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Aspects of Quark Confinement

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Abstract

Permanent quark confinement is one of the various possible phenomena in a non-Abelian gauge field theory. Here we consider the theory in a box with periodic boundary conditions. The edges of the box are taken to infinity in the end. Various kinds of twists can be included in the boundary conditions, corresponding to the amount of electric or magnetic flux trapped in the box. An exact duality relation gives restrictions on the dynamical properties of these fluxes. Instantons further affect this dynamics. We speculate on their role in spontaneous chiral symmetry breakdown.

1. Introduction

It has long been speculated that topological features of non-Abelian gauge theories are responsible for the permanent confinement of quarks inside hadrons. On the other hand there exist different models with non-Abelian gauge fields which do not seem to confine their fundamental fermions, for example the Georgi–Glashow model and the Weinberg–Salam model. If we consider a space–time bounded by the walls of a box then the question whether or not such a thing as confinement takes place is a question about the limit for the box becoming infinitely large. In statistical physics this is called the thermodynamic limit. What we want to show is that in this limit phase transitions might occur, from one into another mode. One of those modes is a confinement mode, another is a "Higgs mode", a third is a "Coulomb" or "Georgi–Glashow mode".

Crucial for understanding this situation is the consideration of electric or magnetic flux. This flux is most easily defined if we give the box periodic boundary conditions. Twists in these boundary conditions, not unlike the twists of a Moebius strip, allow one to define magnetic and electric flux, trapped in the various principle directions in the box.

Another type of topological twist gives rise to the well-known instanton field configurations. Although these are twists in a four-dimensional space–time, they do interfere in a non-trivial way with the magnetic and electric twists. A consequence is that the string tension parameter cannot be a smooth non-vanishing function of the instanton angle θ. We will explain how this comes about, and how our observation might be related to spontaneous chiral symmetry breaking.

The first part of this lecture coincides with lectures given elsewhere [1]. Here we elaborate more on the effects due to instantons.

2. The periodic box

We give space–time the topology $S_1 \otimes S_1 \otimes S_1 \otimes \mathbb{R}$, where $\mathbb{R}$ stands for the time axis. Time runs from $-\infty$ to ∞. Later $\mathbb{R}$ is also replaced by S_1 in the Euclidan direction, Inside the box, space–time is flat. These are no quarks yet (at best they are external sources, to be or not to be bound together by string-like configurations).

Now in the continuum theory the gauge fields themselves are representations of SU(N)/Z(N), where Z(N) is the center of the group SU(N):

$$Z(N) = \{e^{2\pi i n/N} I; \quad n = 0, \dots, N-1\} \tag{2.1}$$

This is because any gauge transformation of the type (2.1) leaves $A_\mu(x)$ invariant. A consequence of this is the existence of another class of topological quantum numbers in this box besides the familiar Pontryagin number. Consider the most general possible periodic boundary condition for $A_\mu(x)$ in the box. Take first a plane $\{x_1, x_2\}$ in the 1, 2 direction with fixed values of x_3 and x_4. One may have

$$\begin{aligned} A_\mu(a_1, x_2) &= \Omega_1(x_2) A_\mu(0, x_2) \\ A_\mu(x_1, a_2) &= \Omega_2(x_1) A_\mu(x_1, 0) \end{aligned} \tag{2.2}$$

Here, a_1, a_2 are the periods.

ΩA_μ is short for

$$\Omega A_\mu \Omega^{-1} + \frac{1}{gi} \Omega \partial_\mu \Omega^{-1} \tag{2.3}$$

The periodicity conditions for $\Omega_{1,2}(x)$ follow by considering eq. (2.2) at the corners of the box:

$$\Omega_1(a_2)\Omega_2(0) = \Omega_2(a_1)\Omega_1(0) Z \tag{2.4}$$

where Z is some element of Z(N).

One may now perform continuous gauge transformations on $A_\mu(x)$,

$$A_\mu(x_1, x_2) \to \Omega(x_1, x_2) A_\mu(x_1, x_2), \tag{2.5}$$

where $\Omega(x_1, x_2)$ (non-periodic) can be arranged either such that $\Omega_2(x_1) \to I$ or such that $\Omega_2(x_1) \to I$, but not both, because Z in eq. (2.4) remains invariant under eq. (2.5) as one can easily verify. We call this element $Z(1, 2)$ because the 12 plane was chosen. By continuity $Z(1, 2)$ cannot depend on x_3 or x_4. For each ($\mu\nu$) direction such a Z element exist, to be labeled by integers

$$n_{\mu\nu} = -n_{\nu\mu}, \tag{2.6}$$

definded modulo N. Clearly this gives

$$N^{d(d-1)/2} = N^6 \tag{2.7}$$

topological classes of gauge field configurations. Note that these classes disappear if a field in the fundamental representation of SU(N) is added to the system (these fields would make unacceptable jumps at the boundary). Indeed, to understand quark confinement it is necessary to understand pure gauge systems without quarks first.

