# Chapter 4

# Algebraic set theory and CZF

This Chapter is meant to make good on the claim that $\Pi W$-pretoposes form a natural context for models of constructive-predicative set theories, like **CZF**.

Aczel's set theory **CZF** is introduced in the first Section. **CZF** provides not only a setting in which one can practice Bishop-style constructive mathematics in manner very similar to ordinary mathematics, but it also has a precise justification as a constructive theory. In [2] (see also [3] and [4]), Aczel interpreted his theory in Martin-Löf type theory with W-types and one universe, a theory which is indisputably constructive, and, in this sense, **CZF** has the best possible credentials for deserving the epithet "constructive".[1]

The connection with $\Pi W$-pretoposes goes via algebraic set theory. Algebraic set theory is a flexible categorical framework for studying set theories of very different stripes. How this theory can be used to model **CZF** in $\Pi W$-pretoposes is the subject of Moerdijk and Palmgren's article [61]. This will be recapitulated in Section 2.

In Section 3, I explain how a recent model of **CZF** discovered independently by Streicher and Lubarsky falls within this framework. The model is then further investigated and shown to validate some interesting principles incompatible with either classical logic or the powerset axiom.

## 4.1 Introduction to CZF

This Section provides an introduction to Aczel's set theory **CZF**. A good reference for **CZF** is [7].

Like ordinary formal set theory, **CZF** is a first-order theory with one non-logical symbol $\epsilon$. But unlike ordinary set theory, its underlying logic is intuitionistic. To

[1] For the interpretation to work, the universe need not be closed under W-types. And one needs only one W-type, which is then used to build a universe of well-founded sets.

formulate its axioms, I will use the following abbreviations:

$$\exists x\epsilon a\,(\ldots) \;:=\; \exists x\,(x\epsilon a \wedge \ldots),$$
$$\forall x\epsilon a\,(\ldots) \;:=\; \forall x\,(x\epsilon a \rightarrow \ldots).$$

Recall that a formula is called *bounded* when all the quantifiers it contains are of one of these two forms. Finally, I write $\mathrm{B}(x\epsilon a, y\epsilon b)\,\phi$ to mean:

$$\forall x\epsilon a\,\exists y\epsilon b\,\phi \wedge \forall y\epsilon b\,\exists x\epsilon a\,\phi.$$

Its axioms are the (universal closures of) the following formulas, in which $\phi$ is arbitrary, unless otherwise stated.

**(Extensionality)** $\forall x\,(x\epsilon a \leftrightarrow x\epsilon b) \rightarrow a = b$

**(Pairing)** $\exists y\,\forall x\,(x\epsilon y \leftrightarrow (x = a \vee x = b))$

**(Union)** $\exists y\,\forall x\,(x\epsilon y \leftrightarrow \exists z\,(x\epsilon z \wedge z\epsilon a))$

**(Set Induction)** $\forall x\,(\forall y\epsilon x\,\phi(y) \rightarrow \phi(x)) \rightarrow \forall x\,\phi(x)$

**(Infinity)** $\exists a\,(\exists\, x\, x\epsilon a \wedge \forall x\epsilon a\,\exists y\epsilon a\, x\epsilon y)$

**($\Delta_0$-Separation)** $\exists y\,\forall x\,(x\epsilon y \leftrightarrow (\phi(x) \wedge x\epsilon a))$ for all *bounded* formulas $\phi$ not containing $v$ as a free variable

**(Strong Collection)** $\forall x\epsilon a\,\exists y\,\phi(x, y) \rightarrow \exists b\,\mathrm{B}(x\epsilon a, y\epsilon b)\,\phi(x, y)$

**(Subset Collection)** $\exists c\,\forall z\,(\forall x\epsilon a\,\exists y\epsilon b\,\phi(x, y, z) \rightarrow \exists d\epsilon c\,\mathrm{B}(x\epsilon a, y\epsilon d)\,\phi(x, y, z))$

Set Induction is constructive version of the Axiom of Foundation (or Regularity Axiom). Such a reformulation is in order, because the axiom as usually stated implies the Law of Excluded Middle. Strong Collection can be considered as a strengthening of the Replacement Axiom. The Subset Collection Axiom has a more palatable formulation (equivalent to it over the other axioms), called Fullness. Write $\mathbf{mv}(a, b)$ for the class of all multi-valued functions from a set $a$ to a set $b$, i.e. relations $R$ such that $\forall x\epsilon a\,\exists y\epsilon b\,(a, b)\epsilon R$ (pairs of sets can be coded by the standard trick).

**(Fullness)** $\exists z\,(z \subseteq \mathbf{mv}(a, b) \wedge \forall x\epsilon\mathbf{mv}(a, b)\,\exists c\epsilon z\,(c \subseteq x))$

Using this formulation, it is also easier to see that Subset Collection implies Exponentiation, the statement that the functions from a set $a$ to a set $b$ form a set.

