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Subleading and non-holomorphic corrections to $N = 2$ BPS black hole entropy

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Abstract: BPS black hole degeneracies can be expressed in terms of an inverse Laplace transform of a partition function based on a mixed electric/magnetic ensemble, which involves a non-trivial integration measure. This measure has been evaluated for black holes with various degrees of supersymmetry and for N=4 supersymmetric black holes all results agree. It generally receives contributions from non-holomorphic corrections. An explicit evaluation of these corrections in the context of the effective action of the FHSV model reveals that these are related to, but quantitatively different from, the non-holomorphic corrections to the topological string, indicating that the relation between the twisted partition functions of the latter and the effective action is more subtle than has so far been envisaged. The effective action result leads to a duality invariant BPS free energy and arguments are presented for the existence of consistent non-holomorphic deformations of special geometry that can account for these effects. A prediction is given for the measure based on semiclassical arguments for a class of N=2 black holes. Furthermore an attempt is made to confront some of the results of this paper with a recent proposal for the microstate degeneracies of the STU model.

Keywords: Supersymmetry and Duality, Black Holes in String Theory, Extended Supersymmetry, Topological Strings.

Contents

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1. Introduction

The degeneracy of BPS states of certain wrapped brane/string configurations defines a microscopic entropy which, in quite a number of cases, has been successfully compared to the macroscopic entropy of supersymmetric black hole solutions in the corresponding effective supergravity theories. Agreement is usually obtained in the limit where charges are large [1], because in that limit one can make use of the Cardy formula for the underlying conformal field theory. The macroscopic entropy is not necessarily identified with a quarter of the horizon area, since there are corrections associated with higher-derivative couplings [2–4]. More recently, it was proposed that the entropy of four-dimensional BPS black holes with $N = 2$ supersymmetry is related to a partition function based on a mixed ensemble defined in terms of magnetic charges and electrostatic potentials. Discarding non-holomorphic corrections this partition function equals the modulus square of the topological string partition function [5]. On the basis of this relation it was concluded that the microscopic black hole degeneracies can be retrieved from the topological string partition function by an inverse Laplace transform. This observation gave new impetus to studying the relation between microscopic and macroscopic descriptions of black holes on the one hand, and the relation between black hole degeneracies and the topological string on the other. (See, for instance, [6 – 15].)

As was readily understood the proper definition of the inverse Laplace integral is subtle for reasons of convergence and in view of ambiguities in choosing the integration contours. The issue of non-holomorphicity did not enter into the original proposal. Early discussions can be found in [16 – 18]. Non-holomorphic terms are essential for duality invariance, and indeed such terms were encountered when confronting the asymptotic results from microstate counting with macroscopic results based on effective actions [19, 20, 11]. They involve terms originating from higher-order interactions that contain the square of the Riemann tensor, such as the ones that were determined in [21 – 23], which are part of the effective field theory. The presence of non-holomorphic corrections can also be inferred from the relation with the topological string, where they are encoded in the so-called holomorphic anomaly equations [24].

At an early stage there were strong indications that the inverse Laplace transform must involve a non-trivial integration measure (which will contribute to the subleading entropy corrections for large black holes in the limit of large charges), so that (subleading) non-holomorphic corrections can always be factored out from the mixed partition function and absorbed into this measure. Therefore a further understanding of these matters will ultimately depend on how well the measure factor can be understood. A strong argument in favour of the measure was based on the invariance under duality, as the partition function for the mixed ensemble does not transform simply under electric/magnetic duality. An alternative starting point [25, 11] can be based on an ensemble of electric and magnetic charges, which is manifestly invariant under duality. From this set-up the previous formulation based on the mixed partition function can be reobtained in the semiclassical approximation, but, as it turns out, it is now accompanied by a non-trivial measure factor. Independently, a direct evaluation of the mixed partition function from specific microscopic degeneracy formulae also revealed the presence of a measure factor [9], and it was shown that for large charges these measure factors were in fact equal [11, 26].

Somewhat unfortunately, the examples studied in [19, 9, 20, 11] did not pertain to genuine $N = 2$ supersymmetric string compactifications (the work reported in [2 – 4] is an exception to this), but to compactifications with $N = 4$ supersymmetry. The latter were then treated in the context of an $N = 2$ supersymmetric truncation with minor modification such as to account for the four extra graviphotons (leading to eight extra charges) and moduli provided by the two additional gravitino supermultiplets.[1] The purpose of the present paper is to study applications that pertain to genuine $N = 2$ supersymmetric models in four space-time dimensions, where such modifications are unnecessary. The problem with generic $N = 2$ supersymmetric compactifications is, however, that there are not many cases where it is possible to make direct comparisons with microstate counting and, at the same time, exact duality invariance is rather rare. There are a few models which stand out in this respect, such as the FHSV model [28] and the STU model [29, 30],

[1]This is in contrast with the work on small black holes reported in [27].

