GENERATING SYSTEMS AND REPRESENTABILITY FOR SYMPLECTIC CAPACITIES

DUŠAN JOKSIMOVIĆ AND FABIAN ZILTENER

Abstract. We consider the problem by K. Cieliebak, H. Hofer, J. Latschev, and F. Schlenk (CHLS) that is concerned with finding a minimal generating system for (symplectic) capacities on a given symplectic category. We show that under some mild hypotheses every countably Borel-generating set of (normalized) capacities has cardinality bigger than the continuum. This appears to be the first result regarding the problem of CHLS, except for a result by D. McDuff, stating that the ECH-capacities are monotonely generating for the category of ellipsoids in dimension 4.

Under the same mild hypotheses we also prove that almost no normalized capacity is domain- or target-representable. This provides some solutions to two central problems by CHLS.

In addition, we prove that every finitely differentiably generating system of symplectic capacities on a given symplectic category is uncountable, provided that the category contains a one-parameter family of symplectic manifolds that is "strictly volume-increasing" and "embedding-capacity-wise constant". It follows that the Ekeland-Hofer capacities and the volume capacity do not finitely differentiably generate all generalized capacities on the category of ellipsoids. This answers a variant of a question by CHLS.

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1. Introduction and main results

1.1. The Problems. Let $m, k \in \mathbb{N}_0 := \{0, 1, \ldots\}$. We define $\Omega^{m,k}$ to be the following category:

- Its objects are pairs (M, ω), where M is a manifold[1] of dimension m, and ω is a differential k-form on M.
- Its morphisms are embeddings[2] that intertwine the differential forms.

Recall that a subcategory $\mathcal{C}'$ of a category $\mathcal{C}$ is called isomorphism-closed iff every isomorphism of $\mathcal{C}$ starting at some object of $\mathcal{C}'$ is a morphism of $\mathcal{C}'$.[3]

Definition 1. *A* weak (m, k)-(differential) form category *is a subcategory* $\mathcal{C} = (\mathcal{O}, \mathcal{M})$ *of* $\Omega^{m,k}$ [4]*, such that if* $(M, \omega) \in \mathcal{O}$ *and* $a \in (0, \infty)$ *then* $(M, a\omega) \in \mathcal{O}$*. We call such a* $\mathcal{C}$ *a* (m, k)-form-category *iff it is also isomorphism-closed.*

A (weak) symplectic category (in dimension $2n$) *is a (weak)* $(2n, 2)$*-form-category whose objects are symplectic manifolds.*

Examples 2 ((weak) (m, k)-form-category)**.**

(i) Let $\mathfrak{M}$ be a diffeomorphism class of smooth manifolds of dimension m. The full subcategory of $\Omega^{m,k}$ whose objects (M, ω) satisfy $M \in \mathfrak{M}$, is a (m, k)-form-category.

(ii) Let (M, ω) be an object of $\Omega^{m,k}$. We define $\mathcal{O}_{M,\omega}$ to be the set of all pairs $(U, \omega|U)$, where U ranges over all open subsets of M. We define $\mathcal{M}_{M,\omega}$ to be the set of all triples $\big(U, V, \varphi|U\big)$, where U and V range over all open subsets of M and φ over all isomorphisms of (M, ω), such that $\varphi(U) \subseteq V$. Hence the morphisms between two objects are the restrictions of global form-preserving diffeomorphisms. The pair $\mathrm{Op}_{M,\omega} := \big(\mathcal{O}_{M,\omega}, \mathcal{M}_{M,\omega}\big)$ is a weak (m, k)-form-category, which is not isomorphism-closed, hence not a (m, k)-form-category.

Remark 3 (isomorphism-closedness)**.** Symplectic categories were first defined in [CHLS07, 2.1. Definition, p. 5]. In that definition isomorphism-closedness is not assumed. However, this condition is needed in order to avoid the following set-theoretic issue in the definition of the notion of a symplectic capacity on a given symplectic category $\mathcal{C}$.

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[1] In this article "manifold" refers to a smooth (C^∞) real finite-dimensional manifold. It is allowed to be disconnected and to have boundary.

[2] By an embedding we mean an injective smooth immersion with continuous inverse. We do not impose any condition involving the boundaries of the two manifolds.

[3] In particular, it ends at some object of $\mathcal{C}'$.

[4] Here $\mathcal{O}$ and $\mathcal{M}$ denote the classes of objects and morphisms of $\mathcal{C}$, respectively.

This article is based on ZFC, the Zermelo-Fraenkel axiomatic system together with the axiom of choice. A category is a pair consisting of classes of objects and morphisms. Formally, in ZFC there is no notion of a "class" that is not a set. The system *can* handle a "class" that is determined by a wellformed formula, such as the "class" of all sets or the "class" of all symplectic manifolds, by rewriting every statement involving the "class" as a statement involving the formula.

However, it is not possible in ZFC to define the "class" of all maps between two classes, even if the target class is a set. In particular, it is a priori not possible to define the "class" of all symplectic capacities on a given symplectic category. Our assumption that $\mathcal{C}$ is isomorphism-closed makes it possible to define this "class" even as a set, see below.

We now define the notion of a (generalized) capacity on a given (m,k)-form-category. Let S be a set. By $|S|$ we denote the *(von Neumann) cardinality of* S, i.e., the smallest (von Neumann) ordinal that is in bijection with S. For every pair of sets S, S' we denote by ${S'}^S$ the set of maps from S to S'. For every pair of cardinals α, β [5] we also use β^α to denote the cardinality of β^α. Recursively, we define $\beth_0 := \mathbb{N}_0$, and for every $i \in \mathbb{N}_0$, the cardinal $\beth_{i+1} := 2^{\beth_i}$.[6]

We denote by B^m_r ($\overline{B}^m_r$) the open (closed) ball of radius r around 0 in $\mathbb{R}^m$, and

$$B := B^{2n}_1, \quad Z^{2n}_r := B^2_r \times \mathbb{R}^{2n-2}, \quad Z := Z^{2n}_1.$$

We equip B^{2n}_r and Z^{2n}_r with the standard symplectic form ω_{st}. Let $\mathcal{C} = (\mathcal{O}, \mathcal{M})$ be a (m,k)-form-category. We define the set

$$\mathcal{O}_0 := \big\{(M,\omega) \in \mathcal{O} \,\big|\, \text{The set underlying } M \text{ is a subset of } \beth_1.\big\}. \tag{1}$$

Definition 4. *A* generalized capacity on $\mathcal{C}$ *is a map*

$$c : \mathcal{O}_0 \to [0,\infty]$$

with the following properties:

(i) ***(monotonicity)*** *If (M,ω) and (M',ω') are two objects in $\mathcal{O}_0$ between which there exists a $\mathcal{C}$-morphism, then*

$$c(M,\omega) \le c(M',\omega').$$

(ii) ***(conformality)*** *For every $(M,\omega) \in \mathcal{O}_0$ and $a \in (0,\infty)$ we have*

$$c(M, a\omega) = a\, c(M,\omega).$$

Assume now that $k = 2$, $m = 2n$ for some integer n, and that $\mathcal{O}_0$ contains some objects B_0, Z_0 that are isomorphic to B, Z. Let c be a generalized capacity on $\mathcal{C}$. We call c a capacity *iff is satisfies:*

(iii) ***(non-triviality)*** $c(B_0) > 0$ *and* $c(Z_0) < \infty$.[7]

We call it normalized *iff it satisfies:*

(iv) ***(normalization)*** $c(B_0) = c(Z_0) = \pi$.[8]

[5] i.e., cardinalities of some sets

[6] $\beth$ (bet) is the second letter of the Hebrew alphabet.

[7] These conditions do not depend on the choices of B_0, Z_0, since c is is invariant under isomorphisms by monotonicity.

[8] In [CHLS07, 2.1. Definition, p. 5] only the condition $c(B) = 1$ is imposed here (in the context of a symplectic category). Our first main result, Theorem 15, holds even with our stronger definition.

We denote by

$$\mathcal{C}ap(\mathcal{C}), \quad \mathcal{NC}ap(\mathcal{C})$$

the sets of generalized and normalized capacities on $\mathcal{C}$. If $\mathcal{C}$ is a symplectic category then we call a (generalized/ normalized) capacity on $\mathcal{C}$ also a (generalized/ normalized) symplectic capacity*.*

Example 5 (embedding capacities)**.** Let $\mathcal{C} = (\mathcal{O}, \mathcal{M})$ be a (m,k)-form-category and (M,ω) an object of $\Omega^{m,k}$. We define $\mathcal{O}_0$ as in (1) and the *domain-embedding capacity for* (M,ω) *on* $\mathcal{C}$ to be the map

$$c_{M,\omega} := c^{\mathcal{C}}_{M,\omega} : \mathcal{O}_0 \to [0,\infty],$$

$$c_{M,\omega}(M',\omega') := \sup\left\{ c \in (0,\infty) \,\middle|\, \exists\, \Omega^{m,k}\text{-morphism } (M, c\omega) \to (M',\omega') \right\}. \tag{2}$$

For brevity we also just call this the *embedding capacity for* (M,ω). We define the *target-embedding capacity for* (M,ω) *on* $\mathcal{C}$ to be the map

$$c^{M,\omega} := c^{M,\omega}_{\mathcal{C}} : \mathcal{O}_0 \to [0,\infty],$$

$$c^{M,\omega}(M',\omega') := \inf\left\{ c \in (0,\infty) \,\middle|\, \exists\, \Omega^{m,k}\text{-morphism } (M',\omega') \to (M, c\omega) \right\}. \tag{3}$$

These are generalized capacities.[9] Assume that $k = 2$ and $m = 2n$ for some n. We define the *Gromov width on* $\mathcal{C}$ to be

$$w := \pi c^{\mathcal{C}}_{B,\omega_{\mathrm{st}}}. \tag{4}$$

If $B, Z \in \mathcal{O}$, then by Gromov's nonsqueezing theorem the Gromov width is a normalized capacity.

Capacities on the category of all symplectic manifolds of a fixed dimension were introduced by I. Ekeland and H. Hofer in [EH89, EH90]. They measure how much a given symplectic manifold does not embed into another one. For an overview over symplectic capacities we refer to [CHLS07, Sch18, Sch05] and references therein.

Remarks.
- $\mathcal{C}ap(\mathcal{C})$ and $\mathcal{NC}ap(\mathcal{C})$ are indeed sets, since $\mathcal{O}_0$ is a set.
- Heuristically, let us denote by $\widetilde{\mathcal{C}ap}(\mathcal{C})$ the "subclass" of "$[0,\infty]^{\mathcal{O}}$" consisting of all "maps" satisfying (i,ii) above. Formally, the restriction from $\mathcal{O}$ to $\mathcal{O}_0$ induces a bijection between $\widetilde{\mathcal{C}ap}(\mathcal{C})$ and $\mathcal{C}ap(\mathcal{C})$.[10] This means that our definition of a generalized capacity corresponds to the intuition behind the usual "definition". Here we use isomorphism-closedness of $\mathcal{C}$. Compare to Remark 3.

[9] In the definition on p. 13 in [CHLS07] (M,ω) is assumed to be an object of $\mathcal{C}$, and the morphisms in the definitions of the embedding capacities are asked to be $\mathcal{C}$-morphisms. However, in [CHLS07, Example 2, p. 14] the definition is applied with a (M,ω) that is not an object of $\mathcal{C}$. In order to make that example work, one needs to allow for $\Omega^{m,k}$-morphisms in the definition of the embedding capacities.

[10] This follows from the fact that every object of $\Omega^{m,k}$ is isomorphic to one whose underlying set is a subset of $\beth_1$, and the assumption that $\mathcal{C}$ is isomorphism-closed. To prove the fact, recall that by definition, the topology of every manifold M is second countable. Using the axiom of choice, it follows that its underlying set has cardinality $\leq \beth_1$. This means that there exists an injective map $f : M \to \beth_1$. Consider now an object (M,ω) of $\Omega^{m,k}$. Pushing forward the manifold structure and ω via a map f, we obtain an object of $\Omega^{m,k}$ isomorphic to (M,ω), whose underlying set is a subset of $\beth_1$. This proves the fact.

- Isomorphism-closedness of $\mathcal{C}$ implies that there is a canonical bijection between $\mathcal{C}ap(\mathcal{C})$ and the set of generalized capacities that we obtain by replacing $\mathcal{O}_0$ by any subset of $\mathcal{O}$ that contains an isomorphic copy of each element of $\mathcal{O}$. Such a set can for example be obtained by replacing $\beth_1$ in (1) by any set of cardinality at least $\beth_1$.[11] This means that our definition of $\mathcal{C}ap(\mathcal{C})$ is natural.

Let $f : [0,\infty]^\ell \to [0,\infty]^{\ell'}$ be a function. We call it *homogeneous* iff it is positively 1-homogoneous, i.e., iff $f(ax) = af(x)$ for every $a \in (0,\infty)$ and $x \in [0,\infty]^\ell$.[12] We equip $[0,\infty]^\ell$ with the partial order given by $x \leq y$ iff $x_i \leq y_i$ for every $i \in \{1,\dots,\ell\}$. We call f *monotone* iff it preserves this partial order. As pointed out by K. Cieliebak, H. Hofer, J. Latschev, and F. Schlenk (CHLS) in [CHLS07], if $\ell' = 1$, f is homogeneous and monotone, and $c_1,\dots,c_\ell$ are generalized capacities, then $f \circ (c_1,\dots,c_\ell)$ is again a generalized capacity. Homogeneity and monotonicity are preserved under compositions.

Examples. The following functions are homogoneous and monotone:

- maximum, minimum
- For every $a \in [0,\infty)^\ell$ and $p \in \mathbb{R} \setminus \{0\}$ the function
$$f_{a,p}(x) := \sqrt[p]{\sum_{i=1}^{\ell} a_i x_i^p}.\,^{13}$$
In the case $a = \left(\frac{1}{\ell},\dots,\frac{1}{\ell}\right)$, $p = 1$ the function $f_{a,p}$ is the arithmetic mean, and in the case $a = \left(\frac{1}{\ell},\dots,\frac{1}{\ell}\right)$, $p = -1$ it is the harmonic mean.
- For every $p \in [0,\infty)^\ell$ satisfying $\sum_{i=1}^{\ell} p_i = 1$ the function
$$x \mapsto \prod_{i=1}^{\ell} x_i^{p_i}.$$
In the case $p = \left(\frac{1}{\ell},\dots,\frac{1}{\ell}\right)$ this is the geometric mean.

Let $\mathcal{G}$ be a subset of $\mathcal{C}ap(\mathcal{C})$. By a *finite homogeneous monotone combination of* $\mathcal{G}$ we mean an expression of the form $f \circ (c_1,\dots,c_\ell)|\mathcal{O}_0$, where $\ell \in \mathbb{N}_0$, $f : [0,\infty]^\ell \to [0,\infty]$ is homogeneous and monotone, and $c_1,\dots,c_\ell \in \mathcal{G}$. We define the *set CHLS-generated by* $\mathcal{G}$ to be the set of all maps $c : \mathcal{O}_0 \to [0,\infty]$ that are the pointwise limit of a sequence of finite homogeneous monotone combinations of $\mathcal{G}$. Since pointwise limits preserve homogeneity and monotonicity, the set CHLS-generated by $\mathcal{G}$, consists again of generalized capacities.

In [CHLS07, Problem 5, p. 17] a *generating system for the (generalized) symplectic capacities on* $\mathcal{C}$ is defined to be a subset $\mathcal{G}$ of $\mathcal{C}ap(\mathcal{C})$ [14], whose CHLS-generated

[11] This follows from an argument as in the last footnote.

[12] Here we use the convention $a \cdot \infty := \infty$ for every $a \in (0,\infty)$.

[13] Here we use the conventions $\infty + a = \infty$ for every $a \in [0,\infty]$, $\infty^p = \infty$ for every $p > 0$, and $0^p := \infty$ and $\infty^p := 0$ for every $p < 0$.

[14] CHLS use the common definition of a symplectic capacity that does not deal with the set-theoretic issue mentioned in Remark 3. In particular, they do not explicitly state that $\mathcal{G}$ should be a subset of $\mathcal{C}ap(\mathcal{C})$ (which is a subset of $[0,\infty]^{\mathcal{O}_0}$), but presumably they implicitly ask for this.

set is the whole of $\mathcal{C}ap(\mathcal{C})$. They also propose more restrictive notions of "generation", for example one in which the only allowed combining functions f are the maximum and minimum.

The set CHLS-generated by $\mathcal{G}$ is obtained by combining capacities in a lot of ways. One may therefore expect that few capacities suffice to generate all the other capacities. It is tempting to even look for a generating set of capacities that is minimal, in the sense that none of its subsets is generating. This problem was posed by CHLS:

Problem 6 ([CHLS07], Problem 5, p.17)**.** *For a given symplectic category $\mathcal{C}$, find a minimal generating system $\mathcal{G}$ for the (generalized) symplectic capacities on $\mathcal{C}$.*

A concrete instance of this problem is the following.

Question 7. *Does there exist a countable*[15] *(minimal) generating system for the capacities on a given symplectic category $\mathcal{C}$?*

To our knowledge, up to now, Problem 6 has been completely open, except for a result by D. McDuff, which states that the ECH-capacities are generating in a weaker sense for the category of ellipsoids in dimension 4, see Theorem 28 below.[16]

Our first main result, Theorem 15 below, answers Question 7 in the negative for a weak notion of a "generating system", in dimension at least 4. The theorem states that under some mild hypotheses on $\mathcal{C}$ the cardinality of the set of generalized (or normalized) capacities is $\beth_2$. Its last part implies that a set of $[0,\infty]$-valued functions of cardinality at most $\beth_1$ countably Borel-generates a set of cardinality at most $\beth_1$ in the sense of Definition 14 below.

As an immediate consequence, every countably Borel-generating system for $\mathcal{C}ap(\mathcal{C})$ (or $\mathcal{NC}ap(\mathcal{C})$) has cardinality bigger than the continuum. See Corollary 16 below.

Countable Borel-generation is a weak notion of generation. In particular, it is weaker than the notion of generation proposed by CHLS after [CHLS07, Problem 5, p. 17], in which only the maximum and minimum are allowed as combining functions. Compare to Remarks 13 and 18 below. Hence Corollary 16 makes a statement about many systems of capacities.

This corollary diminishes the hope of finding manageable generating systems of (generalized) symplectic capacities.

Our first main result also has immediate applications to two questions that CHLS prominently posed as Problems 1 and 2 in their article [CHLS07]. To explain these problems, let $n \in \mathbb{N}$ and $\mathcal{C} = (\mathcal{O}, \mathcal{M})$ be a symplectic category in dimension $2n$.

Definition 8 (representability)**.** *Let c be a capacity on $\mathcal{C}$. We call c* domain-/ target-representable *iff there exists a symplectic manifold (M,ω), for which $c = c_{M,\omega}$/ $c = c^{M,\omega}$. We call it* connectedly target-representable *iff there exists a* connected *symplectic manifold (M,ω), for which $c = c^{M,\omega}$.*

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[15]By this we mean finite or countably infinite.

[16]There are of course some trivial cases in which Question 7 is easy, e.g. the case in which there are only finitely many $\mathcal{C}$-isomorphism classes.

Remark. *By definition, the topology of a manifold is second countable. Without this condition every capacity would be target-representable, if all objects of $\mathcal{C}$ are connected, see* [CHLS07, Example 2, p. 14].

Question 9 (target-representability, [CHLS07], p. 14, Problem 1)**.** *Which (generalized) capacities on $\mathcal{C}$ are connectedly target-representable?*[17]

Question 10 (domain-representability, [CHLS07], p. 14, Problem 2)**.** *Which (generalized) capacities on $\mathcal{C}$ are domain-representable?*[18]

In particular, one may wonder about the following:

Question 11. *Is* every *generalized capacity connectedly target-representable?*

If the answer to this question is "yes", then this simplifies the study of capacities, since we may then identify every capacity with some symplectic manifold that target-represents it.

Apart from some elementary examples, up to now, the answers to Questions 9,10, and 11 appear to be completely unknown. In order to answer Question 11 negatively, it seems that we need to understand all symplectic embeddings from objects of $\mathcal{C}$ to all connected symplectic manifolds. At first glance this looks like a hopeless enterprise.

