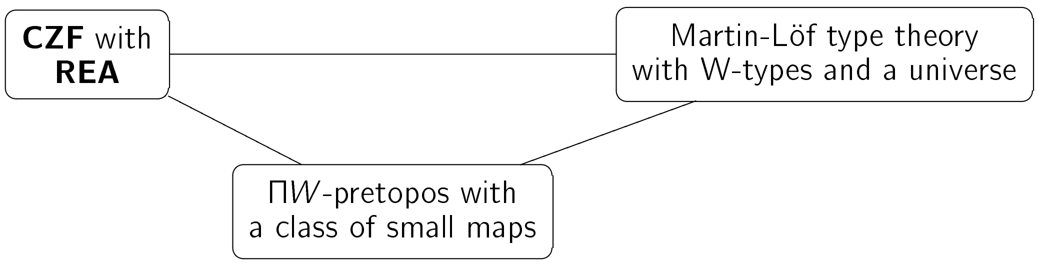
Chapter 5

Coinduction in categories

In the preceeding Chapters, I have been exploiting the connections between the three concepts in the following picture.



They all concern basic notions (the set theory **CZF**, Martin-Löf type theory, locally cartesian closed pretoposes) extended with additional structure (**REA**, W-types and the existence of certain initial algebras, respectively) to incorporate inductive definitions. The idea of Federico De Marchi and me was to investigate a possible "non-well-founded" or "coinductive" analogue to this picture.

The question we asked ourselves is whether a set theory like **CZF** with the Anti-Foundation Axiom instead of the Axiom of Foundation, has similar strong relations with categories or type theories equipped with coinductive types, as does **CZF** + **REA** with categories and type theories with inductive types. Categories with what I have called M-types (see Chapter 2) seem the appropriate analogue to investigate. Where W-types are the initial algebras for polynomial functors, M-types are their final coalgebras. As we have seen in the Chapter 1, W-types frequently consist of well-founded trees, while M-types consist of general ("non-well-founded"[1]) trees. Type theory with coinductive types (M-types) instead of W-types was introduced by Federico De Marchi in [26], and the relation between categories with M-types and type theory with coinductive types was investigated there.

[1] The phrase "non-well-founded" is a bit confusing: it does *not* mean "not well-founded". It means rather something like "not necessarily well-founded". The function of the word "non-well-founded" is more to warn the reader that one is thinking of arbitrary trees and is not restricting oneself to the well-founded case.

A result by Lindström [52] connected type theory and non-well-founded set theory: she discovered how one can model non-well-founded set theory in Martin-Löf type theory with one universe. Somewhat surprisingly, she did not need any kind of coinductive types. A similar phenomenon will arise in the next Chapter where I will discuss models of non-well-founded set theory in categories. On this point, the analogy with the inductive (well-founded) picture does not seem to be perfect: categorical or type-theoretic W-types are necessary to build interpretations of well-founded set theory in [2] and [61].

In this Chapter, I will be more concerned with categories possessing M-types in themselves. In particular, I will prove existence results for M-types and closure properties of categories with M-types (glueing, coalgebras for a cartesian comonad and (pre)sheaves). In some cases, the results for categories with M-types are better than the ones for ΠW-pretoposes, on which they occasionally shed some light. As discussed, these closure properties have proved most important in topos theory and led to the formulation of various independence results. Hopefully, these closure properties of categories with M-types will prove helpful in investigating non-well-founded set theories and type theories.

This Chapter reports joint work with Federico De Marchi, and has been submitted for publication.

5.1 Preliminaries

Throughout this Chapter, $\mathcal{E}$ will denote a locally cartesian closed pretopos with a natural number object.

Recall from Chapter 1 that one associates to a morphism $f\colon B \longrightarrow A$ in $\mathcal{E}$, a polynomial functor $P_f\colon \mathcal{E} \longrightarrow \mathcal{E}$, which is defined as

$$P_f(X) = \Sigma_{a \in A} X^{B_a}$$

or, more formally, as

$$P_f(X) = \Sigma_A (A \times X \xrightarrow{p_1} A)^{(B \xrightarrow{f} A)},$$

where the exponential is taken in the slice category $\mathcal{E}/A$. The final coalgebra for P_f is called the M-type for f, whenever it exists, and denoted by M_f. The intuition is that f represents a signature, with the elements a in A representing term constructors of arity B_a. The elements of the M-type are then (possibly infinite) terms over this signature. Another intuition is that they are trees where nodes are labelled by elements a in A and edges by elements b in B, in such a way that $f^{-1}(a)$ enumerates the edges into a node labelled by a.

One says that $\mathcal{E}$ *has M-types*, if final coalgebras exist for every polynomial functor. A ΠM-*pretopos* will be a locally cartesian closed pretopos with a natural number object and M-types. It is the purpose of this Chapter to prove the closure of ΠM-pretoposes under slicing, formation of coalgebras for a cartesian comonad and (pre)sheaves.

As already pointed out, by Lambek's lemma (Lemma A.14), the P_f-coalgebra structure map of an M-type M_f for a morphism $f: B \longrightarrow A$,

$$\tau_f: M_f \longrightarrow P_f(M_f)$$

is an isomorphism, and therefore has a section, denoted by $\sup_f$ (or just sup, when f is understood). Furthermore, because there is a natural transformation $\rho: P_f \longrightarrow A$, where A is the constant functor sending objects to A and morphisms to the identity on A, whose component on an object X sends $(a, t) \in P_f(X)$ to $a \in A$, τ also determines a *root map*

$$M_f \xrightarrow{\tau_f} P_f(M_f) \xrightarrow{\rho} A,$$

which, by an abuse of notation, will again be denoted by ρ. I will also abuse terminology by calling the components ρ_X of the natural transformation "root maps". I am confident that this will not generate any confusion.

Given a pullback diagram in $\mathcal{E}$

$$\begin{array}{ccc} B' & \xrightarrow{\beta} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ A' & \xrightarrow[\alpha]{} & A, \end{array}$$

one can think of α as a morphism of signatures, since the fibre over each $a' \in A'$ is isomorphic to the fibre over $\alpha(a') \in A$. It is therefore reasonable to expect, in such a situation, an induced morphism between $M_{f'}$ and M_f, when these exist.

In fact, as already pointed out in [60], such a pullback square induces a natural transformation $\widetilde{\alpha}: P_{f'} \longrightarrow P_f$ such that

$$\rho\widetilde{\alpha} = \alpha\rho. \tag{5.1}$$

Post-composition with $\widetilde{\alpha}$ turns any $P_{f'}$-coalgebra into one for P_f. In particular, this happens for $M_{f'}$, thus inducing a unique coalgebra homomorphism as in

$$\begin{array}{ccc} M_{f'} & \xrightarrow{\alpha_!} & M_f \\ {\scriptstyle \tau_{f'}}\downarrow & & \\ P_{f'}(M_{f'}) & & \downarrow{\scriptstyle \tau_f} \\ {\scriptstyle \widetilde{\alpha}}\downarrow & & \\ P_f(M_{f'}) & \xrightarrow[P_f(\alpha_!)]{} & P_f(M_f). \end{array} \tag{5.2}$$

Notice that, by (5.1), the morphism $\alpha_!$ preserves the root maps.