In order to have a fully satisfactory theory of inductively defined sets in **CZF**, Aczel proposed to extend **CZF** with the Regular Extension Axiom.[2] A set $A$ is called *regular*,

[2] The extension is a good one in that the Regular Extension Axiom is validated by the interpretation of **CZF** in Martin-Löf type theory with W-types and one universe closed under W-types. This is a stronger type theory than the one needed for **CZF** proper, but still indisputably constructive.

when it is transitive, and for every $R \epsilon \mathbf{mv}(a, A)$, where $a \epsilon A$, there is a bounding set $b \epsilon A$ such that $\mathsf{B}(x \epsilon a, y \epsilon b)\,(x, y) \epsilon R$. The Regular Extension Axiom (**REA** for short) says:

**(REA)** $\forall x\, \exists r\, (x \epsilon r \wedge r \text{ is regular})$

For instance, this allows one to prove that, working inside **CZF**, the category of sets has W-types (see [7]). In fact:

**Theorem 4.1.1** *The category of sets and functions of* **CZF** + **REA** *is a* $\Pi W$*-pretopos.*

## 4.2 Introduction to algebraic set theory

Algebraic set theory, as introduced by Joyal and Moerdijk in their book [47], is a flexible categorical framework for studying formal set theories. The idea is that a uniform categorical approach should be applicable to set theories with very different flavours: classical or constructive, predicative or impredicative, well-founded or non-well-founded, etcetera.

The approach relies on the notion of a *small map*. In a category, whose objects and morphisms are thought of as general classes and functional relations (possibly of the size of a class) or general sets and functions, certain morphisms are singled out because their *fibres* possess a special set-theoretic property, typically that of being relatively small in some precise sense. One could think of being a set as opposed to being a proper class, finite as opposed to infinite, countable as opposed to uncountable, but also of being a small type as opposed to a type outside a particular type-theoretic universe.

The flexibility of the approach resides in the fact that the axioms for the class of small maps are not fixed once and for all: these are determined by the particular set theory or set-theoretic notion one is interested in. This is something we will actually see, because in this thesis, two different sets of axioms will be introduced. But in this Chapter the axioms for the class of small maps I will work with are those of Moerdijk and Palmgren in [61]. This choice is determined by two things: my interest in the predicative-constructive set theory **CZF** and my wish to see the category of setoids as a natural example.[3]

This Section recaps definitions and results from [61].

Let $S$ be a class of maps in an ambient category $\mathcal{E}$, which I assume to be a $\Pi W$-pretopos.

[3]For different axiom systems, see [47], [9] and other references at the "Algebraic Set Theory" website: `http://www.phil.cmu.edu/projects/ast/`. And also Chapter 6.

**Definition 4.2.1** $S$ is called *stable* if it satisfies the following axioms:

**(S1)** (Pullback stability) In a pullback square

$$\begin{array}{ccc} D & \longrightarrow & C \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ B & \underset{p}{\longrightarrow} & A \end{array} \tag{4.1}$$

$g$ belongs to $S$, whenever $f$ does.

**(S2)** (Descent) If in a pullback diagram as in (4.1), $p$ is epi, then $f$ belongs to $S$, whenever $g$ does.

**(S3)** (Sum) If two maps $f\colon B \longrightarrow A$ and $f'\colon B' \longrightarrow A'$ belong to $S$, then so does $f + f'\colon A + A' \longrightarrow B + B'$.

These axioms express that maps belong to $S$ by virtue of the properties of their fibres.

**Definition 4.2.2** A class $S$ is called a *locally full subcategory*, if it is stable and also satisfies the following axiom:

**(S4)** In a commuting triangle

$$\begin{array}{ccc} C & \xrightarrow{\;g\;} & B \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle f} & \\ & A & \end{array}$$

where $f$ belongs to $S$, $g$ belongs to $S$ if and only if $h$ does.

**Remark 4.2.3** If **(S1)** holds and all identities belong to $S$, **(S4)** is equivalent to the conjunction of the following two statements:

**(S4a)** Maps in $S$ are closed under composition.

**(S4b)** If $f\colon X \longrightarrow Y$ belongs to $S$, the diagonal $X \longrightarrow X \times_Y X$ in $\mathcal{E}/Y$ also belongs to $S$.

When thinking in terms of type constructors, this means that **(S4)** expresses that smallness is closed under dependent sums and (extensional) equality types. I will actually require the class of small maps to be closed under all type constructors, hence the next definition.

For any object $X$ in $\mathcal{E}$, I write $S_X$ for the full subcategory of $\mathcal{E}/X$ whose objects belong to $S$. An object $X$ is called *small*, when the unique map $X \longrightarrow 1$ is small.

**Definition 4.2.4** A locally full subcategory $S$ in a $\Pi W$-pretopos $\mathcal{E}$ is called a *class of small maps*, if for any object $X$ of $\mathcal{E}$, $S_X$ is a $\Pi W$-pretopos, and the inclusion functor

$$S_X \rightarrowtail \mathcal{E}/X$$

preserves the structure of a $\Pi W$-pretopos.

**Lemma 4.2.5** *(See [61], Lemma 3.4.) A locally full subcategory $S$ in a $\Pi W$-pretopos $\mathcal{E}$ is class of small maps iff it has the following five properties:*

**(F1)** $1_X \in S$ *for every object* $X$ *in* $\mathcal{E}$.

**(F2)** $0 \longrightarrow X$ *is in* $S$, *and if* $Y \longrightarrow X$ *and* $Z \longrightarrow X$ *are in* $S$ *then so is* $Y + Z \longrightarrow X$.

**(F3)** *For an exact diagram in* $\mathcal{E}/X$,

$$\begin{array}{ccccc} R & \rightrightarrows & Y & \twoheadrightarrow & Y/R \\ & \searrow & \downarrow & \swarrow & \\ & & X & & \end{array}$$

*if* $R \longrightarrow X$ *and* $Y \longrightarrow X$ *belong to* $S$ *then so does* $Y/R \longrightarrow X$.

