6.2.1 an abstract categorical reformulation of the classical argument given by Aczel in his book [5]. In order for that to work, he had to assume that the functor preserves weak pullbacks (and so did I, in my reformulation). Later, in a joint paper with Nax Mendler [6], they gave a different construction of final coalgebras, which allowed them to drop this assumption. A translation of that argument in my setting, would reveal that the construction relies heavily on the exactness properties of the ambient category of classes. Since the functors in the following examples always preserve weak pullbacks, I prefer sticking to the original version of the result (thus making weaker assumptions on the category $\mathcal{C}$), without bothering the reader with a (presently unnecessary) second version, which, however, I believe can be proved.

More recently, the work of Adámek et al. [8] has shown that every endofunctor on the category of classes is small-based, thereby proving that it has a final coalgebra (by Aczel and Mendler's result). Their proof makes a heavy use of set theoretic machinery, which would be interesting to analyse in the present setting.

6.2.2 Small maps

I am now going to consider on $\mathcal{C}$ a class of *small maps*. This will allow us to show that certain polynomial functors, as well as the powerclass functor, are small-based, and therefore we will be able to apply Theorem 6.2.1 to obtain a final coalgebra for them.

From now on, $\mathcal{C}$ will denote a Heyting pretopos with an (indexed) natural number object. Recall that such categories have all finite colimits, and these are stable under pullback (see Lemma A.12).

As I explained in Chapter 4, there are various axiomatisations for a class of small maps, starting with that of Joyal and Moerdijk in [47]. In this Chapter, I will follow the formulation of Awodey et al. [9] and Awodey and Warren [10]. A comparison with the original approach by Joyal and Moerdijk and the approach in Chapter 4, will appear in Remark 6.2.5 below.

A class $\mathcal{S}$ of arrows in $\mathcal{C}$ is called a class of *small maps* if it satisfies the following axioms:

(S1) $\mathcal{S}$ is closed under composition and identities;

(S2) if in a pullback square

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ C & \longrightarrow & D \end{array}$$

$f \in \mathcal{S}$, then $g \in \mathcal{S}$;

(S3) for every object C in $\mathcal{C}$, the diagonal $\Delta_C : C \longrightarrow C \times C$ is in $\mathcal{S}$;

(S4) given an epi $e : C \longrightarrow D$ and a commutative triangle

$$\begin{array}{ccc} C & \overset{e}{\twoheadrightarrow} & D \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & A, & \end{array}$$

if f is in $\mathcal{S}$, then so is g;

(S5) if $f : C \longrightarrow A$ and $g : D \longrightarrow A$ are in $\mathcal{S}$, then so is their copairing

$$[f, g] : C + D \longrightarrow A.$$

I have chosen labels that were also used in Chapter 4, but I do not think this will lead to any confusion.

An arrow in $\mathcal{S}$ will be called *small*, and objects X will be called *small* in case the unique arrow $X \longrightarrow 1$ is small. A *small subobject* R of an object A is a subobject $R \rightarrowtail A$ in which R is small. A *small relation* between objects A and B is a subobject $R \rightarrowtail A \times B$ such that its composite with the projection on A is small (notice that this does not mean that R is a small subobject of $A \times B$).

On a class of small maps, I also require representability of small relations by means of a *powerclass* object:

(P1) for any object C in $\mathcal{C}$ there is an object $\mathcal{P}_s(C)$ and a natural correspondence between maps $I \longrightarrow \mathcal{P}_s(C)$ and small relations between I and C.