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which exhibit both exact S- and T-dualities and for which microstate degeneracy formulae have recently been proposed [31]. For $N = 2$ models based on compact Calabi-Yau spaces, the measure factor has recently been evaluated at strong topological string coupling [15]. We will show that this result disagrees with the semiclassical prediction relevant at weak coupling. We will comment on this at the end of section 4, where we also compare to results for the measure factor in $N = 4, 8$ models.

Special attention is devoted to the issue of non-holomorphic corrections, which contribute to the measure factor. As it turns out, the existence of a semiclassical free energy for BPS black holes (which plays an important role in the variational principle for the attractor equations) indicates that these corrections must be encoded in a single real homogeneous function. For $N = 4$ black holes this form of the free energy has been used successfully [19, 11], but in that case the non-holomorphic corrections are severely restricted, so that the consequences of this approach were rather minor. Therefore we further investigate the consequences of this approach in the context of the FHSV model by concentrating on the requirements posed by the exact dualities of this model. As an example we derive the subleading corrections to the function that encodes the effective action and explicitly compare the result to the topological string for the genus-1 and genus-2 contributions. As it turns out the results are clearly different.

Hence the precise relationship between the non-holomorphic terms in the effective action and those in the topological string partition functions is not entirely clear. In fact we will present further evidence that the relation between the functions that encode the effective action and the partition functions of the topological string is more subtle than has previously been envisaged. In [32, 24] it was shown that certain string amplitudes are related to the twisted partition functions of the topological string. These results, however, do not necessarily imply that the effective action should also have such a direct relationship, in view of the fact that the effective action encompasses only the one-particle irreducible diagrams and not the connected diagrams. As is well known the relation between these two sets of diagrams proceeds through a Legendre transform. Interestingly enough, a Legendre transform is also involved when one wishes to realize the duality transformations in a manifest way in a field-theoretic context. Here it is important to realize that the action is not manifestly invariant under symmetries that are induced by electric/magnetic duality [33, 34]. In order to obtain manifestly invariant quantities, one may, for instance, apply a Legendre transform and consider the Hamiltonian instead. However, in the context of special geometry it is suggestive to consider the Legendre transform that leads from complex to real special geometry. In that case one obtains the so-called Hesse potential, which is related to the black hole free energy and which is manifestly duality invariant (this was discussed in [11]). The above scenario for explaining the discrepancy is admittedly a bit speculative and it is beyond the scope of this paper to try and work this out further. Obviously, this aspect has a bearing on the original conjecture of [5].

Returning to the black holes, there are two aspects that have come under intense scrutiny lately which will not enter into our analysis. The first aspect concerns the dependence of the microstate degeneracies on the asymptotic values of the scalar moduli, i.e., on the values of the scalar fields at spatial infinity (see, for instance, [35, 15, 36 – 38]). This

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dependence is associated with the appearance or disappearance of multicentered black hole configurations [39, 40] for a given total charge. The second aspect concerns the so-called entropy enigma, a surprising phenomenon that may arise at weak topological string coupling [15]. It is based on the fact that there exist multicentered black hole solutions that carry an entropy that is vastly larger than the entropy of singlecentered solutions carrying the same charges. The occurence of this phenomenon would imply a breakdown of the conjecture of [5], which was supposed to work at weak coupling. It would be difficult to reconcile this with the fact that the predictions for large black holes have always been in agreement with semiclassical reasoning. Evidence against such a breakdown has recently been given in [41]. The approach followed in this paper will not take into account the two aspects just described and we will assume that semiclassical arguments do make sense.

This paper is organized as follows. Section 2 contains a brief review of the derivation of the measure factor from a duality invariant perspective. Subsequently the non-holomorphic corrections are incorporated in the black hole free energy and we discuss the semiclassical approximation. Section 3 describes the consequences of S- and T-duality invariance for a class of models that contain in particular the FSHV and the STU models. In section 4 the measure factors for the mixed partition function are evaluated for these models in the semiclassical approximation. In section 5 non-holomophic corrections are studied for the FHSV model and compared to the results for the topological string. Subsequently non-holomorphic deformations of special geometry are discussed. Section 6 deals with the STU model and describes an attempt to reconcile the macroscopic and microscopic results for the BPS black hole entropy in that model. Section 7 presents our conclusions.