Again, extensive use will be made of the language of paths. Recall the observation made in Chapter 2, that the notion of path can be defined in the internal logic of $\mathcal{E}$ for any P_f-coalgebra

$$X \xrightarrow{\gamma} P_f X.$$

The idea is that a finite sequence of odd length $\langle x_0, b_0, x_1, b_1, \ldots, x_n\rangle$ is called a *path* in (X, γ), if every x_i is in X, every b_i is in B and for every $i < n$ one has

$$x_{i+1} = \gamma(x_i)(b_i). \tag{5.3}$$

More precisely, if $\gamma(x_i) = (a_i, t_i)$, then one is asking that $f(b_i) = a_i$ and $x_{i+1} = t_i(b_i)$. An element $x \in X$ is called a *child* of $y \in X$, when there is a path $\langle y, b, x\rangle$.

In the particular case when X is the final coalgebra M_f, a path $\langle m_0, b_0, \ldots, m_n\rangle$ in this sense coincides precisely with a path in the usual sense in the non-well-founded tree m_0. I will therefore say that such a path *lies* in m_0, and by extension, a path $\langle x_0, b_0, \ldots, x_n\rangle$ lies in $x_0 \in X$ for any coalgebra (X, γ). All paths in a coalgebra (X, γ) are collected into a subobject

$$\text{Paths}(\gamma) \subseteq (X + B + 1)^{\mathbb{N}}.$$

Any morphism of coalgebras $\alpha\colon (X, \gamma) \longrightarrow (Y, \delta)$ induces a morphism

$$\alpha_*\colon \text{Paths}(\gamma) \longrightarrow \text{Paths}(\delta) \tag{5.4}$$

between the objects of paths in the respective coalgebras. A path $\langle x_0, b_0, \ldots, x_n\rangle$ is sent by α_* to $\langle \alpha(x_0), b_0, \ldots, \alpha(x_n)\rangle$. Furthermore, given a path $\tau = \langle y_0, b_0, \ldots, y_n\rangle$ in Y and an x_0 such that $\alpha(x_0) = y_0$, there is a unique path σ starting with x_0 such that $\alpha_*(\sigma) = \tau$. (Proof: define x_{i+1} inductively for every $i < n$ using (5.3) and put $\sigma = \langle x_0, b_0, \ldots, x_n\rangle$.)

In fact, in order to introduce the concept of path, one needs even less than a coalgebra: it is sufficient to have a common environment in which to read equation (5.3). Given a map $f\colon B \longrightarrow A$ in $\mathcal{E}$, consider the category P_f-prtclg of *P_f-proto-coalgebras*. Its objects are pairs of maps

$$(\gamma, m) = X \xrightarrow{\gamma} Y \xleftarrow{m} P_f(X), \tag{5.5}$$

where m is monic. An arrow between (γ, m) and (γ', m') is a pair of maps (α, β) making the following commute:

$$\begin{array}{ccccc} X & \xrightarrow{\gamma} & Y & \xleftarrow{m} & P_f(X) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle P_f(\alpha)} \\ X' & \xrightarrow[\gamma']{} & Y' & \xleftarrow[m']{} & P_f(X'). \end{array}$$

Notice that there is an obvious inclusion functor

$$I\colon P_f-\text{coalg} \longrightarrow P_f-\text{prtclg}, \tag{5.6}$$

mapping a coalgebra $\gamma \colon X \longrightarrow P_f(X)$ to the pair $(\gamma, \mathrm{id}_{P_f X})$. Proto-coalgebras do not seem to be very interesting in themselves, but they will be very helpful for studying M-types.

For a proto-coalgebra as in (5.5), one can introduce the notion of a path in the following way. I shall call an element $x \in X$ *branching* if $\gamma(x)$ lies in the image of m. Then, I call a sequence of odd length $\sigma = \langle x_0, b_0, x_1, b_1, \ldots, x_n \rangle$ a *path* if it satisfies the properties:

1. $x_i \in X$ is branching for all $i < n$

2. $b_i \in B_{a_i}$ for all $i < n$

3. $t_i(b_i) = x_{i+1}$ for all $i < n$

where (a_i, t_i) is the (unique) element in $P_f X$ such that $\gamma(x_i) = m(a_i, t_i)$. An element $x \in X$ is called *coherent*, if all paths starting with x end with a branching element. So, all coherent elements are automatically branching, and their children, identified through m, are themselves coherent. So the object $Coh(\gamma)$ of coherent elements has a P_f-coalgebra structure. In fact, this is the biggest coalgebra which one can embed in (γ, m), i.e. a coreflection of the latter for the inclusion functor I of (5.6).

Proposition 5.1.1 *The assignment $(\gamma, m) \longmapsto Coh(\gamma)$ mapping any P_f-proto-coalgebra to the object of coherent elements in it, determines a right adjoint Coh to the functor $I \colon P_f - \mathrm{coalg} \longrightarrow P_f - \mathrm{prtclg}$.*

Proof. Consider a proto-coalgebra

$$X \xrightarrow{\gamma} Y \xleftarrow{m} P_f(X),$$

and build the object $Coh(\gamma)$ of coherent elements in X. Because any coherent element $x \in Coh(\gamma)$ is also branching, one can find a (necessarily unique) pair (a, t) such that $\gamma(x) = m(a, t)$. By defining $\chi(x) = (a, t)$, I equip $Coh(\gamma)$ with a P_f-coalgebra structure (notice that, x being coherent, so are the elements in the image of t). The coalgebra $(Coh(\gamma), \chi)$ clearly fits in a commutative diagram

$$\begin{array}{ccc} Coh(\gamma) & \xrightarrow{\quad i \quad} & X \\ \chi \downarrow & & \downarrow \gamma \\ P_f(Coh(\gamma)) \xrightarrow[P_f i]{} P_f(X) & \xrightarrow[m]{} & Y. \end{array}$$

Let now (X', χ') be any other P_f-coalgebra. Then, given a coalgebra morphism

$$\begin{array}{ccc} X' & \xrightarrow{\phi} & Coh(\gamma) \\ \downarrow{\scriptstyle \chi'} & & \downarrow{\scriptstyle \chi} \\ P_f(X') & \xrightarrow[P_f\phi]{} & P_f(Coh(\gamma)), \end{array}$$

the pair $(i\phi, mP_f(i\phi))$ clearly determines a proto-coalgebra morphism from $I(X', \chi')$ to (γ, m). Conversely, any proto-coalgebra morphism

$$\begin{array}{ccccc} X' & \xrightarrow{\chi'} & P_f(X') & = & P_f(X') \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle P_f(\alpha)} \\ X & \xrightarrow[\gamma]{} & Y & \xleftarrow[m]{} & P_f(X) \end{array}$$

has the property that $\alpha(x')$ is branching for any $x' \in X'$. Using an opportune extension to proto-coalgebras of the morphism α_* described in (5.4) above, one can then easily check that elements in the image of α are coherent. Hence, α factors through the object $Coh(\gamma)$, inducing a coalgebra morphism from (X', χ') to $(Coh(\gamma), \chi)$.