As we shall see, the new topological classes will imply the existence of new vacuum parameters besides the well-known instanton [2] angle θ. The latter still exists in our box, and will be associated with a topological quantum number ν, an arbitrary integer.

3. Order and disorder loop integrals

To elucidate the physical significance of the topological numbers $n_{\mu\nu}$ we first concentrate on gauge field theory in a three dimensional periodic box with time running from $-\infty$ to ∞. To be specific we will choose the temporal gauge,

$$A_4 = 0 \tag{3.1}$$

(this is the gauge in which rotation towards Euclidean space is particularly elegant). Space has the topology $(S_1)^3$. There is an infinite set of homotopy classes of closed oriented curves C in this space: C may wind any number of times in each of the three principal directions. For each curve C at each time t there is a quantum mechanical operator $A(C, t)$ defined by

$$A(C,t) = \operatorname{Tr} P \exp \int_C ig\mathbf{A}(\mathbf{x},t)\cdot d\mathbf{x}, \tag{3.2}$$

called Wilson loop or order parameter. Here P stands for path ordering of the factors $\mathbf{A}(\mathbf{x}, t)$ when the exponents are expanded. The ordering is done with respect to the matrix indices. The $\mathbf{A}(\mathbf{x}, t)$ are also operators in Hilbert space, but for different $\mathbf{x}$, same t, all $A(\mathbf{x}, t)$ commute with each other. By analogy with ordinary electromagnetism we say that $A(C)$ *measures* magnetic flux *through* C, and in the same time *creates* an electric flux line *along* C. Since $A(C)$ is gauge-invariant under purely periodic gauge transformations, our versions of magnetic and electric flux are gauge-invariant. Therefore they are not directly linked to the gauge *co*variant curl $G^a_{\mu\nu}(\mathbf{x})$.

There exists a dual analogon of $A(C)$ which will be called $B(C)$ or disorder loop operator [3]. C is again a closed oriented curve in $(S_1)^3$. A simple definition of $B(C)$ could be made by postulating its equal-time commutation rules with $A(C)$:

$$[A(C), A(C')] = 0$$
$$[B(C), B(C')] = 0;$$
$$A(C)B(C') = B(C')A(C)\exp 2\pi i n/N, \tag{3.3}$$

where n is the number of times C' winds around C in a certain direction. Note that n is only well defined if either C or C' is in the trivial homotopy class (that is, can be shrunk to a point by continuous deformations). Therefore, if C' is in a nontrivial class we must choose C to be in a trivial class. Since these commutation rules (3.3) determine $B(C)$ only up to factors that commute with A and B, we could make further requirements, for instance that $B(C)$ be a unitary operator.

An explicit definition of $B(C)$ can be given as follows. First we go to the temporal gauge, $A_0 = 0$. We then must distinguish a "large Hilbert space" $\mathscr{H}$ of all field configurations $A(x)$ from a "physical Hilbert space" $H \subset \mathscr{H}$. This H is defined to be the subspace of $\mathscr{H}$ of all gauge invariant states:

$$H = \{|\psi\rangle; \langle \mathbf{A}(\mathbf{x})|\psi\rangle = \langle \Omega\mathbf{A}(\mathbf{x})|\psi\rangle\} \tag{3.4}$$

where Ω is any infinitesimal gauge transformation in 3 dim. space. Often we will also write Ω for the corresponding rotation in $\mathscr{H}$:

$$H = \{|\psi\rangle; \Omega|\psi\rangle = |\psi\rangle, \Omega \text{ infinitesimal}\}. \tag{3.5}$$

Now consider a pseudo-gauge transformation $\Omega^{[C']}$ defined to be a genuine gauge transformation at all points $\mathbf{x} \notin C'$, but singular on C'. For any closed path $x(\theta)$ with $0 < \theta < 2\pi$ twisting n times around C' we require

$$\Omega^{[C']}(x(2\pi)) = \Omega^{[C']}(x(0))\, e^{2\pi i n/N} \tag{3.6}$$

This discontinuity is not felt by the fields $A(\mathbf{x}, t)$ which are invariant under $Z(N)$. They do feel the singularity at C' however. We define $B(C')$ as $\Omega^{[C']}$ but with the singularity at C' smoothened; this corresponds to some form of regularization, and implies that the operator differs from an ordinary gauge transformation. Therefore, even for $|\psi\rangle \in H$ we have

$$B(C')|\psi\rangle \neq |\psi\rangle \tag{3.7}$$

For any regular gauge transformation Ω we have an Ω' such that

$$\Omega\Omega^{[C']} = \Omega^{[C']}\Omega' \tag{3.8}$$

Therefore if $|\psi\rangle \in H$ then $B(C')|\psi\rangle \in H$, and $B(C')$ is gauge-invariant. We say that $B(C')$ *measures* electric flux *through* C' and *creates* a magnetic flux line *along* C'.