2. The BPS black hole free energy and the partition function

At the field-theoretic level it is known that the attractor equations that determine the values of the moduli at the black hole horizon [42 − 44], follow from a variational principle. This variational principle is described in terms of a so-called entropy function. There exists an entropy function for extremal black holes [45, 46], where the attractor mechanism is induced by the restricted space-time geometry of the horizon, and one for BPS black holes [11], where the attractor mechanism follows from supersymmetry enhancement at the horizon. For $N = 2$ supergravity the relation between these entropy functions has been clarified in [47]. To preserve the variational principle when non-holomorphic corrections are present, it follows that these corrections must enter into the BPS free energy in a well-defined way. Requiring the existence of a free energy seems desirable from the point of view of semiclassical arguments and the relation with black hole thermodynamics, and it should be interesting to derive this result directly from an effective action. However, no effective $N = 2$ supersymmetric action is known to date that incorporates the non-holomorphic terms, although partial results are known for $N = 1$ [48] and from the string amplitudes that are related to the topological string [32]. We will discuss this last relationship in section 5. At any rate, the results of this paper indicate that, indeed, one can safely proceed by checking the internal consistency at the level of the entropy function, guided by the (partially established) relation with the full effective action. This is the underlying strategy of this paper.

In the first subsection we discuss the definition of the free energy, and its relation with the black hole partition function and the BPS entropy function, for a given set of degeneracies and a corresponding locally supersymmetric effective action. The second subsection describes the non-holomorphic contributions to the free energy, and the third subsection deals with the semiclassical approximation.

2.1 BPS free energy and partition functions

We consider charged black holes in the context of $N = 2$ supergravity in four space-time dimensions, which contains $n + 1$ abelian vector gauge fields, labeled by indices $I, J = 0, 1, \ldots, n$, so that black hole solutions can carry $2(n + 1)$ possible electric and magnetic charges. The theory describes the supergravity fields and n vector multiplets (the extra index $I = 0$ accounts for the gauge field that belongs to the supergravity multiplet), and possibly a number of hypermultiplets which will only play an ancillary role. A partition sum over a canonical ensemble of corresponding BPS black hole microstates is defined as follows,

$$Z(\phi, \chi) = \sum_{\{p,q\}} d(p, q)\, \mathrm{e}^{\pi[q_I\phi^I - p^I\chi_I]}\,, \tag{2.1}$$

where $d(p, q)$ denotes the degeneracy of the black hole microstates with given magnetic and electric charges equal to p^I and q_I, respectively. This expression is consistent with electric/magnetic duality, provided that the electro- and magnetostatic potentials (ϕ^I, χ_I) transform as a symplectic vector, just as the charges (p^I, q_I), while the degeneracies $d(p, q)$ transform as functions of the charges under the duality. In case that the duality is realized as a symmetry, then the $d(p, q)$ should be invariant.

Viewing $Z(\phi, \chi)$ as an analytic function in ϕ^I and χ_I, the degeneracies $d(p, q)$ can be retrieved by an inverse Laplace transform,

$$d(p, q) \propto \int \mathrm{d}\phi^I\, \mathrm{d}\chi_I\; Z(\phi, \chi)\, \mathrm{e}^{\pi[-q_I\phi^I + p^I\chi_I]}\,, \tag{2.2}$$

where the integration contours run, for instance, over the intervals $(\phi - \mathrm{i}, \phi + \mathrm{i})$ and $(\chi - \mathrm{i}, \chi + \mathrm{i})$ (we are assuming an integer-valued charge lattice). Obviously, this makes sense as long as $Z(\phi, \chi)$ is formally periodic under shifts of ϕ and χ by multiples of 2i.