It is now easy to check that the two constructions are mutually inverse, thereby describing the desired adjunction. □

A particular subcategory of proto-coalgebras arises when one has another endofunctor F on $\mathcal{E}$ and an injective natural transformation $m: P_f \rightarrowtail F$. In this case, any F-coalgebra $\chi: X \longrightarrow FX$ can easily be turned into the P_f-proto-coalgebra (χ, m_X). This determines a functor $\widehat{m}: F-\text{coalg} \longrightarrow P_f-\text{prtclg}$, which is clearly faithful.

Proposition 5.1.2 *The adjunction $I \dashv Coh$ of Proposition 5.1.1 restricts to an adjunction $m_* \dashv Coh\ \widehat{m}$, if $m_*: P_f-\text{coalg} \longrightarrow F-\text{coalg}$ takes $\chi: X \longrightarrow P_f X$ to $(X, m_X\chi)$.*

Proof. Consider a P_f-coalgebra (Z, γ) and an F-coalgebra (X, χ). Then, a simple diagram chase, using the naturality of m, shows that F-coalgebra morphisms from $m_*(Z, \gamma)$ to (X, χ) correspond bijectively to morphisms of proto-coalgebras from $I(Z, \gamma)$ to $\widehat{m}(X, \chi)$, hence by Proposition 5.1.1 to P_f-coalgebra homomorphisms from (Z, γ) to $Coh(\widehat{m}(X, \chi))$. □

5.2 Existence results for M-types

The crucial point in showing that ΠM-pretoposes are closed under the various constructions I am going to consider, will always be that of showing existence of M-types.

The machinery to do so will be set up in this Section. But the results are not just useful for that. They are, I think, valuable in themselves and raise interesting questions.

Traditionally, one can recover non-well-founded trees from well-founded ones, whenever the signature has one specified constant. In fact, the constant allows for the definition of truncation functions, which cut a tree at a certain depth and replace all the term constructors at that level by that specified constant. The way to recover non-well-founded trees is then to consider sequences of trees $(t_n)_{n>0}$ such that each t_n is the truncation at depth n of t_m for all $m > n$. Each such sequence is viewed as the sequence of approximations of a non-well-founded tree.

Recall that the context is that of a Π-pretopos $\mathcal{E}$ with nno. In this context, I call a map $f\colon B \longrightarrow A$ *pointed*, when the signature it represents has a specified constant symbol, i.e. if there exists a global element $\bot\colon 1 \longrightarrow A$ such that the following is a pullback:

$$\begin{array}{ccc} 0 & \longrightarrow & B \\ \downarrow & & \downarrow f \\ 1 & \underset{\bot}{\longrightarrow} & A. \end{array}$$

The next two statements make clear that, instead of starting with well-founded trees, i.e. with the W-type for f, one can build these approximations from any fixpoint of P_f.

Lemma 5.2.1 *If for some pointed f in $\mathcal{E}$, P_f has a fixpoint, then it also has a final coalgebra.*

Proof. Assume X is an algebra whose structure map $\sup\colon P_f X \longrightarrow X$ is an isomorphism. Observe, first of all, that X has a global element

$$\bot\colon 1 \longrightarrow X, \tag{5.7}$$

namely $\sup_\bot(t)$, where $\bot$ is the point of f and t is the unique map $B_\bot = 0 \longrightarrow X$.

Define, by induction, the following truncation functions $tr_n\colon X \longrightarrow X$:

$$\begin{aligned} tr_0 &= \bot \\ tr_{n+1} &= \sup \circ P_f(tr_n) \circ \sup^{-1} \end{aligned}$$

Using these maps, one can define an object M, consisting of sequences $(\alpha_n \in X)_{n>0}$ with the property:

$$\alpha_n = tr_n(\alpha_m) \text{ for all } n < m.$$

Now, one defines a morphism $\tau\colon M \longrightarrow P_f M$ as follows. Given a sequence $\alpha = (\alpha_n) \in M$, observe that $\rho(\alpha_n)$ is independent of n and is some element $a \in A$. Hence, each α_n is of the form $\sup_a(t_n)$ for some $t_n\colon B_a \longrightarrow X$, and I define $t\colon B_a \longrightarrow M$ by putting

$t(b)_n = t_{n+1}(b)$ for every $b \in B_a$; then $\tau(\alpha) = (a, t)$. Thus, M has the structure of a P_f-coalgebra, and I claim it is the terminal one.

To show this, given another coalgebra $\chi: Y \longrightarrow P_f Y$, I wish to define a map of coalgebras $\widehat{p}: Y \longrightarrow M$. This means defining maps $\widehat{p}_n: Y \longrightarrow X$ for every $n > 0$, with the property that $\widehat{p}_n = tr_n \widehat{p}_m$ for all $n < m$. Intuitively, $\widehat{p}_n$ maps a state of Y to its "unfolding up to level n", which I can mimic in X. Formally, they are defined inductively by

$$\begin{aligned} \widehat{p}_0 &= \bot \\ \widehat{p}_{n+1} &= \sup \circ P_f(\widehat{p}_n) \circ \chi. \end{aligned}$$

It is now easy to show, by induction on n, that $\widehat{p}_n = tr_n \widehat{p}_m$ for all $m > n$. For $n = 0$, both sides of the equation become the constant map $\bot$. Supposing the equation holds for a fixed n and any $m > n$, then for $n+1$ and any $m > n$ one has $\widehat{p}_{n+1} = \sup P_f(\widehat{p}_n)\chi = \sup P_f(tr_n \widehat{p}_m)\chi = \sup P_f(tr_n)\sup^{-1}\sup P_f(\widehat{p}_m)\chi = tr_{n+1}\widehat{p}_{m+1}$.