In particular, the identity on $\mathcal{P}_s(C)$ determines a small relation $\in_C \subseteq \mathcal{P}_s(C) \times C$. One should think of $\mathcal{P}_s(C)$ as the object of all small subobjects of C; the relation $\in_C$ then becomes the membership relation between elements of C and small subobjects of C. The association $C \mapsto \mathcal{P}_s(C)$ defines a covariant functor (in fact, a monad) on $\mathcal{C}$. I further require the two following axioms:

(I) The natural number object $\mathbb{N}$ is small;

(R) There exists a *universal small map* $\pi\colon E \longrightarrow U$ in $\mathcal{C}$, such that any other small map $f\colon A \longrightarrow B$ fits in a diagram

$$\begin{array}{ccccc} A & \longleftarrow & \bullet & \longrightarrow & E \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} \\ B & \underset{q}{\twoheadleftarrow} & \bullet & \longrightarrow & U \end{array}$$

where both squares are pullbacks and q is epi.

It can now be proved that a class $\mathcal{S}$ satisfying these axioms induces a class of small maps on each slice $\mathcal{C}/C$. Moreover, the reindexing functor along a small map $f\colon C \longrightarrow D$ has a right adjoint $\Pi_f\colon \mathcal{C}/C \longrightarrow \mathcal{C}/D$. In particular, it follows that all small maps are exponentiable in $\mathcal{C}$ (see [10]).

Remark 6.2.5 The axioms that I have chosen for the class of small maps subsume all of the Joyal-Moerdijk axioms in [47, pp. 6–8], except for the collection axiom (**A7**). In particular, the Descent Axiom (**A3**) can be seen to follow from axioms (**S1**) – (**S5**) and (**P1**).

Conversely, the axioms of Joyal and Moerdijk imply all of the present axioms except for (**S3**) and (**I**). The results in Section 6.3 will imply that, by adding these axioms, a model of the weak set theory $\mathbf{CZF}_0$ can be obtained in the setting of [47].

The axioms given here are in a similar manner incomparable in strength with the axioms in Chapter 4. A class of small maps as defined here need not satisfy the axioms called (**F4**) and (**F5**) in the statement of Lemma 4.2.5, while a (representable) class of small maps in the sense of Chapter 4 need not satisfy (**S3**) and (**S4**).

6.2.3 Final coalgebras in categories with small maps

From now on, I shall consider on $\mathcal{C}$ a class of small maps $\mathcal{S}$. Using their properties, I am able to prove the existence of the M-type for every small map $f\colon D \longrightarrow C$, as well as the existence of a final $\mathcal{P}_s$-coalgebra.

Recall that a polynomial functor P_f induced by an exponentiable map $f\colon D \longrightarrow C$ in a cartesian category $\mathcal{C}$ is indexed, see Theorem 2.1.3. In fact, it can be presented as the composite $P_f = \Sigma_C \Pi_f D^*$ of three indexed functors preserving pullbacks. It is therefore also immediate that P_f preserves pullbacks. Of course, the *indexed M-type* of f is the indexed final coalgebra of P_f (if necessary, see Appendix A for the definition of an indexed final coalgebra).

Theorem 6.2.6 *If $f\colon D \longrightarrow C$ is a small map in $\mathcal{C}$, then f has an (indexed) M-type.*

Proof. In order to obtain an (indexed) final P_f-coalgebra, I want to apply Theorem 6.2.1, and for this, what remains to be checked is that P_f is small-based. To this end, I first need to find an exponentiable coalgebra $(\overline{E}, \overline{\varepsilon})$, and then to verify condition (6.6).

The universal small map $\pi\colon E \longrightarrow U$ in $\mathcal{C}$ is exponentiable, as I noticed after the presentation of axiom **(R)**. Hence, unwinding the construction preceding Lemma 6.1.10, I obtain an exponentiable object in $P_f-\mathbb{C}$oalg. Using the internal language of $\mathcal{C}$, I can describe $(\overline{E}, \overline{\varepsilon})$ as follows.