Identifying the logarithm of $Z(\phi, \chi)$ with a free energy, it is expected that this expression has a field-theoretic counterpart, because the electrostatic and magnetostatic fields appear as some of the scalar moduli in the field-theoretic description. Indeed, such a free energy function exists and it is contained in the so-called BPS entropy function. Stationary points of this entropy function are subject to the attractor equations which fix the value of the moduli at the black hole horizon, and the value of the entropy function at the stationary point equals the macroscopic entropy. The latter is a function of the charges and it equals the Legendre transform of the free energy. The BPS entropy function was originally proposed in [49] for actions that are at most quadratic in space-time derivatives and its generalization to higher derivatives was discussed in [11]. It is natural to identify the partition function (2.1) with the exponent of the relevant free energy, which is contained in the entropy function. In the case at hand, where one considers functions of real potentials (ϕ^I, χ_I), this free energy equals twice the so-called Hesse potential $\mathcal{H}$, which depends on the holomorphic function that encodes the $N = 2$ supergravity theory of the vector multiplet sector [50]. In the notation of [11], we write

$$\sum_{\{p,q\}} d(p,q)\, \mathrm{e}^{\pi[q_I\phi^I - p^I\chi_I]} \sim \sum_{\text{shifts}} \mathrm{e}^{2\pi\, \mathcal{H}(\phi/2,\chi/2)}\,. \tag{2.3}$$

The Hesse potential is a macroscopic quantity which does not in general exhibit the periodicity that is characteristic for the partition function. Therefore, the right-hand side of (2.3) requires an explicit periodicity sum over discrete imaginary shifts of the ϕ^I and χ_I.[2] In the inverse Laplace integral (2.2) we expect that this periodicity sum can be incorporated into the integration contour.

It is in general difficult to find an explicit representation for the Hesse potential. The standard way to encode the effective supergravity theory (as far as the vector multiplet sector is concerned), is in terms of a holomorphic function of the complex scalar fields Y^I, and the resulting geometric structure is known as special geometry. Here one identifies a symplectic vector by combining the scalars Y^I with the holomorphic derivatives F_I of the function $F(Y)$, which transforms under duality precisely as the charges (p^I, q_I). Of course, this leaves several options for parametrizing the models, and the obvious one that leaves the symplectic structure intact is to choose real variables equal to the electro- and magnetostatic potentials [51],

$$\phi^I = Y^I + \bar{Y}^I\,, \qquad \chi_I = F_I + \bar{F}_{\bar{I}}\,. \tag{2.4}$$

In these variables one obtains the Hesse potential as a Legendre transform of the imaginary part of $F(Y)$ with respect to the imaginary part of the Y^I. This is precisely equal to one-half of the free energy $\mathcal{F}(Y,\bar{Y})$, defined in complex coordinates, that we will discuss momentarily. Substitution of these relations leads to,

$$\sum_{\{p,q\}} d(p,q)\, \mathrm{e}^{\pi[q_I(Y^I+\bar{Y}^I) - p^I(F_I+\bar{F}_I)]} \sim \sum_{\text{shifts}} \mathrm{e}^{\pi\, \mathcal{F}(Y,\bar{Y})}\,, \tag{2.5}$$

but now the definition of the shifts has become very subtle as they still refer to imaginary values of ϕ^I and χ_I. This subtlety should again be reflected in the choice of the integration contours in the inverse Laplace transform. We emphasize that at this point we are assuming that $F(Y)$ is a holomorphic function which is homogeneous of second degree, although so far we did not make use of this. The equation (2.5) is the conjectured relation between the microscopic data, defined in terms of the degeneracies $d(p,q)$, and the field-theoretic data, encoded in the free energy $\mathcal{F}$. In this section we will derive the expression for this free energy in terms of derivatives of the function F in the presence of subleading and non-holomorphic corrections, and discuss some consequences of this result. The expression

[2] In case that the Hesse potential exhibits a periodicity with a multiple of the periodicity interval, then the sum over the imaginary shifts will have to be modded out appropriately such as to avoid overcounting.

for the free energy follows from the requirement that the attractor equations are based on a variational principle. The reason for adopting this procedure is that in the presence of non-holomorphic corrections, the effective action is not fully known and hence cannot be used directly to define the free energy. We already discussed this strategy at the beginning of this section.

Postponing the discussion of various subtleties and generalizations, we consider a variable change from the real variables (χ^I, ϕ_I) to the complex variables Y^I in the integral (2.2), replacing $Z(\phi, \chi)$ by $\exp[2\pi\,\mathcal{H}(\phi/2, \chi/2)]$, and subsequently by $\exp[\pi\,\mathcal{F}(Y, \bar{Y})]$ when changing variables. This leads to the integral,