**(F4)** *For any* $Y \longrightarrow X$ *and* $Z \longrightarrow X$ *in* $S$, *their exponent* $(Z \longrightarrow X)^{(Y \longrightarrow X)}$ *in* $\mathcal{E}/X$ *belongs to* $S$.

**(F5)** *For a commutative diagram*

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

*with all maps in* $S$, *the W-type* $W_X(f)$ *taken in* $\mathcal{E}/X$ *(which is a map in* $\mathcal{E}$ *with codomain* $X$*) belongs to* $S$.

**Definition 4.2.6** A stable class (locally full subcategory, class of small maps) $S$ is called *representable*, if there is a map $\pi\colon E \longrightarrow U$ in $S$ such that any map $f\colon B \longrightarrow A$ in $S$ fits into a double pullback diagram of the form

$$\begin{array}{ccccc} B & \longleftarrow & B' & \longrightarrow & E \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} \\ A & \underset{p}{\twoheadleftarrow} & A' & \longrightarrow & U \end{array}$$

where $p$ is epi, as indicated.

Representability formulates the existence of a weak version of a universe. The map $\pi$ in the definition of representability is often called the *universal small map*, even though it is not unique (not even up to isomorphism). In the internal logic of $\mathcal{E}$, representability means that a map $f\colon B \longrightarrow A$ belongs to $S$ iff it holds that

$$\forall a \in A \exists u \in U \colon B_a \cong E_u.$$

In particular, it means that one can talk about "being small" in the internal logic of $\mathcal{E}$.

The axioms for a class of small maps that I have given so far form the basic definition. The definition can be extended by adding various choice or collection principles. There is the *collection axiom* **(CA)** in the sense of Joyal and Moerdijk in [47]:

**(CA)** For any small map $f\colon A \longrightarrow X$ and epi $C \longrightarrow A$, there exists a quasi-pullback of the form

$$\begin{array}{ccccc} B & \longrightarrow & C & \twoheadrightarrow & A \\ {\scriptstyle g}\downarrow & & & & \downarrow{\scriptstyle f} \\ Y & & \twoheadrightarrow & & X \end{array}$$

where $Y \longrightarrow X$ is epi and $g\colon B \longrightarrow Y$ is small.

As discussed in [61], the collection axiom can be reformulated using the notion of a *collection map*. Informally, a map $g\colon D \longrightarrow C$ in $\mathcal{E}$ is a collection map, whenever it is true (in the internal logic of $\mathcal{E}$), that for any map $f\colon F \longrightarrow D_c$ covering some fibre of $g$, there is another fibre $D_{c'}$ covering $D_c$ via a map $p\colon D_{c'} \longrightarrow D_c$ which factors through $f$.

**Definition 4.2.7** A morphism $g\colon D \longrightarrow C$ in $\mathcal{E}$ is a *collection map*, when for any map $T \longrightarrow C$ and any epi $E \longrightarrow T \times_C D$ there is a diagram of the form

$$\begin{array}{ccccccccc} D & \longleftarrow & D \times_C T' & \longrightarrow & E & \twoheadrightarrow & T \times_C D & \longrightarrow & D \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ C & \longleftarrow & T' & & \twoheadrightarrow & & T & \longrightarrow & C \end{array}$$

where the middle square is a quasi-pullback with an epi on the bottom, while the two outer squares are pullbacks. A map $g\colon D \longrightarrow C$ over $A$ is a *collection map over $A$*, if it is a collection map in $\mathcal{E}/A$.

Observe that a collection map is a categorical notion, and does not refer to or depend on a class of small maps.

**Proposition 4.2.8** *(See [61], Proposition 4.5.) A map $D \longrightarrow C$ is a collection map over $C$ if, and only if, it is a choice map.*

In case the class of small maps is representable, the collection axiom is equivalent to stating that the universal small map $\pi\colon E \longrightarrow U$ is a collection map. (This is imprecise, but in a harmless way: if one universal small map is a collection map, they all are.)

In [61], Moerdijk and Palmgren work with a much stronger axiom: what they call the *axiom of multiple choice* **(AMC)**. Internally it says that for any small set $B$ there is a collection map $D \longrightarrow C$ where $D$ and $C$ are small, and $C$ is inhabited, together with a map $D \longrightarrow B$ making $D \longrightarrow B \times C$ into a surjection.

**Definition 4.2.9** A class of small maps $S$ satisfies the *axiom of multiple choice* **(AMC)**, iff for any map $B \longrightarrow A$ in $S$, there exists an epi $A' \longrightarrow A$ and a quasi-pullback of the form

$$\begin{array}{ccccc} D & \longrightarrow & & \longrightarrow & B \\ \downarrow & & & & \downarrow \\ C & \twoheadrightarrow & A' & \twoheadrightarrow & A \end{array}$$

where $D \longrightarrow C$ is a small collection map over $A'$ and $C \longrightarrow A'$ is a small epi.

**Proposition 4.2.10** *(See [61], Proposition 4.3.) The axiom of multiple choice implies the collection axiom.*

The idea of Moerdijk and Palmgren in [61] is to generalise Aczel's interpretation of **CZF** into Martin-Löf type theory with W-types and one universe, to an interpretation of **CZF** into any $\Pi W$-pretopos $\mathcal{E}$ with a representable class of small maps, where one expects to recover Aczel's syntactic construction in case $\mathcal{E}$ is $\mathcal{S}etoids$. In that light one should see the following two results:

**Theorem 4.2.11** *(See [61], Section 12.) When intensional Martin-Löf type theory is equipped with W-types and one universe, the category of setoids is equipped with a representable class of small maps satisfying* **(AMC)**.