The object $\overline{U}$ on which $\overline{E}$ lives is described as

$$\overline{U} = \{(u \in U, t\colon E_u \longrightarrow P_f(E_u))\},$$

and $\overline{E}$ is now defined as

$$\overline{E} = \{(u \in U, t\colon E_u \longrightarrow P_f(E_u), e \in E_u)\}.$$

The coalgebra structure $\overline{\varepsilon}\colon \overline{E} \longrightarrow P_f^{\overline{U}} \overline{E}$ takes a triple (u, t, e) (with $te = (c, r)$) to the pair $(c, s\colon D_c \longrightarrow \overline{E})$, where the map s takes an element $d \in D_c$ to the triple $(u, t, r(d))$.

Given a coalgebra (A, α), the canonical cocone from $(\overline{E}, \overline{\varepsilon})$ to it takes the following form. The internal category $\mathbb{K}^{(A,\alpha)}$ is given by

$$\begin{aligned} K_0^{(A,\alpha)} &= \{(u \in U, t\colon E_u \to P_f(E_u), m\colon E_u \to A) \mid P_f(m)t = \alpha m\}; \\ K_1^{(A,\alpha)} &= \{(u, t, m, u', t', m', \phi\colon E_u \to E_{u'}) \mid (u, t, m), (u', t', m') \in K_0^{(A,\alpha)}, \\ &\qquad t'\phi = P_f(\phi)t \text{ and } m'\phi = m\}. \end{aligned}$$

(Notice that, in writing the formulas above, I have used the functor P_f in the internal language of $\mathcal{C}$; I can safely do that because the functor is indexed. I shall implicitly follow the same reasoning in the proof of Theorem 6.3.4 below, in order to build an (indexed) final $\mathcal{P}_s$-coalgebra.)

The diagram $L^{(A,\alpha)}$ is specified by a coalgebra over $K_0^{(A,\alpha)}$, but for my purposes I only need to consider its carrier, which is

$$UL^{(A,\alpha)} = \{(u, t, m, e) \mid (u, t, m) \in K_0^{(A,\alpha)} \text{ and } e \in E_u\}.$$

Condition (6.6) says that the colimit of this internal diagram in $\mathcal{C}$ is A, but this is implied by the conjunction of the two following statements, which I am now going to prove:

1. For all $a \in A$ there exists $(u, t, m, e) \in UL^{(A,\alpha)}$ such that $me = a$;
2. If (u_0, t_0, m_0, e_0) and (u_1, t_1, m_1, e_1) are elements of $UL^{(A,\alpha)}$ such that $m_0 e_0 = m_1 e_1$, then there exist $(u, t, m, e) \in UL^{(A,\alpha)}$ and coalgebra maps $\phi_i\colon E_u \longrightarrow E_{u_i}$ $(i = 0, 1)$ such that $m_i \phi_i = m$ and $\phi_i e = e_i$.

Condition 2 is trivial: given (u_0, t_0, m_0, e_0) and (u_1, t_1, m_1, e_1), Lemma 6.2.2 allows one to fill a square

$$\begin{array}{ccc} (P,\gamma) & \longrightarrow & (E_{u_0}, t_0) \\ \downarrow & & \downarrow m_0 \\ (E_{u_1}, t_1) & \xrightarrow[m_1]{} & (A,\alpha), \end{array}$$

in such a way that the underlying square in $\mathcal{C}$ is a pullback (hence, P is a small object). Therefore, (P,γ) is isomorphic to a coalgebra (E_u, t), and, under this isomorphism, the span

$$(E_{u_0}, t_0) \longleftarrow (P,\gamma) \longrightarrow (E_{u_1}, t_1)$$

takes the form

$$(E_{u_0}, t_0) \xleftarrow{\phi_0} (E_u, t) \xrightarrow{\phi_1} (E_{u_1}, t_1).$$

Moreover, since $m_0 e_0 = m_1 e_1$, there is an $e \in E_u$ such that $\phi_i e = e_i$. Then, defining m as any of the two composites $m_i \phi_i$, the element (u, t, m, e) in $UL^{(A,\alpha)}$ satisfies the desired conditions.