$$\begin{aligned} d(p,q) &\propto \int \mathrm{d}\left(Y+\bar{Y}\right)^I \ \mathrm{d}\left(F+\bar{F}\right)_I \ \mathrm{e}^{\pi\,\Sigma(Y,\bar{Y},p,q)} \\ &\propto \int \mathrm{d}Y^I\,\mathrm{d}\bar{Y}^I \ \Delta\left(Y,\bar{Y}\right) \ \mathrm{e}^{\pi\,\Sigma(Y,\bar{Y},p,q)} \ , \end{aligned} \tag{2.6}$$

where $\Delta(Y, \bar{Y})$ denotes the Jacobian associated with the change of integration variables $(\phi, \chi) \to (Y, \bar{Y})$,

$$\Delta(Y, \bar{Y}) = |\det[\,\mathrm{Im}\,2\,F_{KL}]| \ , \tag{2.7}$$

and Σ denotes the BPS entropy function which decomposes according to

$$\Sigma\left(Y, \bar{Y}, p, q\right) = \mathcal{F}\left(Y, \bar{Y}\right) - q_I\left(Y^I + \bar{Y}^I\right) + p^I\left(F_I + \bar{F}_I\right) \ . \tag{2.8}$$

Here p^I and q_I couple to the corresponding magneto- and electrostatic potentials (c.f. (2.4)) at the horizon in a way that is consistent with electric/magnetic duality. Furthermore, $\mathcal{F}(Y, \bar{Y})$ represents the free energy alluded to earlier. In the following we will consider its definition.

The free energy $\mathcal{F}$ has the property that its variations take the form,

$$\delta\mathcal{F} = \mathrm{i}\left(Y^I - \bar{Y}^I\right)\,\delta\left(F_I + \bar{F}_I\right) - \mathrm{i}\left(F_I - \bar{F}_I\right)\,\delta\left(Y^I + \bar{Y}^I\right) \ , \tag{2.9}$$

so that the variation of the entropy function Σ with respect to the Y^I, while keeping the charges fixed, yields the black hole attractor equations,

$$Y^I - \bar{Y}^I = \mathrm{i}p^I \ , \qquad F_I\left(Y\right) - \bar{F}_I\left(\bar{Y}\right) = \mathrm{i}q_I \ . \tag{2.10}$$

These equations determine the values of the Y^I at the black hole horizon in terms of the charges. Under the mild assumption that the matrix $N_{IJ} = 2\,\mathrm{Im}\,F_{IJ}$ is non-degenerate, it thus follows that stationary points of Σ must satisfy the attractor equations.

One can now evaluate the integral (2.6) in the semiclassical approximation and show that the answer takes the form,

$$d(p,q) = \mathrm{e}^{\mathcal{S}_{\text{macro}}(p,q)} \ , \tag{2.11}$$

where $\mathcal{S}_{\text{macro}}(p, q)$ equals the value of $\pi\,\Sigma$ taken at the saddle point. This is a gratifying result as we correctly recover the classical result, provided a free energy function exists with the required properties. In principle, we should have included the measure factor (2.7) when expanding around the saddle point but these contributions are suppressed in the limit of large charges, where all the charges and the fields Y^I are scaled uniformly.

Before continuing and discussing the free energy in further detail, we wish to emphasize that the scalar fields belonging to the vector multiplets are projectively defined in the underlying superconformal framework used for constructing the effective supergravity theory. On the other hand, the fields Y^I must have been given an intrinsic normalization as follows from the observation that both sides of the attractor equations scale differently in view of the fact that the charges are constant. This is also obvious from the equation $q_I Y^I - p^I F_I = -\mathrm{i}(\bar{Y}^I F_I - Y^I \bar{F}_I)$, which holds generally at the attractor point. Indeed we have adopted a normalization condition on the Y^I such that they are no longer subject to these projective redefinitions.[3] In the case that the function $F(Y)$ is holomorphic and homogenous of second degree, the expression for the free energy is known and equal to $\mathcal{F}(Y,\bar{Y}) = -\mathrm{i}(\bar{Y}^I F_I - Y^I \bar{F}_I)$. Indeed this expression satisfies (2.9) by virtue of the homogeneity of the function $F(Y)$ [49].