The following application of the first main result may therefore come as a surprise. Namely, under some mild hypotheses the answer to Question 11 is "no". In fact, it remains "no", even if we ask the question only for *normalized* capacities and drop the word "connectedly". Perhaps all the more unexpectedly, *almost no* normalized capacity is target-representable. By this we mean that the set target-representable normalized capacities has strictly smaller cardinality than the set of all normalized capacities (and both are infinite). Similarly, almost no normalized capacity is domain-representable. See Corollary 17 below. This provides some answers to Questions 9,10, and 11.

Consider now Problem 6 in the more general context in which $\mathcal{C} = (\mathcal{O}, \mathcal{M})$ is only a *weak* (m, k)-form-category. In order to take care of the set-theoretic problems mentioned above, we assume that $\mathcal{C}$ is small, i.e., that $\mathcal{O}$ is a set. In this setting we define the notion of a (generalized) capacity as in Definition 4, with $\mathcal{O}_0$ replaced by $\mathcal{O}$.

Example (Ekeland-Hofer capacities)**.** For every object (M, ω) of $\Omega^{m,k}$ we define the weak (m, k)-form-category $\mathrm{Op}_{M,\omega}$ as in Example 2(ii). Hence $\mathcal{O}$ consists of the topology (= set of all open subsets) of M, and $\mathcal{M}$ of all restrictions of global form-preserving diffeomorphisms. Consider the case in which $k = 2$ and $(M = V, \omega)$ is a symplectic vector space.[19] Let $i \in \mathbb{N}_0$. The i-th Ekeland-Hofer capacity c_i^{EH} is a capacity on $\mathrm{Op}_{V,\omega}$, which is defined as a certain min-max involving the symplectic action, see [EH89, EH90] or [CHLS07, p. 7]. The capacity c_1^{EH} is normalized; the other Ekeland-Hofer capacities are not normalized.

[17] In [CHLS07] this question is asked, based on the definition of the embedding capacities on p. 13 in that article. However, [CHLS07, Example 2, p. 14] suggests that CHLS are interested in this question with the modified definition given in Example 5. Compare to footnote 9.

[18] A remark similar to footnote 17 applies.

[19] Hence m is even.

The Ekeland-Hofer capacities are hard to compute. Their values are known for ellipsoids and polydisks, see [EH90, Proposition 4, p. 562] and [EH90, Proposition 5, p. 563].

Let (V,ω) be a symplectic vector space. Recall that a (bounded, open, full) ellipsoid in V is a set of the form $p^{-1}((-\infty,0))$, where $p:V\to\mathbb{R}$ is a quadratic polynomial function whose second order part is positive definite. We equip each ellipsoid E with the restriction of ω to E. Consider the important full subcategory $\mathit{Ell}_V:=\mathit{Ell}_{V,\omega}$ of $\mathrm{Op}_{V,\omega}$, consisting of ellipsoids. The objects of Ell_V are uniquely determined by the Ekeland-Hofer capacities, up to isomorphism, see [CHLS07, FACT 10, p. 27]. Therefore the following question seems natural:

Question 12 ([CHLS07], Problem 15, p.28)**.** *Do the Ekeland-Hofer capacities together with the volume capacity form a generating system of the set of all generalized*[20] *capacities on the category of ellipsoids* Ell_V*?*[21]

In the case $\dim V=4$ this question was answered negatively by D. McDuff, see [McD09, Corollary 1.4].

Our second main result answers Question 12 in the negative in dimension at least 4, provided that we interpret "generating" to mean "finitely-differentiably generating". (See Definition 21 below.) In fact, every finitely-differentiably generating system on the category of ellipsoids is uncountable, see Corollary 26 below.

1.2. **Main results: Theorem 15 (cardinalities of the set of capacities and of the generated set) and Theorem 24 (uncountability of every generating set under a very mild hypothesis).** To state our first main result, we need the following. Let (X,τ) be a topological space. Recall that the (τ-)Borel σ-algebra of X is the smallest σ-algebra containing the topology of X. We call its elements *(τ-)Borel sets.*

Remark 13 (Borel sets)**.** Consider the real line $X=\mathbb{R}$. The axiom of choice (AC) implies that there exist subsets of $\mathbb{R}$ that are not Lebesgue-measurable, hence not Borel-measurable. However, all subsets occurring in practice are Borel. Furthermore, for any concretely described subset of $\mathbb{R}$, it appears to be difficult to prove (using AC) that it is indeed not Borel-measurable.[22]

Let now (X,τ) and (X',τ') be topological spaces. A function $f:X\to X'$ is called *(τ,τ')-Borel-measurable* iff the pre-image under f of every τ'-Borel set in X' is a τ-Borel set.[23] In particular, every continuous function is Borel-measurable. Borel-measurability is preserved under composition. It is preserved under pointwise limits of sequences if X' is metrizable. This yields many examples of Borel-measurable functions. In fact, all functions occurring in practice are Borel-measurable.

[20] In the paper the question is stated without the word "generalized", but from the discussion that precedes the question it is clear that the authors ask it for generalized capacities.

[21] One needs to include the volume capacity, since the Ekeland-Hofer capacities do not generate this capacity in the sense of [CHLS07], see [CHLS07, Example 10, p. 28].

[22] An example of such a subset A was provided by N. Luzin. It can be obtained from [Kec95, Exercise (27.2), p. 209] via [Kec95, Exercise (3.4)(ii), p. 14]. This set is $\boldsymbol{\Sigma}^1_1$-analytic, see [Kec95, Definitions (22.9), p. 169, (21.13), p. 156]. It follows from a theorem by Souslin, [Kec95, (14.2) Theorem, p. 85] and the definition of $\boldsymbol{\Sigma}^1_1$-analyticity that A is not Borel.

[23] This happens if and only if the pre-image under f of every element of τ' is a τ-Borel set.

Let S, S' be sets. We denote

$$S'^S := \{\text{function from } S \text{ to } S'\}.$$

For every subset $\mathcal{G} \subseteq S'^S$ we denote by

$$\mathrm{ev}_{\mathcal{G}} : S \to S'^{\mathcal{G}}, \quad \mathrm{ev}_{\mathcal{G}}(s)(u) := u(s), \tag{5}$$

the evaluation map. If (X, τ) is a topological space then we denote by τ_S the product topology on X^S.

Definition 14 (countably Borel-generated set)**.** *Let S be a set, (X, τ) a topological space, and $\mathcal{G} \subseteq X^S$. We define the* set countably (τ-)Borel-generated by $\mathcal{G}$ *to be*

$$\langle \mathcal{G} \rangle := \{ f \circ \mathrm{ev}_{\mathcal{G}_0} \,\big|\, \mathcal{G}_0 \subseteq \mathcal{G} \text{ countable, } f : X^{\mathcal{G}_0} \to X\text{: } (\tau_{\mathcal{G}_0}, \tau)\text{-Borel-measurable} \} \subseteq X^S.$$

For every subset $\mathcal{F} \subseteq X^S$ we say that $\mathcal{G}$ countably (τ-)Borel-generates at least $\mathcal{F}$ *iff $\mathcal{F} \subseteq \langle \mathcal{G} \rangle$.*

We denote by $\mathrm{int}\, S$ the interior of a subset S of a topological space. Let V be a vector space, $S \subseteq V$, $A \subseteq \mathbb{R}$, and $n \in \mathbb{N}_0$. We denote $AS := \{ av \,\big|\, a \in A,\ v \in S \}$. In the case $A = \{a\}$ we also denote this set by aS. We call S *strictly starshaped around 0* iff $[0, 1)S \subseteq \mathrm{int}\, S$. For every $i \in \{1, \ldots, n\}$ we denote by $\mathrm{pr}_i : V^n = V \times \cdots \times V \to V$ the canonical projection onto the i-th component. For every multilinear form ω on V we denote

$$\omega^{\oplus n} := \sum_{i=1}^{n} \mathrm{pr}_i^* \,\omega.$$

For every $r \in (1, \infty)$ we define the closed spherical shell of radii $1, r$ in $\mathbb{R}^m$ to be

$$Sh_r^m := \overline{B}_r^m \setminus B_1^m.$$

We equip $Sh_r := Sh_r^{2n}$ with the standard symplectic form ω_{st}. The first main result of this article is the following.

Theorem 15 (cardinalities of the set of (normalized) capacities and of the generated set)**.** *The following statements hold:*

(i) Let $k, n \in \{2, 3, \ldots\}$ with k even, and $\mathcal{C} = (\mathcal{O}, \mathcal{M})$ be a (kn, k)-form-category. Then the cardinalitiy of $\mathcal{C}ap(\mathcal{C})$ equals $\beth_2$, provided that there exist
- *a (real) vector space V of dimension k,*
- *a volume form Ω on V,[24]*
- *a nonempty compact submanifold K of V^n (with boundary) that is strictly starshaped around 0,*
- *a number $r \in \left(1, \sqrt[kn]{2}\right)$,*

such that defining $M_a := (r + a)K \setminus \mathrm{int}\, K$ and equipping this manifold with the restriction of $\Omega^{\oplus n}$, we have

$$M_a \sqcup M_{-a} \in \mathcal{O}, \quad \forall a \in (0, r - 1).\,^{25}$$

[24] By this we mean a nonvanishing top degree skewsymmetric form.

[25] Here $A \sqcup B$ denotes the disjoint union of two sets A, B. This can be defined in different ways, e.g. as the set consisting of all pairs $(0, a), (1, b)$, with $a \in A$, $b \in B$, or alternatively pairs $(1, a), (2, b)$. Based on this, we obtain two definitions of the disjoint union of two objects of $\Omega^{kn,k}$. The disjoint union defined in either way is isomorphic to the one defined in the other way. Since we assume $\mathcal{C}$ to be isomorphism-closed, the above spherical shell condition does not depend on the choice of how we define the disjoint union.

(ii) *Let* $n \in \{2, 3, \ldots\}$ *and* $\mathcal{C} = (\mathcal{O}, \mathcal{M})$ *be a* $(2n, 2)$*-form-category that contains the objects* B *and* Z*. The cardinality of* $\mathcal{NC}ap(\mathcal{C})$ *equals* $\beth_2$*, provided that there exists* $r \in \left(1, \sqrt[2n]{2}\right)$ *satisfying*

$$Sh_{r-a} \sqcup Sh_{r+a} \in \mathcal{O}, \qquad \forall a \in (0, r-1). \tag{6}$$

(iii) *Let* S *be a set and* (X, τ) *a separable metrizable topological space. If a subset of* X^S *has cardinality at most* $\beth_1$*, then the set it countably* τ*-Borel-generates has cardinality at most* $\beth_1$*.*

This result has the following immediate application. We define $\mathcal{O}_0$ as in (1), and τ_0 to be the standard topology on $[0, \infty]$, w.r.t. which it is homeomorphic to the interval $[0, 1]$.

Corollary 16 (cardinality of a generating set)**.** *(i) Under the hypotheses of Theorem 15(i) every subset of* $[0, \infty]^{\mathcal{O}_0}$ *that countably* τ_0*-Borel-generates at least* $\mathcal{C}ap(\mathcal{C})$ *has cardinality (strictly) bigger than* $\beth_1$*.*

(ii) *Under the hypotheses of Theorem 15(ii) every subset of* $[0, \infty]^{\mathcal{O}_0}$ *that countably* τ_0*-Borel-generates at least* $\mathcal{NC}ap(\mathcal{C})$ *has cardinality (strictly) bigger than* $\beth_1$*.*

These statements hold in particular for $\mathcal{C}$ given by the category of all symplectic manifolds of some fixed dimension, which is at least 4. This answers Question 7 negatively for this category.

Another direct consequence of Theorem 15 is the following. We say that *almost no* element of a given infinite set has a given property iff the subset of all elements with this property has smaller cardinality than the whole set.

Corollary 17 (representability)**.** *Under the hypotheses of Theorem 15(ii) almost no normalized capacity on* $\mathcal{C}$ *can be domain- or target-represented.*

It follows that under the hypotheses of Theorem 15(ii) there are as many normalized capacities that can be neither domain- nor target-represented, as there are normalized capacities overall (namely $\beth_2$). This provides some answers to Questions 9,10, and 11.

Proof of Corollary 17. The set of isomorphism classes of $2n$-dimensional manifolds together with 2-forms has cardinality $\beth_1$. This follows from Corollary 57 below. The image of this set under the map $[(M, \omega)] \mapsto c_{M,\omega}$ is the set of all domain-representable capacities. It follows that at most $\beth_1$ normalized capacities can be domain-represented. A similar statement holds for target-representation. The statement of Corollary 17 now follows from Theorem 15(ii). □

The proof technique for Corollary 16 can potentially also be used to show that certain sets of capacities do not recognize morphisms, see Remark 20 below.

Remarks.

- As Corollary 16 holds for (kn, k)-form categories with k even and $n \geq 2$, the fact that generating systems of capacities are large, is not a genuinely symplectic phenomenon.

- The proof of Theorem 15(ii) shows that the cardinality of the set of *discontinuous* normalized capacities is $\beth_2$. This improves the result by K. Zehmisch and the second author that discontinuous capacities exist, see [ZZ13].[26]
- The statements of Theorem 15(i,ii) and thus of Corollary 16 hold in a more general setting, see Theorem 42 and Proposition 43 below. In particular, let V, Ω be as in Theorem 15(i), $j \in \{1, 2, \ldots\}$, and for each $a \in \mathbb{R}$ let M_a be the complement of j disjoint open sets in some compact submanifold of V^n. The cardinality of $\mathcal{C}ap(\mathcal{C})$ equals $\beth_2$, provided that $M_a \sqcup M_{-a} \in \mathcal{O}$ [27] for every a, the volumes of the open sets are all equal (also for different a), the volume of each M_a is small enough and strictly increasing in a, and each M_a is 1-connected.
- Morally, Corollary 16 implies that every generating set of capacities has as many elements as there are capacities. More precisely, we denote by ZF the Zermelo-Fraenkel axiomatic system, and ZFC := ZF + AC. We claim that ZFC is consistent with the statement that under the hypotheses of Theorem 15(i) every subset of $[0, \infty]^{\mathcal{O}_0}$ that countably Borel-generates at least $\mathcal{C}ap(\mathcal{C})$ has the same cardinality as $\mathcal{C}ap(\mathcal{C})$ (namely $\beth_2$) [28].
  To see this, assume that the generalized continuum hypothesis (GCH) holds. This means that for every infinite cardinal α there is no cardinal strictly between α and 2^α. In particular, there is no cardinal strictly between $\beth_1$ and $\beth_2 = 2^{\beth_1}$. Hence under the hypotheses of Theorem 15(i) by Corollary 16(i) every subset of $[0, \infty]^{\mathcal{O}_0}$ that countably Borel-generates at least $\mathcal{N}\mathcal{C}ap(\mathcal{C})$ has cardinality at least $\beth_2$. By Theorem 15(i) this equals the cardinality of $\mathcal{C}ap(\mathcal{C})$. Since GCH is consistent with ZFC [29], the claim follows.

Remark 18 (comparison of different notions of generating systems)**.** Let $\mathcal{C}$ be a symplectic category and $\mathcal{G}$ a generating system of symplectic capacities on $\mathcal{C}$ in the sense of [CHLS07, Problem 5, p. 17] (see p. 5), with the extra condition that each combining function $f : [0, \infty]^n \to [0, \infty]$ is Borel-measurable. Then $\mathcal{G}$ countably Borel-generates $\mathcal{C}ap(\mathcal{C})$. (See Definition 14.)[30] This holds in particular if $\mathcal{G}$ is a generating system in the more restrictive sense proposed by CHLS after [CHLS07, Problem 5, p. 17], in which only the maximum and minimum are allowed as combining functions.

[26] The proof of [ZZ13, Theorem 1.2] actually shows that the spherical shell capacities used in that proof are all different. This implies that the set of discontinuous normalized symplectic capacities has cardinality at least $\beth_1$.

[27] In particular we assume here that M_a is a smooth submanifold of V^n.

[28] provided that ZF is consistent

[29] provided that ZF is consistent

[30] To see this, let $c \in \mathcal{C}ap(\mathcal{C})$. We choose a sequence of combining functions and capacities in $\mathcal{G}$ as in the definition of generating system in [CHLS07, Problem 5, p. 17]. We define $\mathcal{G}_0$ to be the set of all these capacities. Each combining function gives rise to a Borel-measurable function from $[0, \infty]^{\mathcal{G}_0}$ to $[0, \infty]$. Its restriction to the image of $\mathrm{ev}_{\mathcal{G}_0}$ is measurable w.r.t. the σ-algebra induced by the Borel σ-algebra. By assumption the sequence of these restrictions converges pointwise. The limit f is again measurable. Since its target space is $[0, \infty]$, an argument involving approximations by simple functions shows that f extends to a Borel-measurable function on $[0, \infty]^{\mathcal{G}_0}$. Hence $\mathcal{G}_0$ and f satisfy the conditions of Definition 14, as desired.

Definition 14 relaxes the conditions in the definition of a generating system in the sense of [CHLS07] in two ways:

- The combining functions are allowed to depend on countably many variables, i.e., elements of the generating system, not just on finitely many variables.
- The assumption that the combining functions are homogeneous and monotone is omitted.

Let $\mathcal{C}$ be a (m,k)-form category and $\mathcal{G} \subseteq \mathcal{C}ap(\mathcal{C})$. We say that $\mathcal{G}$ recognizes morphisms iff for each pair of objects $(M,\omega),(M',\omega')$ of $\mathcal{C}$ the following holds. Assume that $c(M,\omega) \leq c(M',\omega')$, for every $c \in \mathcal{G}$. Then there exists a $\mathcal{C}$-morphism from (M,ω) to (M',ω'). CHLS asked the following as a central question in the case in which $\mathcal{C}$ is a symplectic category and $\mathcal{G} = \mathcal{C}ap(\mathcal{C})$ (see [CHLS07, Question 1, p. 20]):

Question 19. *Does $\mathcal{G}$ recognize morphisms?*

Remark 20 (generation and recognition of morphisms)**.** Suppose the following:

(*) Every set that monotonely generates in the sense of the definition on p. 16 below, has cardinality bigger than $\beth_1$.

Let $\mathcal{G}$ be a set of cardinality at most $\beth_1$. Then $\mathcal{G}$ does not recognize morphisms. Hence the answer to Question 19 is "no". To see this, observe that by our assumption (*) the set $\mathcal{G}$ does not monotonely generate. Therefore, by Proposition 59 below, it does not recognize morphisms.

By Corollary 16, under the hypotheses of Theorem 15(i), condition (*) is satisfied with "monotonely generates" replaced by "countably τ_0-Borel-generates". Therefore potentially, the proof technique for Corollary 16 may be adapted, in order to provide a negative answer to Question 19 under suitable conditions on $\mathcal{C}$ that do not involve (*).

Our *second main result* provides a condition that morally speaking, is weaker than those of Theorem 15(i,ii), and under which every generating set of capacities is still uncountable. Here we use the following more restrictive meaning of "generating".

Definition 21 (Finitely differentiably generating set)**.** *Let S be a set, and $\mathcal{F},\mathcal{G} \subseteq [0,\infty]^S$. We say that $\mathcal{G}$* finitely-differentiably generates at least *$\mathcal{F}$ iff the following holds. For every $F \in \mathcal{F}$ there exists a finite subset $\mathcal{G}_0 \subseteq \mathcal{G}$ and a differentiable map $f : [0,\infty]^{\mathcal{G}_0} \to [0,\infty]$, such that $F = f \circ ev_{\mathcal{G}_0}$.*[31]

Let now $k,n \in \mathbb{N} := \{1,2,\ldots\}$ and (M,ω) be an object of $\Omega^{kn,k}$. We call ω *maxipotent* iff $\omega^{\wedge n} = \omega \wedge \cdots \wedge \omega$ does not vanish anywhere.

Remark 22 (maxipotency and nondegeneracy)**.** Let V be a (real) vector space and $k \in \mathbb{N}$. We call a k-linear form ω on V *nondegenerate* iff interior multiplication with ω is an injective map from V to the space of $(k-1)$-linear forms. Let $k,n \in \mathbb{N}$ and assume that $\dim V = kn$. We call a skewsymmetric k-form ω on V *maxipotent*

[31]Here we view $[0,\infty]$ as a compact 1-dimensional manifold with boundary. Its Cartesian power is a manifold with boundary and corners. The map f is only assumed to be differentiable one time, with possibly discontinuous derivative.

iff $\omega^{\wedge n} \neq 0$. Every maxipotent form on V is nondegenerate. The converse holds if and only if $k = 1$, $k = 2$, or $n = 1$.