As for condition 1, fix an element $a \in A$. One can build a subobject $\langle a \rangle$ of A inductively, as follows:

$$\begin{aligned} \langle a \rangle_0 &= \{a\}; \\ \langle a \rangle_{n+1} &= \bigcup_{a' \in \langle a \rangle_n} t(D_c) \text{ where } \alpha a' = (c, t \colon D_c \longrightarrow A). \end{aligned}$$

Then, each $\langle a \rangle_n$ is a small object, because it is a small-indexed union of small objects. For the same reason (since, by axiom **(I)**, $\mathbb{N}$ is a small object) their union $\langle a \rangle = \bigcup_{n \in \mathbb{N}} \langle a \rangle_n$ is small, and it is a subobject of A. It is not hard to see that the coalgebra structure α induces a coalgebra α' on $\langle a \rangle$ (in fact, $\langle a \rangle$ is the smallest subcoalgebra of (A,α) containing a, i.e. the subcoalgebra *generated* by a), and, up to isomorphism, this is a coalgebra $t \colon E_u \longrightarrow P_f E_u$, with embedding $m \colon E_u \longrightarrow A$. Via the isomorphism $E_u \cong \langle a \rangle$, the element a becomes an element $e \in E_u$ such that $me = a$. Hence, one gets the desired 4-tuple (u, t, m, e) in $UL^{(A,\alpha)}$.

This concludes the proof of the theorem. □

Theorem 6.2.7 *The powerclass functor $\mathcal{P}_s$ has an (indexed) final coalgebra.*

Proof. It is easy to check that $\mathcal{P}_s$ is the component on 1 of an indexed functor, and that it maps pullbacks to weak pullbacks.

Therefore, once again, I just need to verify that $\mathcal{P}_s$ is small-based. I proceed exactly like in the proof of Theorem 6.2.6 above, except for the construction of the coalgebra $(\langle a \rangle, \alpha')$ generated by an element $a \in A$ in 1. Given a $\mathcal{P}_s$-coalgebra (A,α), I construct the subcoalgebra of (A, α) generated by a as follows. First, I define inductively the subobjects

$$\begin{aligned} \langle a \rangle_0 &= \{a\}; \\ \langle a \rangle_{n+1} &= \bigcup_{a' \in \langle a \rangle_n} \alpha(a'). \end{aligned}$$

Each $\langle a \rangle_n$ is a small object, and so is their union $\langle a \rangle = \bigcup_{n \in \mathbb{N}} \langle a \rangle_n$. The coalgebra structure α' is again induced by restriction of α on $\langle a \rangle$. □

6.3 The final $\mathcal{P}_s$-coalgebra as a model of AFA

The standing assumption in this Section is that $\mathcal{C}$ is a Heyting pretopos with an (indexed) natural number object and a class $\mathcal{S}$ of small maps. In the last Section, I proved that in this case the $\mathcal{P}_s$-functor has a final coalgebra in $\mathcal{C}$. Now I will explain how this final coalgebra can be used to model various set theories with the Anti-Foundation Axiom. First I work out the case for the weak constructive theory $\mathbf{CZF}_0$, and then indicate how the same method can be applied to obtain models for stronger, better known or classical set theories.

The presentation of $\mathbf{CZF}_0$ follows that of Aczel and Rathjen in [7]; the same theory appears under the name **BCST*** in the work of Awodey and Warren in [10]. It is a first-order theory whose underlying logic is intuitionistic; its non-logical symbols are a binary relation symbol ϵ and a constant ω, to be thought of as membership and the set of (von Neumann) natural numbers, respectively. Two more symbols will be added for sake of readability, as I proceed to state the axioms. Notice that, as in Chapter 4, in order to make a distinction between the membership relation of the set theory and that induced by the powerclass functor, I shall denote the former by ϵ and the latter by $\in$.