**Theorem 4.2.12** *(See [61], Theorem 7.1.) Let $\mathcal{E}$ be a $\Pi W$-pretopos equipped with a representable class of small maps $S$ satisfying* **(AMC)**. *Then $\mathcal{E}$ contains a model of the set theory* **CZF** + **REA**.

## 4.3 A realisability model of CZF

To illustrate the framework of algebraic set theory, I will show here how the models of **CZF** obtained by Streicher in [80] and by Lubarsky in [53] fit into it. Actually, I will show that the models are the same.

Using category theory and some known results on the effective topos, it will be an easy exercise to establish the validity of a lot of constructivist principles in the model. Their collective consistency is new. Finally, I show that **CZF** is consistent with a general uniformity principle:

$$\forall x\, \exists y \epsilon a\, \phi(x, y) \to \exists y \epsilon a\, \forall x\, \phi(x, y),$$

which appears to be new.[4]

Our ambient category $\mathcal{E}$ is the effective topos RT $= \mathcal{E}ff$. Recall that a set is called *subcountable*, when it is covered by a subset of the natural numbers. Since the effective topos is a topos with nno $N$, the notion also makes sense in the internal logic of the effective topos: $Y$ is subcountable, when

$$\exists X \in \mathcal{P}N\, \exists g \colon X \longrightarrow Y \colon\ g \text{ is a surjection.}$$

Also recall that the effective topos is the exact completion of its subcategory of projectives, the partitioned assemblies, as discussed in the previous Chapter.

**Lemma 4.3.1** *The following are equivalent for a morphism $f \colon A \longrightarrow B$ in $\mathcal{E}ff$.*

1. *In the internal logic of $\mathcal{E}ff$ it is true that all fibres of $f$ are subcountable.*
2. *The morphism $f$ fits into a diagram of the following shape*

$$\begin{array}{ccccc} Y \times N & \leftarrowtail & X & \twoheadrightarrow & A \\ & \searrow & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ & & Y & \twoheadrightarrow & B, \end{array}$$

   *where the square is a quasi-pullback.*

3. *The morphism $f$ fits into a diagram of the following shape*

$$\begin{array}{ccccc} Q \times N & \leftarrowtail & P & \twoheadrightarrow & A \\ & \searrow & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ & & Q & \twoheadrightarrow & B, \end{array}$$

   *where the square is a quasi-pullback, $P$ is a $\neg\neg$-closed subobject of $Q \times N$ and $g$ is a choice map between partitioned assemblies.*

[4] The model, and my results, are obviously related to earlier work by Friedman in [29], but especially his unpublished work as reported in Myhill's paper [62]. I must confess I find it hard to get a clear picture of Friedman's work and therefore I am having difficulties in establishing its precise relation to mine. Still, I think I can safely say that the set theories studied there are weaker that **CZF** in not containing Subset Collection, there is no result on the regular extension axiom or the presentation axiom, and the relationship to subcountable morphisms in the effective topos.

**Proof.** The equivalence of 1 and 2 is a standard exercise in translating internal logic into diagrammatic language, and vice versa. That 3 implies 2 is trivial.

2 ⇒ 3: Because every object is covered by a partitioned assembly, $X$ can be covered by a partitioned assembly $Q$. Now $Q \times N$ is also a partitioned assembly, since $N$ is a partitioned assembly and partioned assemblies are closed under products. Now the subobject $Z = Q \times_Y X$ of $Q \times N$ can be covered by a $\neg\neg$-closed subobject $P$ of $Q \times N$. The idea is easy: the subobject $Z \subseteq Q \times N$ can be identified with a function $Z\colon Q \times N \longrightarrow \mathcal{P}N$ such that there is a realiser for

$$\vdash Z(q, n) \to [q] \wedge [n].$$

Then form $P = \{\, (q, n) \mid n_1 \in Z(q, n_0) \,\}$, which is a partitioned assembly with $[(q, n)] = n$, and actually a $\neg\neg$-closed subobject of $Q \times N$. $P$ covers $Z$, clearly. The diagram

$$\begin{array}{ccc} Z & \rightarrowtail & Q \times N \\ \uparrow\!\!\uparrow & \nearrow & \\ P & & \end{array}$$

does not commute, but composing with the projection $Q \times N \longrightarrow Q$ it does. (What I am basically using here is Shanin's Principle, a principle valid in the internal logic of $\mathcal{E}ff$, see [65], Proposition 1.7.) Finally, $g\colon P \subseteq Q \times N \longrightarrow Q$, as a morphism between partitioned assemblies, is a choice map. □

Let $S$ be the class of maps having any of the equivalent properties in this lemma. This class of maps was already identified by Joyal and Moerdijk in [47] and baptised "quasi-modest", but I prefer simply "subcountable". Joyal and Moerdijk prove many useful properties of these subcountable morphisms, but they are not put to any use in [47]. Here I will show that it leads to a model of **CZF**, actually the same one as contained in both [80] and [53].

First I want to prove that $S$ is a class of small maps. To do so, it will be useful to introduce the the category of bases over a partitioned assembly $X$. When $X$ is a partitioned assembly, consider the full subcategory $\mathcal{B}ase_X$ of $\mathcal{E}ff/X$ consisting of the $\neg\neg$-closed subobjects of $X \times N \longrightarrow X$. The point is that $\mathcal{B}ase_X$ has the structure of a weak $\Pi W$-pretopos, and the inclusion of $\mathcal{B}ase_X$ in $\mathcal{P}asm/X$ preserves this structure. (These are not exactly trivial, but entirely innocent generalisations of things we have seen before.)