However, in reality, the function $F(Y)$ will depend also on an extra complex field Υ which is equal to the lowest-dimensional component of the square of the Weyl multiplet. The presence of this field encodes interactions in the effective field theory proportional to the square of the Weyl tensor. Supersymmetry requires the function $F(Y,\Upsilon)$ to remain holomorphic and homogeneous of second degree,

$$F\left(\lambda Y, \lambda^2 \Upsilon\right) = \lambda^2 F\left(Y, \Upsilon\right) . \tag{2.14}$$

The BPS free energy takes the following form in the presence of Υ-dependent terms,

$$\mathcal{F}\left(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}\right) = -\mathrm{i}\left(\bar{Y}^I F_I - Y^I \bar{F}_I\right) - 2\mathrm{i}\left(\Upsilon F_\Upsilon - \bar{\Upsilon} \bar{F}_\Upsilon\right) , \tag{2.15}$$

where $F_\Upsilon = \partial F/\partial \Upsilon$. And again, this free energy satisfies (2.9) by virtue of the (modified) homogeneity property (2.14), where $F(Y)$ and $F_I(Y)$ are everywhere replaced by $F(Y,\Upsilon)$ and $F_I(Y,\Upsilon)$, and where Υ is kept fixed under the variation. Note that the definition (2.15) is consistent with electric/magnetic duality [52]. Furthermore, an encouraging feature is that the expression (2.15) follows directly when evaluating the Hesse potential based on the holomorphic function F in the presence of Υ-dependent terms, without making any reference to the attractor equations [11].

The BPS attractor equations impose a constant real value for Υ, namely $\Upsilon = -64$. This implies that the terms proportional to positive powers of Υ encode subleading contributions

[3] To be specific, the original (projectively defined) fields X^I and the normalized fields Y^I are related by [49],

$$Y^I = \frac{\bar{Z}\, X^I}{\sqrt{\mathrm{i}\,(\bar{X}^I F_I(X) - \bar{F}_I(\bar{X}) X^I)}} \,, \tag{2.12}$$

where

$$Z = \frac{p^I F_I(X) - q_I X^I}{\sqrt{\mathrm{i}\,(\bar{X}^I F_I(X) - \bar{F}_I(\bar{X}) X^I)}} \,. \tag{2.13}$$

This latter quantity is sometimes referred to as the holomorphic BPS mass. Note that the Y^I are invariant under uniform complex rescalings of the underlying variables X^I.

to the entropy. The reason for this is that the attractor equations and the entropy function scale uniformly under simultaneous scale transformations of the Y^I and Υ fields according to (2.14), provided we scale the charges accordingly. The fact that the attractor equations fix Υ to a constant affects this scaling property. This phenomenon has been successfully demonstrated in [4], following earlier work in [2, 3].

2.2 Non-holomorphic corrections

A more subtle issue concerns the non-holomorphic corrections to the entropy function. Already at an early stage [16] it was clear that non-holomorphic corrections were required for manifest S-duality in $N = 4$ supersymmetric heterotic string compactifications, which have dual realizations as type-II string theory on $K3 \times \mathrm{T}^2$, or M-theory on $K3 \times \mathrm{T}^2 \times S^1$. Non-holomorphic modifications signal departures from the Wilsonian action and are caused by integrating out the massless modes. These modifications are required in order to preserve the physical symmetries which cannot be fully realized at the level of the Wilsonian action. An early example of this can be found in [48], where it was shown that the gauge coupling constants become moduli dependent with non-holomorphic corrections. Applying the $N = 2$ attractor equations to this particular situation reveals the need for non-holomorphic modifications [16]. Specifically, requiring the vector (Y^I, F_I) to transform consistently under S-duality monodromies, an S-duality invariant entropy was obtained. The results of this analysis were also in accord with the results for the non-holomorphic terms found in the corresponding effective action [21]. Subsequently, but much later, it was demonstrated in [19] how these results emerge from a semiclassical approximation of the microscopic degeneracy formula for $N = 4$ dyons [53−58]. However, as we already alluded to in section 1, the $N = 4$ supersymmetric models are of limited use for studying the general situation as their Υ-dependence in $F(Y, \Upsilon)$ is severely restricted.

Nevertheless, there is one question that can be addressed already at this stage, namely, whether one can still derive the attractor equations from a variational principle in the presence of the non-holomorphic corrections and define a closed expression for the BPS entropy function and the free energy introduced earlier. To investigate this question let us evaluate the variation of the free energy $\mathcal{F}$ defined in (2.15) minus the right-hand side of its expected variation (2.9), without making any further assumptions on the function F,

$$
\begin{aligned}
\delta\mathcal{F} - \mathrm{i}\left(Y^I - \bar{Y}^I\right)\,\delta\left(F_I + \bar{F}_I\right) + \mathrm{i}\left(F_I - \bar{F}_{\bar{I}}\right)\,\delta\left(Y^I + \bar{Y}^I\right) = \\
-\mathrm{i}\left(2\,\Upsilon\delta F_\Upsilon + Y^I\,\delta F_I - F_I\,\delta Y^I\right) + \text{h.c.}\,. \qquad (2.16)
\end{aligned}
$$