The conventions of Chapter 4 are assumed to be in place. In particular, I use the following abbreviations:

$$\begin{aligned} \exists x \epsilon a \, (\ldots) &:= \exists x \, (x \epsilon a \wedge \ldots), \\ \forall x \epsilon a \, (\ldots) &:= \forall x \, (x \epsilon a \to \ldots). \end{aligned}$$

The axioms for $\mathbf{CZF}_0$ are (the universal closures) of the following statements:

(Extensionality) $\forall x \, (x \epsilon a \leftrightarrow x \epsilon b) \to a = b$

(Pairing) $\exists y \, \forall x \, (x \epsilon y \leftrightarrow (x = a \vee x = b))$

(Union) $\exists y \, \forall x (x \epsilon y \leftrightarrow \exists z \, (x \epsilon z \wedge z \epsilon a))$

(Emptyset) $\exists y\, \forall x\, (x \epsilon y \leftrightarrow \bot)$

(Intersection) $\exists y\, \forall x\, (x \epsilon y \leftrightarrow (x \epsilon a \wedge x \epsilon b))$

(Replacement) $\forall x \epsilon a\, \exists! y\, \phi \rightarrow \exists b\, \forall y\, (y \epsilon b \leftrightarrow \exists x \epsilon\ a\, \phi)$

Two more axioms will be added, but before I do so, I want to point out that all instances of Δ_0-separation follow from these axioms, i.e. one can deduce all instances of

(Δ_0-Separation) $\exists y\, \forall x\, (x \epsilon y \leftrightarrow (x \epsilon a \wedge \phi(x)))$

where ϕ is a formula in which y does not occur and all quantifiers are bounded. Furthermore, in view of the above axioms, I can introduce a new constant $\emptyset$ to denote the empty set, and a function symbol s which maps a set x to its "successor" $x \cup \{x\}$. This allows one to formulate concisely our last axioms:

(Infinity-1) $\emptyset \epsilon \omega \wedge \forall x \epsilon \omega\, (sx \epsilon \omega)$

(Infinity-2) $\psi(\emptyset) \wedge \forall x \epsilon \omega\, (\psi(x) \rightarrow \psi(sx)) \rightarrow \forall x \epsilon \omega\, \psi(x)$.

It is an old observation by Rieger that models for set theory can be obtained as fixpoints for the powerclass functor (see [76]). The same is true in the context of algebraic set theory (see, [19] for a similar result):

Theorem 6.3.1 *Every $\mathcal{P}_s$-fixpoint in $\mathcal{C}$ provides a model of* $\mathbf{CZF}_0$.

Proof. Suppose there is a fixpoint $E\colon V \longrightarrow \mathcal{P}_s V$, with inverse I. Call y the *name* of a small subobject $A \subseteq V$, when $E(y)$ is its corresponding element in $\mathcal{P}_s(V)$. One interprets the formula $x \epsilon y$ as an abbreviation of the statement $x \in E(y)$ in the internal language of $\mathcal{C}$. Then, the verification of the axioms for $\mathbf{CZF}_0$ goes as follows.

Extensionality holds because two small subobjects $E(x)$ and $E(y)$ of V are equal if and only if, in the internal language of $\mathcal{C}$, $z \in E(x) \leftrightarrow z \in E(y)$. The pairing of two elements x and y represented by two arrows $1 \longrightarrow V$, is given by $I(l)$, where l is the name of the (small) image of their copairing $[x, y]\colon 1 + 1 \longrightarrow V$. The union of the sets contained in a set x is interpreted by applying the multiplication of the monad $\mathcal{P}_s$ to $(\mathcal{P}_s E)(E(x))$. The intersection of two elements x and y in V is given by $I(E(x) \cap E(y))$, where the intersection is taken in $\mathcal{P}_s(V)$. The least subobject $0 \subseteq V$ is small, and its name $\emptyset\colon 1 \longrightarrow V$ models the empty set.

For the Replacement axiom, consider a, and suppose that for every $x \epsilon a$ there exists a unique y such that ϕ. Then, the subobject $\{y \mid \exists x \epsilon a\, \phi\}$ of V is covered by $E(a)$, hence small. Applying I to its name, one obtains the image of ϕ.