We now want to find a conserved variety of Non-Abelian gauge-invariant magnetic flux in the 3-direction in the 3 dimensional periodic box. One might be tempted to look for some curve C enclosing the box in the 12 direction so that $A(C)$ measures the flux through the box. That turns out not to work because such a flux is not guaranteed to be conserved. It is better to consider a curve C' in the 3-direction winding over the torus exactly once:

$$C' = \{\mathbf{x}(s), 0 \leqslant s \leqslant 1; \mathbf{x}(1) = \mathbf{x}(0) + (0, 0, a_3)\} \tag{3.9}$$

$B(C')$ creates one magnetic flux line. But $B(C')$ also changes the number n_{12} into $n_{12} + 1$. This is because $\Omega^{[C']}$ makes a $Z(N)$ jump according to (3.6). If $\Omega_{1,2}(\mathbf{x})$ in (2.2) are still defined to be continuous then Z in (2.4) changes by one unit. Clearly, n_{12} measures the number of times an operator of the type $B(C')$ has acted, i.e., the number of magnetic flux lines created. n_{12} is also conserved by continuity. We simply define

$$n_{ij} = \epsilon_{ijk} m_k \tag{3.10}$$

with m_k the total magnetic flux in the k-direction. Note that $\mathbf{m}$ corresponds to the usual magnetic flux (apart from a numerical constant) in the Abelian case. Here, $\mathbf{m}$ is only defined as an integer modulo N.

4. Non-Abelian gauge-invariant electric flux in the box

As in the magentic case, there exists no simple curve C such that the total electric flux through C, measured by $B(C)$, corresponds to a conserved total flux through the box. We consider a curve C winding once over the torus in the 3-direction and consider the electric flux creation operator $A(C)$. But first we must study some new conserved quantum numbers.

Let $|\psi\rangle$ be a state in the before mentioned little Hilbert space H. Then, according to eq. (3.5), $|\psi\rangle$ is invariant under *infinitesimal* gauge transformations Ω. But we also have some non-trivial homotopy classes of gauge transformations Ω. These are the pseudoperiodic ones:

$$\Omega(a_1, x_2, x_3) = \Omega(0, x_2, x_3)Z_1$$
$$\Omega(x_1, a_2, x_3) = \Omega(x_1, 0, x_3)Z_2$$
$$\Omega(x_1, x_2, a_3) = \Omega(x_1, x_2, 0)Z_3$$
$$Z_{1,2,3} \in \text{center } Z(N) \text{ of } SU(N), \tag{4.1}$$

and also those Ω which are periodic but do carry a non-trivial

Pontryagin number ν. A little problem arises when we try to combine these two topological features. the $Z_{1,2,3}$ an be labeled by three integers $k_{1,2,3}$ between 0 and N.

$$Z_t = e^{2\pi i k_t/N} \tag{4.2}$$

But how is ν defined? The best definition is obtained if we consider a field configuration in a *four* dimensional space, obtained by multiplying the box $(S_1)^3$ with a line segment:

$$0 \leqslant t \leqslant 1$$

Now choose a boundary condition: $A(t=1) = \Omega A(t=0)$. Then, if the fields in between are continuous, then

$$P \equiv g^2 \int G_{\mu\nu}\tilde{G}_{\mu\nu} d^4x/32\pi^2 \tag{4.3}$$

is uniquely determined by Ω. On S_4 this would be the integer ν. Now however, it needs not to be integer anymore because of the twists in the periodic boundary conditions for $(S_1)^3$. We find that the required boundary condition, with an Ω satisfying eq. (4.1), can easily be fulfilled by a field A_μ in an Abelian subalgebra of the gauge system. The integral (4.3) is then easy to work out:

$$P = \frac{(\mathbf{mk})}{N} + \nu \tag{4.4}$$

where ν is integer and $\mathbf{m}$ is the magnetization defined in the previous section. Notice that ν is only well defined if $\mathbf{m}$ and $\mathbf{k}$ are given as genuine integers, not modulo N. Taking this warning to heart, we write $\Omega[\mathbf{k},\nu]$ for any Ω in the homotopy class $[\mathbf{k},\nu]$.