The right-hand side of the above equation should either vanish, or become proportional to the variation of a new term, which can then be absorbed into $\mathcal{F}$. Inspection shows that there are two obvious solutions. When the function F is homogeneous of second degree and holomorphic, (2.14) implies,

$$
2\,\Upsilon F_\Upsilon + Y^I F_I = 2\,F\,, \qquad (2.17)
$$

so that $2\,\Upsilon\,\delta F_\Upsilon + Y^I\,\delta F_I - F_I\,\delta Y^I = 0$. In that case the right-hand side of (2.16) vanishes, confirming the result quoted earlier for the holomorphic case. Alternatively, we may relax the holomorphicity requirement and assume that F (or part of F) is not holomorphic but purely imaginary, so that we can write $F = 2\,\mathrm{i}\Omega(Y,\bar{Y},\Upsilon,\bar{\Upsilon})$ with Ω a real homogeneous function of second degree, which therefore satisfies $2\,\Upsilon\Omega_\Upsilon + 2\,\bar{\Upsilon}\Omega_{\bar{\Upsilon}} + Y^I\Omega_I + \bar{Y}^{\bar{I}}\Omega_{\bar{I}} = 2\,\Omega$. In that case the right-hand side of (2.16) vanishes as well. Hence we may write,

$$F = F^{(0)}(Y,\Upsilon) + 2\mathrm{i}\,\Omega(Y,\bar{Y},\Upsilon,\bar{\Upsilon})\,, \tag{2.18}$$

where the attractor equations (2.10) retain the same form, irrespective of the presence of the non-holomorphic terms. The decomposition (2.18) is not unique. When the function Ω is harmonic, i.e. when it can be written as the sum of a holomorphic and an anti-holomorphic function, then one may absorb the holomorphic part into the first term. The anti-holomorphic part will then not contribute as it will vanish under the holomorphic derivatives which enter the attractor equations and the free energy. In practice we will require that $F^{(0)}$ is independent of Υ.

We are not aware of any other general solutions. These two solutions are the ones that have been discussed before and are consistent with all known cases. The second option seems to take the form of a consistent non-holomorphic deformation of special geometry, as we shall further discuss in section 5.

2.3 Semiclassical approximation

Having determined the free energy with possible non-holomorphic deformations we return to the inverse Laplace integral (2.6). This integral, defined in the first line of (2.6), is expressed in terms of Σ given in (2.8) with the associated free energy given in (2.15) with $\Upsilon = -64$. In the presence of non-holomorphic corrections, the function F appearing in these expressions is the non-holomorphic one introduced in (2.18). These non-holomorphic modifications will also introduce an explicit modification in the integration measure Δ, as follows,

$$\begin{aligned} d(p,q) &\propto \int \mathrm{d}\left(Y^I + \bar{Y}^{\bar{I}}\right)\, \mathrm{d}\left(F_I + \bar{F}_{\bar{I}}\right)\; \mathrm{e}^{\pi\,\Sigma(Y,\bar{Y},p,q)} \\ &\propto \int \mathrm{d}Y^I\, \mathrm{d}\bar{Y}^{\bar{I}}\; \Delta^-\left(Y,\bar{Y}\right)\; \mathrm{e}^{\pi\,\Sigma(Y,\bar{Y},p,q)}\,, \end{aligned} \tag{2.19}$$

where we now introduce two Jacobian factors, $\Delta^\pm\left(Y,\bar{Y}\right)$, defined by

$$\Delta^\pm\left(Y,\bar{Y}\right) = \left|\det\left[\mathrm{Im}[\,2\,F_{KL} \pm 2\,F_{K\bar{L}}]\right]\right|\,. \tag{2.20}$$

Observe that the mixed derivative satisfies,

$$F_{I\bar{J}} = -\bar{F}_{\bar{J}I}\,. \tag{2.21}$$

because of the fact that the non-holomorphic terms are characterized by the real function Ω. Obviously, the mixed derivatives vanish when the function $\Omega(Y,\bar{Y},\Upsilon,\bar{\Upsilon})$ is harmonic. When this is not the case, we must adopt indices $\bar{I},\bar{J},\ldots$ to refer specifically to non-holomorphic coordinates and derivatives.