I leave to the reader the verification that $\widehat{p}$ is the unique P_f-coalgebra morphism from Y to M. □

Theorem 5.2.2 *If fixpoints exist in $\mathcal{E}$ for all P_f (with f pointed), then $\mathcal{E}$ has M-types.*

Proof. Let $f: B \longrightarrow A$ be a map. I freely add a point to the signature represented by f, by considering the composite

$$f_\bot: B \xrightarrow{f} A \overset{i}{\rightarrowtail} A + 1 \tag{5.8}$$

(with the point $j = \bot: 1 \longrightarrow A + 1$). Notice that the obvious pullback

$$\begin{array}{ccc} B & \xrightarrow{\text{id}} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_\bot} \\ A & \underset{i}{\rightarrowtail} & A + 1 \end{array}$$

determines a (monic) natural transformation $i_!: P_f \longrightarrow P_{f_\bot}$ by (5.2); hence, by Proposition 5.1.2, the functor $(i_!)_*: P_f-\text{coalg} \longrightarrow P_{f_\bot}-\text{coalg}$ has a right adjoint. Now observe that $P_{f_\bot}$ has a fixpoint, by assumption, hence a final coalgebra by Lemma 5.2.1. This will be preserved by the right adjoint of $(i_!)_*$, hence P_f has a final coalgebra. □

This proof gives a categorical counterpart of the standard set-theoretic construction: add a dummy constant to the signature, build infinite trees by sequences of approximations, then select the actual M-type by taking those infinite trees which involve only term constructors from the original signature. This last passage is performed by the coreflection functor of Proposition 5.1.2, since branching elements are trees in the M-type of $f_\perp$ whose root is not $\perp$, and coherent ones are trees with no occurrence of $\perp$ at any point.

From this last theorem, one readily deduces the following result, first pointed out by Abbott, Altenkirch and Ghani [1].

Corollary 5.2.3 *Every ΠW-pretopos is a ΠM-pretopos.*

Proof. Since the W-type associated to a (pointed) map f is a fixpoint for P_f, $\mathcal{E}$ also has all M-types by the previous theorem. □

Remark 5.2.4 This result shows that there is a substantial class of examples of ΠM-pretoposes. It is an open problem to find a non-syntactic example of a ΠM-pretopos that is not a ΠW-pretopos.

In Chapter 2, we have seen some examples of categories which have M-types, but are not ΠM-pretoposes; for instance, the category of modest sets, or that of assemblies (or ω-sets). The only reason these categories are not examples of ΠM-pretoposes is that they fail to be exact. However, notice that exactness is not necessary for the proofs. In fact, regularity would be sufficient to establish all the closure properties.

Although Theorem 5.2.2 is clearly helpful in proving that categories have M-types, it is even more so, when combined with the following observation.

Lemma 5.2.5 *Any prefixpoint $\alpha\colon P_f X \longrightarrow X$, that is, an algebra whose structure map is monic, has a subalgebra that is a fixpoint.*

Proof. Any prefixpoint $\alpha\colon P_f X \longrightarrow X$ can be seen as a P_f-proto-coalgebra

$$X \xrightarrow{\mathrm{id}} X \xleftarrow{\alpha} P_f X.$$

Its coreflection $Coh(\mathrm{id}, \alpha)$, defined in Proposition 5.1.1, is a P_f-coalgebra $\gamma\colon Y \longrightarrow P_f Y$ (in fact, the largest) fitting in the following commutative square:

$$\begin{array}{ccc} Y & \stackrel{i}{\rightarrowtail} & X \\ {\scriptstyle\gamma}\downarrow & & \uparrow{\scriptstyle\alpha} \\ P_f Y & \underset{P_f i}{\rightarrowtail} & P_f X. \end{array}$$

Now, consider the image under the functor $I\colon P_f - \mathrm{coalg} \longrightarrow P_f - \mathrm{prtclg}$ of the coalgebra $P_f(\gamma)\colon P_f Y \longrightarrow P_f^2 Y$. The morphism of proto-coalgebras

$$\begin{array}{ccccc} P_f Y & \xrightarrow{P_f\gamma} & P_f^2 Y & \xleftarrow{\mathrm{id}} & P_f^2 Y \\ {\scriptstyle \alpha P_f i}\downarrow & & \downarrow{\scriptstyle \alpha P_f(\alpha) P_f^2 i} & & \downarrow{\scriptstyle P_f(\alpha) P_f^2 i} \\ X & \xrightarrow[\mathrm{id}]{} & X & \xleftarrow[\alpha]{} & P_f X \end{array}$$

transposes through the adjunction $I \dashv Coh$ to a morphism $\phi\colon (P_f Y, P_f \gamma) \longrightarrow (Y, \gamma)$, which is a right inverse of $\gamma\colon (Y, \gamma) \longrightarrow (P_f Y, P_f \gamma)$ by the universal property of (Y, γ). Hence, I have $\gamma\phi = P_f(\phi\gamma) = \mathrm{id}$, proving that γ and ϕ are mutually inverse. □

Putting together Theorem 5.2.2 and Lemma 5.2.5, one gets at once the following:

Corollary 5.2.6 *If $\mathcal{E}$ has prefixpoints for every polynomial functor, then $\mathcal{E}$ has M-types.*

As an application of the techniques in this Section, I present the following result, which is to be compared with the one by Santocanale in [78]. An immediate corollary of his Theorem 4.5 is that M-types exist in every locally cartesian closed pretopos with a natural number object, for maps of the form $f\colon B \longrightarrow A$ where A is a finite sum of copies of 1. Notice that such an object A has *decidable equality*, i.e. the diagonal $\Delta\colon A \longrightarrow A \times A$ has a complement in the subobject lattice of $A \times A$. I extend the statement above to *all* maps whose codomain has decidable equality.

Proposition 5.2.7 *When $f\colon B \longrightarrow A$ is a morphism in $\mathcal{E}$ whose codomain A has decidable equality, then the M-type for f exists.*

Proof. Without loss of generality, one may assume that f is pointed; in fact, if one replaces A by $A_\perp = A + 1$ and f by $f_\perp$ as in (5.8), then $A_\perp$ also has decidable equality, and the existence of an M-type for the composite $f_\perp$ implies that of an M-type for f (see the proof of Theorem 5.2.2). Then, by Lemma 5.2.5 and Lemma 5.2.1, it is enough to show that P_f has a prefixpoint.

Let S be the object of all finite sequences of the form

$$\langle a_0, b_0, a_1, b_1, \ldots, a_n \rangle$$

where $f(b_i) = a_i$ for all $i < n$. (Like paths in a coalgebra, this object S can be constructed using the internal logic of $\mathcal{E}$.) Now, let V be the object of all decidable subobjects of S (these can be considered as functions $S \longrightarrow 1 + 1$). Define the map $m\colon P_f V \longrightarrow V$ taking a pair $(a, t\colon B_a \longrightarrow V)$ to the subobject P of S defined by the following clauses:

1. $\langle a_0 \rangle \in P$ iff $a_0 = a$.
2. $\langle a_0, b_0 \rangle * \sigma \in P$ iff $a_0 = a$ and $\sigma \in t(b_0)$.