Let (M, ω) be a maxipotent object of $\Omega^{kn,k}$. We equip M with the orientation induced by $\omega^{\wedge n}$ and define

$$\mathrm{Vol}(M) := \mathrm{Vol}(M, \omega) := \frac{1}{n!} \int_M \omega^{\wedge n}. \tag{7}$$

Remark 23 (volume)**.** Assume that k is odd. Then we have $\omega \wedge \omega = 0$, and therefore $\mathrm{Vol}(M, \omega) = 0$ in the case $n \geq 2$.

Our second main result is the following.

Theorem 24 (uncountability of every generating set under a very mild hypothesis)**.** *Let $\mathcal{C} = (\mathcal{O}, \mathcal{M})$ be a small*[32] *(kn, k)-form-category. Then every subset of $\mathcal{C}ap(\mathcal{C})$ that finitely differentiably generates (at least) $\mathcal{C}ap(\mathcal{C})$, is uncountable, provided that there exists an interval A of positive length and a function $M : A \to \mathcal{O}$, such that*

$$M_a := M(a) \text{ is maxipotent for every } a \in A, \tag{8}$$

$$\mathrm{Vol} \circ M \text{ is continuous and strictly increasing,} \tag{9}$$

$$c_{M_a}(M_{a'}) = 1, \forall a, a' \in A : \, a \leq a'. \tag{10}$$

Remarks.

- Condition (8) ensures that the volume of each M_a is well-defined. Hence condition (9) makes sense.
- Condition (10) means that M is "embedding-capacity-wise constant", in the sense that the composition of the map $\big\{(a, a') \in A^2 \,\big|\, a \leq a'\big\} \ni (a, a') \mapsto (M_a, M_{a'})$ with the "embedding capacity map" $(X, X') \mapsto c_X(X')$ is constant.
- Assume that there exists a function M satisfying (8,9). Then we have $n > 0$. If $n \geq 2$, then k is even. This follows from Remark 23. Assume that there exists a function satisfying (8,9,10). Then we have $k > 0$. If each M_a is compact, then $n \neq 1$. This follows from Moser's isotopy argument.

Example 25. Let $n \geq 2$ and A be an interval of positive length. We denote by $\mathcal{U}$ the set of all open subsets of $\mathbb{R}^{2n}$ that contain B_1^{2n} and are contained in Z_1^{2n}. We equip each element of $\mathcal{U}$ with the restriction of the form ω_{st}. Let $M : A \to \mathcal{U}$ be an increasing map in the sense that $a \leq a'$ implies that $M(a) \subseteq M(a')$. If M also satisfies (9) then it satisfies all conditions of Theorem 24. The inequality "$\leq$" in condition (10) follows from Gromov's nonsqueezing theorem.

Corollary 26 (uncountability of every generating set for ellipsoids)**.** *Let V be a symplectic vector space of dimension at least 4. Then every subset of $\mathcal{C}ap(\mathit{Ell}_V)$ that finitely-differentiably generates $\mathcal{C}ap(\mathit{Ell}_V)$, is uncountable.*

Proof. This follows from Theorem 24 and Example 25 by considering the ellipsoids

$$M_a := \left\{ x = \big(x_1, \ldots, x_n\big) \in \mathbb{R}^{2n} = (\mathbb{R}^2)^n \,\middle|\, \sum_{i=1}^{n-1} \|x_i\|^2 + \frac{\|x_n\|^2}{a} < 1 \right\},$$

[32] A category is called small iff the objects and the morphisms form sets.

for $a \in A := [1, \infty)$. Here $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^2$. Our hypothesis $n \geq 2$ guarantees that the inequality "$\leq$" in (10) holds. $\square$

In particular, the Ekeland-Hofer capacities together with the volume capacity do not finitely-differentiably generate the set of all generalized capacities on Ell_V. This provides a negative answer to the variant of Question 12, involving the notion of finite-differentiable generation.

Remark. The hypotheses of Theorems 15(i) and 24 do not imply each other. However morally, the hypotheses of Theorem 15(i) are more restrictive than those of Theorem 24. On the other hand, Theorem 15 directly implies Corollary 16, the conclusion of which is stronger than that of Theorem 24.

1.3. **Ideas of the proofs.** The idea of the proof of Theorem 15(i) is the following. Recall the definition (2) of the embedding capacity $c_M := c_{M,\omega}$. We choose/ define V, Ω, K, r, M_a as in the hypothesis of the theorem. We define $W_a := M_a \sqcup M_{-a}$. For each $A \in \mathcal{P}((0, r-1))$ [33] we define

$$c_A := \sup_{a \in A} c_{W_a}.$$

This is a symplectic capacity, satisfying

$$c_A(W_a) = 1, \quad \forall a \in A, \tag{11}$$

$$\sup_{a \in (0,r-1) \setminus A} c_A(W_a) < 1. \tag{12}$$

The second statement follows from an argument involving the helicity of an exact k-form on a manifold of dimension $kn - 1$. (To build some intuition, see the explanations on p. 21 and the Figures 1,2,3.) Helicity generalizes contact volume. It is related to the volume induced by an exact k-form on an kn-manifold via a variant of Stokes' Theorem. The conditions (11,12) imply that $c_A \neq c_{A'}$ if $A \neq A' \in \mathcal{P}((0, r-1))$. Since the cardinality of $\mathcal{P}((0, r-1))$ equals $\beth_2$, it follows that the cardinality of $\mathcal{C}ap(\mathcal{C})$ is at least $\beth_2$.

On the other hand, we denote by S the set of equivalence classes of symplectic manifolds. This set has cardinality $\beth_1$. Since $\mathcal{C}ap(\mathcal{C})$ can be viewed as a subset of $[0, \infty]^S$, it has cardinality at most $\beth_2$, hence equal to $\beth_2$.

A refined version of this argument shows Theorem 15(ii), i.e., that $|\mathcal{NC}ap(\mathcal{C})| = \beth_2$. For this we normalize each capacity c_A, by replacing it by the maximum of c_A and the Gromov width.

The proof of Theorem 15(iii) is based on the fact that the set of Borel-measurable maps from a second countable space to a separable metrizable space has cardinality at most $\beth_1$. The proof of this uses the following well-known results:

- Every map f with target a separable metric space is determined by the pre-images under f of balls with rational radii around points in a countable dense subset.
- The σ-algebra generated by a collection of cardinality at most $\beth_1$ has itself cardinality at most $\beth_1$. The proof of this uses transfinite induction.

[33] Here $\mathcal{P}(S)$ denotes the power set of a set S.

The idea of the proof of Theorem 24 is to exploit the fact that every monotone function on an interval is differentiable almost everywhere. It follows that for every countable set $\mathcal{G}$ of symplectic capacities, there exists a point $a_0 \in A$ at which the function $a \mapsto c(M_a)$ is differentiable, for every $c \in \mathcal{G}$. On the other hand, our hypotheses imply that the map $a \mapsto c_{M_{a_0}}(M_a)$ is not differentiable at a_0. It follows that $\mathcal{G}$ does not finitely differentiably generate $c_{M_{a_0}}$.

Remark (helicity)**.** In [ZZ13] K. Zehmisch and F. Ziltener used helicity to show that the spherical capacity is discontinuous on some smooth family of ellipsoidal shells. This argument is related to the proof of Theorem 15(i,ii).

1.4. **Related work.** In this subsection we recall a result by D. McDuff, which states that the ECH-capacities are monotonely generating for the category of ellipsoids in dimension 4. On ellipsoids, these capacities are given by the following. Let $n, j \in \mathbb{N}_0$. We define the map

$$\mathcal{N}_j^n : [0,\infty)^n \to [0,\infty),$$
$$\mathcal{N}_j^n(a) := \min \left\{ b \in [0,\infty) \,\middle|\, j+1 \le \# \left\{ m \in \mathbb{N}_0^n \,\middle|\, m \cdot a = \sum_{i=1}^n m_i a_i \le b \right\} \right\}.$$

Remark. The sequence $\big(\mathcal{N}_j^n(a)\big)_{j \in \mathbb{N}_0}$ is obtained by arranging all the nonnegative integer combinations of $a_1, \ldots, a_n$ in increasing order, with repetitions.

We define the ellipsoid

$$E(a) := \left\{ x = \big(x_1, \ldots, x_n\big) \in \mathbb{R}^{2n} = (\mathbb{R}^2)^n \,\middle|\, \sum_{i=1}^n \frac{\|x_i\|^2}{a_i} < 1 \right\}.$$

(Here $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^2$.) We equip this manifold with the standard symplectic form.

Let (V, ω) be a symplectic vector space. We denote by $\mathcal{O}_{V,\omega}$ the set of all open ellipsoids in V, equipped with the restriction of ω, by $\mathcal{M}_{V,\omega}$ the set of all symplectic embeddings between elements of $\mathcal{O}_{V,\omega}$, and $ELL_{V,\omega} := (\mathcal{O}_{V,\omega}, \mathcal{M}_{V,\omega})$. For every $j \in \mathbb{N}_0$ we define the map

$$c_j^{V,\omega} : \mathcal{O}_{V,\omega} \to [0,\infty), \tag{13}$$

by setting $c_j^{V,\omega}(E) := \mathcal{N}_j^n(a)$, where $a \in [0,\infty)^n$ is such that E is affinely symplectomorphic to $E(a)$. This number is well-defined, i.e., such an a exists (see [MS98, Lemma 2.43]) and $\mathcal{N}_j^n(a)$ does not depend on its choice. The latter is true, since if $E(a)$ and $E(a')$ are affinely symplectomorphic, then a and a' are permutations of each other. (See [MS98, Lemma 2.43].) The following result is due to M. Hutchings.

Theorem 27. *If* $\dim V = 4$ *then for every* $j \in \mathbb{N}_0$ *the map* $c_j^{V,\omega}$ *is a generalized capacity.*

Proof. Homogeneity follows from the definition of $\mathcal{N}_j^n$. Monotonicity was proved by M. Hutchings in [Hut11, Proposition 1.2, Theorem 1.1]. □

Remark. $c_j^{V,\omega}$ is the restriction of the j-th ECH-capacity to $ELL_{V,\omega}$, see [Hut11, Proposition 1.2].

D. McDuff proved that the set of all $c_j^{V,\omega}$ (with $j \in \mathbb{N}_0$) monotonely generates all generalized capacities. To explain this, let S, S' be sets. We fix $(0,\infty)$-actions on S and S' and call a map $f : S \to S'$ *(positively 1-)homogeneous* iff it is $(0,\infty)$-equivariant. We equip the interval $(0,\infty)$ with multiplication and let it act on the extended interval $[0,\infty]$ via multiplication.

Recall that a preorder on a set S is a reflexive and transitive relation on S. We call a map f between two preordered sets *monotone (or increasing)* if it preserves the preorders, i.e., if $s \leq s'$ implies that $f(s) \leq f(s')$. Let $(S, \leq)$ be a preordered set. We fix an order-preserving $(0,\infty)$-action on S. We define the set of *(generalized) capacities on S* to be

$$\mathcal{C}ap(S) := \left\{c \in [0,\infty]^S \,\middle|\, c \text{ monotone and } (0,\infty)\text{-equivariant}\right\}. \tag{14}$$

We equip the set $[0,\infty]^S$ with the preorder

$$x \leq x' \iff x(s) \leq x'(s),\ \forall s \in S.$$

Let $\mathcal{G} \subseteq \mathcal{C}ap(S)$. We say that $\mathcal{G}$ *monotonely generates* iff for every $c \in \mathcal{C}ap(S)$ there exists a monotone function $F : [0,\infty]^{\mathcal{G}} \to [0,\infty]$, such that $c = F \circ \mathrm{ev}_{\mathcal{G}}$. We say that $\mathcal{G}$ *homogeneously and monotonely generates* iff the function F above can also be chosen to be homogeneous.

Remark. The set $\mathcal{G}$ monotonely generates if and only if it homogeneously and monotonely generates. The "only if"-direction follows by considering the monotonization (see p. 46 below) of the restriction of F as above to $\mathrm{im}(\mathrm{ev}_{\mathcal{G}})$. Here we use that every $c \in \mathcal{C}ap(S)$ is homogeneous, and thus $F|\,\mathrm{im}(\mathrm{ev}_{\mathcal{G}})$ is homogeneous, as well as Remark 60 below.

Let (V, ω) be a symplectic vector space. Recall the definition (13) of the capacity $c_j^{V,\omega}$. The next result easily follows from D. McDuff's solution of the embedding problem for ellipsoids in dimension 4.

Theorem 28 (monotone generation for ellipsoids in dimension 4)**.** *If* $\dim V = 4$ *then the set of all* $c_j^{V,\omega}$ *(with* $j \in \mathbb{N}_0$*) monotonely generates (the generalized capacities on the category of ellipsoids* $ELL_{V,\omega}$*).*

This theorem provides a positive answer to the variant of Question 7 with "generating" in the sense of CHLS replaced by "monotonely generating". Monotone generation is (possibly nonstrictly) weaker than generation in the sense of CHLS, since the pointwise limit of monotone functions is monotone. To deduce the theorem from McDuff's result, we characterize monotone generation in terms of almost order-recognition, see Section B.

1.5. **Organization of this article.** In Section 2 we formulate Theorem 42, which states that the cardinality of the set of (normalized) capacities is $\beth_2$ for every (kn,k)-form-category containing a suitable family of objects $(W_a, \omega_a)_{a\in A_0}$. This result generalizes Theorem 15(i,ii). A crucial hypothesis is the following. We denote by I_a the set of connected components of the boundary of W_a, and $I := (I_a)_{a\in A_0}$. Then the collection of boundary helicities associated with $(W_a, \omega_a)_{a\in A_0}$ is an I-collection. We introduce the notions of helicity and of an I-collection in this section. We also state Proposition 43, which provides sufficient criteria for the helicity hypothesis of Theorem 42.

In Sections 3 and 4 we prove Theorem 42 and Proposition 43. Section 5 contains the proof of Theorem 15(iii), which states that every set of cardinality at most $\beth_1$ countably Borel-generates a set of cardinality at most $\beth_1$.

Section 6 is devoted to the proof of our second main result, Theorem 24, stating that every finitely differentiably generating set of capacities is uncountable under some very mild hypothesis.

In Section A we prove an auxiliary result, which state that the set of diffeomorphism classes of manifolds has cardinality $\beth_1$. We also show that the same holds for the set of all equivalence classes of (M, ω), where M is a manifold and ω a differential form on M.

Finally, in Section B we deduce Theorem 28 (monotone generation for ellipsoids) from McDuff's characterization of the existence of symplectic embeddings between ellipsoids.

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2. Proof of Theorem 15(i,ii) (cardinality of the set of capacities)

In this section we prove Theorem 15(i,ii), based on a more general result, Theorem 42 below. That result states that the set of (normalized) capacities on a given (kn, k)-category $\mathcal{C}$ has cardinality $\beth_2$, provided that $\mathcal{C}$ contains a suitable family of objects $(W_a, \omega_a)_{a \in A_0}$. A crucial hypothesis is that the collection of boundary helicities associated with $(W_a, \omega_a)_{a \in A_0}$, is an I-collection.

We also state Proposition 43, which provides sufficient conditions for this hypothesis to be satisfied.

2.1. (Boundary) helicity of an exact differential form.

In this subsection we introduce the notion of helicity of an exact form, and based on this, the notion of boundary helicity.

Let $k, n \in \mathbb{N}_0$ be such that $n \geq 2$, N a closed[34] $(kn-1)$-manifold, O an orientation on N, and σ an exact k-form on N.

Definition 29 (helicity)**.** *We define the helicity of (N, O, σ) to be the integral*

$$h(N, O, \sigma) := \int_{N,O} \alpha \wedge \sigma^{\wedge(n-1)}, \tag{15}$$

where α is an arbitrary primitive of σ, and $\int_{N,O}$ denotes integration over N w.r.t. O.

We show that this number is well-defined, i.e., it does not depend on the choice of the primitive α. Let α and α' be primitives of σ. Then $\alpha' - \alpha$ is closed, and therefore

$$(\alpha' - \alpha) \wedge \sigma^{\wedge(n-1)} = (-1)^{k-1} d\left((\alpha' - \alpha) \wedge \alpha \wedge \sigma^{\wedge(n-2)}\right).$$

[34]This means compact and without boundary.

Here we used that $n \geq 2$. Using Stokes' Theorem and our assumption that N has no boundary, it follows that

$$\int_{N,O} (\alpha' - \alpha) \wedge \sigma^{\wedge(n-1)} = 0.$$

Therefore, the integral (15) does not depend on the choice of α.

Remark 30 (case k odd, case $n = 1$)**.** The helicity vanishes if k is odd. This follows from the equality

$$\alpha \wedge (d\alpha)^{n-1} = \frac{1}{2} d\left(\alpha^{\wedge 2} \wedge (d\alpha)^{n-2}\right),$$

which holds for every even-degree form α, and from Stokes' Theorem. The helicity is not well-defined in the case $n = 1$. Namely, in this case $\dim N = k - 1$, and therefore every $(k-1)$-form is a primitive of the k-form 0. Hence the integral (15) depends on the choice of a primitive.

Remark 31 (orientation)**.** Denoting by $\overline{O}$ the orientation opposite to O, we have

$$h(N, \overline{O}, \sigma) = -h(N, O, \sigma).$$

Remark 32 (rescaling)**.** For every $c \in \mathbb{R}$ we have

$$h\big(N, O, c\sigma\big) = c^n h\big(N, O, \sigma\big).$$

This follows from a straight-forward argument.

Remark 33 (naturality)**.** Let N and N' be closed $(kn-1)$-manifolds, O an orientation on N, σ an exact k-form on N, and $\varphi : N \to N'$ a (smooth) embedding. We denote

$$\varphi_*(N, O, \sigma) := \big(\varphi(N), \varphi_* O, \varphi_* \sigma\big)$$

(push-forwards of the orientation and the form). A straight-forward argument shows that

$$h\big(\varphi_*\big(N, O, \sigma\big)\big) = h(N, O, \sigma).$$

Remark 34 (helicity of a vector field)**.** In the case $k = 2$ and $n = 2$ the integral (15) equals the helicity of a vector field V on a three-manifold N, which is dual to the two-form σ, via some fixed volume form. See [AK98, Definition 1.14, p. 125]. This justifies the name “helicity” for the map h.

The helicity of the boundary of a compact manifold equals the volume of the manifold. This is a crucial ingredient of the proofs of the main results and the content of the following lemma. Let M be a manifold, $N \subseteq M$ a submanifold, and ω a differential form on M. We denote by ∂M the boundary of M, and

(16) $$\omega_N := \text{pullback of } \omega \text{ by the canonical inclusion of } N \text{ into } M.$$

If O is an orientation on M and N is contained in ∂M, then we define

(17) $$O_N := O_N^M := \text{orientation of } N \text{ induced by } O.$$

Let $k, n \in \mathbb{N}_0$, such that $n \geq 2$, (M, O) be a compact oriented (smooth) manifold of dimension kn and ω an exact k-form on M.

Lemma 35 (volume = helicity)**.** *The following equality holds:*

$$\int_{M,O} \omega^{\wedge n} = h\big(\partial M, O_{\partial M}, \omega_{\partial M}\big).$$

Remark. The left hand side of this equality is $n!$ times the signed volume of M associated with O and ω.

