B

Derivation of the external magnetic field of a homogeneously magnetized cylinder

Here, we derive the external magnetic field of a homogeneously magnetized cylinder of infinite length. The cylinder is magnetized due to an constant magnetic field perpendicular to it. The applied field is high, so remanent magnetization can be neglected, and no currents are present. Then, the internal and external magnetic fields, $\mathbf{H}$, and inductions, $\mathbf{B}$, are found by solving the Laplace equation of the magnetic scalar potential ϕ_m [1].

$$\nabla^2 \phi_m = 0 \tag{B.1}$$

with:

$$\mathbf{H} = -\nabla \phi_m \tag{B.2}$$

and:

$$\mathbf{B} = \mu_0(-\nabla \phi_m + \mathbf{M}) \tag{B.3}$$

with μ_0 the permeability of free space. Because the cylinder is perpendicular to the applied field, the potential is independent of the coordinate y, with the $\hat{y}$-direction in the length direction of the cylinder. The general solutions of the Laplace equation are the so-called cylindrical harmonics:

$$\phi_m = A + B\ln(r) + \sum_{n=0}^{\infty}\{C_n r^n \cos(n\phi) + D_n r^{-n}\cos(n\phi) + E_n r^n \sin(n\phi) + F_n r^{-n}\sin(n\phi)\} \tag{B.4}$$

Now, the constant A, B, C, D, E, F have to be found for the regions inside (ϕ_{m1}) and outside (ϕ_{m2}) the cylinder. This is done by applying several boundary conditions.

1. The far field is parallel to $\cos(\phi)\hat{r}$ and is not influenced by the disturbing cylinder, which means:

$$\lim_{r\to\infty} \mathbf{H}_2 = \frac{B_0}{\mu_0\mu_2}\cos(\phi)\hat{r} \tag{B.5}$$

2. The cylinder is homogeneously magnetized, which means:

$$\mathbf{M}_1 = M_1(H_1)\cos(\phi)\hat{r} \tag{B.6}$$

3. The tangential component of **H** is continuous across any surface:

$$H_{1\phi} = H_{2\phi} \tag{B.7}$$

with $H_{i\phi} = \mathbf{H}_i \cdot \hat{\phi}$.

4. The normal component of **B** is continuous across the surface of the cylinder:

$$B_{1r} = B_{2r} \tag{B.8}$$

with $B_{ir} = \mathbf{B}_i \cdot \hat{r}$.

Because the magnetic potential is uniform and cannot get infinite at any point and by applying boundary condition 1, the external potential is written as:

$$\phi_{m2} = \sum_{n=0}^{\infty} \{D_{2n} r^{-n} \cos(n\phi) + F_{2n} r^{-n} \sin(n\phi)\} - \frac{B_0}{\mu_0 \mu_2} r \cos(\phi) \tag{B.9}$$

The internal potential is only restricted by its uniformity:

$$\phi_{m1} = \sum_{n=0}^{\infty} \{C_{1n} r^{n} \cos(n\phi) + E_{1n} r^{n} \sin(n\phi)\} \tag{B.10}$$

Now, the boundary conditions 2, 3 and 4 are applied at the cylinder surface $r = a$. Only the $n = 1$ components have to be taken into account, because of the $\cos(\phi)$ term. Boundary condition 3 becomes:

$$\frac{\delta \phi_{m1}}{\delta \phi}|_{r=a} = \frac{\delta \phi_{m2}}{\delta \phi}|_{r=a} \tag{B.11}$$

or:

$$-C_{11} a \sin(\phi) + E_{11} a \cos(\phi) = -D_{21} a^{-1} \sin(\phi) + F_{21} a^{-1} \cos(\phi) + \frac{B_0}{\mu_0 \mu_2} a \sin(\phi) \tag{B.12}$$

Boundary condition 4 becomes:

$$\mu_0 (-\frac{\delta \phi_{m1}}{\delta r}|_{r=a} + M_1 \cos(\phi)) = -\mu_0 \mu_2 \frac{\delta \phi_{m2}}{\delta r}|_{r=a} \tag{B.13}$$

or:

$$\mu_0 (-C_{11} \cos(\phi) - E_{11} \sin(\phi) + M_1 \cos(\phi)) = \mu_o \mu_2 (D_{21} a^{-2} \cos\phi - F_{21} a^{-2} \sin(\phi)) + B_0 \cos(\phi) \tag{B.14}$$

The coefficients are found to be:

$$C_{11} = \frac{M_1}{1+\mu_2} - \frac{2B_0}{\mu_0 (1+\mu_2)} \tag{B.15}$$

$$D_{21} = \frac{M_1 a^2}{1+\mu_2} + \frac{B_0 (1-\mu_2) a^2}{\mu_0 (1+\mu_2) \mu_2} \tag{B.16}$$

$$E_{21} = 0 \tag{B.17}$$

$$F_{21} = 0 \tag{B.18}$$

The magnetic potential equations become:

$$\phi_{m1} = \frac{\mu_0 M_1 - 2B_0}{\mu_0(1+\mu_2)} r\cos(\phi) \tag{B.19}$$

$$\phi_{m2} = \frac{\mu_0\mu_2 M_1 + B_0(1-\mu_2)}{\mu_0\mu_2(1+\mu_2)} \frac{\cos(\phi)a^2}{r} - \frac{B_0}{\mu_0\mu_2} r\cos(\phi) \tag{B.20}$$