Notice that not only do the $A_\mu(x)$ transform smoothly under $\Omega[\mathbf{k},\nu]$, since they are invariant under the $Z(N)$ transformations of eq. (4.1), but also their boundary conditions do not change. These Ω commute therefore with the magnetic flux $\mathbf{m}$. If two Ω satisfy the same equation (4.1) and have the same ν, they may act differently on states of the big Hilbert space $\mathscr{H}$, but since they differ only be regular gauge transformations they act identically on states in H, defined in eq. (3.5). We may simultaneously diagonalize the Hamiltonian H, the magnetic flux $\mathbf{m}$, and $\Omega[\mathbf{k},\nu]$:

$$\Omega[\mathbf{k},\nu]\,|\psi\rangle = e^{i\omega(\mathbf{k},\nu)}|\psi\rangle, \tag{4.5}$$

where $\omega(\mathbf{k},\nu)$ are strictly conserved numbers. Now the Ω operators form a group. Defining for each Ω the number p as in (4.4) we have

$$\Omega[\mathbf{k}_1,p_1]\,\Omega[\mathbf{k}_2,p_2] = \Omega[\mathbf{k}_1+\mathbf{k}_2,p_1+p_2] \tag{4.6}$$

so

$$\omega(\mathbf{k}_1,\nu_1) + \omega(\mathbf{k}_2,\nu_2) = \omega(\mathbf{k}_1+\mathbf{k}_2,\nu_2+\nu_2) \tag{4.7}$$

and

$$\omega(\mathbf{k}+N\mathbf{l},\nu) = \omega(\mathbf{k},\nu+(\mathbf{lm})) \tag{4.8}$$

if $\mathbf{l}$ is an integer. We find that ω must be linear in $\mathbf{k}$ and ν:

$$\omega(\mathbf{k},\nu) = \frac{2\pi}{N}(\mathbf{ek}) + \frac{\theta}{N}(\mathbf{mk}) + \theta\nu \tag{4.9}$$

where e_i are integer numbers defined modulo N, and θ is the familiar instanton angle, defined to lie between 0 and 2π.

Now let us turn back to $A(C)$ defined in eq. (3.2). If C is the curve considered in the beginning of this section, $A(C)$ is not invariant under $\Omega[\mathbf{k},\nu]$ because

$$A(C) \to \mathrm{Tr}\,\Omega(\mathbf{x}_1)\left[P\exp\int_C ig\mathbf{A}\,d\mathbf{x}\right]\Omega^{-1}(\mathbf{x}_1+\mathbf{a}_3)$$
$$= e^{-2\pi i k_3/N}A(C) \tag{4.10}$$

Therefore,

$$A(C)\Omega[\mathbf{k},\nu]\,|\psi\rangle = \Omega[\mathbf{k},\nu]\,e^{-2\pi i k_3/N}A(C)|\psi\rangle \tag{4.11}$$

If

$$\Omega[\mathbf{k},\nu]\,|\psi\rangle = e^{i\omega(\mathbf{k},\nu)}|\psi\rangle, \tag{4.12}$$

and

$$A(C)|\psi\rangle = |\psi'\rangle, \tag{4.13}$$

then

$$\Omega[\mathbf{k},\nu]\,|\psi'\rangle = e^{i\omega(\mathbf{k},\nu)+2\pi i k_3/N}\,|\psi'\rangle \tag{4.14}$$

Therefore $A(C)$ increases e_3 by one unit:

$$e_3A(C)|\psi\rangle = A(C)(e_3+1)|\psi\rangle \tag{4.15}$$

e_3 is a good indicator for electric flux in the 3-direction, up to a constant. It is strictly conserved. However if we let θ run from 0 to 2π then $\mathbf{e}$ turns into $\mathbf{e}+\mathbf{m}$. It is therefore physically perhaps more appropriate to identify

$$\mathbf{e} + \frac{\theta}{2\pi}\mathbf{m} \tag{4.16}$$

as being the total electric flux in the three directions of the box.