Finally, the Infinity axioms follow from the axiom **(I)**. The morphism $\emptyset\colon 1 \longrightarrow V$ together with the map $s\colon V \longrightarrow V$ which takes an element x to $x \cup \{x\}$, yields a morphism $\alpha\colon \mathbb{N} \longrightarrow V$. Since $\mathbb{N}$ is small, so is the image of α, as a subobject of V, and applying I to its name one gets an ω in V which validates the axioms Infinity-1 and Infinity-2. □

This theorem shows that every fixpoint for the functor $\mathcal{P}_s$ models a very basic set theory. Now, by demanding further properties of the fixpoint, one can deduce the validity of more axioms. For example, in [47], it is shown how the initial $\mathcal{P}_s$-algebra (which is a fixpoint, after all) models the Axiom of Foundation. Here, I show that a final $\mathcal{P}_s$-coalgebra satisfies the Anti-Foundation Axiom. To formulate this axiom, I define the following notions. A (directed) graph consists of a pair of sets (n, e) such that $n \subseteq e \times e$. A colouring of such a graph is a function c assigning to every node $x \epsilon n$ a set $c(x)$ such that

$$c(x) = \{c(y) \mid (x, y) \epsilon e\}.$$

This can be formulated solely in terms of ϵ using the standard encoding of pairs and functions. In ordinary set theory (with classical logic and the Foundation Axiom), the only graphs that have a colouring are well-founded trees and these colourings are then necessarily unique.

The Anti-Foundation Axiom says:

(AFA) Every graph has a unique colouring.

Proposition 6.3.2 *If $\mathcal{C}$ has an (indexed) final $\mathcal{P}_s$-coalgebra, then this is a model for the theory* $\mathbf{CZF}_0$+**AFA**.

Proof. I clearly have to check just **AFA**, since any final coalgebra is a fixpoint. To this end, note first of all that, because (V, E) is an indexed final coalgebra, one can think of it as a final $\mathcal{P}_s$-coalgebra in the internal logic of $\mathcal{C}$.

So, suppose one has a graph (n, e) in V. Then, n (internally) has the structure of a $\mathcal{P}_s$-coalgebra $\nu\colon n \longrightarrow \mathcal{P}_s n$, by sending a node $x \,\epsilon\, n$ to the (small) set of nodes $y \,\epsilon\, n$ such that $(x, y) \,\epsilon\, e$. The colouring of n is now given by the unique $\mathcal{P}_s$-coalgebra map $\gamma\colon n \longrightarrow V$. □

By Theorem 6.2.7, it then follows at once:

Corollary 6.3.3 *Every Heyting pretopos with a natural number object and class of small maps contains a model of* $\mathbf{CZF}_0$+**AFA**.

This result can be extended to theories stronger than $\mathbf{CZF}_0$. For example, to the set theory **CST** introduced by Myhill in [62]. This theory is closely related to (in fact, intertranslatable with) $\mathbf{CZF}_0$+**Exp**, where **Exp** is (the universal closure of) the following axiom.

(Exponentiation) $\exists t\,(f\epsilon t \leftrightarrow \mathrm{Fun}(f, x, y))$

Here, the predicate $\mathrm{Fun}(f, x, y)$ expresses the fact that f is a function from x to y, and it can be formally written as the conjunction of $\forall a\epsilon x\, \exists! b\epsilon y\, (a, b)\epsilon f$ and $\forall z\epsilon f\, \exists a\epsilon x, b\epsilon y\, (z = (a, b))$.

Theorem 6.3.4 *Assume the class $\mathcal{S}$ of small maps also satisfies*

(E) *The functor Π_f preserves small maps for any f in $\mathcal{S}$.*

Then, $\mathcal{C}$ contains a model of **CST**+**AFA**.