(Here, $*$ is the symbol for concatenation.) P is obviously decidable, so m is well-defined. To see that it is monic, suppose $P = m(a, t)$ and $P' = m(a', t')$ are equal. Then,

$$\langle a \rangle \in P \Longrightarrow \langle a \rangle \in P' \Longrightarrow a = a',$$

and, for every $b \in B_a$ and $\sigma \in S$,

$$\begin{aligned} \sigma \in t(b) &\iff \langle a, b\rangle * \sigma \in P \\ &\iff \langle a, b\rangle * \sigma \in P' \\ &\iff \sigma \in t'(b), \end{aligned}$$

so $t = t'$ and m is monic. Hence, (V, m) is a prefixpoint for P_f and the proof is finished. □

It is an interesting question whether this result can be generalised even further. However, it is my feeling that not all M-types can be proved to exist in general. Unfortunately, the lack of examples of Π-pretoposes with natural number object, but without W-types makes it hard to give counterexamples.

Remark 5.2.8 To obtain a concrete description of the M-type for a map f with a codomain with decidable equality, one should start with the objects S and V constructed in the proof of Proposition 5.2.7. Then one should deduce a fixpoint V' from V, as in Corollary 5.2.6. This means selecting the coherent elements of V, and these turn out to be those decidable subobjects P of S satisfying the following properties:

1. $\langle a\rangle \in P$ for a unique $a \in A$;
2. if $\langle a_0, b_0, \ldots, a_n\rangle \in P$, then there exists a unique a_{n+1} for any $b_n \in B_{a_n}$ such that $\langle a_0, b_0, \ldots, a_n, b_n, a_{n+1}\rangle \in P$.

Next, one should turn this fixpoint into the M-type for f (as in Lemma 5.2.1), but this step is redundant, since the choice of V is such that V' already is the desired M-type.

5.3 Closure properties

After these preliminaries, I establish closure of ΠM-pretoposes under slicing, coalgebras for a cartesian comonad, presheaves and sheaves.

5.3.1 M-types and slicing

I start by considering preservation of the ΠM-pretopos structure under slicing. Let I be an object in a Π-pretopos with nno $\mathcal{E}$. Then, it is well-known that the slice category $\mathcal{E}/I$ has again the same structure, and the reindexing functor $x^*: \mathcal{E}/I \longrightarrow \mathcal{E}/J$ for any map $x: J \longrightarrow I$ in $\mathcal{E}$ preserves it. So, I can focus on showing the existence of M-types in $\mathcal{E}/I$. Their preservation under reindexing immediately follows from some results on indexed categories (see Lemma A.19 and Lemma A.21). Therefore, I shall concentrate on the existence of M-types in slice categories, proving a "local existence" result, from which I derive a global statement.

Let us consider a map

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ & {\scriptstyle\beta}\searrow \quad \swarrow{\scriptstyle\alpha} & \\ & I & \end{array} \tag{5.9}$$

in $\mathcal{E}/I$. I shall denote by P_f the polynomial functor determined by f (or, more precisely, by Σf) in $\mathcal{E}$, and by P_f^I the polynomial endofunctor determined in $\mathcal{E}/I$. The functor $P_f: \mathcal{E} \longrightarrow \mathcal{E}$ can be extended to a functor $P_f: \mathcal{E} \longrightarrow \mathcal{E}/I$; in fact, $P_f X$ lives over A via the root map, and the composite $\alpha\rho: P_f X \longrightarrow I$ defines the desired extension.

Lemma 5.3.1 *There is an injective natural transformation of endofunctors on $\mathcal{E}/I$*

$$c: P_f^I \longrightarrow P_f \Sigma_I.$$

Proof. For an object $\xi: X \longrightarrow I$ in $\mathcal{E}/I$ and $i \in I$:

$$P_f^I(X \xrightarrow{\xi} I)_i = \{(a, t: B_a \longrightarrow X) \mid \alpha(a) = i, \forall b \in B_a: \beta t(b) = i\}$$

and

$$P_f(\Sigma_I(X \xrightarrow{\xi} I)) = \{(a, t: B_a \longrightarrow X) \mid \alpha(a) = i\}.$$

The first in clearly contained in the second. Naturality is readily checked. □

Using the map c of Lemma 5.3.1, one can build an M-type for f in $\mathcal{E}/I$, whenever M_f exists in $\mathcal{E}$.

Theorem 5.3.2 *Let $\mathcal{E}$ be a locally cartesian closed pretopos with a natural number object and I an object in $\mathcal{E}$. Consider a map $f: B \longrightarrow A$ over I, such that the functor $P_f: \mathcal{E} \longrightarrow \mathcal{E}$ has a final coalgebra. Then, f has an M-type in $\mathcal{E}/I$.*

Proof. Let $\tau_f: M_f \longrightarrow P_f M_f$ be the M-type associated to f in $\mathcal{E}$. M_f can be considered as an object over I, by taking the composite μ of the root map $\rho: M_f \longrightarrow A$ with the map $\alpha: A \longrightarrow I$, and (M_f, τ_f) then becomes the final $P_f\Sigma_I$-coalgebra, as one can easily check. The adjunction determined by the natural transformation $c: P_f^I \longrightarrow P_f \Sigma_I$ as in Proposition 5.1.2 takes the final $P_f\Sigma_I$-coalgebra (M_f, τ_f) to its coreflection M_f^I, and because right adjoints preserve limits, this is the final P_f^I-coalgebra. □

Remark 5.3.3 The injective natural transformation c of Lemma 5.3.1 identifies as branching elements in $P_f\Sigma_I$ those obtained by applying a term constructor in A to elements living in its same fibre over I.

The coreflection process used to build M_f^I out of the M-type (M_f, τ_f), helps to understand which elements of the latter do actually belong to the former. Trees in M_f^I are coherent for the notion of branching determined by P_f^I, hence, not only the children of the root node live in its same fibre over I, but all the children of the children do too, and so on. In other words, M_f^I consists of those trees in M_f all nodes of which live in the same fibre over I. As such, the object M_f^I can also be described as the equaliser

$$\begin{array}{ccccc} M_f^I & \rightarrowtail & M_f & \xrightarrow{\langle \mathrm{id},\alpha\rangle_!} & M_{f\times I}, \\ & & & \searrow^{\langle \mathrm{id},\alpha\rho\rangle} \quad \nearrow_{\chi} & \\ & & & M_f \times I & \end{array}$$

where χ is the map coinductively defined as

$$\chi(\mathrm{sup}_a t, i) = \mathrm{sup}_{(a,i)}(\chi\langle t, i\rangle).$$

As an immediate consequence of Theorem 5.3.2, one gets the following:

Corollary 5.3.4 *For any given object I of a ΠM-pretopos $\mathcal{E}$, the slice category $\mathcal{E}/I$ is again a ΠM-pretopos.*

Remark 5.3.5 This last result could have also been proved directly by combining Corollary 5.2.6 and Lemma 5.3.1. However, the proof of Theorem 5.3.2 shows that the construction is actually simpler. More specifically, notice that, in this case, one obtains the M-type for a map f directly after the coreflection, and it is not necessary to add any dummy variable, nor to build sequences of approximations.