Proof of Lemma 35. Choosing a primitive α of ω, we have

$$\omega^{\wedge n} = d\big(\alpha \wedge \omega^{\wedge(n-1)}\big),$$

and therefore, by Stokes' Theorem,

$$\int_{M,O} \omega^{\wedge n} = \int_{\partial M, O_{\partial M}} \alpha \wedge \omega^{\wedge(n-1)} = h(\partial M, O_{\partial M}, \omega_{\partial M}).$$

This proves Lemma 35. □

This lemma has the following consequence. We denote

$$I_M := \big\{\text{connected component of } \partial M\big\}.$$

Definition 36 (boundary helicity)**.** *We define the* boundary helicity of (M, O, ω) *to be the map*

$$h_M := h_{M,O,\omega} : I_M \to \mathbb{R}, \quad h_{M,O,\omega}(i) := h\big(i, O_i, \omega_i\big),$$

Corollary 37 (volume = helicity)**.** *The following equality holds:*

$$\int_{M,O} \omega^{\wedge n} = \sum_{i \in I_M} h\big(i, O_i, \omega_i\big).$$

Proof. This directly follows from Lemma 35. □

2.2. I**-collections.** An I-collection is collection $f = (f_a)_{a \in A_0}$ of real-valued functions with finite domains, such that the supremum of a certain set of numbers is less than 1. The set consists of all numbers C for which $A \cup B$ is nonempty, where A and B are certain sets of partitions, which depend on f and C. I-collections will occur in the generalized main result, Theorem 42 below. Namely, one hypothesis of this result is that the boundary helicities of a certain collection of manifolds and forms, are an I-collection.

Definition 38. *Let I and I' be finite sets. A (I, I')-partition is a partition $\mathcal{P}$ of the disjoint union $I \sqcup I'$, such that*

$$\forall J \in \mathcal{P} : \; |J \cap I| = 1. \tag{18}$$

Let $f : I \to \mathbb{R}$, $f' : I' \to \mathbb{R}$, and $C \in (0, \infty)$. For every $J \subseteq I \sqcup I'$ we define

$$\sum\nolimits_{J,f,f',C} := -C \sum_{i \in J \cap I} f(i) + \sum_{i' \in J \cap I'} f'(i'). \tag{19}$$

A (f, f', C)-partition *is a (I, I')-partition $\mathcal{P}$ such that*

$$\sum\nolimits_{J,f,f',C} \geq 0, \quad \forall J \in \mathcal{P}. \tag{20}$$

Definition 39. *Let I^+, I^-, I' be finite sets. We denote $I := I^+ \sqcup I^-$.* A $\big(I^+, I^-, I'\big)$-partition *is a partition $\mathcal{P}$ of $I \sqcup I'$ with the following properties:*

(a) There exists a unique element of $\mathcal{P}$ that intersects both I^+ and I^- in exactly one point.
(b) All other $J \in \mathcal{P}$ intersect I in exactly one point.

Let $f^\pm : I^\pm \to \mathbb{R}$, $f' : I' \to \mathbb{R}$, and $C \in (0,\infty)$. We denote by $f := f^+ \sqcup f^- : I \to \mathbb{R}$ the disjoint union of functions.[35] *A* (f^+, f^-, f', C)-partition *is a* (I^+, I^-, I')-*partition satisfying (20).*

Remark 40. Every (I^+, I^-, I')-partition $\mathcal{P}$ satisfies

$$|\mathcal{P}| = |I| - 1.$$

Let A_0 be an interval and I a collection of finite sets indexed by A_0, i.e., a map from A_0 to the class of all finite sets. We denote $I_a := I(a)$. Let $f = \big(f_a : I_a \to \mathbb{R}\big)_{a\in A_0}$ be a collection of functions. We define

$$C_0^f := \sup\big\{C \in (0,\infty) \,\big|\, \exists a, a' \in A_0 :\ a > a',\ \exists\big(f_a, f_{a'}, C\big)\text{-partition}\big\}, \tag{21}$$

$$\begin{aligned} C_1^f \quad := \sup\big\{C \in (0,\infty) \,\big|\ \ & \exists a, a' \in A_0 \cap (0,\infty) :\ a < a', \\ & \exists\big(f_a, f_{-a}, f_{a'}, C\big)\text{-partition}\big\}. \end{aligned} \tag{22}$$

Here we use the convention that $\sup \emptyset := 0$.

Definition 41 (I-collection)**.** *We call f an I-collection iff the following holds:*

$$C_0^f < 1, \tag{23}$$

$$C_1^f < 1. \tag{24}$$

Remark. *The condition of being an I-collection is invariant under rescaling by some positive constant.*

2.3. **Cardinality of the set of capacities in a more general setting, sufficient conditions for being an I-collection, proof of Theorem 15(i,ii).** Theorem 15(i,ii) is a special case of the following more general result. We call a k-form ω on a kn-manifold *maxipotent* iff $\omega^{\wedge n} = \omega \wedge \cdots \wedge \omega$ does not vanish anywhere. [36]In this case we denote

$$O_\omega := \text{orientation on } M \text{ induced by } \omega^{\wedge n}.$$

Recall that B, Z denote the unit ball and the standard symplectic cylinder, ω_{st} the standard symplectic form, $c_{M,\omega}$ the embedding capacity for (M,ω) as in (2), and w the Gromov width.

Theorem 42 (cardinality of the set of (normalized) capacities, more general setting)**.** *The following holds:*

(i) Let $k, n \in \{2,3,\ldots\}$ with k even, and $\mathcal{C} = (\mathcal{O}, \mathcal{M})$ be a (kn,k)-form-category. Then the cardinality of $\mathcal{C}ap(\mathcal{C})$ equals $\beth_2$, provided that there exist an interval A_0 around 0 of positive length, and a collection $(M_a, \omega_a)_{a\in A_0}$ of objects of $\Omega^{kn,k}$, such that for every $a \in A_0$, M_a is nonempty, compact, and 1-connected,[37] *ω_a is maxipotent and exact, and the following holds:*
   (a) $(W_a, \eta_a) := (M_a \sqcup M_{-a}, \omega_a \sqcup \omega_{-a}) \in \mathcal{O}$, for every $a \in A_0 \cap (0,\infty)$.

[35]This is the function defined by $f(i) := f^\pm(i)$ if $i \in I^\pm$.

[36]See Remark 22 for the relation between maxipotency and nondegeneracy.

[37]This means connected and simply connected.

(b) We denote by I_a the set of connected components of ∂M_a, and $I := (I_a)_{a\in A_0}$. The collection of boundary helicities $f := \big(h_{M_a,O_{\omega_a},\omega_a}\big)_{a\in A_0}$ is an I-collection.

(ii) Let $n \in \{2, 3, \dots, \}$ and $\mathcal{C} = (\mathcal{O}, \mathcal{M})$ be a symplectic category that contains the objects B and Z. Then the cardinality of $\mathcal{NC}ap(\mathcal{C})$ equals $\beth_2$, provided that there exist A_0 and $(M_a, \omega_a)_{a\in A_0}$ as in (i), such that also the following holds:

(a) $\sup_{a\in A_0} w(M_a, \omega_a) < 1$

(b) $\sup_{a\in A_0} c_{M_a,\omega_a}(Z, \omega_{\mathrm{st}}) \le \pi$

We will prove this theorem in Section 3. The idea of the proof is to consider the family of capacities

$$c_A := \sup_{a\in A} c_{W_a,\eta_a}, \quad A \in \mathcal{P}\big(A_0 \cap (0,\infty)\big).$$

Hypothesis (ib) implies that there exists $c_0 < 1$ such that for all $a \ne a' \in (0, \infty)$ and $c \ge c_0$, the pair $(W_a, c\eta_a)$ does not embed into $(W_{a'}, \eta_{a'})$. See the explanations below. It follows that

$$\sup\big\{c_A(W_a, \eta_a) \,\big|\, a \in A_0 \cap (0, \infty) \setminus A\big\} < 1, \quad \forall A.$$

Since also $c_A(W_a, \eta_a) = 1$, for every $a \in A$, it follows that

$$c_A \ne c_{A'}, \quad \text{if } A \ne A'.$$

Since the cardinality of $\mathcal{P}((0, \infty))$ equals $\beth_2$, it follows that the cardinality of $\mathcal{C}ap(\mathcal{C})$ is at least $\beth_2$. On the other hand, we denote by S the set of equivalence classes of symplectic manifolds. This set has cardinality $\beth_1$. Since $\mathcal{C}ap(\mathcal{C})$ can be viewed as a subset of $[0, \infty]^S$, it has cardinality at most $\beth_2$, hence equal to $\beth_2$.

A refined version of this argument shows that $|\mathcal{NC}ap(\mathcal{C})| = \beth_2$. For this we normalize each capacity c_A, by replacing it by the maximum of c_A and the Gromov width. Hypothesis (iia) guarantees that the modified capacities are still all different from each other. Hypothesis (iib) guarantees that they are normalized.

To understand the reason why no big multiple of (W_a, η_a) embeds into $(W_{a'}, \eta_{a'})$, consider the case in which each M_a is a spherical shell in a symplectic vector space, with inner radius 1 and outer radius $r + a$ for some fixed $r > 1$. Assume that $(M_a, c\omega_a)$ embeds into $(M_{a'}, \omega_{a'})$ in such a way that the image of the inner boundary sphere of M_a wraps around the inner boundary sphere of $M_{a'}$. By Corollary 37 (Stokes' Theorem for helicity) and Remark 31 the difference of the helicities of these spheres equals the enclosed volume on the right hand side. Since this volume is nonnegative, it follows that $c \ge 1$. Using our hypothesis (ib) that the collection of boundary helicities is an I-collection, it follows that $a \le a'$.

It follows that if $a > a'$ then no multiple of W_a (symplectically) embeds into $W_{a'}$ in such a way that the inner boundary sphere of M_a wraps around one of the two inner boundary spheres of $W_{a'}$. Figure 1 illustrates this. In contrast with this, Figure 2 shows a possible embedding. In this case our helicity hypothesis (ib) implies that the rescaling factor is small.

If $a < a'$ then M_a embeds into $M_{a'}$ (without rescaling). However, there is not enough space left for M_{-a}. See Figure 3.

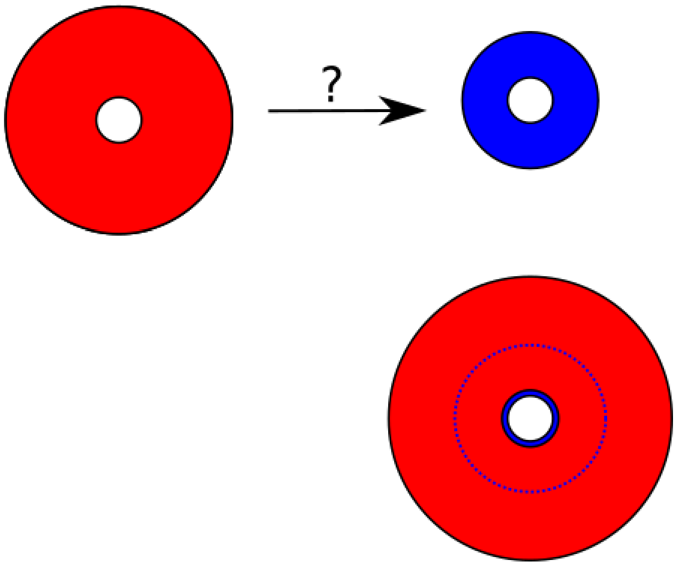

FIGURE 1. If $a > a'$ then no multiple of the red spherical shell M_a (symplectically) embeds into the blue shell $M_{a'}$ in such a way that the inner boundary sphere of the red shell wraps around the inner boundary sphere of the blue shell, since our helicity hypothesis (ib) forces the rescaling factor to be at least 1.

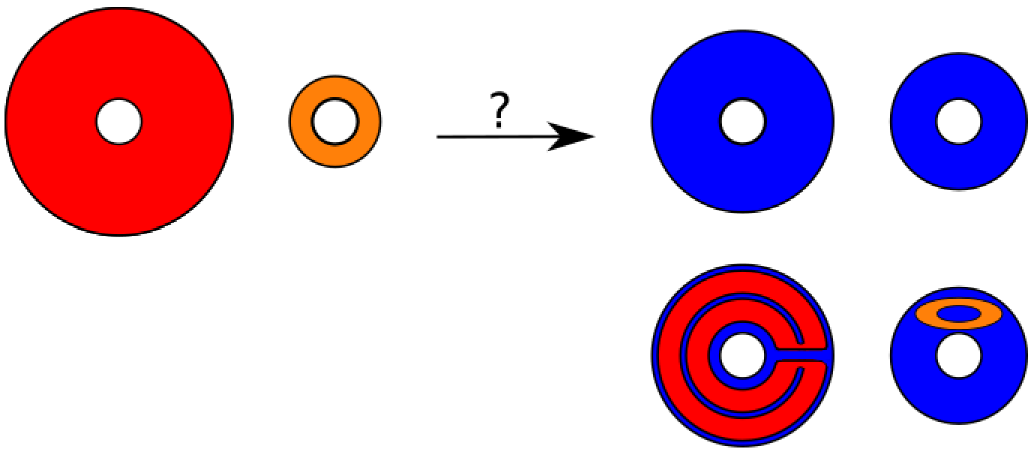

FIGURE 2. A possible embedding of $(W_a, c\eta_a)$ into $(W_{a'}, \eta_{a'})$ in the case $a > a'$. The constant c needs to be small (even if a is close to a'), since the volume of the hole enclosed by the image of M_a equals minus c times the helicity of the inner boundary sphere of M_a. Here we use again our helicity hypothesis (ib).

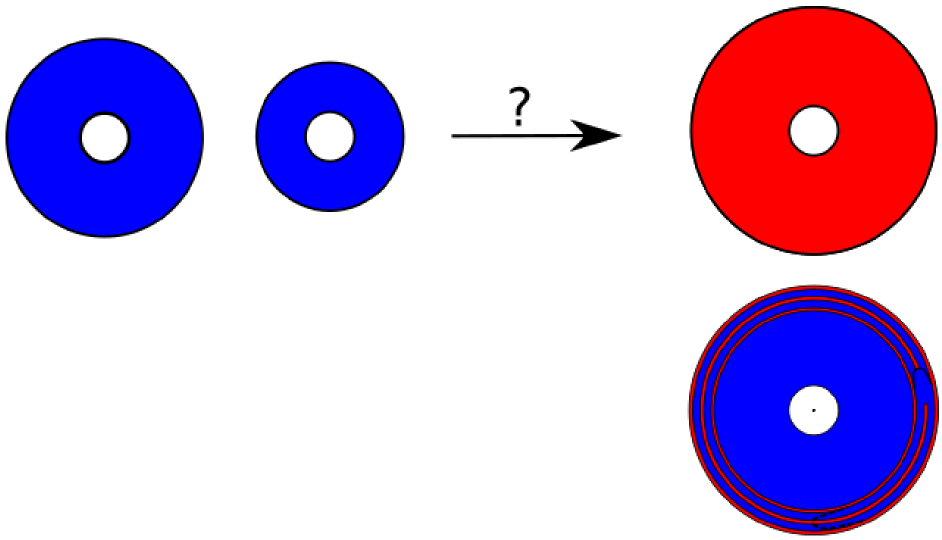

FIGURE 3. An attempt for an embedding of W_a into $M_{a'}$ in the case $a < a'$ (without rescaling). The image of M_{-a} overlaps itself, since there is not enough space left in $M_{a'}$.

In the proof of Theorem 15(i) we will use the following sufficient criterion for condition (ib) of Theorem 42. For every finite set S and every function $f : S \to \mathbb{R}$ we denote

$$\sum f := \sum_{s \in S} f(s). \tag{25}$$

Let A_0 be an interval, $I := (I_a)_{a \in A_0}$ a collection of finite sets, and $f = \big(f_a : I_a \to \mathbb{R}\big)_{a \in A_0}$ a collection of functions. We define the disjoint unions of I and f to be

$$\bigsqcup I := \bigsqcup_{a \in A_0} I_a := \big\{(a,i) \,\big|\, a \in A_0,\ i \in I_a\big\},$$
$$\bigsqcup f : \bigsqcup I \to \mathbb{R}, \quad \bigsqcup f(a,i) := f_a(i).$$

Proposition 43 (sufficient conditions for being an I-collection)**.** *The collection f is an I-collection if there exists $\ell \in \mathbb{N}_0$, such that the following holds:*

(i) For all $a \in A_0$ we have

$$|I_a| = \ell, \tag{26}$$
$$f_a \geq -1, \tag{27}$$
$$f_a^{-1}(-1) \neq \emptyset, \tag{28}$$
$$|f_a^{-1}((0,\infty))| = 1, \tag{29}$$
$$\sum f_a \leq 1. \tag{30}$$

(ii) For all $a, a' \in A_0$ we have

$$\sum f_a > \sum f_{a'}, \quad \textit{if } a > a'. \tag{31}$$

(iii) We have

$$\sup\Big(\mathrm{im}\Big(\bigsqcup f\Big) \cap (-\infty, 0]\Big) < -1 + \inf_{a \in A_0} \sum f_a. \tag{32}$$

If $\ell \geq 4$ then we have

$$\sup \bigsqcup f < 2 \inf \Big(\operatorname{im} \Big(\bigsqcup f \Big) \cap (0, \infty) \Big) + 1. \tag{33}$$

Remark. *The conditions (28,29) imply that $\ell \geq 2$.*

We will prove this proposition in Section 4.

Proof of Theorem 15(i,ii). **(i):** We choose V, Ω, K, r as in the hypothesis. We define

$$\omega := \Omega^{\oplus n}.$$

Since by hypothesis, k is even and Ω is a volume form, the form ω is maxipotent, i.e., $\omega^{\wedge n}$ is a volume form. We denote by O the orientation on V^n induced by this form. Since by hypothesis, K is nonempty and strictly starshaped around 0, its interior contains 0. It follows that

$$C := \int_{K,O} \omega^{\wedge n} > 0.\text{[38]}$$

By hypothesis, we have

$$a_1 := \min \big\{ r - 1, \ \sqrt[kn]{2} - r \big\} > 0. \tag{34}$$

We choose $a_0 \in (0, a_1)$ and define $A_0 := [-a_0, a_0]$. For every $a \in A_0$ we define

$$M_a := (r + a)K \setminus \operatorname{int} K, \tag{35}$$

$$\omega_a := C^{-\frac{1}{n}} \omega \big| M_a, \tag{36}$$

$$I_a := \big\{ \text{connected component of } \partial M_a \big\}, \tag{37}$$

$$I := (I_a)_{a \in A_0}.$$

The form ω_a is well-defined, since $C > 0$. We check the hypotheses of Theorem 42(i). Let $a \in A_0$. The set M_a is compact. Since K is strictly starshaped around 0, M_a is a smooth submanifold of V^n that continuously deformation retracts onto ∂K. The manifold ∂K is homeomorphic to the sphere S_1^{kn-1}. Since by hypothesis $k, n \geq 2$, this sphere is 1-connected. Hence the same holds for M_a. The form Ω is exact. Hence the same holds for ω and thus for ω_a.

Condition (ia) is satisfied by our hypothesis and the rescaling property for a (kn, k)-form-category. We show that the collection of boundary helicities

$$f := \big(f_a := h_{M_a, O_{\omega_a}, \omega_a} \big)_{a \in A_0} \tag{38}$$

satisfies (ib). We check the hypotheses of Proposition 43. Let $s \in (0, \infty)$. We denote by O^s the orientation on $\partial(sK)$ induced by O and sK. By Lemma 35 we have

$$h \left(\partial(sK), O^s, \omega_{\partial(sK)} \right) = \int_{sK,O} \omega^{\wedge n} = C s^{kn}. \tag{39}$$

For every connected component i of ∂M_a we denote by O_i the orientation of i induced by O, M_a. Using (39,36) and Remarks 32,31, we obtain

$$h \left(i, O_i, (\omega_a)_i \right) = \begin{cases} (r + a)^{kn}, & \text{for } i = \partial\big((r + a)K\big), \\ -1, & \text{for } i = \partial K. \end{cases} \tag{40}$$

[38]Here we view V^n as a manifold and ω as a differential form on it.

Here we used that the orientation of ∂K induced by O and M_a is the opposite of O^1. It follows that

(41)
$$\sum f_a := \sum_{i\in I_a} f_a(i) = -1+(r+a)^{kn} \in \big[-1+(r-a_0)^{kn}, -1+(r+a_0)^{kn}\big],\ \forall a\in A_0.$$

Since $a_0 < a_1 \leq r-1$, we have $-1+(r-a_0)^{kn} > 0$. Hence by (41), we have $\inf_{a\in A_0}\sum f_a > 0$. Using (40), it follows that condition (32) is satisfied.

Since $a_0 < a_1 \leq \sqrt[kn]{2}-r$, we have $-1+(r+a_0)^{kn} < 1$. Using (41), it follows that $\sup_{a\in A_0}\sum f_a < 1$. Hence inequality (30) is satisfied. The collection f also satisfies the other hypotheses of Proposition 43. Applying this proposition, it follows that f is an I-collection. Hence condition (ib) is satisfied.