Now, a change it made to cartesian coordinates, with $z = r\cos(\phi)$ and $x = r\sin(\phi)$:

$$\phi_{m1} = \frac{\mu_0 M_1 - 2B_0}{\mu_0(1+\mu_2)} z \tag{B.21}$$

$$\phi_{m2} = \frac{\mu_0\mu_2 M_1 + B_0(1-\mu_2)}{\mu_0\mu_2(1+\mu_2)} \frac{a^2}{x^2+z^2} z - \frac{B_0}{\mu_0\mu_2} z \tag{B.22}$$

The magnetic field equations become:

$$\mathbf{H}_1 = -\frac{\delta\phi_{m1}}{\delta z}\hat{z} = -\frac{\mu_0 M_1 - 2B_0}{\mu_0(1+\mu_2)}\hat{z} \tag{B.23}$$

$$\begin{aligned}\mathbf{H}_2 &= -\frac{\delta\phi_{m2}}{\delta x}\hat{x} - \frac{\delta\phi_{m2}}{\delta z}\hat{z} \\ &= \frac{\mu_0\mu_2 M_1 + B_0(1-\mu_2)}{\mu_0\mu_2(1+\mu_2)} a^2\left(\frac{z^2-x^2}{(x^2+z^2)^2}\hat{z} + \frac{2xz}{(x^2+z^2)^2}\hat{x}\right) + \frac{B_0}{\mu_0\mu_2}\hat{z}\end{aligned} \tag{B.24}$$

and the magnetic induction:

$$\mathbf{B}_1 = -\mu_0\frac{\delta\phi_{m1}}{\delta z}\hat{z} + \mu_0 M_1\hat{z} = -\frac{\mu_0\mu_2 M_1 - 2B_0}{1+\mu_2}\hat{z} \tag{B.25}$$

$$\begin{aligned}\mathbf{B}_2 &= \mu_0\mu_2\left(-\frac{\delta\phi_{m2}}{\delta x}\hat{x} - \frac{\delta\phi_{m2}}{\delta z}\hat{z}\right) \\ &\frac{\mu_0\mu_2 M_1 + B_0(1-\mu_2)}{1+\mu_2} a^2\left(\frac{z^2-x^2}{(x^2+z^2)^2}\hat{z} + \frac{2xz}{(x^2+z^2)^2}\hat{x}\right) + B_0\hat{z}\end{aligned} \tag{B.26}$$

In the MR-scanner, the applied field around the iso-center is constant. During scanning, only very small, negligible variations around the main field are present. In that case, we may write M_1 as proportional to B_0 using the dimensionless, field dependent parameter $\xi(B_0)$:

$$M_1 = \frac{\xi(B_0)B_0}{\mu_0\mu_2} \tag{B.27}$$

Now, equation B.26 can be written as:

$$\mathbf{B}_2 = \frac{\xi - \chi_2}{\pi(2+\chi_2)} B_0 A \left(\frac{z^2-x^2}{(x^2+z^2)^2}\hat{z} + \frac{2xz}{(x^2+z^2)^2}\hat{x}\right) + B_0\hat{z} \tag{B.28}$$

with $\chi_2 = \mu_2 - 1$ the magnetic susceptibility of the environment and A the cross-sectional area of the cylinder. For $\chi_2 << 1$, it becomes:

$$\mathbf{B}_2 \approx \frac{\xi - \chi_2}{2\pi} B_0 A \left(\frac{z^2 - x^2}{(x^2 + z^2)^2} \hat{z} + \frac{2xz}{(x^2 + z^2)^2} \hat{x} \right) + B_0 \hat{z} \tag{B.29}$$

In MRI small frequency differences around the Larmor frequency are analyzed. The frequency differences depend on the field variations induced by the disturbing cylinder $\Delta f = 2\pi\gamma\Delta B$, with $\Delta B = |\mathbf{B}_2| - |\mathbf{B}_0|$ and γ the gyromagnetic ratio.

$$|\mathbf{B}_2| = B_0 \left(\left(1 + \tfrac{(\xi-\chi_2)A}{2\pi} \tfrac{z^2-x^2}{(x^2+z^2)^2} \right)^2 + \left(\tfrac{(\xi-\chi_2)A}{2\pi} \tfrac{2xz}{(x^2+z^2)^2} \right)^2 \right)^{\frac{1}{2}} \tag{B.30}$$

$$= B_0 \left(1 + \tfrac{(\xi-\chi_2)A}{\pi} \tfrac{z^2-x^2}{(x^2+z^2)^2} + (\tfrac{(\xi-\chi_2)A}{2\pi})^2 \tfrac{1}{(x^2+z^2)^2} \right)^{\frac{1}{2}} \tag{B.31}$$

$$\simeq B_0 \left(1 + \tfrac{(\xi-\chi_2)A}{\pi} \tfrac{z^2-x^2}{(x^2+z^2)^2} \right)^{\frac{1}{2}} \tag{B.32}$$

$$\simeq B_0 \left(1 + \tfrac{(\xi-\chi_2)A}{2\pi} \tfrac{z^2-x^2}{(x^2+z^2)^2} \right) \tag{B.33}$$

Above, at the first approximation, it is assumed that the cross-sectional area of the cylinder is small compared to the field-of-view of the imaging slice, which implies that frequency differences close to the cylinder have negligible effect on the total distortion. At the second, a Taylor-expansion of $\sqrt{1+u} \approx 1 + \frac{u}{2} - \frac{u^2}{8}$... is used. ΔB becomes:

$$\Delta B(x, z) = B_0 \frac{(\xi - \chi_2)A}{2\pi} \frac{z^2 - x^2}{(x^2 + z^2)^2} \tag{B.34}$$

References

1. Reitz JR, Milford FJ, Christy RW. Foundations of Electromagnetic Theory. Reading, MA USA: Addison-Wesley Publishing Company, 4th edition, 1993.