Proof. We already saw how the final $\mathcal{P}_s$-coalgebra (V, E) models $\mathbf{CZF}_0$+**AFA**. Now, **(E)** implies that A^B is small, if A and B are, so, $E(y)^{E(x)}$ is always small. This gives rise to a small subobject of V, by considering the image of the morphism that sends a function $f \in E(y)^{E(x)}$ to the element in V representing its graph. The image under I of the name of this small object is the desired exponential t. □

Another example of a stronger theory which can be obtained by imposing further axioms for small maps is provided by $\mathbf{IZF}^-$, which is intuitionistic **ZF** without the Foundation Axiom. It is obtained by adding to $\mathbf{CZF}_0$ the following axioms:

(Powerset) $\exists y\, \forall x(x\epsilon y \leftrightarrow \forall z\epsilon x(z\epsilon a))$

(Full Separation) $\exists y\, \forall x(x\epsilon y \leftrightarrow (x\epsilon a \wedge \phi(x)))$

(Collection) $\forall x\epsilon a\, \exists y\, \phi(x, y) \rightarrow \exists b \forall x\epsilon a\, \exists y\epsilon b\, \phi(x, y)$

(In Full Separation, y is not allowed to occur in ϕ.)

By now, the proof of the following theorem should be routine (if not, the reader should consult [19]):

Theorem 6.3.5 *Assume the class of small maps $\mathcal{S}$ also satisfies*

(P2) *if $X \longrightarrow B$ belongs to $\mathcal{S}$, then so does $\mathcal{P}_s(X \longrightarrow B)$;*

(M) *every monomorphism is small;*

(C) *for any two arrows $p: Y \longrightarrow X$ and $f: X \longrightarrow A$ where p is epi and f belongs to $\mathcal{S}$, there exists a quasi-pullback square of the form*

$$\begin{array}{ccccc} Z & \longrightarrow & Y & \overset{p}{\twoheadrightarrow} & X \\ {\scriptstyle g}\downarrow & & & & \downarrow{\scriptstyle f} \\ B & & \underset{h}{\twoheadrightarrow} & & A \end{array}$$

where h is epi and g belongs to $\mathcal{S}$.

Then, $\mathcal{C}$ contains a model of **IZF**$^-$+**AFA**.

Corollary 6.3.6 *If the pretopos $\mathcal{C}$ is Boolean, then classical logic is also true in the model, which will therefore validate* **ZF**$^-$+**AFA**, *Zermelo-Fraenkel set theory with Anti-Foundation instead of Foundation.*

Finally, one can build a model for a non-well-founded version of Aczel's set theory **CZF**, discussed in Chapter 4. The set theory **CZF**$^-$+**AFA** is obtained by dropping Set Induction and replacing it by **AFA**, and was studied by M. Rathjen in [71, 72]. It is obtained by adding to **CZF**$_0$ the axiom **AFA**, as well as the following:

(Strong Collection) $\forall x \epsilon a\, \exists y\, \phi(x, y) \to \exists b\, \mathrm{B}(x \epsilon a, y \epsilon b)\, \phi(x, y)$

(Subset Collection) $\exists c\, \forall z\, (\forall x \epsilon a\, \exists y \epsilon b\, \phi(x, y, z) \to \exists d \epsilon c\, \mathrm{B}(x \epsilon a, y \epsilon d)\, \phi(x, y, z)$

Here $\mathrm{B}(x \epsilon a, y \epsilon b)\, \phi$ abbreviates:

$$\forall x \epsilon a\, \exists y \epsilon b\, \phi \wedge \forall y \epsilon b\, \exists x \epsilon a\, \phi.$$