Therefore, all hypotheses of Theorem 42(i) are satisfied. Applying this theorem, it follows that the cardinality of $\mathcal{C}ap(\mathcal{C})$ equals $\beth_2$. This proves Theorem 15(i).

To prove (ii), assume that the hypotheses of this part of the theorem are satisfied. We choose $r \in \big(1, \sqrt[2n]{2}\big)$ satisfying (6). We define $V := \mathbb{R}^2$, Ω to be the standard area form on $\mathbb{R}^2$, $K := \overline{B}_1^{2n}$, and a_1 as in (34). We choose $a_0 \in (0, a_1)$, and define $A_0 := [-a_0, a_0]$ and (M_a, ω_a) as in (35,36). The tripel (V, Ω, K) satisfies the conditions of part (i) of Theorem 15. Hence by what we proved above, the collection $(M_a, \omega_a) = \big(Sh_{r+a}, C^{-\frac{1}{n}}\omega_{\mathrm{st}}\big|M_a\big)$, $a\in A_0$, satisfies the conditions of Theorem 42(i).

We check the condition (iia). We define I_a and f_a as in (37,38). For every $a \in A_0$ we have

$$\begin{aligned}\int_{M_a} \omega_a^{\wedge n} &= \sum f_a \qquad \text{(by Corollary 37)}\\ &= -1+(r+a)^{2n} \qquad \text{(by (41))},\\ &\leq -1+(r+a_0)^{2n}.\end{aligned}$$

Since $\int_B \omega_{\mathrm{st}}^{\wedge n} = \pi^n$, it follows that

$$w(M_a, \omega_a) \leq \sqrt[n]{-1+(r+a_0)^{2n}}.$$

Using the inequalities $a_0 < a_1 \leq \sqrt[2n]{2} - r$, it follows that

$$\sup_{a\in A_0} w(M_a, \omega_a) \leq \sqrt[n]{-1+(r+a_0)^{2n}} < 1.$$

Hence condition (iia) is satisfied.

We check (iib). Let $a \in A_0$. Then we have $r + a \geq r - a_0 > r - a_1 \geq 1$. Hence denoting $s := \frac{r+a+1}{2}$, the shell Sh_{r+a} contains the sphere S_s^{2n-1}. Using skinny nonsqueezing ([SZ12, Corollary 5, p. 8]) and the inequalities $n \geq 2$, $s > 1$, it follows that $(M_a, b\omega_a)$ does not symplectically embed into Z for any $b \geq 1$. Hence (iib) holds.

Therefore, all hypotheses of Theorem 42(ii) are satisfied. Applying this part of the theorem, it follows that the cardinality of $\mathcal{NC}ap(\mathcal{C})$ equals $\beth_2$. This proves Theorem 15(ii). $\square$

3. Proof of Theorem 42 (cardinality of the set of capacities, more general setting)

As mentioned, the idea of proof of Theorem 42 is that our helicity hypothesis (ib) and Stokes' Theorem for helicity imply that for $a \neq a'$ only small multiples of (W_a, η_a) embed into $(W_{a'}, \eta_{a'})$. The idea behind this is that every embedding φ of M_a into $M_{a'}$ gives rise to a partition of the disjoint union of the sets of connected components of ∂M_a and $\partial M_{a'}$. The elements of this partition consist of components that lie in the same connected component of the complement of $\varphi(\operatorname{Int} M)$. Here $\operatorname{Int} M$ denotes the interior of M as a manifold with boundary, and we identify each component of ∂M_a with its image under φ.

Stokes' Theorem for helicity implies that the inequality (20) is satisfied. Together with a similar argument in which we consider embeddings of W_a into $M_{a'}$, it follows that the partition satisfies the conditions of Definitions 38,39. Combining this with our helicity hypothesis (ib), it follows that indeed only small multiples of W_a embed into $W_{a'}$.

Lemmata 46 and 48 below will be used to make this argument precise. To formulate the first lemma, we need the following.

Remark 44 (pullback of relation)**.** Let S', S be sets, R a relation on S, and $f : S' \to S$ a map. Denoting by $\times$ the Cartesian product of maps, the set

$$R' := f^* R := (f \times f)^{-1}(R)$$

is a relation on S'. If R is reflexive/ symmetric/ transitive, then the same holds for R'.

Let X be a topological space. We define

$$\mathcal{C}_X := \big\{\text{path-connected subset of } X\big\} \tag{42}$$

and the relation $\sim_X$ on $\mathcal{C}_X$ by

$$A \sim_X B :\iff \exists \text{ continuous path starting in } A \text{ and ending in } B. \tag{43}$$

This is an equivalence relation.

Let M and M' be topological manifolds of the same dimension, and $\varphi : M \to M'$ a topological embedding, i.e., a homeomorphism onto its image. We denote by $\operatorname{Int}(M)$ and ∂M the interior and the boundary of M as a manifold with boundary. We denote

$$I := I_M := \big\{\text{connected component of } \partial M\big\}, \quad I' := I_{M'} \tag{44}$$

$$P := M' \setminus \varphi(\operatorname{Int}(M)). \tag{45}$$

We define

$$\Phi : \mathcal{P}(M) \to \mathcal{P}(M'), \quad \Phi(A) := \text{ image of } A \text{ under } \varphi,$$
$$\Psi : I \sqcup I' \to \mathcal{P}(P), \quad \Psi := \Phi \text{ on } I, \quad \Psi := \text{id on } I',$$
$$\sim^\varphi := \Psi^* \sim_P,$$
$$\mathcal{P}^\varphi := \text{partition of } I \sqcup I' \text{ associated with } \sim^\varphi .$$

Remark 45 (partition induced by embedding)**.** For every path-component P_0 of P we define

$$
\begin{aligned}
(46) \qquad J^\varphi(P_0) &:= \Psi^{-1}(\mathcal{P}(P_0)) \\
&= \left\{ i \in I \,\middle|\, \Phi(i) \in \mathcal{P}(P_0) \right\} \sqcup (I' \cap \mathcal{P}(P_0)).
\end{aligned}
$$

The map

$$J^\varphi : \left\{ \text{path-component } P_0 \text{ of } P : \; J^\varphi(P_0) \neq \emptyset \right\} \to \mathcal{P}^\varphi$$

is well-defined and a bijection.

For every field F and $i \in \mathbb{N}_0$ we denote by $H_i(M; F)$ the degree i singular homology of M with coefficients in F.

Lemma 46 (partition associated with an embedding)**.** *Assume that M, M' are compact, M' is connected, $\partial M' \neq \emptyset$, and that there exists a field F, for which $H_1(M'; F)$ vanishes. Then the following holds:*

(i) If M is nonempty and connected then $\mathcal{P}^\varphi$ is a $(I_M, I_{M'})$-partition.
(ii) If M consists of precisely two connected components M^+ and M^- then $\mathcal{P}^\varphi$ is a $\left(I_{M^+}, I_{M^-}, I_{M'}\right)$-partition.

Recall that the first statement means that condition (18) is satisfied, i.e., $|J \cap I_M| = 1$ for every $J \in \mathcal{P}^\varphi$. The idea of proof of the inequality ≤ 1 is the following. Each J corresponds to a path-component P_0 of the complement of $\varphi(\operatorname{Int} M)$. Suppose that there exists J that intersects I_M in at least two points i_0, i_1 (= components of ∂M). Then there is a path in P_0 joining $\varphi(i_0)$ and $\varphi(i_1)$. By connecting this path with a path in $\varphi(M)$ with the same endpoints, we obtain a loop in M' that intersects i_0 and i_1 in one point each. See Figure 4.

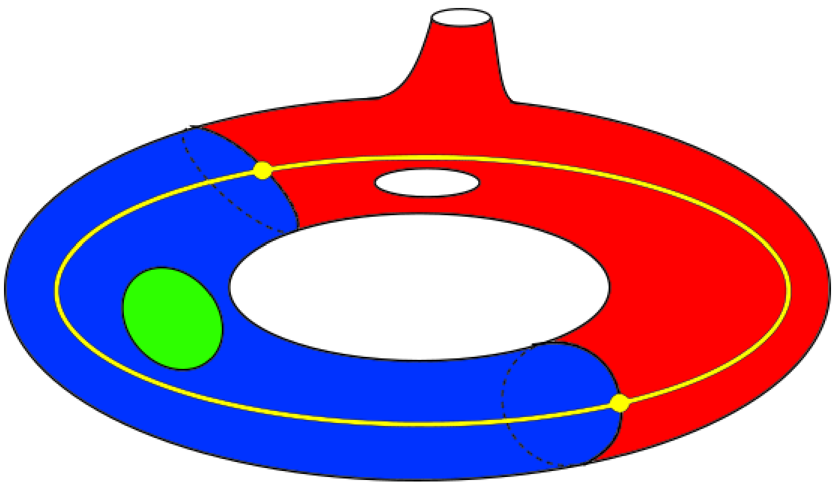

FIGURE 4. The blue region is the image of M under φ, and the red and green regions are the path-components of the complement of $\varphi(\operatorname{Int} M)$. The red region contains the images of two connected components i_0, i_1 of the boundary of M. The yellow loop intersects these images in one point each.

Hence the algebraic intersection number of this loop with i_0 equals 1. In particular, it represents a nonzero first homology class. Hence the hypothesis that the first homology of M' vanishes, is violated. It follows that $|J \cap I_M| \leq 1$.

In order to make this argument precise one needs to ensure that the algebraic intersection number equals the "naïve intersection number". For simplicity, we therefore use an alternative method of proof, which is based on a certain Mayer-Vietoris sequence for singular homology. We need the following.

Remark 47 (embedding is open, boundary)**.** We denote by $\partial^X S$ the boundary of a subset S of a topological space X. Let M, M' be topological manifolds of the same dimension n, and $\varphi : M \to M'$ an injective continuous map. By invariance of the domain, in every pair of charts for $\operatorname{Int} M$ and M', the map φ sends every open subset of $\mathbb{R}^n$ to an open subset of $\mathbb{R}^n$. It follows that the set $\varphi(\operatorname{Int} M)$ is open in M'. This implies that

$$\varphi(\partial M) \subseteq \partial^{M'} \varphi(\operatorname{Int} M),$$

and if M is compact, then equality holds.

Suppose now that M is nonempty and compact, $\partial M = \emptyset$, and M' is connected. Then M' has no boundary, either. To see this, observe that $\varphi(M)$ is compact, hence closed in M'. Since $M = \operatorname{Int} M$, as mentioned above, $\varphi(M)$ is also open. Since M' is connected, it follows that $\varphi(M) = M'$. Since in every pair of charts for M and M', φ sends every open subset of $\mathbb{R}^n$ to an open subset of $\mathbb{R}^n$, it follows that $\partial M' = \emptyset$.

Proof of Lemma 46. Assume that M, M' are compact, $M \neq \emptyset$, M' is connected, and $\partial M' \neq \emptyset$. We denote

$$I := I_M, \quad I' := I_{M'}, \quad P := M' \setminus \varphi(\operatorname{Int}(M)),$$

and by k the number of connected components of M.

Claim 1. *We have*

$$|\mathcal{P}^\varphi| = |I| + 1 - k. \tag{47}$$

Proof of Claim 1. Let P_0 be a path-component of P.

Claim 2. *P_0 intersects $\varphi(\partial M)$.*

Proof of Claim 2. By Remark 47 we have $\partial M \neq \emptyset$. Since by hypothesis, M' is connected, there exists a continuous path $x' : [0,1] \to M'$ that starts in P_0 and ends at $\varphi(\partial M)$. Since M is compact, the same holds for ∂M, and hence for $\varphi(\partial M)$. Hence the minimum

$$t_0 := \min \big\{ t \in [0,1] \,\big|\, x'(t) \in \varphi(\partial M) \big\}$$

exists. By Remark 47 the set $\varphi(\operatorname{Int} M)$ is open in M'. It follows that $x'(t_0) \notin \varphi(\operatorname{Int} M)$, and hence $x'([0,t_0]) \subseteq P = M' \setminus \varphi(\operatorname{Int} M)$. (In the case $t_0 = 0$ this holds, since $x'(0) \in P_0 \subseteq P$.) It follows that $x'(t_0) \in P_0$. Since also $x'(t_0) \in \varphi(\partial M)$, it follows that $P_0 \cap \varphi(\partial M) \neq \emptyset$. This proves Claim 2. □

Claim 2 implies that the set $J^\varphi(P_0)$ (defined as in (46)) is nonempty. Hence by Remark 45 we have

$$\big|\big\{\text{path-component of } P\big\}\big| = |\mathcal{P}^\varphi|. \tag{48}$$

By M. Brown's Collar Neighbourhood Theorem [Bro62] (see also [Con71, Theorem, p. 180]) there exists an open subset V of M and a (strong) deformation retraction h of V onto ∂M. We define

$$A := \varphi(M), \quad B := M' \setminus \varphi(M \setminus V).$$

Extending $\varphi \circ h_t \circ \varphi^{-1} : \varphi(V) \to \varphi(V)$ by the identity, we obtain a map $h' : [0,1] \times B \to B$. Since by Remark 47, the restriction of φ to $\mathrm{Int}\, M$ is open, the map h' is continuous, and therefore a deformation retraction of B onto P.

We choose a field F as in the hypothesis, and denote by H_i singular homology in degree i with coefficients in F. Since P is a deformation retract of B, these spaces have isomorphic H_0. Combining this with (48), it follows that

$$\begin{aligned} |\mathcal{P}^\varphi| &= \big|\{\text{path-component of } P\}\big| \\ &= \dim H_0(P) \\ &= \dim H_0(B). \end{aligned} \tag{49}$$

The interiors of A and B cover M'. Therefore, the Mayer-Vietoris Theorem implies that there is an exact sequence

$$\ldots \to H_1(M') \to H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(M') \to 0.$$

Since by hypothesis, $H_1(M') = 0$, it follows that

$$\dim H_0(B) = \dim H_0(A \cap B) + \dim H_0(M') - \dim H_0(A). \tag{50}$$

Since $A \cap B = \varphi(V)$ and φ is a homeomorphism onto its image, we have $H_0(A \cap B) \cong H_0(V)$. Since V deformation retracts onto ∂M, we have $H_0(V) \cong H_0(\partial M)$, hence $H_0(A \cap B) \cong H_0(\partial M)$. Since ∂M is a topological manifold, its path-components are precisely its connected components. Recalling the definition (44) of I, it follows that

$$\dim H_0(A \cap B) = |I|. \tag{51}$$

Since by hypothesis M' is connected, we have

$$\dim H_0(M') = 1. \tag{52}$$

Since $A := \varphi(M)$, we have $H_0(A) \cong H_0(M)$, and therefore

$$\dim H_0(A) = k.$$

Combining this with (49,50,51,52), equality (47) follows. This proves Claim 1. □

Remark 45 and Claim 2 imply that every element of $\mathcal{P}^\varphi$ intersects I.

We prove (i). Assume that M is connected. Then by Claim 1, we have $|\mathcal{P}^\varphi| = |I|$. It follows that $|J \cap I| = 1$, for every $J \in \mathcal{P}^\varphi$. Hence $\mathcal{P}^\varphi$ is a (I, I')-partition. This proves (i).

Assume now that $M^\pm$ are as in the hypothesis of (ii). By Claim 1 we have $|\mathcal{P}^\varphi| = |I| - 1$. Since every element of $\mathcal{P}^\varphi$ intersects I, it follows that there exists a unique $J_0 \in \mathcal{P}^\varphi$, such that $|J_0 \cap I| = 2$, and

$$|J \cap I| = 1, \quad \forall J \in \mathcal{P}^\varphi \setminus \{J_0\}. \tag{53}$$

By Remark 45 there exists a unique path-component P_0 of P, such that $J_0 = J^\varphi(P_0)$.

Claim 3. *We have*

$$J_0 \cap I^- \neq \emptyset \neq J_0 \cap I^+.$$

Proof of Claim 3. We denote by P_0^+ the path-component of $M' \setminus \varphi(\mathrm{Int}(M^+))$ containing P_0. Assume by contradiction that $P_0^+ \cap \varphi(M^-) = \emptyset$. Then we have

$$P_0^+ = P_0, \quad J^{\varphi|M^+}(P_0^+) = J^\varphi(P_0) = J_0, \quad J_0 \cap I = J_0 \cap I^+.$$

Since $|J_0 \cap I| = 2$, we obtain a contradiction with (i), with I, φ replaced by $I^+, \varphi|M^+$. Hence we have

$$P_0^+ \cap \varphi(M^-) \neq \emptyset.$$

It follows that there exists a continuous path $x' : [0,1] \to M' \setminus \varphi(\mathrm{Int}(M^+))$ that starts at P_0 and ends at $\varphi(M^-)$. Since M is compact, the same holds for $\varphi(M^-)$. Hence the minimum

$$t_0 := \min\big\{t \in [0,1] \,\big|\, x'(t) \in \varphi(M^-)\big\}$$

exists. By Remark 47 the set $\varphi(\mathrm{Int}\, M^-)$ is open. It follows that $x'(t_0) \notin \varphi(\mathrm{Int}\, M^-)$, hence $x'([0,t_0]) \subseteq P$, and therefore

$$x'(t_0) \in P_0. \tag{54}$$

On the other hand $x'(t_0) \in \varphi(M^-) \subseteq \overline{\varphi(\mathrm{Int}\, M^-)}$, and therefore

$$x'(t_0) \in \partial^{M'} \varphi(\mathrm{Int}\, M^-) = \varphi(\partial M^-).$$

Here we used Remark 47. Combining this with (54), it follows that $P_0 \cap \varphi(\partial M^-) \neq \emptyset$, and therefore $J_0 \cap I^- \neq \emptyset$.

An analogous argument shows that $J_0 \cap I^+ \neq \emptyset$. This proves Claim 3. □

By Claim 3 and (53) $\mathcal{P}^\varphi$ is a $\big(I^+, I^-, I'\big)$-partition. This proves (ii) and completes the proof of Lemma 46. □

The second ingredient of the proof of Theorem 42 is the following. Let $k, n \in \mathbb{N}_0$ with $n \geq 2$, M, M' be compact (smooth) manifolds of dimension kn, ω, ω' exact maxipotent k-forms on M, M', $c \in (0, \infty)$, and $\varphi : M \to M'$ a (smooth) orientation preserving embedding that intertwines $c\omega$ and ω'. We denote by O, O' the orientations of M, M' induced by ω, ω'. Recall Definitions 29,36 of (boundary) helicity.

Lemma 48 (helicity inequality)**.** *Condition (20) holds with $\mathcal{P} = \mathcal{P}^\varphi$, $f = h_{M,O,\omega}$, $f' = h_{M',O',\omega'}$, and $C := c^n$.*

The reason for this is that the left hand side of (20) is the volume of the path-component of the complement of $\varphi(\mathrm{Int}\, M)$, determined by J. To make this precise, we need the following.

Remark 49. Let X and X' be topological spaces and $f : X \to X'$ be continuous. Recall the definitions (42,43) of $\mathcal{C}_X$ and $\sim_X$.

(i) The map

$$f_* : \mathcal{C}_X \to \mathcal{C}_{X'}, \quad f_*(A) := f(A),$$

is well-defined. Furthermore, we have

$$f_* \times f_*(\sim_X) \subseteq \sim_{X'} .$$

(ii) Assume that $X = X'$ and for every $x \in X$ there exists a continuous path from x to $f(x)$. Then for every pair $A, B \in \mathcal{C}_X$ we have

$$f_*(A) \sim_{f(X)} f_*(B) \Rightarrow A \sim_X B.$$

This follows from transitivity of $\sim_X$.