5.3.2 M-types and coalgebras

In this Section, I turn my attention to the construction of categories of coalgebras for a cartesian comonad (G, ϵ, δ). See [55], Chapter VI, for the definition of a comonad and a coalgebra for a comonad. By a *cartesian* comonad, I mean here that the functor G is cartesian. As for the slicing case, I already know that most of the structure of a ΠM-pretopos is preserved by taking coalgebras for G:

Theorem 5.3.6 *If $\mathcal{E}$ is a locally cartesian closed pretopos with natural number object, then so is $\mathcal{E}_G$ for a cartesian comonad $G = (G, \epsilon, \delta)$ on $\mathcal{E}$.*

Proof. Theorem 4.2.1 on page 173 of [44] gives us that $\mathcal{E}_G$ is cartesian, in fact locally cartesian closed, and that it has a natural number object. The two additional requirements of having finite disjoint sums and being exact are easily verified, using in particular that the forgetful functor $U\colon \mathcal{E}_G \longrightarrow \mathcal{E}$ creates finite limits. □

The aim of this Subsection is to prove that $\mathcal{E}_G$ inherits M-types from $\mathcal{E}$, in case they exist in that category. The question whether ΠW-pretoposes are closed under taking coalgebras for a cartesian comonad, is still open.

Given a morphism f of coalgebras, this induces a polynomial functor $P_f : \mathcal{E}_G \longrightarrow \mathcal{E}_G$, while its underlying map Uf determines the endofunctor P_{Uf} on $\mathcal{E}$. The two are related as follows:

Proposition 5.3.7 *Let $f : (B, \beta) \longrightarrow (A, \alpha)$ be a map of G-coalgebras. Then, there is an injective natural transformation*

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{P_{Uf}} & \mathcal{E} \\ {\scriptstyle G}\downarrow & \overset{i}{\rightarrowtail} & \downarrow{\scriptstyle G} \\ \mathcal{E}_G & \xrightarrow[P_f]{} & \mathcal{E}_G, \end{array}$$

whose mate under the adjunction $U \dashv G$, I shall denote by

$$j : UP_f \longrightarrow P_{Uf}U : \mathcal{E}_G \longrightarrow \mathcal{E}. \tag{5.10}$$

Proof. Recall from [44] that there is the following natural isomorphism

$$\mathcal{E}_G/(A, \alpha) \cong (\mathcal{E}/A)_{G'}, \tag{5.11}$$

where G' is a cartesian comonad on $\mathcal{E}/A$, which is computed on an object $t : X \longrightarrow A$ in $\mathcal{E}/A$ by taking the following pullback:

$$\begin{array}{ccc} G'X & \rightarrowtail & GX \\ {\scriptstyle G't}\downarrow & & \downarrow{\scriptstyle Gt} \\ A & \xrightarrow[\alpha]{} & GA. \end{array} \tag{5.12}$$

Notice that both horizontal arrows in this pullback are monic, because ϵ_A is a retraction of the G-coalgebra α.

Through the isomorphism (5.11), the object $A \times GX \longrightarrow A$ corresponds to $G'(p_1 : A \times X \to A)$, whereas f corresponds to some map f' in $(\mathcal{E}/A)_{G'}$. Therefore the object $P_f(GX)$ (i.e. the source of the exponential $(A \times GX \longrightarrow A)^f$ in the category $\mathcal{E}_G/(A, \alpha)$) corresponds to the exponential $(G'p_1)^{f'}$. Since $U' : (\mathcal{E}/A)_{G'} \longrightarrow \mathcal{E}/A$ preserves products because G' does, there is the following chain of natural bijections:

$$\begin{array}{rcl} Y & \longrightarrow & G'(p_1^{U'f'}) \\ \hline U'Y & \longrightarrow & p_1^{U'f'} \\ \hline U'Y \times U'f' & \longrightarrow & p_1 \\ \hline U'(Y \times f') & \longrightarrow & p_1 \\ \hline Y \times f' & \longrightarrow & (G'p_1) \\ \hline Y & \longrightarrow & (G'p_1)^{f'}. \end{array}$$

So one deduces $(G'p_1)^{f'} \cong G'(p_1^{U'f'}) = G'(p_1^{Uf})$. The latter fits in the following pullback square, which is an instance of (5.12):

$$\begin{array}{ccc} G'((A \times X \to A)^{Uf}) & \stackrel{i_X}{\rightarrowtail} & G((A \times X \to A)^{Uf}) \\ \downarrow & & \downarrow \\ A & \underset{\alpha}{\rightarrowtail} & GA. \end{array}$$

Now notice that the top-right entry of the diagram is exactly $GP_{Uf}(X)$, hence the map i therein defines the X-th component of a natural transformation of the desired form. □

I am now ready to formulate a local existence result for M-types in categories of coalgebras.

Theorem 5.3.8 *Let $f\colon (B,\beta) \longrightarrow (A,\alpha)$ be a map of G-coalgebras. If the underlying map Uf has an M-type in $\mathcal{E}$, then the functor $P_f\colon \mathcal{E}_G \longrightarrow \mathcal{E}_G$ has a final coalgebra in $\mathcal{E}_G$.*

Proof. The natural transformation i of Proposition 5.3.7 allows one to turn any P_{Uf}-coalgebra into a P_f-proto-coalgebra. In particular, for the M-type $\tau\colon M = M_{Uf} \longrightarrow P_{Uf}M$ in $\mathcal{E}$, one obtains the proto-coalgebra

$$GM \xrightarrow{G\tau} GP_{Uf}M \stackrel{i_M}{\leftarrowtail} P_f GM,$$

whose coreflection $Coh(M) = Coh(G\tau, i_M)$ is final in P_f-coalg. To see this, consider another coalgebra (X, γ) (therefore, X is a G-coalgebra, and $\gamma\colon X \longrightarrow P_f X$ is a G-coalgebra homomorphism). To give a morphism of P_f-coalgebras from (X, γ) to $Coh(M)$ is the same, through $I \dashv Coh$, as giving a map $\psi\colon X \longrightarrow GM$ in $\mathcal{E}_G$ which is a morphism of P_f-proto-coalgebras, i.e. that makes the following commute:

$$\begin{array}{ccccc} X & & \xrightarrow{\gamma} & & P_f X \\ {\scriptstyle\psi}\downarrow & & & & \downarrow{\scriptstyle P_f\psi} \\ GM & \underset{G\tau}{\longrightarrow} & GP_{Uf}M & \underset{i_M}{\leftarrowtail} & P_f GM. \end{array}$$