**Lemma 4.3.2** *The inclusion* $(\mathcal{B}ase_X)_{ex} \subseteq (\mathcal{P}asm/X)_{ex} = \mathcal{E}ff/X$ *is an inclusion of* $\Pi W$*-pretoposes.*

**Proof.** I will skip numbers of uninteresting details: the inclusion is exact, by construction. That it preserves sums is easy to see. The inclusion of $\mathcal{B}ase_X$ in $\mathcal{P}asm/X$

preserves weak $\Pi$, so the inclusion $(\mathcal{B}ase_X)_{ex} \subseteq (\mathcal{P}asm/X)_{ex}$, preserves $\Pi$ by construction (of genuine $\Pi$ in the exact completion out of weak $\Pi$ in the original category). Then it also preserves polynomial functors $P_f$ and hence also W-types by yet another application of Theorem 2.1.5, because subcountables are closed under subobjects. □

**Proposition 4.3.3** *The class $S$ of subcountable maps is a class of small maps in $\mathcal{E}ff$.*

**Proof.** That $S$ is a locally full subcategory can be found in [47]. Now I use Lemma 4.2.5 to see that is a class of small maps.

That it satisfies **(F1)** and **(F2)** is trivial (and can also be found in [47]). It also satisfies **(F3)**; actually, it is easy to see that in any triangle where the top is epi

$$\begin{array}{ccc} B & \twoheadrightarrow & C \\ {\scriptstyle g}\searrow & & \swarrow{\scriptstyle f} \\ & A & \end{array}$$

and $g$ is in $S$, $f$ is also in $S$.

To check **(F4)**, assume $Y \longrightarrow X$ and $Z \longrightarrow X$ are in $S$. Both fit into quasi-pullback squares

$$\begin{array}{ccc} P & \twoheadrightarrow & Y \\ \downarrow & & \downarrow \\ Q & \twoheadrightarrow & X \end{array} \qquad \begin{array}{ccc} R & \twoheadrightarrow & Z \\ \downarrow & & \downarrow \\ S & \twoheadrightarrow & X, \end{array}$$

where $P \longrightarrow Q$ and $R \longrightarrow S$ are morphisms in $\mathcal{B}ase_R$ and $\mathcal{B}ase_S$, respectively, hence choice maps. Actually, one may assume $Q = S$ and $Q \longrightarrow X = S \longrightarrow X$. Then $(P \longrightarrow Q)^{(R \longrightarrow Q)}$ is in $(\mathcal{B}ase_Q)_{ex}$, hence in $S_Q$. But $(Y \longrightarrow X)^{(Z \longrightarrow X)}$ is a subquotient of this, hence in $S_X$.

To check **(F5)**, suppose $f$ fits into a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ \searrow & & \swarrow \\ & X & \end{array}$$

where all arrows are in $S$. Now $X$ can be covered by a partitioned assembly $Y$ via a map

$$Y \overset{p}{\twoheadrightarrow} X,$$

in such a way that in $\mathcal{E}ff/Y$, we have a quasi-pullback diagram

$$\begin{array}{ccc} B' & \twoheadrightarrow & p^*B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle p^*f} \\ A' & \twoheadrightarrow & p^*A, \end{array}$$

where $B' \longrightarrow Y$ and $A' \longrightarrow Y$ are in $\mathcal{B}ase_{X'}$. By the previous lemma, $W_g$ is in $(\mathcal{B}ase_{X'})_{ex}$, hence in $S_{X'}$. By (the proof of) Theorem 3.2.2, $W_{p^*f} \cong p^*W_f$ is a subquotient of $W_g$, hence also subcountable. Then by stability of W-types and axiom **(S2)**, $W_f$ is also subcountable. □

But the class $S$ has more properties:

**Lemma 4.3.4** *The class $S$ also has the following properties:*

**(R)** *The class $S$ is representable.*

**(F)** *All the monos belong to $S$.*

**(Q)** *In any triangle where the top is epi*

$$\begin{array}{ccc} B & \twoheadrightarrow & C \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & A & \end{array}$$

*and $g$ is in $S$, $f$ is also in $S$.*

**(AMC)** *The class $S$ satisfies AMC.*

**Proof.** Properties **(R), (F), (Q)** are all proved in [47]. That it satisfies **(AMC)** is trivial: every $f \in S$ fits into a quasi-pullback diagram

$$\begin{array}{ccc} X & \twoheadrightarrow & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \twoheadrightarrow & B, \end{array}$$

where $g\colon X \longrightarrow Y$ is a small choice map, hence a small collection map over $Y$ (see Proposition 4.2.8). □

Since $S$ is representable and also satisfies **(AMC)**, we know by Theorem 4.2.12 that the effective topos contains a model $V$ of **CZF** + **REA** based on the class of subcountable maps. In the remainder of this Chapter, I will study this model $V$. In effect, I will show that it validates the following list of principles. Since the set of natural numbers $\omega$ is definable in **CZF**, I will freely use this symbol when formulating these principles. I will also use 0 and the successor operation $s$.