Proof of Lemma 48. Let $M, O, \omega, M', O', \omega', c, \varphi$ be as in the hypothesis. We define $I = I_M, I' = I_{M'}$ as in (44). Consider first the **case** in which

$$\varphi(\partial M) \cap \partial M' = \emptyset. \tag{55}$$

Then the set

$$P := M' \setminus \varphi(\operatorname{Int} M)$$

is a smooth submanifold of M'. Let $i \in I$. We denote $\widehat{i} := \varphi(i)$. We define O_N^M as in (17), and abbreviate

$$O_i := O_i^M, \quad O_{\widehat{i}} := (O')_{\widehat{i}}^P.$$

Recall that $\overline{O}$ denotes the orientation opposite to O. Since φ intertwines O, O', and $P, \varphi(M)$ lie on opposite sides of $\widehat{i}$, we have

$$(\varphi|i)_*\overline{O_i} = \overline{(\varphi|i)_* O_i} = O_{\widehat{i}}. \tag{56}$$

Recall the definition (16) of ω_N. Since φ intertwines $c\omega, \omega'$, we have

$$(\varphi|i)_* c\omega_i = \omega'_{\widehat{i}}. \tag{57}$$

We have

$$\begin{aligned}
-c^n h(i, O_i, \omega_i) &= c^n h(i, \overline{O_i}, \omega_i) \qquad \text{(by Remark 31)} \\
&= h\big(i, \overline{O_i}, c\omega_i\big) \qquad \text{(by Remark 32)} \\
&= h\Big((\varphi|i)_*\big(i, \overline{O_i}, c\omega_i\big)\Big) \qquad \text{(by Remark 33)} \\
&= h\big(\widehat{i}, O_{\widehat{i}}, \omega'_{\widehat{i}}\big) \qquad \text{(using } \widehat{i} = \varphi(i), \text{ (56,57)).}
\end{aligned} \tag{58}$$

Let P_0 be a path-component of P. We define $J := J^\varphi(P_0)$ as in (46). Using $h_{M,O,\omega}(i) = h(i, O_i, \omega_i)$ and (58), we have

$$\begin{aligned}
&-c^n \sum_{i \in J \cap I} h_{M,O,\omega}(i) + \sum_{i' \in J \cap I'} h_{M',O',\omega'}(i') \\
&= \sum_{\widehat{i} \in I_{P_0}} h_{P_0, O'|P_0, \omega'|P_0}\big(\widehat{i}\big) \\
&= \int_{P_0, O'|P_0} \omega'^n \qquad \text{(using Corollary 37)} \\
&\geq 0.
\end{aligned}$$

Hence the statement of Lemma 48 holds in the case (55).

Consider now the general situation. Let $(K_i, r_i)_{i \in I}$ be a collection, where for each $i \in I$, K_i is a compact connected neighbourhood of i that is a (smooth) submanifold of M (with boundary), and $r_i : K_i \to i$ is a continuous retraction, such that the sets K_i, $i \in I$, are disjoint. We denote by $\mathrm{int}(K_i)$ the interior of K_i in M. We define

$$\begin{aligned} \widetilde{M} &:= M \setminus \textstyle\bigcup_{i\in I} \mathrm{int}(K_i),\\ \widetilde{\varphi} &:= \varphi|\widetilde{M},\\ \widetilde{I}_i := I_{K_i} \setminus \{i\}, \forall i \in I, &\quad \widetilde{I} := I_{\widetilde{M}}. \end{aligned}$$

We define

$$\widetilde{\ }: \mathcal{P}(I \sqcup I') \to \mathcal{P}\big(\widetilde{I} \sqcup I'\big), \quad \widetilde{J} := (J \setminus I) \cup \bigcup_{i \in J\cap I} \widetilde{I}_i. \tag{59}$$

The set $\widetilde{M}$ is a submanifold of M, and

$$\widetilde{\varphi}(\partial\widetilde{M}) \cap \partial M' = \emptyset. \tag{60}$$

Claim 1.

$$\mathcal{P}^{\widetilde{\varphi}} = \widetilde{\mathcal{P}^{\varphi}} := \big\{\widetilde{J} \,\big|\, J \in \mathcal{P}^{\varphi}\big\}. \tag{61}$$

Proof of Claim 1. We define

$$\widetilde{P} := M' \setminus \varphi(\mathrm{Int}(\widetilde{M})), \quad r : \widetilde{P} \to P,$$

by setting

$$r := \begin{cases} \varphi \circ r_i \circ \varphi^{-1} & \text{on } \varphi(K_i), \text{ with } i \in I,\\ r = \mathrm{id} & \text{on } M' \setminus \varphi(M). \end{cases}$$

Since the sets K_i are disjoint, the map r is well-defined. Since by hypothesis, φ is an embedding between two manifolds of the same dimension, the map r is continuous. Let $i \in I$. Since K_i is path-connected and r_i is a retraction onto the subset i of K_i, the hypotheses of Remark 49(ii) are satisfied with $f = r$. Applying this remark, it follows that for every pair $\widetilde{A}, \widetilde{B}$ of path-connected subsets of $\widetilde{P}$ we have

$$\widetilde{A} \sim_{\widetilde{P}} \widetilde{B} \iff r(\widetilde{A}) \sim_{r(\widetilde{P})=P} r(\widetilde{B}).$$

This implies that if $i_0, i_1 \in I$, $\widetilde{i}_k \in \widetilde{I}_{i_k}$, for $k = 0, 1$, and $i'_0, i'_1 \in I'$ then

$$\widetilde{i}_0 \sim_{\widetilde{\varphi}} \widetilde{i}_1 \iff i_0 \sim_{\varphi} i_1, \quad i'_0 \sim_{\widetilde{\varphi}} i'_1 \iff i'_0 \sim_{\varphi} i'_1, \quad \widetilde{i}_0 \sim_{\widetilde{\varphi}} i'_0 \iff i_0 \sim_{\varphi} i'_0.$$

Equality (61) follows. This proves Claim 1. ☐

We abbreviate

$$h_M := h_{M,O,\omega}.$$

Recall the definition (19). Using (60), by what we already proved, condition (20) holds with I replaced by $\widetilde{I}$, $\mathcal{P} := \mathcal{P}^{\widetilde{\varphi}}$, $f := h_{\widetilde{M}}$, $f' := h_{M'}$, and $C := c^n$. Using Claim 1, it follows that

$$\textstyle\sum_{\widetilde{J}, h_{\widetilde{M}}, h_{M'}, c^n} \geq 0, \quad \forall J \in \mathcal{P}^{\varphi}. \tag{62}$$

We denote by $\partial^X S$ the boundary of a subset S of a topological space X. For every $i \in I$ Remark 31 and Lemma 35 imply that

$$\begin{aligned} h_{\widetilde{M}}(\partial^M K_i) &= -h_{K_i}(\partial^M K_i)\\ &= h_M(i) - \int_{K_i} \omega^{\wedge n}, \end{aligned} \tag{63}$$

where the integral is w.r.t. the orientation $O|K_i$. Let $J \in \mathcal{P}^\varphi$. Recalling the definition (59) of $\widetilde{}$ and using (63), we have

$$\sum_{\widetilde{i}\in\widetilde{J}\cap\widetilde{I}} h_{\widetilde{M}}(\widetilde{i}) = \sum_{i\in J\cap I}\left(h_M(i) - \int_{K_i}\omega^{\wedge n}\right).$$

Combining this with (62) and recalling the definition (19), it follows that

$$\sum\nolimits_{J,h_M,h_{M'},c^n} \geq -c^n \sum_{i\in J\cap I}\int_{K_i}\omega^{\wedge n}.$$

Since this holds for every choice of $(K_i)_{i\in I}$, it follows that $\sum_{J,h_M,h_{M'},c^n} \geq 0$. Hence condition (20) holds with $\mathcal{P} := \mathcal{P}^\varphi$, $f := h_M$, $f' := h_{M'}$, and $C := c^n$. This proves Lemma 48. $\square$

Remark (helicity inequality)**.** Under the hypotheses of this lemma, the set $M' \setminus \varphi(\mathrm{Int}(M))$ need not be a submanifold of M', since $\varphi(\partial M)$ may intersect $\partial M'$. This is the reason for the construction of $\widetilde{M}$ in the proof of this lemma.

We are now ready for the proof of Theorem 42.

Proof of Theorem 42. Assume that there exist $A_0, (M_a,\omega_a)_{a\in A_0}$ as in the hypothesis of (i). Let $a \in A_0 \cap (0,\infty)$. We define

$$(W_a,\eta_a) := (M_a \sqcup M_{-a}, \omega_a \sqcup \omega_{-a}).$$

Since by our hypothesis (ia) $(W_a,\eta_a) \in \mathcal{O}$, the capacity c_{W_a,η_a} makes sense. Let $A \in \mathcal{P}\big(A_0 \cap (0,\infty)\big)$. Recall the definition (1) of $\mathcal{O}_0$. We define the map

$$c_A := \sup_{a\in A} c_{W_a,\eta_a} : \mathcal{O}_0 \to [0,\infty].$$

If $k = 2$ and the ball B lies in $\mathcal{O}$, then we define the map $\widetilde{c}_A$: by

$$\widetilde{c}_A := \max\{c_A, w\} : \mathcal{O}_0 \to [0,\infty]. \tag{64}$$

The maps c_A and $\widetilde{c}_A$ are generalized capacities on $\mathcal{C}$.

Claim 1. *(i) The map $\mathcal{P}\big(A_0\cap(0,\infty)\big) \ni A \mapsto c_A \in \mathcal{C}ap(\mathcal{C})$ is injective.*

Assume now that the hypotheses of Theorem 42(ii) are satisfied.

(ii) The map $\mathcal{P}\big(A_0\cap(0,\infty)\big) \ni A \mapsto \widetilde{c}_A \in \mathcal{C}ap(\mathcal{C})$ is injective.

(iii) For every $A \in \mathcal{P}\big(A_0\cap(0,\infty)\big)$ the capacity $\widetilde{c}_A$ is normalized.

Proof of Claim 1. We denote

$$h_M := h_{M,O,\omega}, \quad f_a := h_{M_a}, \quad f := (f_a)_{a\in A_0},$$

and define C_0^f, C_1^f as in (21,22). Let $a \neq a' \in A_0 \cap (0,\infty)$, and $c \in (0,\infty)$, such that there exists a $\mathcal{C}$-morphism φ from $(W_a, c\eta_a)$ to $(W_{a'},\eta_{a'})$.

Case A: There exist such a φ and $b \in \{a,-a\}$, $b' \in \{a',-a'\}$, such that $b > b'$ and $\varphi(M_b) \subseteq M_{b'}$. We denote

$$M := M_b, \quad \omega := \omega_b, \quad M' := M_{b'}, \quad \omega' := \omega_{b'}, \quad I := I_M, \quad I' := I_{M'}.$$

Let $d \in A_0$. By hypotheses M_d is nonempty, compact, and 1-connected. Since by hypothesis $n \geq 2 > 0$ and ω_d is maxipotent and exact, we have $\partial M_d \neq \emptyset$. Hence the hypotheses of Lemma 46(i) are satisfied. Applying this lemma, it follows that

$\mathcal{P}^\varphi$ is a (I,I')-partition. By Lemma 48 the set $\mathcal{P}^\varphi$ is a $\big(h_M,h_{M'},c^n\big)$-partition. It follows that

$$c^n\leq C_0^f. \tag{65}$$

Consider now the **case** that is complementary to Case A. Then $a<a'$ and there exists a morphism φ from $(W_a,c\eta_a)$ to $(W_{a'},\eta_{a'})$, such that $\varphi(W_a)\subseteq M_{a'}$. Lemmata 46(ii) and 48 imply that $\mathcal{P}^\varphi$ is a $\big(h_{M_a},h_{M_{-a}},h_{M_{a'}},c^n\big)$-partition. It follows that $c^n\leq C_1^f$. Combining this with (65), in any case we have

$$c^n\leq C:=\max\left\{C_0^f,C_1^f\right\}.$$

It follows that

$$\begin{aligned}&\sup\big\{c\in(0,\infty)\,\big|\,\exists a\neq a'\in A_0\cap(0,\infty)\,\exists\text{ morphism }(W_a,c\eta_a)\to(W_{a'},\eta_{a'})\big\}\\ \leq&\sqrt[n]{C}\\ <&1\qquad\text{(using our hypothesis (ib) and Definition 41).}\end{aligned}$$

It follows that

$$c_A(W_{a'},\eta_{a'})<1,\quad\forall A\in\mathcal{P}\big(A_0\cap(0,\infty)\big),\quad a'\in A_0\cap(0,\infty)\setminus A.\text{[39]} \tag{66}$$

Let $A\neq A'\in\mathcal{P}\big(A_0\cap(0,\infty)\big)$. Assume first that $A'\setminus A\neq\emptyset$. We choose $a'\in A'\setminus A$. Since $c_{A'}(W_{a'},\eta_{a'})\geq 1$,[40] inequality (66) implies that $c_A\neq c_{A'}$. This also holds in the case $A\setminus A'\neq\emptyset$, by an analogous argument. This proves statement (i).

We prove **(ii)**. Combining inequality (66) with our hypothesis (iia), we have

$$\widetilde{c}_A(W_{a'},\eta_{a'})<1,\quad\forall A\in\mathcal{P}\big(A_0\cap(0,\infty)\big),\quad a'\in A_0\cap(0,\infty)\setminus A.$$

Hence an argument as above shows that the map $\mathcal{P}\big(A_0\cap(0,\infty)\big)\ni A\mapsto\widetilde{c}_A$ is injective. This proves (ii).

We prove (iii). Let $A\in\mathcal{P}\big(A_0\cap(0,\infty)\big)$. By our definition (64) we have

$$\pi=w(B)\leq\widetilde{c}_A(B). \tag{67}$$

Since B symplectically embeds into Z, we have $c_{M,\omega}(B)\leq c_{M,\omega}(Z)$ for every symplectic manifold (M,ω) of dimension $2n$. It follows that

$$\widetilde{c}_A(B)\leq\widetilde{c}_A(Z). \tag{68}$$

Our hypothesis (iib) and Gromov's Nonsqueezing Theorem imply that $\widetilde{c}_A(Z)\leq\pi$. Combining this with (67,68), it follows that $\widetilde{c}_A$ is normalized. This proves (iii) and therefore Claim 1. $\square$

Claim 1(i) implies that

$$|\mathcal{C}ap(\mathcal{C})|\geq\big|\mathcal{P}\big(A_0\cap(0,\infty)\big)\big|=\beth_2, \tag{69}$$

[39] A priori map $c:=c_A$ is only defined on the set $\mathcal{O}_0$. For a general $(M,\omega)\in\mathcal{O}$ we define $c(M,\omega):=c(M_0,\omega_0)$, where (M_0,ω_0) is an arbitrary object of $\mathcal{O}_0$ isomorphic to (M,ω).

[40] In fact equality holds, but we do not use this.

where in the second inequality we used our hypothesis that A_0 is an interval of positive length. On the other hand, by Corollary 57 in the appendix the set $\mathcal{O}_0$ has cardinality at most $\beth_1$. It follows that

$$|\mathcal{C}ap(\mathcal{C})| \leq \left|[0,\infty]^{\mathcal{O}_0}\right| \leq \beth_1^{\beth_1} = \beth_2.$$

Combining this with (69), the statement of Theorem 42(i) follows.

The statement of Theorem 42(ii) follows from an analogous argument, using parts (ii,iii) of Claim 1. This completes the proof of Theorem 42. □

4. Proof of Proposition 43 (sufficient conditions for being an I-collection)

Proof of Proposition 43. Let $I = (I_a), f = (f_a)$ be as in the hypothesis. To simplify notation, we canonically identify the collection f with its disjoint union $\bigsqcup f : \bigsqcup I \to \mathbb{R}$.

Claim 1. *Let $a, a' \in A_0$. If $a > a'$ then for every partition $\mathcal{P}$ of $I_a \sqcup I_{a'}$ there exists $J \in \mathcal{P}$, such that*

$$\sum_{i \in J \cap I_a} f(i) > \sum_{i' \in J \cap I_{a'}} f(i'). \tag{70}$$

Proof of Claim 1. This follows from hypothesis (31). □

By hypothesis (26) there exists k, such that $|I_a| = k+1$, for every $a \in A_0$. By hypothesis (29) for every $a \in A_0$ the set $f_a^{-1}((0,\infty))$ contains a unique element p_a. Hypotheses (30,27) imply that

$$f(p_a) \leq k+1, \quad \forall a \in A_0. \tag{71}$$

Recalling the notation (25), we have

$$\inf_{a \in A_0} \sum f_a > 0, \qquad \text{(using (32,28))} \tag{72}$$

$$f(p_a) > 1, \quad \forall a \in A_0 \qquad \text{(using (72,28))}. \tag{73}$$

Claim 2. *If $k = 1$ or 2 then the inequality (33) holds.*

Proof. For every $a \in A_0$ we have

$$\begin{aligned}
f(p_a) &= \sum f_a - \sum_{n \in I_a \setminus \{p_a\}} f(n) \\
&\geq \inf_b \sum f_b + 1 - (k-1) \sup \big(\mathrm{im}(f) \cap (-\infty, 0]\big) \qquad \text{(using (28))} \\
&> k + (2-k) \inf_b \sum f_b \qquad \text{(using (32))} \\
&\geq k \qquad \text{(using that } k = 1 \text{ or } 2, \text{ and (72))}.
\end{aligned}$$

Using (71), it follows that (33) holds. This proves Claim 2. □

We now check the conditions (23,24) of Definition 41.

Condition (23): Let $a, a' \in A_0$ be such that $a > a'$, $C \in (0,\infty)$ and $\mathcal{P}$ be a $\big(f_a, f_{a'}, C\big)$-partition. If $C \geq 1$ then Claim 1 implies that condition (20) in Definition 38 with $I := I_a$, $I' := I_{a'}$ is violated. It follows that $C < 1$.

We denote by J_0 the unique element of $\mathcal{P}$ containing p_a.

Claim 3. *We have $p_{a'} \in J_0$.*

Proof of Claim 3. By Definition 38 we have $|J_0 \cap I_a| = 1$. It follows that $J_0 \cap I_a = \{p_a\}$. Therefore, by condition (20) applied to $J := J_0$, we have

$$Cf(p_a) \leq \sum_{i' \in J_0 \cap I_{a'}} f(i').$$

Since $Cf(p_a) > 0$ and $p_{a'}$ is the only point in $I_{a'}$ at which f is positive, Claim 3 follows. □

Claim 4. *We have $f_{a'}^{-1}(-1) \subseteq J_0$.*

Proof of Claim 4. Let $J \in \mathcal{P} \setminus \{J_0\}$. By (18) the set $J \cap I_a$ consists of a unique element i. Hypothesis (27) and the inequality $C < 1$ imply that $Cf(i) > -1$. Combining this with (20), it follows that

$$\sum_{i' \in J \cap I_{a'}} f(i') > -1. \tag{74}$$

Since J and J_0 are disjoint, Claim 3 implies that $p_{a'} \notin J$. Therefore, (74) implies that $J \cap I_{a'} \cap f^{-1}(-1) = \emptyset$. Since this holds for every $J \in \mathcal{P} \setminus \{J_0\}$, and $\mathcal{P}$ covers $I_{a'}$, it follows that $I_{a'} \cap f^{-1}(-1) \subseteq J_0$. This proves Claim 4. □

Claims 3,4 and hypothesis (28) imply that $|J_0 \cap I_{a'}| \geq 2$. Since $|I_a| = |I_{a'}| = k+1$ and $p_a \in J_0 \cap I_a$, it follows that

$$\left|\left(I_a \sqcup I_{a'}\right) \setminus J_0\right| \leq 2k - 1. \tag{75}$$

The condition (18) implies that $\left|\mathcal{P} \setminus \{J_0\}\right| = |I_a| - 1 = k$. Since the elements of $\mathcal{P} \setminus \{J_0\}$ are disjoint and their union is contained in $\left(I_a \sqcup I_{a'}\right) \setminus J_0$, using (75), it follows that there exists $J_1 \in \mathcal{P} \setminus \{J_0\}$ satisfying $|J_1| \leq 1$. Since $|J_1 \cap I_a| = 1$, it follows that

$$J_1 \cap I_{a'} = \emptyset. \tag{76}$$

The facts $J_1 \neq J_0$, and that p_a lies in J_0 and is the only point of I_a at which f is positive, imply that $\sum_{i \in J_1 \cap I_a} f(i) \leq \sup\left(\operatorname{im}(f) \cap (-\infty, 0]\right)$. Using (76) and recalling the definition (19), it follows that

$$\sum\nolimits_{J_1, f_a, f_{a'}, C} \geq -C \sup\left(\operatorname{im}(f) \cap (-\infty, 0]\right). \tag{77}$$

Summing up the inequality (20) over all $J \in \mathcal{P} \setminus \{J_1\}$ and adding (77), we obtain

$$-C \sum f_a + \sum f_{a'} \geq -C \sup\left(\operatorname{im}(f) \cap (-\infty, 0]\right).$$

It follows that

$$\begin{aligned} C\left(-\sup\left(\operatorname{im}(f) \cap (-\infty, 0]\right) + \inf_a \sum f_a\right) &\leq \sum f_{a'} \\ &\leq 1 \qquad \text{(using hypothesis (30)).} \end{aligned}$$

Combining this with hypothesis (32), it follows that $C_0^f < 1$. Hence f satisfies (23).