This transposes, through $U \dashv G$, to the following diagram in $\mathcal{E}$, where j is the natural transformation defined in (5.10):

$$\begin{array}{ccccc} UX & \xrightarrow{U\gamma} & UP_f X & \xrightarrow{j_X} & P_{Uf}UX \\ {\scriptstyle\widehat{\psi}}\downarrow & & & & \downarrow{\scriptstyle P_{Uf}\widehat{\psi}} \\ M & & \underset{\tau}{\longrightarrow} & & P_{Uf}M. \end{array}$$

But finality of M implies that there is precisely one such $\widehat{\psi}$ for any coalgebra (X, γ), hence finality is proved. □

Corollary 5.3.9 *If $\mathcal{E}$ is a ΠM-pretopos and $G = (G, \epsilon, \delta)$ is a cartesian comonad on $\mathcal{E}$, then the category $\mathcal{E}_G$ of (Eilenberg-Moore) coalgebras for G is again a ΠM-pretopos.*

Remark 5.3.10 Notice that Corollary 5.3.9 could also be deduced by Corollary 5.2.6, in conjunction with Proposition 5.3.7. However, analogously to what happens in the slicing case, Theorem 5.3.8 shows that one does not need to perform the whole construction, since the coreflection step gives directly the final coalgebra.

Remark 5.3.11 In particular, this result shows stability of ΠM-pretoposes under the glueing construction, since this is a special case of taking coalgebras for a cartesian comonad (see [44]).

5.3.3 M-types and presheaves

In this Section, I concern myself with the formation of presheaves for an internal category in a ΠM-pretopos. My aim is to show that the resulting category is again a ΠM-pretopos.

So consider an internal category $\mathcal{C}$ in a ΠM-pretopos $\mathcal{E}$, with object of objects $\mathcal{C}_0$ (see Appendix A for the definition of an internal category). By using the fact that the category of presheaves $\mathrm{Psh}(\mathcal{C})$ is the category of coalgebras for a cartesian comonad on the slice category $\mathcal{E}/\mathcal{C}_0$ (see for instance [44], Example A.4.2.4 (b)), I get at once

Proposition 5.3.12 *The presheaf category $\mathrm{Psh}(\mathcal{C})$ is a ΠM-pretopos.*

Unwinding the proof, it is possible to give a concrete description of the M-type in presheaf categories, along the lines of the description of W-types in [61]. I will just give the description and leave the verifications to the reader.

First of all, I need to introduce the functor $|\cdot|\colon \mathrm{Psh}(\mathcal{C}) \longrightarrow \mathcal{E}$ which takes a presheaf $\mathcal{A}$ to its "underlying set" $|\mathcal{A}| = \{(a, C) \mid a \in \mathcal{A}(C)\}$. This is just the composite of the forgetful functor $U\colon \mathrm{Psh}(\mathcal{C}) \longrightarrow \mathcal{E}/\mathcal{C}_0$ with $\Sigma_{\mathcal{C}_0}\colon \mathcal{E}/\mathcal{C}_0 \longrightarrow \mathcal{E}$.

Let $f\colon \mathcal{B} \longrightarrow \mathcal{A}$ be a morphism of presheaves. Then, the "fibre" $\mathcal{B}_a$ of f over $a \in \mathcal{A}(C)$ for an object C in $\mathcal{C}$ is a presheaf, whose action on D is described in the internal language of $\mathcal{E}$ as

$$\mathcal{B}_a(D) = \{(\beta, b) \mid \beta\colon D \longrightarrow C, a \cdot \beta = f(b)\}$$

and restriction along a morphism $\delta\colon D' \longrightarrow D$ is defined as

$$(\beta, b) \cdot \delta = (\beta\delta, b \cdot \delta).$$

Now the presheaf morphism f also induces a map

$$f'\colon \Sigma_{(a,C) \in |\mathcal{A}|} |\mathcal{B}_a| \longrightarrow |\mathcal{A}|$$

whose fibre over (a, C) is precisely $|\mathcal{B}_a|$. Consider the M-type $M_{f'}$ in $\mathcal{E}$: the M-type $\mathcal{M}$ for f in presheaves will be built by selecting the right elements from this M-type.

Elements $T \in M_{f'}$ are of the form

$$T = \sup_{(a,C)} t,$$

where $(a, C) \in |\mathcal{A}|$ and $t: \mathcal{B}_a \longrightarrow M_{f'}$. $M_{f'}$ can be considered as an object in $\mathcal{E}/\mathcal{C}_0$, when one maps such a T to C. Write $\mathcal{N}(C)$ for the fibre over $C \in \mathcal{C}_0$. $\mathcal{N}$ actually possesses the structure of a presheaf, because for any $T \in \mathcal{N}(C)$ and $\alpha: C' \longrightarrow C$,

$$T \cdot \alpha = \sup_{a',C'} t\widetilde{\alpha},$$

where $a' = a \cdot \alpha$ and $\widetilde{\alpha}$ is the obvious morphism $|\mathcal{B}_{a'}| \longrightarrow |\mathcal{B}_a|$, defined by sending (β, b) to $(\alpha\beta, b)$.

Out of this presheaf $\mathcal{N}$, one has to select the coherent elements (the trees called *natural* in [60]). Call a tree S *composable*, when all subtrees $T = \sup_{(a,C)} t$ of S satisfy

$$t(\beta, b) \in \mathcal{N}(\mathrm{dom}(\beta)).$$

Call S *coherent* or *natural*, when all subtrees $T = \sup_{(a,C)} t$ of S in addition satisfy that

$$t(\beta, b) \cdot \gamma = t(\beta\gamma, b \cdot \gamma).$$

These notions can be defined using the language of paths. Let $\mathcal{M}$ be the subobject of $\mathcal{N}$ consisting of the coherent elements. It is a presheaf, and, as the reader can verify, the M-type for f in presheaves. So, in effect, I have proved:

Theorem 5.3.13 *Consider a map $f: \mathcal{B} \longrightarrow \mathcal{A}$ in $\mathrm{Psh}(\mathcal{C})$. If the induced map f' has an M-type in $\mathcal{E}$, then f has an M-type in $\mathrm{Psh}(\mathcal{C})$.*

5.3.4 M-types and sheaves

In this Section, I wish to show that ΠM-pretoposes are closed under taking sheaves. I approach this question in the following manner: I show that ΠM-pretoposes are closed under reflective subcategories with cartesian reflector (by the way, the question whether the corresponding result for ΠW-pretoposes holds, is still open). It is well-known that in topos theory categories of sheaves are such subcategories of the category of presheaves. Within a predicative metatheory, the construction of a sheafification functor, a cartesian left adjoint for the inclusion of sheaves in presheaves, runs into some problems. Solutions have been proposed in [61] and [15]. Here, I will simply assume that this problem can be solved. Then closure of ΠM-pretoposes under sheaves follows from closure under reflective subcategories, because I have just shown that ΠM-pretoposes are closed under taking presheaves for an internal site.