**Theorem 4.3.5** *The following principles are valid in the model $V$:*

**(Full Separation)** $\exists y\, \forall x\, (x \epsilon y \leftrightarrow (\phi(x) \wedge x \epsilon a))$ *for* all *formulas $\phi$ not containing $v$ as a free variable.*

**(All sets subcountable)** *All sets are subcountable.*

**(Non-existence of $\mathcal{P}\omega$)** *The powerset of the set of natural numbers does not exist.*

**(Axiom of Countable Choice)** $\forall i \epsilon \omega\, \exists x\, \psi(i, x) \rightarrow \exists a, f : \omega \longrightarrow a\, \forall i \epsilon \omega\, \psi(i, f(i)).$

**(Axiom of Relativised Dependent Choice)** $\phi(x_0) \wedge \forall x\, (\phi(x) \rightarrow \exists y\, (\psi(x, y) \wedge \phi(y))) \rightarrow \exists a\, \exists f : \omega \longrightarrow a\, (f(0) = x_0 \wedge \forall i \in \omega\, \phi(f(i), f(si))).$

**(Presentation Axiom)** *Every set is the surjective image of a base (see below).*

**(Markov's Principle)** $\forall n \epsilon \omega\, [\phi(n) \vee \neg\phi(n)] \rightarrow [\neg\neg\exists n \in \omega\, \phi(n) \rightarrow \exists n \epsilon \omega\, \phi(n)].$

**(Independence of Premisses)** $(\neg\theta \rightarrow \exists x\, \psi) \rightarrow \exists x\, (\neg\theta \rightarrow \psi).$

**(Church's Thesis)** $\forall n \epsilon \omega\, \exists m \epsilon \omega\, \phi(n, m) \rightarrow \exists e \epsilon \omega\, \forall n \epsilon \omega\, \exists m, p \epsilon \omega\, [T(e, n, p) \wedge U(p, m) \wedge \phi(n, m)]$ *for every formula $\phi(u, v)$, where $T$ and $U$ are the set-theoretic predicates which numeralwise represent, respectively, Kleene's $T$ and result-extraction predicate $U$.*

**(Uniformity Principle)** $\forall x\, \exists y \epsilon a\, \phi(x, y) \rightarrow \exists y \epsilon a\, \forall x\, \phi(x, y).$

**(Unzerlegbarkeit)** $\forall x\, (\phi(x) \vee \neg\phi(x)) \rightarrow \forall x\, \phi \vee \forall x\, \neg\phi.$

Most of these principles also hold in the realisability models of Rathjen [70], except for the subcountability of all sets, and the general Uniformity Principle. In order to show all of this, I need to give a concrete description. In our case that is somewhat easier than in [61], since the axiom **(Q)** is valid here.

On $\mathcal{E}ff$, one can define the *powerclass functor* $\mathcal{P}_s$. The idea is that $\mathcal{P}_s(X)$ is the set of all subcountable subsets of $X$. This one can easily construct in terms of the universal small map $\pi : E \longrightarrow U$:

$$\mathcal{P}_s(X) = \{ R \in \mathcal{P}X \mid \exists u \in U : R \cong E_u \}.$$

$\mathcal{P}_s$ is obviously a subfunctor of the powerobject functor $\mathcal{P}$ (which exists in any topos), and inherits an elementhood relation $\in_X \subseteq \mathcal{P}_s(X) \times X$ from $\mathcal{P}$.

The model for **CZF** is the initial algebra for the functor $\mathcal{P}_s$, which happens to exist. This means that it is a fixpoint $V$ and there are mutually inverse mappings $I : \mathcal{P}_s(V) \longrightarrow V$ and $E : V \longrightarrow \mathcal{P}_s(V)$. The internal elementhood relation $\epsilon$ on $V$ is defined in terms of $\in$ as follows:

$$x \epsilon y \Leftrightarrow x \in E(y).$$

One can see that the model $V$ exists by slightly modifying the work of Moerdijk and Palmgren in [61]. Call a map $\pi\colon E \longrightarrow U$ a *weak representation* for a class of small maps $S$, when a morphism belongs to $S$, if and only if, there is a diagram of the following form:

$$\begin{array}{ccccc} B & \longleftarrow & B' & \longrightarrow & E \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} \\ A & \underset{p}{\twoheadleftarrow} & A' & \longrightarrow & U \end{array}$$

where the left square is a quasi-pullback, and the right square is a genuine pullback. This expresses that every small map is locally a quotient of $\pi$. Moerdijk and Palmgren show how the initial $\mathcal{P}_s$-algebra can be constructed from $\pi$.

Our class of small maps has a weak representation of a relatively easy form:

$$\begin{array}{ccc} \in_N & \rightarrowtail & \in_N \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ \mathcal{P}_{\neg\neg}(N) & \rightarrowtail & \mathcal{P}(N). \end{array}$$

Therefore $\pi$ is a morphism between assemblies, where $\mathcal{P}_{\neg\neg}(N) = \nabla\mathcal{P}\mathbb{N}$, i.e. the set of all subsets $A$ of the natural numbers, where $A$ is realised by any natural number, and $\in_N = \{(n, A) \mid n \in A\}$, where $(n, A)$ is realised simply by $n$.

According to Moerdijk and Palmgren, the initial $\mathcal{P}_s$-algebra can be constructed by first taking the W-type associated to $\pi$ and then dividing out, internally, by bisimulation:

$$\mathrm{sup}_A(t) \sim \mathrm{sup}_{A'}(t') \quad \Leftrightarrow \quad \forall a \in A\, \exists a' \in A'\colon ta \sim t'a' \text{ and } \forall a' \in A'\, \exists a \in A\colon ta \sim t'a'.$$

The W-type associated to $\pi$ can be calculated in the category of assemblies, and is the following. The underlying set consists of well-founded trees where the edges are labelled by natural numbers, in such a way that the edges into a fixed node are labelled by distinct natural numbers. The decorations (realisers) of such trees $\mathrm{sup}_A(t)$ are those $n \in \mathbb{N}$ such that $n \cdot a \downarrow$ for all $a \in A$ and $n \cdot a$ is a decoration of $t(a)$.