Condition (24): Let $a, a' \in (0, \infty)$, such that $a < a'$, $C \in (0, \infty)$ and $\mathcal{P}$ be a $\left(f_a, f_{-a}, f_{a'}, C\right)$-partition. We denote by $J_0 \in \mathcal{P}$ the unique element that contains p_a. We will show that $\mathcal{P}$ and J_0 look like in Figure 5.

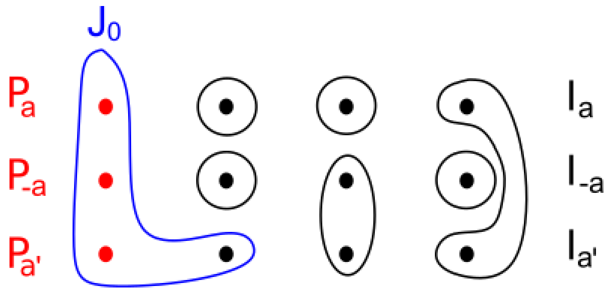

FIGURE 5. The dots in the first row constitute the set I_a, which contains the point p_a, and similarly for I_{-a} and $I_{a'}$. The blue and black sets denote the elements of the partition $\mathcal{P}$. We show below that except for p_a, the blue set J_0 also contains $p_{-a}, p_{a'}$, and an element of $I_{a'}$ at which f takes on the value -1. Note that J_0 intersects both I_a and I_{-a} in exactly one point, and that the other elements of $\mathcal{P}$ intersect $I_a \sqcup I_{-a}$ in exactly one point.

Claim 5. *We have $p_{a'}, p_{-a} \in J_0$.*

Proof of Claim 5. We show that $p_{a'} \in J_0$. Conditions (a,b) of Definition 39 with $I^\pm := I_{\pm a}$ imply that $J_0 \cap I_{\pm a}$ is empty or a singleton. Combining this with the fact that $p_a \in J_0$, hypothesis (27), and (73), we obtain

$$\sum_{i \in J_0 \cap (I_a \sqcup I_{-a})} f(i) > 0.$$

Using condition (20) with $J = J_0$, it follows that $p_{a'} \in J_0$.

To show that $p_{-a} \in J_0$, let $J \in \mathcal{P} \setminus \{J_0\}$. Since $p_{a'} \in J_0$, it does not lie in J. It follows that $\sum_{i' \in J \cap I_{a'}} f(i') \leq 0$. Using (20) with $I = I_a \sqcup I_{-a}$, it follows that

$$\sum_{i \in J \cap (I_a \sqcup I_{-a})} f(i) \leq 0. \tag{78}$$

Conditions (a,b) of Definition 39 with $I^\pm := I_{\pm a}$ imply that $J \cap I_{\pm a}$ is empty or a singleton. Using hypothesis (27) and (78), it follows that $J \cap I_{-a}$ is empty or consists of one element i, satisfying $f(i) \leq 1$. Using (73), it follows that $p_{-a} \notin J$. Since this holds for every $J \in \mathcal{P} \setminus \{J_0\}$, it follows that $p_{-a} \in J_0$. This proves Claim 5. □

Claim 6. *We have $C < 1$.*

Proof of Claim 6. By Remark 40 we have $|\mathcal{P}| = 2k+1$. Since $|I_{a'}| = k+1$, $k \geq 1$, and the elements of $\mathcal{P}$ are disjoint, it follows that there exists $J_1 \in \mathcal{P}$, such that

$$J_1 \cap I_{a'} = \emptyset. \tag{79}$$

Claim 5 implies that $J_1 \neq J_0$, and hence that $p_a, p_{-a} \notin J_1$. By Definition 39(b) we have

$$J_1 \cap I_a \sqcup I_{-a} = \{n\}, \quad \text{for some point } n. \tag{80}$$

By (32) we have

$$f(n) < -1 + \inf_{b \in A_0} \sum f_b. \tag{81}$$

Denoting

$$\sum\nolimits_J := \sum_{i \in J \cap (I_a \sqcup I_{-a})} f(i), \quad \sum\nolimits_J' := \sum_{i' \in J \cap I_{a'}} f(i'),$$

we have

$$\begin{aligned}
1 &\geq \sum f_{a'} \qquad \text{(using 30)} \\
&= \sum_{J \in \mathcal{P}} \sum\nolimits_J' \\
&= \sum_{J \in \mathcal{P}} \Big(-C \sum\nolimits_J + \sum\nolimits_J' \Big) + C \sum \big(f_a + f_{-a} \big) \\
&\geq -C \sum\nolimits_{J_1} + \sum\nolimits_{J_1}' + 2C \inf_{b \in A_0} \sum f_b \qquad \text{(using (20) with } J \in \mathcal{P} \setminus \{J_1\}) \\
&> C \left(1 + \inf_{b \in A_0} \sum f_b \right) \qquad \text{(using (80,81,79))}.
\end{aligned}$$

Using (72), it follows that $C < 1$. This proves Claim 6. □

Claim 7. *We have $f_{a'}^{-1}(-1) \subseteq J_0$.*

Proof of Claim 7. Let $J \in \mathcal{P} \setminus \{J_0\}$. By Claim 5 we have $p_{-a} \in J_0$. Since also $p_a \in J_0$, by Definition 39(b), it follows that $\big| J \cap (I_a \sqcup I_{-a}) \big| = 1$. Using hypothesis (27) and (20), it follows that

$$\begin{aligned}
\sum_{i' \in J \cap I_{a'}} f(i') &\geq -C \\
&> -1 \qquad \text{(by Claim 6)}.
\end{aligned} \tag{82}$$

By Claim 5 we have $p_{a'} \in J_0$. Hence this point does not lie in J. Therefore, (82) implies that $J \cap I_{a'} \cap f^{-1}(-1) = \emptyset$. Since this holds for every $J \in \mathcal{P} \setminus \{J_0\}$, and $\mathcal{P}$ covers $I_{a'}$, it follows that $I_{a'} \cap f^{-1}(-1) \subseteq J_0$. This proves Claim 7. □

Claim 5 and Definition 39(a) imply that $J_0 \cap \big(I_a \sqcup I_{-a} \big) = \{p_a, p_{-a}\}$, and therefore,

$$\sum_{i \in J_0 \cap \big(I_a \sqcup I_{-a} \big)} f(i) = f(p_a) + f(p_{-a}). \tag{83}$$

Claim 7 and hypothesis (28) imply that

$$\sum_{i' \in J_0 \cap I_{a'}} f(i') \leq f(p_{a'}) - 1.$$

Combining this with (83) and (20) with $J = J_0$, it follows that

$$C(f(p_a) + f(p_{-a})) \leq f(p_{a'}) - 1.$$

It follows that

$$\begin{aligned}
C &\leq \frac{f(p_{a'}) - 1}{f(p_a) + f(p_{-a})} \\
&\leq \frac{\sup_b f(p_b) - 1}{2 \inf_b f(p_b)} \\
&< 1 \qquad \text{(using (33))}.
\end{aligned}$$

Here in the case $k = 1$ or 2 we use Claim 2. It follows that $C_1^f < 1$. Hence f satisfies (24). This completes the proof of Proposition 43. $\square$

5. Proof of Theorem 15(iii) (cardinality of a generating system)

The proof of Theorem 15(iii) is based on the following lemma. For every set S we denote by $\mathcal{P}(S)$ its power set. For every subcollection $\mathcal{C} \subseteq \mathcal{P}(X)$ we denote by $\sigma(\mathcal{C})$ the σ-algebra generated by $\mathcal{C}$. It is given by

$$\sigma(\mathcal{C}) := \bigcap_{\mathcal{A}\,\sigma\text{-algebra on } X:\, \mathcal{C}\subseteq\mathcal{A}} \mathcal{A}.$$

A measurable space is a pair $(X, \mathcal{A})$, where X is a set and $\mathcal{A}$ a σ-algebra on X. Let $(X, \mathcal{A}), (X', \mathcal{A}')$ be measurable spaces. A map $f : X \to X'$ is called $(\mathcal{A}, \mathcal{A}')$-*measurable* iff $f^{-1}(A') \in \mathcal{A}$, for all $A' \in \mathcal{A}'$. We denote

$$\mathcal{M}(\mathcal{A}, \mathcal{A}') := \big\{(\mathcal{A}, \mathcal{A}')\text{-measurable map: } X \to X'\big\}.$$

Lemma 50 (cardinality of the set of measurable maps)**.** *Let X, X' be sets and $\mathcal{C} \subseteq \mathcal{P}(X), \mathcal{C}' \subseteq \mathcal{P}(X')$ be subcollections. Assume that $|\mathcal{C}| \leq \beth_1$, $|\mathcal{C}'| \leq \beth_0 = \aleph_0$, and*

$$\forall x' \in X' : \bigcap_{C' \in \mathcal{C}' :\, x' \in C'} C' = \{x'\}. \tag{84}$$

We define $\mathcal{A} := \sigma(\mathcal{C})$, $\mathcal{A}' := \sigma(\mathcal{C}')$. Then $\mathcal{M}(\mathcal{A}, \mathcal{A}')$ has cardinality at most $\beth_1$.

For the proof of this lemma we need the following.

Lemma 51 (cardinality of σ-algebra)**.** *Let X be a set and $\mathcal{C} \subseteq \mathcal{P}(X)$ be a subcollection of cardinality at most $\beth_1$. Then $\sigma(\mathcal{C})$ has cardinality at most $\beth_1$.*

The proof of this lemma is based on the following. Let S be a set, $F : \mathcal{P}(S) \to \mathcal{P}(S)$, such that

$$A \subseteq F(A), \quad \forall A \in \mathcal{P}(S). \tag{85}$$

Let $A \in \mathcal{P}(S)$. We define $\langle F, A\rangle$, the *set generated by* F, A, to be the smallest fixed point of F containing A. This is the set given by

$$\langle F, A\rangle = \bigcap \big\{B \in \mathcal{P}(S) \,\big|\, A \subseteq B = F(B)\big\}.$$[41]

Lemma 52 (cardinality of generated set)**.** *The set $\langle F, A\rangle$ has cardinality at most $\beth_1$, if the following conditions are satisfied:*

(a) F is monotone, i.e., $B \subseteq C$ implies that $F(B) \subseteq F(C)$.
(b) $|A| \leq \beth_1$.
(c) If $|B| \leq \beth_1$ then $|F(B)| \leq \beth_1$, for every $B \in \mathcal{P}(S)$.
(d) If $B \in \mathcal{P}(S)$ satisfies

$$F(C) \subseteq B, \quad \forall \text{ countable subset } C \subseteq B, \tag{86}$$

then B is a fixed point of F.

[41]This intersection is well-defined, since the collection of all admissible B is nonempty. It contains $B = S$.

Proof of Lemma 52. We denote by ω_1 the smallest uncountable (von Neumann) ordinal, i.e., the set of countable ordinals. We define $A_0 := A$, and using transfinite recursion, for every $\alpha \leq \omega_1$, we define

$$A_\alpha := \begin{cases} F(A_\beta), & \text{if } \alpha = \beta + 1, \\ \bigcup_{\beta<\alpha} A_\beta, & \text{if } \alpha \neq 0 \text{ is a limit ordinal.} \end{cases} \tag{87}$$

(A limit ordinal is an ordinal for which there does not exist any ordinal β for which $\alpha = \beta + 1$.)

Claim 1. *We have*

$$\langle F, A \rangle \subseteq A_{\omega_1}.$$

Proof of Claim 1. Since $A_0 \subseteq A_{\omega_1}$, it suffices to show that A_{ω_1} is a fixed point of F.

Claim 2. *Condition* (86) *is satisfied with* $B = A_{\omega_1}$.

Proof of Claim 2. Let $C \subseteq A_{\omega_1}$ be a countable subset. The definition (87), condition (85), and transfinite induction imply that for every pair α, β of ordinals, we have

$$\alpha \leq \beta \Rightarrow A_\alpha \subseteq A_\beta. \tag{88}$$

We choose a collection $(\alpha_c)_{c\in C}$ of countable ordinals, such that $c \in A_{\alpha_c}$, for every $c \in C$. The ordinal

$$\alpha := \sup_{c\in C} \alpha_c := \bigcup_{c\in C} \alpha_c$$

is countable, and therefore less than ω_1. For every $c \in C$, we have $\alpha_c \leq \alpha$, and thus by (88), $A_{\alpha_c} \subseteq A_\alpha$. It follows that $C \subseteq A_\alpha$, and therefore,

$$\begin{aligned} F(C) &\subseteq F(A_\alpha) && \text{(using (a))} \\ &= A_{\alpha+1} && \text{(using (87))} \\ &\subseteq A_{\omega_1} && \text{(using } \alpha + 1 < \omega_1 \text{ and (88)).} \end{aligned}$$

This proves Claim 2. □

By this claim and (d) the set A_{ω_1} is a fixed point of F. This proves Claim 1. □

For every ordinal α we denote by $P(\alpha)$ the statement "$|A_\alpha| \leq \beth_1$".

Claim 3. *The statement $P(\alpha)$ is true for all $\alpha \leq \omega_1$.*

Proof of Claim 3. We prove this by transfinite induction. Let $\alpha \leq \omega_1$ and assume that the statement holds for all $\beta < \alpha$. If $\alpha = 0$ then $P(0)$ holds by our hypothesis (b). If $\alpha = \beta + 1$ for some β then $P(\alpha)$ holds by (87) and our hypothesis (c). If $\alpha \neq 0$ is a limit ordinal, then $P(\alpha)$ holds by (87), our induction hypothesis, and the fact $|\alpha| \leq |\omega_1| \leq \beth_1$. This completes the inductive step. Claim 3 now follows from transfinite induction. □

Lemma 52 follows from Claims 1 and 3. □

Proof of Lemma 51. This follows from Lemma 52 with

$$S := \mathcal{P}(X),\ A := \mathcal{C},\ F(\mathcal{D}) := \left\{ \bigcup \mathcal{E} \,\middle|\, \mathcal{E} \subseteq \mathcal{D} \text{ countable} \right\} \cup \left\{ X \setminus E \,\middle|\, E \in \mathcal{D} \right\}.$$

To see that (d) holds, let $B = \mathcal{D} \in \mathcal{P}(S)$ be such that (86) holds. It suffices to show that $\mathcal{D}$ is closed under countable unions and complements. Let $\mathcal{E} \subseteq \mathcal{D}$ be a countable subcollection. We have

$$\begin{aligned} \bigcup \mathcal{E} &\in F(\mathcal{E}) \\ &\subseteq \mathcal{D} \qquad \text{(using (86))}. \end{aligned}$$

Hence $\mathcal{D}$ is closed under countable unions. Let now $E \in \mathcal{D}$. We have

$$\begin{aligned} X \setminus E &\in F(\{E\}) \\ &\subseteq \mathcal{D} \qquad \text{(using (86))}. \end{aligned}$$

Hence $\mathcal{D}$ is closed under complements. It follows that $\mathcal{D}$ is a fixed point of F. This proves (d) and completes the proof of Lemma 51. □

Proof of Lemma 50. Recall that for every pair of sets S, S' we denote by ${S'}^S$ the set of maps from S to S'. Let $f \in \mathcal{M}(\mathcal{A}, \mathcal{A}')$ and $x' \in X'$. Our hypothesis that $|\mathcal{C}'| \leq \aleph_0$ and (84) imply that the set $\{x'\}$ is a countable intersection of elements of $\mathcal{C}'$. Hence it lies in $\mathcal{A}'$. It follows that $f^{-1}(x') \in \mathcal{A}$. The following map is therefore well-defined:

$$\iota : \mathcal{M}(\mathcal{A}, \mathcal{A}') \to \mathcal{A}^{X'}, \quad \iota(f)(x') := f^{-1}(x').$$

We define the map

$$\begin{aligned} \varphi &: \mathcal{M}(\mathcal{A}, \mathcal{A}') \to \mathcal{A}^{\mathcal{C}'}, \quad \varphi(f)(C') := f^{-1}(C'), \\ \psi &: \mathcal{A}^{\mathcal{C}'} \to \mathcal{A}^{X'}, \quad \psi(A)(x') := \bigcap_{C' \in \mathcal{C}' :\, x' \in C'} A(C'). \end{aligned}$$

Our hypothesis $|\mathcal{C}'| \leq \aleph_0$ implies that $\psi(A)(x')$ is a countable intersection of elements of $\mathcal{A}$, hence an element of $\mathcal{A}$. It follows that ψ is well-defined. For every $f \in \mathcal{M}(\mathcal{A}, \mathcal{A}')$ and $x' \in X'$, we have

$$\begin{aligned} \iota(f)(x') &= f^{-1}(x') \\ &= f^{-1}\left(\bigcap_{C' \in \mathcal{C}' :\, x' \in C'} C' \right) \qquad \text{(by (84))} \\ &= \bigcap_{C' \in \mathcal{C}' :\, x' \in C'} f^{-1}(C') \\ &= \big(\psi(\varphi(f))\big)(x'). \end{aligned}$$

Hence the equality $\iota = \psi \circ \varphi$ holds. Since ι is injective, it follows that φ is injective. Our hypothesis that $|\mathcal{C}| \leq \beth_1$ and Lemma 51 imply that $\big|\mathcal{A} = \sigma(\mathcal{C})\big| \leq \beth_1$. Since $|\mathcal{C}'| \leq \aleph_0$, it follows that $\big|\mathcal{A}^{\mathcal{C}'}\big| \leq \beth_1$. Since φ maps $\mathcal{M}(\mathcal{A}, \mathcal{A}')$ to $\mathcal{A}^{\mathcal{C}'}$, it follows that $\big|\mathcal{M}(\mathcal{A}, \mathcal{A}')\big| \leq \beth_1$. This proves Lemma 50. □

In the proof of Theorem 15(iii) we will also use the following.

Remarks 53.

(i) Every countable product of second countable topological spaces is second countable.

(ii) Let (X, τ) be a topological space and $\mathcal{B}$ a basis of τ. Then the following inequality holds:

$$|\tau| \leq 2^{|\mathcal{B}|}$$

Proof of Theorem 15(iii). Let $\mathcal{G}_0$ be a countable subset of X^S. We equip $X^{\mathcal{G}_0}$ with the product topology $\tau_{\mathcal{G}_0}$. We define $\mathcal{A}_{\mathcal{G}_0}, \mathcal{A}$ to be the Borel σ-algebras of $\tau_{\mathcal{G}_0}, \tau$.