5. Free energy of a given flux configuration

Let us write down the free energy F of a given state $(\mathbf{e},\mathbf{m},\theta)$ at temperature $T = 1/k\beta$:

$$e^{-\beta F} = \mathrm{Tr}_H\,P_e(\mathbf{e})P_m(\mathbf{m})P_\theta(\theta)\,e^{-\beta\mathrm{H}} \tag{5.1}$$

Here H is the Hamiltonian and H the little Hilbert space. P are projection operators. $P_m(\mathbf{m})$ is simply defined to select a given set of $n_{ij} = \epsilon_{ijk}m_k$, the three space-like indices of eq. (2.6). $P_e(\mathbf{e})P_\theta(\theta)$ is defined by selecting states $|\psi\rangle$ with

$$\Omega[\mathbf{k},\nu]\,|\psi\rangle = \exp\left(\frac{2\pi i}{N}(\mathbf{k},\mathbf{e}) + \frac{\theta i}{N}(\mathbf{mk}) + i\theta\nu\right)|\psi\rangle \tag{5.2}$$

Therefore

$$P_e(\mathbf{e})P_\theta(\theta) = \frac{1}{N^3}\sum_{\mathbf{k},\nu}\exp\left(-\frac{2\pi i}{N}(\mathbf{ke}) - \frac{\theta i}{N}(\mathbf{mk}) - i\theta\nu\right)$$
$$\Omega[\mathbf{k},\nu] \tag{5.3}$$

Now $e^{-\beta\mathrm{H}}$ is the evolution operator in imaginary time direction at interval β, expressed by a functional integral over a Euclidean box with sides (a_1,a_2,a_3,β):

$$\langle \mathbf{A}_{(1)}(\mathbf{x})|e^{-\beta\mathrm{H}}|A_{(2)}(\mathbf{x})\rangle = \int DA\,e^{S(A)}\Big/_{\mathbf{A}(\mathbf{x},0)=\mathbf{A}_{(2)}(\mathbf{x})}^{\mathbf{A}(\mathbf{x},\beta)=\mathbf{A}_{(1)}(\mathbf{x})} \tag{5.4}$$

We may fix the gauge for $\mathbf{A}_{(2)}(\mathbf{x})$ for instance by choosing

$$A_{(2)3}(\mathbf{x}) = 0$$
$$A_{(2)2}(x,y,0) = 0$$
$$A_{(2)1}(x,0,0) = 0 \tag{5.5}$$

We already had $A_4(\mathbf{x},t) = 0$. Since only states in H are considered, we insert also a projection operator

$$\int_{\Omega\in I} D\Omega$$

where I is the trivial homotopy class.

"Trace" means that we integrate over all $A_{(1)} = A_{(2)}$ therefore we get periodic boundary conditions in the 4-direction. Insertion of $\int_{\Omega \in I} D\Omega$ means that we have periodicity up to gauge transformations, in the completely unique gauge

$$A_4(\mathbf{x},\beta) = A_3(\mathbf{x},0) = A_2(x,y,0,0) = A_1(x,0,0,0) = 0 \tag{5.6}$$

Equation (5.3) tells us that we have to consider twisted boundary conditions in the 41, 42, 43 directions and Fourier transform:

$$e^{-\beta F(\mathbf{e},\mathbf{m},\theta,\mathbf{a},\beta)} = \frac{1}{N^3}\sum_{\mathbf{k},\nu}\exp\left(-\frac{2\pi i}{N}(\mathbf{ke}) - i\theta\left(\nu + \frac{\mathbf{mk}}{N}\right)\right) \times W\{\mathbf{k},\mathbf{m},\nu,a_\mu\} \tag{5.7}$$

Here $W\{\mathbf{k},\mathbf{m},\nu,a_\mu\}$ is the Euclidean functional integral with boundary conditions fixed by choosing $n_{ij} = \epsilon_{ijk}m_k$; $n_{i4} = k_i$; $a_4 = \beta$, and a Pontryagin number ν. Because of the gauge choice (2.2) this functional integral must include integration over the Ω belonging to the given homotopy classes as they determine the boundary conditions such as (2.2).

The definition of W is completely Euclidean symmetric. In the next section I show how to make use of this symmetry with respect to rotation over 90° in Euclidean space.

6. Duality

The Euclidean symmetry in eq. (5.7) suggests to consider the following SO(4) rotation:

$$\begin{bmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{bmatrix}$$

Let us introduce a notation for the first two components of a vector:

$$x_\mu = (\mathbf{x}, x_4),$$
$$\tilde{x} = (x_1, x_2),$$
$$\hat{x} = (x_2, x_1). \tag{6.2}$$

We have, from eq. (22.7):

$$\exp[-\beta F(\tilde{e}, e_3, \tilde{m}, m_3, \theta, \tilde{a}, a_3, \beta)] = \frac{1}{N^2}\sum_{\tilde{k},\tilde{l}} \exp\left[\frac{2\pi i}{N}\left(-(\tilde{k}\tilde{e}) + (\tilde{l}\tilde{m})\right) - a_3 F(\tilde{l}, e_3, \tilde{k}, m_3, \theta, \hat{a}, \beta, a_3)\right] \tag{6.3}$$

Notice that in this formula the transverse electric and magnetic fluxes are Fourier transformed and interchange positions. Notice also that, apart from a sign difference, there is a complete electric–magnetic symmetry in this expression, in spite of the fact that the definition of F in terms of W was not so symmetric. Equation (6.3) is an exact property of our system. No approximation was made, We refer to it as "duality".