In order for a class of small maps to give a model Subset Collection, the class has to satisfy a rather involved axiom that will be called **(F)**. In order to formulate it, I need to introduce some notation. For two morphisms $A \longrightarrow X$ and $B \longrightarrow X$, $M_X(A, B)$ will denote the poset of multi-valued functions from A to B over X, i.e. jointly monic spans in $\mathcal{C}/X$,

$$A \twoheadleftarrow P \longrightarrow B$$

with $P \longrightarrow X$ small and the map to A epic. By pullback, any $f\colon Y \longrightarrow X$ determines an order preserving function

$$f^*\colon M_X(A, B) \longrightarrow M_Y(f^*A, f^*B).$$

Theorem 6.3.7 *Assume the class of small maps $\mathcal{S}$ also satisfies* **(C)** *as in Theorem 6.3.5, and the following axiom:*

(F) *for any two small maps $A \longrightarrow X$ and $B \longrightarrow X$, there exist an epi $p\colon X' \longrightarrow X$, a small map $f\colon C \longrightarrow X'$ and an element $P \in M_C(f^*p^*A, f^*p^*B)$, such that for any $g\colon D \longrightarrow X'$ and $Q \in M_D(g^*p^*A, g^*p^*B)$, there are morphisms $x\colon E \longrightarrow D$ and $y\colon E \longrightarrow C$, with $gx = fy$ and x epi, such that $x^*Q \geq y^*P$.*

Then, $\mathcal{C}$ contains a model of **CZF**$^-$+**AFA**.

Proof. Any fixpoint for $\mathcal{P}_s$ will model Strong Collection in virtue of property **(C)** of the class of small maps.

Because of **(F)** the fixpoint will also model the axiom called Fullness in Chapter 4. But Fullness is equivalent to Subset Collection over $\mathbf{CZF}_0$ and Strong Collection (see [7]). □

To illustrate that these are not empty theorems, I wish to conclude this Chapter by presenting several cases to which they can be applied. Following [47], one can find several examples of categories endowed with classes of small maps satisfying some of the discussed axioms. I cannot study them in detail, but I would at least like to present them briefly. For a more complete treatment, the reader is advised to look at [47]. A thorough study of the properties of the resulting models is the subject for future research.

The most obvious example is clearly the category of classes, where the notion of smallness is precisely that of a class function having as fibres just sets. This satisfies all the presented axioms. Along the same lines, one can consider the category of sets, where the class of small maps consists of those functions whose fibres have cardinality at most κ, for a fixed infinite regular cardinal κ. This satisfies axioms **(S1-5)**, **(P1)**, **(I)**, **(R)**, **(M)** and **(C)**, but not **(E)**. However, if κ is also inaccessible, then **(E)** is satisfied, as well as **(P2)** and **(F)**.

Consider the topos $\mathrm{Sh}(\mathcal{C})$ of sheaves over a site $\mathcal{C}$, with pullbacks and a subcanonical topology. Then, for an infinite regular cardinal κ greater than the number of arrows in $\mathcal{C}$, define the notion of smallness (relative to κ) following [47], Chapter IV.3. This satisfies the axioms **(S1-5)**, **(P1)**, **(I)** and **(R)**. Moreover, if κ is inaccessible, it satisfies also **(P2)**, **(M)**, **(C)**.

Finally, on the effective topos $\mathcal{E}ff$ one can define a class of small maps in at least two different ways. For the first, consider the global section functor $\Gamma\colon \mathcal{E}ff \longrightarrow \mathcal{S}ets$, and fix a regular cardinal κ. Then, say that a map $f\colon X \longrightarrow Y$ is small if it fits in a quasi-pullback

$$\begin{array}{ccc} P & \twoheadrightarrow & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Q & \twoheadrightarrow & Y \end{array}$$

where P and Q are projectives and $\Gamma(g)$ is κ-small in $\mathcal{S}ets$. With this definition, the class of small maps satisfies all the basic axioms **(S1-5)**, **(P1)**, **(I)** and **(R)**, as well as **(C)** and **(M)**. If κ is inaccessible, it also satisfies **(P2)**.

Alternatively, one can take the class of small maps in $\mathcal{E}ff$ investigated in Chapter 4. This notion of smallness satisfies all the axioms apart from **(P2)**.