Claim 1. *The set $\mathcal{M}(\mathcal{A}_{\mathcal{G}_0}, \mathcal{A})$ has cardinality at most $\beth_1$.*

Proof of Claim 1. Our assumption that τ is separable and metrizable, implies that it is second countable. Hence by Remark 53(i), the same holds for $\tau_{\mathcal{G}_0}$. Hence by Remark 53(ii), we have

$$|\tau_{\mathcal{G}_0}| \leq 2^{\aleph_0} = \beth_1. \tag{89}$$

We have $\mathcal{A}_{\mathcal{G}_0} = \sigma(\tau_{\mathcal{G}_0})$. Since τ is separable, there exists a countable τ-dense subset A of X. We define $\mathcal{C}$ to be the collection of all open balls with rational radius around points in A. Since A is dense, every element of τ is a union of elements of $\mathcal{C}$. Since A is countable, the set $\mathcal{C}$ is countable. It follows that $\mathcal{A} = \sigma(\tau) = \sigma(\mathcal{C})$. Since τ is separable and metrizable, the condition (84) with $\mathcal{C}'$ replaced by $\mathcal{C}$ is satisfied. Using (89), it follows that the hypotheses of Lemma 50 are satisfied with $\mathcal{C}, \mathcal{C}'$ replaced by $\tau_{\mathcal{G}_0}, \mathcal{C}$. Applying this lemma, it follows that $\big|\mathcal{M}(\mathcal{A}_{\mathcal{G}_0}, \mathcal{A})\big| \leq \beth_1$. This proves Claim 1. □

Let $\mathcal{G}$ be a subset of X^S of cardinality at most $\beth_1$. By Definition 14 the set countably Borel-generated by $\mathcal{G}$ is given by

$$\langle \mathcal{G} \rangle := \big\{ f \circ ev_{\mathcal{G}_0} \,|\, \mathcal{G}_0 \subseteq \mathcal{G} \text{ countable, } f \in \mathcal{M}(\mathcal{A}_{\mathcal{G}_0}, \mathcal{A}) \big\}.$$

The set of all countable subsets of $\mathcal{G}$ has cardinality at most $\beth_1^{\aleph_0} = \beth_1$. Using Claim 1, it follows that

$$|\langle \mathcal{G} \rangle| \leq \beth_1^2 = \beth_1.$$

This proves Theorem 15(iii). □

6. Proof of Theorem 24 (uncountability of every generating set under a very mild hypothesis)

Proof of Theorem 24. Let $\mathcal{C} = (\mathcal{O}, \mathcal{M}), A, M$ be as in the hypothesis. W.l.o.g. we may assume that A is open. Our hypothesis (9) implies that the map $\mathrm{Vol}^{\frac{1}{n}} \circ M : A \to \mathbb{R}$ is continuous and strictly increasing. Hence it is injective with image $\widetilde{A}$ given by an interval. We define

$$\widetilde{M} := M \circ \left(\mathrm{Vol}^{\frac{1}{n}} \circ M\right)^{-1} : \widetilde{A} \to \mathcal{O}.$$

Let $\widetilde{a}_0 \in \widetilde{A}$. We define

$$g_{\widetilde{a}_0} := c_{\widetilde{M}_{\widetilde{a}_0}} \circ \widetilde{M} : \widetilde{A} \to \mathbb{R}.$$

Claim 1. *This map is not differentiable at $\widetilde{a}_0$.*

Proof of Claim 1. We have

$$\mathrm{Vol}^{\frac{1}{n}} \circ \widetilde{M} = \mathrm{id}.$$

It follows that

$$g_{\widetilde{a}_0}(\widetilde{a}) \leq \frac{\widetilde{a}}{\widetilde{a}_0}, \quad \forall \widetilde{a} \in \widetilde{A} \cap (0, \widetilde{a}_0). \tag{90}$$

Our hypothesis (10) implies that

$$g_{\widetilde{a}_0}(\widetilde{a}) = 1, \quad \forall \widetilde{a} \in \widetilde{A} \cap [\widetilde{a}_0, \infty).$$

Combining this with (90), it follows that $g_{\widetilde{a}_0}$ is not differentiable at $\widetilde{a}_0$. This proves Claim 1. $\square$

Let now $\mathcal{G}$ be a countable subset of $\mathcal{C}ap(\mathcal{C})$. Let $c \in \mathcal{G}$. The inequality "$\geq$" in our hypothesis (10) implies that the map $c \circ M$ is increasing. It follows that the same holds for $c \circ \widetilde{M}$. Therefore, by Lebesgue's Monotone Differentiation Theorem the map $c \circ \widetilde{M}$ is differentiable[42] almost everywhere, see e.g. [Tao11, p. 156, Theorem 1.6.25]. Since $\mathcal{G}$ is countable, it follows that the set of all points in $\widetilde{A}$ at which the function $c \circ \widetilde{M}$ is differentiable, for every $c \in \mathcal{G}$, has full Lebesgue measure. Since A has positive length, the same holds for $\widetilde{A}$. It follows that there exists a point $\widetilde{a}_0 \in \widetilde{A}$ at which $c \circ \widetilde{M}$ is differentiable, for every $c \in \mathcal{G}$.

Let $\mathcal{G}_0$ be a finite subset of $\mathcal{G}$, and $f : [0,\infty]^{\mathcal{G}_0} \to [0,\infty]$ a differentiable function. We define $\mathrm{ev}_{\mathcal{G}_0}$ as in (5). Since $c \circ \widetilde{M}$ is differentiable at $\widetilde{a}_0$ for every $c \in \mathcal{G}_0$, the same holds for the map $\mathrm{ev}_{\mathcal{G}_0} \circ \widetilde{M} : \widetilde{A} \to [0,\infty]^{\mathcal{G}_0}$. It follows that the composition $f \circ \mathrm{ev}_{\mathcal{G}_0} \circ \widetilde{M}$ is differentiable at $\widetilde{a}_0$. Using Claim 1, it follows that

$$f \circ \mathrm{ev}_{\mathcal{G}_0} \circ \widetilde{M} \neq g_{\widetilde{a}_0} = c_{\widetilde{M}_{\widetilde{a}_0}} \circ \widetilde{M},$$

and therefore that $f \circ \mathrm{ev}_{\mathcal{G}_0} \neq c_{\widetilde{M}_{\widetilde{a}_0}}$. Hence $\mathcal{G}_0$ does not finitely differentiably generate $c_{\widetilde{M}_{\widetilde{a}_0}}$. This proves Theorem 24. $\square$

Appendix A. Cardinality of the set of equivalence classes of pairs of manifolds and forms

In this section we prove that the set of diffeomorphism types of smooth manifolds has cardinality at most $\beth_1$. We also prove that the same holds for the set of all equivalence classes of pairs (M,ω), where M is a manifold, and ω is a differential form on M. We used this in the proof of Theorem 42, to estimate the cardinality of the set of (normalized) capacities from above.

In order to deal with certain set-theoretic issues, we explain how to make the class of all diffeomorphism types a set. Let A, B be sets and $S : A \to B$ a map. Let $a \in A$. We denote $S_a := S(a)$. Recall that in ZFC "everything" is a set, in particular S_a. Recall also that the disjoint union of S is defined to be

$$\bigsqcup S := \big\{ (a,s) \,\big|\, s \in S_a \big\}.$$

We denote

$$H^n := \big\{ x \in \mathbb{R}^n \,\big|\, x_n \geq 0 \big\}.$$

Let S be a set. By an atlas on S we mean a subset

$$\mathcal{A} \subseteq \bigsqcup_{U \in \mathcal{P}(S)} (H^n)^U,$$

such that

$$\bigcup_{(U,\varphi) \in \mathcal{A}} U = S,$$

[42] in the usual sense

for every $(U,\varphi) \in \mathcal{A}$ the map φ is injective, and for all $(U,\varphi),(U',\varphi') \in \mathcal{A}$ the set $\varphi(U \cap U')$ is open (in H^n) and the transition map

$$\varphi' \circ \varphi^{-1} : \varphi(U \cap U') \to H^n$$

is smooth. We call an atlas maximal iff it is not contained in any strictly larger atlas. By a *(smooth finite-dimensional real) manifold (with boundary)* we mean a pair $M = (S,\mathcal{A})$, where S is a set and $\mathcal{A}$ is a maximal atlas on S, such that the induced topology is Hausdorff and second countable. We denote by $\beth_1$ the (von Neumann) cardinal $2^{\beth_0=\aleph_0}$, and by $\sim$ the diffeomorphism relation on

$$\mathcal{M}_0 := \big\{(S,\mathcal{A}) \,\big|\, S \subseteq \beth_1,\ (S,\mathcal{A}) \text{ is a manifold}\big\}. \tag{91}$$

This means that $M \sim M'$ iff M and M' are diffeomorphic. We define the *set of diffeomorphism types (of manifolds)* to be

$$\mathfrak{M} := \big\{ \sim\text{-equivalence class}\big\}.$$

Remarks 54 (diffeomorphism types)**.**

- The above definition overcomes the set theoretic issue that the "set" of diffeomorphism classes of all manifolds (without any restriction on the underlying set) is not a set (in ZFC).
- Every manifold M is diffeomorphic to one whose underlying set is a subset of $\beth_1$. To see this, note that using second countability and the axiom of choice, the set underlying M has cardinality $\leq \beth_1$. This means that there exists an injective map $f : M \to \beth_1$. Pushing forward the manifold structure via f, we obtain a manifold whose underlying set is a subset of $\beth_1$, as claimed.
- By the last remark, heuristically, there is a canonical bijection between $\mathfrak{M}$ and the "set" of diffeomorphism classes of all manifolds.
- One may understand $\mathfrak{M}$ in a more general way as follows. Let $\mathcal{M}$ be a set consisting of manifolds, such that every manifold is diffeomorphic to some element of $\mathcal{M}$. For example, let S be a set of cardinality at least $\beth_1$ and define $\mathcal{M}$ to be the set of all manifolds whose underlying set is a subset of S. The set $\mathfrak{M}$ is in bijection with the set of all diffeomorphism classes of elements of $\mathcal{M}$.

Proposition 55. *The set $\mathfrak{M}$ has cardinality at most $\beth_1$.*

In the proof of this result we will use the following.

Remark 56 (Whitney's Embedding Theorem)**.** Let $n \in \mathbb{N}_0$ and M be a (smooth) manifold of dimension n. There exists a (smooth) embedding of M into $\mathbb{R}^{2n+1}$ with closed image. To see this, consider the double $\widetilde{M}$ of M, which is obtained by gluing two copies of M along the boundary. By Whitney's Embedding Theorem there exists an embedding of $\widetilde{M}$ into $\mathbb{R}^{2n+1}$ with closed image, see e.g. [Hir94, 2.14. Theorem, p. 55][43]. Composing such an embedding with one of the two canonical inclusions of M in $\widetilde{M}$, we obtain an embedding of M into $\mathbb{R}^{2n+1}$ with closed image, as desired.

[43]In this section of Hirsch's book manifolds are not allowed to have boundary. This is the reason for considering $\widetilde{M}$, rather than M.

Proof of Proposition 55. We define

$$\mathcal{M} := \bigsqcup_{m\in\mathbb{N}_0} \{\text{submanifold of } \mathbb{R}^m\}.$$

Claim 1. *We have* $|\mathcal{M}| \leq \beth_1$.

Proof. Let $n, m \in \mathbb{N}_0$. The topological space $\mathbb{N}_0 \times H^n$ is separable. Since $|\mathbb{R}^m| \leq \beth_1$, it follows that

$$\left|C\big(\mathbb{N}_0 \times H^n, \mathbb{R}^m\big)\right| \leq \beth_1. \tag{92}$$

Let $n \in \mathbb{N}_0$ and $(m, M) \in \mathcal{M}$, such that M is of dimension n. Since M is second countable, there exists a surjective map $\psi : \mathbb{N}_0 \times H^n \to M$ whose restriction to $\{i\} \times H^n$ is an embedding, for every $i \in \mathbb{N}_0$. It follows that M lies in the image of the map

$$C\big(\mathbb{N}_0 \times H^n, \mathbb{R}^m\big) \to \mathcal{P}(\mathbb{R}^m), \quad f \mapsto \mathrm{im}(f).$$

Combining this with (92), it follows that $|\mathcal{M}| \leq \beth_1$. This proves Claim 1. □

Let $n \in \mathbb{N}_0$. We choose an injection $\alpha : \mathbb{R}^{2n+1} \to \beth_1$, and consider the pushforward map

$$\alpha_* : \mathcal{M} \to \mathfrak{M}, \quad \alpha_*(S, \mathcal{A}) := \big[\alpha(S), \alpha_*\mathcal{A}\big].$$

Remark 56 implies that this map is surjective. Using Claim 1, it follows that $|\mathfrak{M}| \leq \beth_1$. This proves Proposition 55. □

We define $\mathcal{M}_0$ as in (91),

$$\Omega(M) := \{\text{differential form on } M\},$$
$$\Omega_0 := \textstyle\bigsqcup_{M\in\mathcal{M}_0} \Omega(M),$$

the equivalence relation $\approx$ on Ω_0 by

$$(M, \omega) \approx (M', \omega) :\iff \exists \text{ diffeomorphism } \varphi : M \to M' : \varphi^*\omega' = \omega,$$
$$\text{and } \overline{\Omega} := \Omega_0/\approx.$$

Remark. Philosophically, this is the "set" of all equivalence classes of pairs (M, ω), where M is an arbitrary manifold and ω is a differential form on M. The above definition makes this idea precise.

Corollary 57. *The set* $\overline{\Omega}$ *has cardinality at most* $\beth_1$.

Proof of Corollary 57. If M, M' are manifolds and $\varphi : M \to M'$ is a diffeomorphism then

$$\varphi^* : \Omega(M') \to \Omega(M) \text{ is a bijection.} \tag{93}$$

We denote by $\Pi : \Omega_0 \to \overline{\Omega}$ and $\pi : \mathcal{M}_0 \to \mathfrak{M}$ the canonical projections, and by $f : \Omega_0 \to \mathcal{M}_0$, $f((M, \omega)) := M$, the forgetful map. We define $F : \overline{\Omega} \to \mathfrak{M}$ to be the unique map satisfying $F \circ \Pi = \pi \circ f$. Let $\mathcal{M} \in \mathfrak{M}$. Choosing $M \in \mathcal{M}$, we have

$$\begin{aligned} F^{-1}(\mathcal{M}) &= \Pi\big((F \circ \Pi)^{-1}(\mathcal{M})\big) \\ &= \Pi\big((\pi \circ f)^{-1}(\mathcal{M})\big) \\ &= \Pi\big(f^{-1}(\mathcal{M})\big) \qquad (\text{using that } \pi^{-1}(\mathcal{M}) = \mathcal{M}) \\ &= \Pi\big(f^{-1}(M) = \Omega(M)\big) \qquad (\text{using (93)}). \end{aligned} \tag{94}$$

Since M is separable and $|TM| \leq \beth_1$, we have $|C(M,TM)| \leq \beth_1$. Using $\Omega(M) \subseteq C(M,TM)$, (94), and Proposition 55, it follows that

$$\left|\overline{\Omega} = \bigcup_{\mathcal{M}\in\mathfrak{M}} F^{-1}(\mathcal{M})\right| \leq \beth_1^2 = \beth_1.$$

This proves Corollary 57. □

Remark. Let $n \geq 2$. Then the set of diffeomorphism types of manifolds of dimension n has cardinality equal to $\beth_1$. To see this, we choose a countable set $\mathcal{M}$ of nondiffeomorphic connected n-manifolds. The map

$$\{0,1\}^{\mathcal{M}} \ni u \mapsto \bigsqcup_{M\in\mathcal{M}:\, u(M)=1} M \in \{\text{n-manifold}\}$$

is injective. Hence the set of diffeomorphism types of manifolds of dimension n has cardinality $\geq \beth_1$. Combining this with Proposition 55, it follows that this cardinality equals $\beth_1$, as claimed.

Appendix B. Proof of Theorem 28 (monotone generation for ellipsoids)

Theorem 28 follows from McDuff's characterization of the existence of symplectic embeddings between ellipsoids, and the fact that monotone generation is equivalent to almost order-recognition. To explain this, let $(S,\leq)$ be a preordered set. We fix an order-preserving $(0,\infty)$-action on S. We define the *order-capacity map* $c^{\leq} : S \times S \to [0,\infty]$ by

$$c^{\leq}(s,s') := \sup\big\{a \in (0,\infty) \,\big|\, as \leq s'\big\}.$$

Remark 58. For every $s \in S$ the map $c^{\leq}(s,\cdot)$ is a capacity, as defined in (14).

Let $\mathcal{G} \subseteq \mathcal{C}ap(S)$. We call $\mathcal{G}$ *almost order-recognizing (or almost order-reflecting)* iff for all $s, s' \in S$ the following holds:

$$c(s) \leq c(s'),\ \forall c \in \mathcal{G} \Rightarrow c^{\leq}(s,s') \geq 1.$$

Remark. A map f between two preordered sets is called *order-reflecting* if $f(s) \leq f(s')$ implies that $s \leq s'$. The set $\mathcal{G}$ is almost order-reflecting iff its evaluation map is "almost" order-reflecting, in the sense that $\mathrm{ev}_{\mathcal{G}}(s) \leq \mathrm{ev}_{\mathcal{G}}(s')$ implies that for every $a_0 \in (0,1)$ there exists an $a \in [a_0,\infty)$, such that $as \leq s'$.

Proposition 59 (characterization of monotone generation)**.** *The set $\mathcal{G}$ monotonely generates if and only if it is almost order-recognizing.*

In the proof of this result we use the following. Let $(X,\leq), (X',\leq')$ be preordered sets, $X_0 \subseteq X$, and $f : X_0 \to X'$. We define the *monotonization of f* to be the map $F : X \to X'$ given by

$$F(x) := \sup\big\{f(x_0) \,\big|\, x_0 \in X_0 :\ x_0 \leq x\big\}.$$

Remarks 60 (monotonization)**.**
(i) The map F is monotone.
(ii) If X and X' are equipped with order-preserving $(0,\infty)$-actions and f is homogeneous, then its monotonization is homogeneous.

(iii) If f is monotone then it agrees with the restriction of F to X_0.

Proof of Proposition 59. "$\Rightarrow$": Assume that $\mathcal{G}$ monotonely generates. Let $s, s' \in S$ be such that $c(s) \leq c(s')$, for every $c \in \mathcal{G}$. This means that

$$\mathrm{ev}_{\mathcal{G}}(s) \leq \mathrm{ev}_{\mathcal{G}}(s'). \tag{95}$$

By Remark 58 and our assumption there exists a monotone function $F \in [0,\infty]^{\mathcal{G}}$, such that

$$c_s := c^{\leq}(s,\cdot) = F \circ \mathrm{ev}_{\mathcal{G}}\,.$$

We have

$$\begin{aligned} 1 &\leq c_s(s) \qquad \text{(since $\leq$ is reflexive and hence $s \leq s$)} \\ &= F \circ \mathrm{ev}_{\mathcal{G}}(s) \\ &\leq F \circ \mathrm{ev}_{\mathcal{G}}(s') \qquad \text{(using (95) and monotonicity of F)} \\ &= c_s(s'). \end{aligned}$$

Hence $\mathcal{G}$ is almost order-reflecting. This proves "$\Rightarrow$".

To prove the implication "$\Leftarrow$", assume that $\mathcal{G}$ is almost order-recognizing. Let $c_0 \in \mathcal{C}ap(S)$.

Claim 1. *For every pair of points $s, s' \in S$, satisfying $\mathrm{ev}_{\mathcal{G}}(s) \leq \mathrm{ev}_{\mathcal{G}}(s')$, we have $c_0(s) \leq c_0(s')$.*

Proof. Since $c(s) \leq c(s')$, for every $c \in \mathcal{G}$, by assumption, we have $c_s(s') \geq 1$. Let $a_0 \in (0,1)$. It follows that there exists $a \in [a_0,\infty)$, such that $as \leq s'$. It follows that

$$a_0 c_0(s) \leq a c_0(s) = c_0(as) \leq c_0(s').$$

Since this holds for every $a_0 \in (0,1)$, it follows that $c_0(s) \leq c_0(s')$. This proves Claim 1. □

We define $f : \mathrm{im}(\mathrm{ev}_{\mathcal{G}}) \to [0,\infty]$ by setting $f(x) := c_0(s)$, where s is an arbitrary point in $\mathrm{ev}_{\mathcal{G}}^{-1}(x) \subseteq S$. By Claim 1 this function is well-defined, i.e., it does not depend on the choice of s. It satisfies

$$f \circ \mathrm{ev}_{\mathcal{G}} = c_0. \tag{96}$$

It follows from this equality and Claim 1 that f is monotone. By Remark 60(i,iii) and equality (96) the monotonization F of f is a monotone function on $[0,\infty]^{\mathcal{G}}$ that satisfies $F \circ \mathrm{ev}_{\mathcal{G}} = c_0$. This proves "$\Leftarrow$" and completes the proof of Proposition 59. □

Proof of Theorem 28. We equip the set of ellipsoids in (V,ω) with the preorder $E \leq E'$ iff there exists a symplectic embedding of E into E'. By [McD11, Theorem 1.1] the condition $c_j^{V,\omega}(E) \leq c_j^{V,\omega}(E')$, for all $j \in \mathbb{N}_0$, implies that aE symplectically embeds into E', for all $a \in (0,1)$. This means that the set of all $c_j^{V,\omega}$ (with $j \in \mathbb{N}_0$) is almost order-recognizing. Hence by Proposition 59 this set monotonely generates. This proves Theorem 28. □

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