7. Long-distance behaviour compatible with duality

Equation (6.3) shows that the instanton angle θ plays no role in duality. It does however affect the physical interpretation of $\mathbf{e}$ as electric flux, see eq. (4.16). From now on we put $\theta = 0$ for simplicity, and omit it. But we come back to this point in Section 8.

Let us now assume that the theory has a mass gap. No massless particles occur. Then asymptotic behavior at large distances will be approached exponentially. Then it is excluded that $F(\mathbf{e},\mathbf{m},\mathbf{a},\beta) \to 0$, exponentially as $\mathbf{a},\beta \to \infty$, for all $\mathbf{e}$ and $\mathbf{m}$, which would clearly contradict eq. (6.3). This means that at least some of the flux configurations must get a large energy content as $\mathbf{a},\beta \to \infty$. These flux lines apparently cannot spread out and because they were created along curves C it is practically inescapable that they get a total energy which will be proportional to their length:

$$E = \lim_{\beta\to\infty} F = \rho a \tag{7.1}$$

However, duality will never enable us to determine whether it is the electric or the magnetic flux lines that behave this way. From the requirement that W in eq. (5.7) is always positive one can deduce the impossibility of a third option, namely that only exotic combinations of electric and magnetic fluxes behave as strings (provided $\theta = 0$).

For futher information we must make the physically quite plausible assumption of "factorizability":

$$F(\mathbf{e},\mathbf{m}) \to F_e(\mathbf{e}) + F_m(\mathbf{m}) \quad \text{if } \mathbf{a},\beta \to \infty \tag{7.2}$$

Suppose that we have confinement in the electric domain:

$$F_e(0,0,1) \to \rho a_3 \tag{7.3}$$

where ρ is the fundamental string constant. Then we can derive from duality the behavior of $F_m(\mathbf{m})$.

First we improve (7.3) by applying statistical mechanics to obtain F_e for large but finite β. One obtains:

$$\exp(-\beta F_e(e_1,e_2,0,\mathbf{a},\beta) + C(a,\beta)) = \sum_{n_1^\pm,\, n_2^\pm} \frac{1}{n_1^+!\,n_2^+!\,n_1^-!\,n_2^-!} \times \gamma_1^{n_1^+ + n_1^-}\gamma_2^{n_2^+ + n_2^-}\delta_N(n_1^+ - n_1^- - e_1)\delta_N(n_2^+ - n_2^- - e_2) \tag{7.4}$$

Here

$$\gamma_1 = \lambda a_2 a_3\, e^{-\beta\rho a_1},$$
$$\gamma_2 = \lambda a_1 a_3\, e^{-\beta\rho a_2},$$
$$\delta_N(x) = \frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi i k x/N} = \begin{cases} 1 & \text{if } x = 0 \pmod N \\ 0 & \text{if } x = \text{other integer number} \end{cases} \tag{7.5}$$

The sum is over all nonnegative interger values of $n_i^\pm$ (the orientations $\pm$ are needed if $N \geqslant 3$). The γ's are Boltzmann factors associated with each string-like flux tube.

We now insert this, with (7.2), into (6.3) putting $e_3 = m_3 = 0$. One obtains

$$\exp(-\beta F_m(m_1,m_2,0,\mathbf{a},\beta)) = C' \exp\left(2\sum_a \gamma_a' \cos(2m_a\pi/N)\right) \tag{7.6}$$

where C' is again a constant and

$$\gamma_1' = \lambda a_1 \beta\, e^{-\rho a_2 a_3}$$
$$\gamma_2' = \lambda a_2 \beta\, e^{-\rho a_1 a_3} \tag{7.7}$$

At $\beta \to \infty$ we get

$$F_m(\tilde{m},0,\mathbf{a},\beta) \to E_m(\tilde{m},0,\mathbf{a}) = \sum_i E_i(m_i,\mathbf{a})$$

with

$$E_1(m_1, \mathrm{a}) = 2\lambda\left(1 - \cos\frac{2\pi m_1}{N}\right) a_1\, \mathrm{e}^{-\rho a_2 a_3} \tag{7.8}$$

and similarly for E_2 and E_2.