Now I have to translate the bisimulation relation in terms of realisers. When using the abbreviation:

$$m \vdash x \,\epsilon\, \mathrm{sup}_A(t) \quad \Leftrightarrow \quad j_0 m \in A \text{ and } j_1 m \vdash x \sim t(j_0 m),$$

it becomes:

$$\begin{aligned} n \vdash \mathrm{sup}_A(t) \sim \mathrm{sup}_{A'}(t') \quad \Leftrightarrow \quad & \forall a \in A\colon j_0 n \cdot a \downarrow \text{ and } j_0 n \cdot a \vdash ta \,\epsilon\, \mathrm{sup}_{A'}(t') \text{ and} \\ & \forall a' \in A'\colon j_1 n \cdot a' \downarrow \text{ and } j_1 n \cdot a' \vdash t'a' \,\epsilon\, \mathrm{sup}_A(t). \end{aligned}$$

Using the Recursion Theorem, it is not hard to see that this defines a subobject $\sim$ of $W_\pi \times W_\pi$, in fact, an equivalence relation on $W_\pi$. The quotient in $\mathcal{E}ff$ is $V$, which is therefore $W_\pi$, with $\sim$ as equality.

Using the description of $\mathcal{P}_s$ as a quotient of $P_\pi$ in [47], one can see that:

$$\mathcal{P}_s(X, =) = \{(A \subseteq \mathbb{N}, t \colon A \longrightarrow X)\},$$

where $n \vdash (A, t) = (A', t')$, when $n$ realises the statement that $t$ and $t'$ they have the same image, i.e.:

$$\forall a \in A \, \exists a' \in A' \colon \, ta = t'a' \text{ and } \forall a' \in A' \, \exists a \in A \colon \, ta = t'a'.$$

$I$ and $E$ map $(A, t)$ to $\mathrm{sup}_A(t)$ and vice versa, whereas the internal elementhood relation is defined by:

$$m \vdash x \,\epsilon\, \mathrm{sup}_A(t) \quad \Leftrightarrow \quad j_0 m \in A \text{ and } j_1 m \vdash x = t(j_0 m),$$

which was not just an abbreviation.

**Proposition 4.3.6** *As an object of the effective topos, $V$ is uniform, i.e. there is a natural number $n$ such that:*

$$n \vdash x = x$$

*for all $x \in V$.*

**Proof.** It is clear that $W_\pi$ is uniform (a solution for $f = \lambda n.f$ decorates every tree), and $V$, as its quotient, is therefore also uniform. □

**Corollary 4.3.7** *The following clauses recursively define what it means that a certain statement is realised by a natural number $n$ in the model $V$:*

$$\begin{aligned}
n \vdash x \,\epsilon\, sup_A(t) &\quad \Leftrightarrow \quad j_0 n \in A \text{ and } j_1 n \vdash x = t(j_0 n). \\
n \vdash sup_A(t) = sup_{A'}(t') &\quad \Leftrightarrow \quad \forall a \in A \colon j_0 n \cdot a \downarrow \text{ and } j_0 n \cdot a \vdash ta \,\epsilon\, sup_{A'}(t')) \text{ and} \\
&\qquad\quad\; \forall a' \in A' \colon j_0 n \cdot a' \downarrow \text{ and } j_1 n \cdot a' \vdash t'a' \,\epsilon\, sup_A(t). \\
n \vdash \phi \wedge \psi &\quad \Leftrightarrow \quad j_0 n \vdash \phi \text{ and } j_1 n \vdash \psi. \\
n \vdash \phi \vee \psi &\quad \Leftrightarrow \quad n = \langle 0, m \rangle \text{ and } m \vdash \phi, \text{ or } n = \langle 1, m \rangle \text{ and } m \vdash \psi. \\
n \vdash \phi \rightarrow \psi &\quad \Leftrightarrow \quad \text{For all } m \vdash \phi, n \cdot m \downarrow \text{ and } n \cdot m \vdash \psi. \\
n \vdash \neg \phi &\quad \Leftrightarrow \quad \text{There is no } m \text{ such that } m \vdash \phi. \\
n \vdash \exists x \, \phi(x) &\quad \Leftrightarrow \quad n \vdash \phi(a) \text{ for some } a \in V. \\
n \vdash \forall x \, \phi(x) &\quad \Leftrightarrow \quad n \vdash \phi(a) \text{ for all } a \in V.
\end{aligned}$$

Therefore the model is the same as the one introduced by Lubarsky in [53]. One could use these clauses to verify that all the principles that are listed in Theorem 4.3.5 are valid, but that is not what I recommend. Instead, it is easier to use that $V$ is fixpoint for $\mathcal{P}_s$, together with properties of the class of subcountable maps $S$ and of the effective topos.

**Proof of Theorem 4.3.5.**

**(Full Separation)** The model $V$ satisfies full separation, because all monos belong to $S$. In more detail, assume $w \in V$ and $\phi(x)$ is a set-theoretic property. $W = E(w)$ is a small subset of $V$, and since monos are small, so is $V = \{x \in W \mid \phi(x)\}$. Then take $v = I(V)$.