On cartesian reflectors and the universal closure operators they induce, the reader should consult [44], Sections A4.3 and A4.4. Very briefly, the story is like this. A category $\mathcal{D}$ is a reflective subcategory of a cartesian category $\mathcal{E}$, when the inclusion functor $i: D \longrightarrow E$ has a left adjoint L such that $Li \cong 1$. Now the inclusion is automatically full and faithful.

When the reflector L is cartesian, as I will always assume, it induces an operator on the subobject lattice of any object X. The operator sends a subobject

$$m: X' \rightarrowtail X$$

to the left side of the pullback square

$$\begin{array}{ccc} c(X') & \longrightarrow & iLX' \\ \downarrow & & \downarrow{\scriptstyle iLm} \\ X & \xrightarrow[\eta_X]{} & iLX. \end{array}$$

This operation is order-preserving, idempotent ($c(c(X')) = c(X')$) and inflationary ($X' \leq c(X')$) and commutes with pullback along arbitrary morphisms. Such operators are called *universal closure operators*. In topos theory, every universal closure operator derives from a cartesian reflector, but in the context of Π-pretoposes that is probably not the case.

The objects in $\mathcal{E}$ that come from $\mathcal{D}$ can be characterised in terms of the closure operator c as follows. Call a mono

$$m: X' \rightarrowtail X$$

dense, when its closure $c(X')$ is the maximal object $X \subseteq X$. An object Y in $\mathcal{E}$ is from $\mathcal{D}$ in case any triangle

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y \\ {\scriptstyle m}\downarrow & \nearrow{\scriptstyle f} & \\ X & & \end{array}$$

with m a dense mono, can be filled uniquely by a map f. These objects are, not accidentally, called the *sheaves* for the closure operator c. Objects Y for which such triangles have at most one filling are called *separated* with respect to c. Also the separated objects form a reflective subcategory of $\mathcal{E}$.

It is well-known that in this setting $\mathcal{D}$ is a locally cartesian closed pretopos with a natural number object. Parts of this result, especially that $\mathcal{D}$ is an lccc, can be found in [44] in the aforementioned Sections: I will also need that i preserves the lccc structure, which can also be found there. The same is true for the separated objects: they are also an lccc (not a pretopos, though), where the inclusion also preserves the lccc structure.

Theorem 5.3.14 *Let $f : B \longrightarrow A$ be a morphism in $\mathcal{E}$.*

1. *When f is a morphism of separated objects, M_f is separated.*
2. *When f is a morphism of sheaves, M_f is a sheaf.*

Proof. I will give the argument for sheaves, but the proof is the same in both cases. Let $M = M_f$ be the M-type in $\mathcal{E}$ associated to f, and obtain the sheaf LM by applying the reflector to M. The object $P_f(LM)$ is also a sheaf, because the inclusion preserves the lccc structure. Because of the universal property of L the diagram

$$\begin{array}{ccc} M & \xrightarrow{\eta_M} & iLM \\ {\scriptstyle \tau}\downarrow & & \vdots \\ P_f(M) & \xrightarrow[P_f(\eta_M)]{} & P_f(iLM) \cong iP_f(LM) \end{array}$$

can be filled. Therefore iLM has the structure of P_f-coalgebra in such a way that η_M is a P_f-coalgebra morphism. By finality of M, there is a P_f-coalgebra morphism $r : iLM \longrightarrow M$ such that $r\eta_M = 1$. So $\eta_M r \eta_M = \eta_M = 1\eta_M$ and the universal property of η_M immediately gives that also $\eta_M r = 1$. So $M \cong iLM$ and M is a sheaf. □

Remark 5.3.15 In both cases, it would have been enough to require that the codomain of f is a sheaf (respectively separated). This essentially because the sheaves and separated objects both form exponential ideals in $\mathcal{E}$.

Remark 5.3.16 In case the universal closure operator is not known to derive from a cartesian reflector, it is still possible to show that the M-type $M = M_f$ for a morphism $f : B \longrightarrow A$ with separated codomain is separated. For that purpose, write $x =_c x'$ for $x, x' \in X$, when $(x, x') \in c(\Delta : X \longrightarrow X \times X)$. An object X is then separated, when

$$x =_c x' \Rightarrow x = x'$$

(see [44], Lemma 4.3.6). To show that M is separated, consider

$$B = \{(\sup_a(t), \sup_{a'}(t')) \in M \times M \mid \sup_a(t) =_c \sup_{a'}(t')\}.$$

B has the structure of a P_f-coalgebra in such a way that composing $B \subseteq M \times M$ with either of the two projections yields a P_f-coalgebra morphism. In other words, B has the structure of a *bisimulation* on M. This is true, simply because whenever $\sup_a(t) =_c \sup_{a'}(t')$, then $a =_c a'$, and hence $a = a'$, because A is separated. And because one therefore also has that $tb =_c t'b$ for every $b \in B_a$.

But because of finality of M, all bisimulations on M are contained in the diagonal of M. Hence

$$\sup_a(t) =_c \sup_{a'}(t') \Rightarrow \sup_a(t) = \sup_{a'}(t') \tag{5.13}$$

and M is separated.

Remark 5.3.17 As a corollary, one obtains that the subcategory of separated objects for a universal closure operator on a ΠW-pretopos $\mathcal{E}$ has W-types. $\mathcal{E}$ has M-types by Corollary 5.2.3, and for morphisms f between separated objects, these M-types are separated by the preceeding remark. But since W-types are subobjects of M-types (see Lemma 2.1.4), and separated objects are easily seen to be closed under subobjects, the W-types associated to such morphisms are separated as well. Another way of showing this fact is by directly proving (5.13) by induction.

Theorem 5.3.14 now directly shows:

Theorem 5.3.18 *If $\mathcal{D}$ is a reflective subcategory of a ΠM-pretopos $\mathcal{E}$ with cartesian reflector, $\mathcal{D}$ is also a ΠM-pretopos.*

Corollary 5.3.19 *f $\mathcal{C}$ is an internal site in a ΠM-pretopos $\mathcal{E}$ such that the inclusion of internal sheaves in presheaves has a cartesian left adjoint (a "sheafification functor"), then the category $\mathrm{Sh}(\mathcal{C})$ of internal sheaves for the site $\mathcal{C}$ in $\mathcal{E}$ is a ΠM-pretopos.*