One reads off from eq. (7.8) that there will be no magnetic confinement, because if we let the box become wider the exponential factor $\mathrm{e}^{-\rho a_2 a_3}$ causes a rapid decrease of the energy of the magnetic flux. Notice the occurrence of the string constant ρ in there.

Of course we could equally well have started from the presumption that there were magnetic confinement. One then would conclude that there would be no electric confinement, because then the electric flux would have an energy given by eq. (7.8).

A third possible mode in the thermodynamic limit is the so-called Coulomb mode. Both electric and magentic charges occur explicitly. The mathematics of this mode is further explained in [1].

8. θ-dependence

In our duality equations (6.3) the instanton angle θ seems to play no special role. However the fact that θ occurs explicitly in the equations of Section 5 does have consequences, which we will now explore.

Let us consider the periodic box in three-space with a given non-vanishing amount of magnetization, parametrized with integers $\mathbf{m}$. Take $\mathbf{e} = 0$, so that no electric fluxes are present. Now vary the instanton angle θ continuously from 0 to 2π. We now see from eq. (5.7) that then

$$F(0, \mathbf{m}, 2\pi, \mathbf{a}, \beta) = F(\mathbf{m}, \mathbf{m}, 0, \mathbf{a}, \beta) \tag{8.1}$$

In other words: we went continuously from the case $\mathbf{e} = 0$ to the case $\mathbf{e} = \mathbf{m}$.

This observation is closely related to a similar feature observed for magnetic monopoles in a Georgi–Glashow model with non-trivial instanton angle: if θ runs continuously from zero to 2π then the *electric* charge of a monopole runs continuously from zero to one unit [4]. For the case that $\theta/2\pi$ is fractional it is reasonable to expect that the monopole has a fractional electric charge. Indeed one can show that attributing fractional electric charge to a magnetic monopole is not inconsistent with Dirac's quantization condition [4].

Let us return to our periodic box and assume that electric confinement takes place at all θ. Then one observes a difficulty. The free energy of the configuration with $\mathbf{e} \neq 0$ will have to be much higher than at $\mathbf{e} = 0$. But if the function $F(0, \mathbf{m}, \theta, \ldots) - F(-\mathbf{m}, \mathbf{m}, \theta, \ldots)$ is to be a continuous function of θ, then it must change sign at some point. A zero is likely to occur at $\theta = \pi$, in contradiction with the hypothesis that the energy of an electric flux tends to infinity in a large box.

There are now two possibilities: either confinement only occurs within a finite interval of θ values, excluding $\theta = \pi$, or the sign flip of this function becomes a discrete jump at $\theta = \pi$. Such jumps are characteristic for a phase transition and indeed this is what I think one can conclude: if confinement persists for all θ, then there is a phase transition at $\theta = \pi$. At present however the details of this phase transition are not very clear to me.

9. Further speculations

It could be that the transition compares with the so called "roughening transition" in lattice models; the nature of the string-like structures in the theory changes abruptly but no clear singularity is seen in the free energy of the original theory. But it is also possible that at the transition point confinement breaks down altogether. In that case the transition is more drastic, also for other properties of the model, such as the particle mass spectrum. Let us speculate for a moment that this is the case.

Then introduce a multiplet of fermions in the fundamental representation of the color gauge group. Besides color they have flavor; the flavor multiplicity is called F. All flavors are given identical mass, for simplicity, but, as a start, we introduce also a γ_5 term in the mass matrix:

$$\mathcal{L} = \ldots - \bar{\psi}_{if}(m_0 + i m_5 \gamma_5)\psi_{if} \tag{9.1}$$

where i counts color and f counts flavor:

$$f = 1, \ldots, F \tag{9.2}$$

Of course the γ_5 term can be rotated away by a chiral rotation, but then θ has to be rotated accordingly. Now in the case of F flavors a rotation of the mass vector (m_0, m_5) by an angle ϕ corresponds to a rotation of θ by ϕF. The arguments of the previous sections all refer to the quark-less case, which corresponds to the limit $m_0^2 + m_5^2 \to \infty$. If a phase transition occurs at $\theta = \pi$, then that transition must also occur at fixed and vanishing θ, if the mass vector (m_0, m_5) is rotated by an angle ϕ approaching π/F. The resulting phase transition boundaries for the case $F = 3$ are sketched in Fig. 1. They are the solid lines, at large values for m_0 or m_5.