**(All sets subcountable)** Before we check the principle that all sets are subcountable in $V$, let us first see how the natural numbers are interpreted in $V$. The empty set $\emptyset$ is interpreted by $I(0)$, where $0 \subseteq V$ is the least subobject of $V$, which is small. $s(x) = I(x \cup \{x\})$ defines an operation on $V$, therefore there is a mapping $i \colon N \longrightarrow V$. This is actually an inclusion, and its image is small (because $N$ is). So if one writes $\omega = I(N)$, then this interprets the natural numbers.

If $x$ is an arbitrary element in $V$, $E(x)$ is small, so (internally in $\mathcal{E}ff$) fits into a diagram like this:

$$\begin{array}{ccc} A & \rightarrowtail & N \\ {\scriptstyle q}\downarrow & & \\ E(x). & & \end{array}$$

One embeds the graph of $q$ in $V$, by defining a morphism $T \colon A \longrightarrow V$, as follows:

$$T(a) = (i(a), q(a)) \in V,$$

where I implicitly use the standard coding of pairs of sets. Since $A$ is small, $T$ can also be considered as an element of $\mathcal{P}_s(V)$. Now $t = I(T)$ is inside $V$ a function that maps a subset of the natural numbers to $x$.

**(Non-existence of $\mathcal{P}\omega$)** The principle that all sets are subcountable immediately implies the non-existence of $\mathcal{P}\omega$, using Cantor's Diagonal Argument.

**(Axiom of Countable Choice), (Axiom of Relativised Dependent Choice)** The Principle of Relativised Dependent Choice $V$ inherits from the effective topos $\mathcal{E}ff$.

**(Presentation Axiom)** Recall that a set $b$ in **CZF** is called a *base*, when every surjection $q \colon x \longrightarrow b$ has a section. To see that every set is the surjective image of a base, notice that in $V$ every set is the surjective image of a $\neg\neg$-closed subset of $\omega$, and these are internally projective in $\mathcal{E}ff$.

**(Markov's Principle), (Independence of Premisses)** These hold in $V$, because these principles are valid in $\mathcal{E}ff$.

**(Church's Thesis)** This is a bit harder: see below for an argument.

**(Uniformity Principle), (Unzerlegbarkeit)** To see that the uniformity principle holds, observe that a realiser for a statement of the form $\forall x\, \exists y \epsilon a\, (\ldots)$ specifies an $y \epsilon a$ that works uniformly for all $x$. Unzerlegbarkeit follows from the uniformity principle, using $a = \{0, 1\}$.

□

**Remark 4.3.8** It may be good to point out that not only does $\mathcal{P}\omega$ not exist in the model, neither does $\mathcal{P}x$ when $x$ consists of only one element, say $x = \{\emptyset\}$. For if it would, so would $(\mathcal{P}x)^{\omega}$, by Subset Collection. But it is not hard to see that $(\mathcal{P}x)^{\omega}$ can be reworked into the powerset of $\omega$.

**Relationship with work of Streicher 4.3.9** In [80], Streicher builds a model of **CZF** which in my terms can be understood as follows. He starts from a well-known map $\rho: E \longrightarrow U$ in the category $\mathcal{A}sm$ of assemblies. Here $U$ is the set of all modest sets, with a modest set $u$ realised by any natural number, and a fibre $E_u$ in assemblies being precisely the modest set $u$. He proceeds to build the W-type associated to $\rho$, takes it as a universe of sets, and then interprets equality as bisimulation. One cannot literally quotient by bisimulation, for which one could pass to the effective topos.

When considering $\rho$ as a morphism in the effective topos, it is not hard to see that it is in fact a "weak representation" for the class of subcountable morphisms $S$: for all fibres of "my" weak representation $\pi$ also occur as fibres of $\rho$, and all fibres of $\rho$ are quotients of fibres of $\pi$. Therefore the model is again the initial $\mathcal{P}_s$-algebra for the class of subcountable morphisms $S$ in the effective topos, by the work of Moerdijk and Palmgren.

**Relationship with work of McCarty 4.3.10** In his PhD thesis [58], McCarty introduced a realisability model $U$ for the constructive, but impredicative set theory **IZF**. $U$ is very similar to the model $V$ I have been investigating, but its exact relation is not immediately obvious. In [48], the authors Kouwenhoven-Gentil and Van Oosten show how also McCarty's model $U$ is the initial $\mathcal{P}_t$-algebra for a class of small maps $T$ in the effective topos. As $S \subseteq T$, and hence $\mathcal{P}_s \subseteq \mathcal{P}_t$, $U$ is also a $\mathcal{P}_s$-algebra, so it is clear that $V$ embeds into $U$. Actually, $V$ consists of those $x \in U$ that $U$ believes to be hereditarily subcountable.

To see this, write

$$A = \{x \in U \,|\, U \models x \text{ is hereditarily subcountable}\}.$$

$A$ is a $\mathcal{P}_s$-subalgebra of $U$, and it will be isomorphic to $V$, once one proves that is initial. It is obviously a fixpoint, so it suffices to show that it is well-founded (see [48]). So let $B \subseteq A$ be a $\mathcal{P}_s$-subalgebra of $A$, and define

$$W = \{x \in U \,|\, x \in A \Rightarrow x \in B\}.$$

It is not hard to see that this is a $\mathcal{P}_t$-subalgebra of $U$, so $W = U$ and $A = B$.

This also shows concerning Church's Thesis, that, as it is valid in McCarty's model $U$ and it concerns only sets that also exist in $V$, it is also valid in $V$. The same applies to what is called Extended Church's Thesis.