It is unlikely that these phase boundaries just stop somewhere. Probably they continue all the way to the origin, where a larger singularity will develop. We now suggest that this singularity at the origin causes chiral symmetry to be spontaneously broken. The mass terms are coupled to the vector $(\bar{\psi}\psi, \bar{\psi}i\gamma_5\psi)$. A smooth behavior of the system at $m_0, m_5 = 0$ would require $\langle\bar{\psi}\psi\rangle = 0$. But because of the singularity and because we have separate phase regions surrounding the origin we may have that the value of $\langle\bar{\psi}\psi\rangle$ depends on the direction in which the origin is approached. It is no longer required to vanish by symmetry. If $\langle\bar{\psi}\psi\rangle$ and $\langle\bar{\psi}\gamma_5\psi\rangle$ would vanish then no singularity would develop at $m_0 = m_5 = 0$. So we seem to obtain necessarily a spontaneous breakdown also of chiral SU(F) ⊗ SU(F) symmetry.

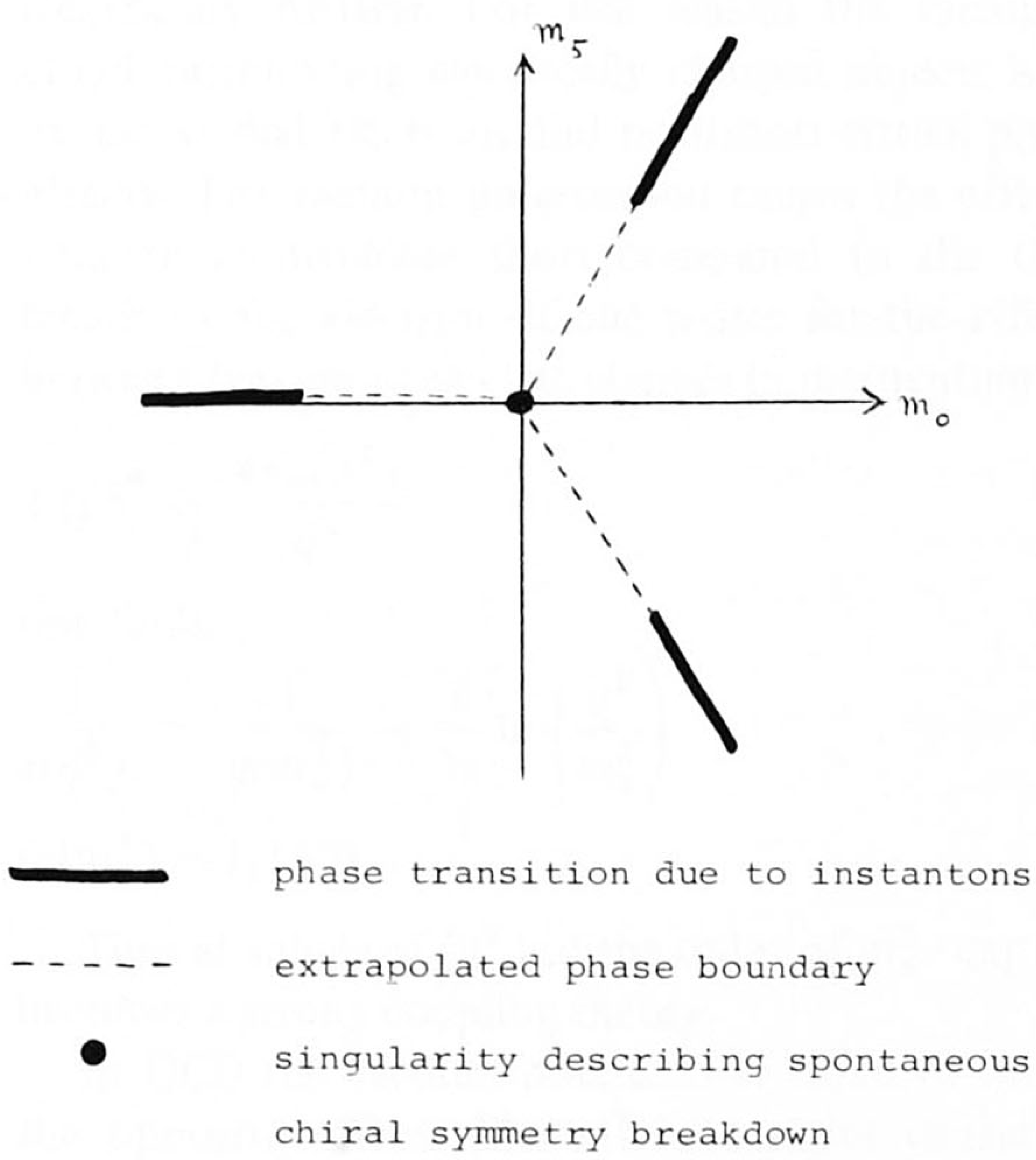

Fig. 1. Singularities in the mass matrix; case of three flavors.

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