Subsequently one evaluates the semiclassical Gaussian integral that emerges when expanding the exponent in the integrand to second order in δY^I and $\delta\bar{Y}^I$ about the attractor point. As it turns out [11], this can be done in two steps, because at the saddle point the semiclassical determinant factorizes into two sub-determinants, one associated with the real and another one with the imaginary values of the Y^I. These two sub-determinants are precisely equal to Δ^+ and Δ^-, respectively, defined in (2.20). Performing the integral only over the imaginary parts of the Y^I partially cancels the Jacobian factor in (2.19), and one is left with the integral,

$$d(p,q) \propto \int \mathrm{d}\phi\, \sqrt{\Delta^-(p,\phi)}\; \mathrm{e}^{\pi[\mathcal{F}_{\mathrm{E}}(p,\phi)-q_I\phi^I]}\;, \tag{2.22}$$

where

$$Y^I = \frac{1}{2}\left(\phi^I + \mathrm{i}p^I\right)\;. \tag{2.23}$$

Hence this result takes the form of the OSV integral [5], with an extra integration measure $\sqrt{\Delta^-}$. In view of the original setting in terms of the Hesse potential, we expect that the integration contours in (2.22) should be taken along the imaginary axes. The free energy associated with the mixed ensemble, $\mathcal{F}_{\mathrm{E}}(p,\phi)$, reads as follows,

$$\mathcal{F}_{\mathrm{E}}(p,\phi) = 4\Big[\mathrm{Im}\,F(Y,\bar{Y},\Upsilon,\bar{\Upsilon}) - \Omega(Y,\bar{Y},\Upsilon,\bar{\Upsilon})\Big]_{Y^I=(\phi^I+\mathrm{i}p^I)/2}\,. \tag{2.24}$$

The remaining attractor equations read, $q_I = \partial\mathcal{F}_{\mathrm{E}}/\partial\phi^I$. We note the presence of the term proportional to Ω, which partially cancels the Ω-dependence in the function F. The reader may verify that, when Ω is harmonic, everything can be expressed in terms of the imaginary part of the properly modified holomorphic function F.

It remains to complete the semiclassical approximation and perform the integral over the ϕ^I. This gives the result,

$$d(p,q) = \sqrt{\left|\frac{\Delta^-(Y,\bar{Y})}{\Delta^+(Y,\bar{Y})}\right|_{\text{attractor}}}\; \mathrm{e}^{\mathcal{S}_{\text{macro}}(p,q)}\;. \tag{2.25}$$

In the absence of non-holomorphic corrections the ratio of the two determinants is equal to unity and one recovers precisely the macroscopic entropy, as in (2.11).

Inverting (2.22) to a partition sum over a mixed ensemble, one finds,

$$\begin{aligned} Z(p,\phi) &= \sum_{\{q\}} d(p,q)\,\mathrm{e}^{\pi\, q_I\phi^I} \\ &\sim \sum_{\text{shifts}} \sqrt{\Delta^-(p,\phi)}\;\mathrm{e}^{\pi\,\mathcal{F}_{\mathrm{E}}(p,\phi)}\;. \end{aligned} \tag{2.26}$$

The function $\mathcal{F}_{\mathrm{E}}$ is not duality invariant and the invariance is only recaptured when completing the saddle-point approximation with respect to the fields ϕ^I. Therefore an evaluation of (2.22) beyond the saddle-point approximation will most likely give rise to a violation of (some of) the duality symmetries.

To discuss the validity of the semiclassical approximation, we recall that the entropy function is homogeneous of degree two under uniform rescalings of the charges, (p^I, q_I), and the fields Y^I and $\sqrt{\Upsilon}$ and their complex conjugates. However, Υ will take a fixed value as a result of the attractor equations. Therefore Υ-dependent terms affect the uniform scaling and, under the assumption that only positive powers of Υ appear, are associated with subleading corrections. The leading terms in the BPS entropy function scale quadratically, and so does the entropy. On the other hand, the leading contributions to the determinant factors scale with zero weight. Hence the latter terms do not have to be expanded about the saddle point as they would yield contributions with negative scaling weights. The semiclassical approximation thus pertains to all terms that scale with non-negative scaling weights. Therefore subleading corrections to the entropy function with zero weight are comparable to the *leading* terms in the determinant factors. Assuming that Ω is at least proportional to Υ or its complex conjugate, we have to include the terms in Ω that are linear in Υ, but we can suppress them in the determinants. In that case the prefactor in (2.25) equals unity.

Hence we expect that the semiclassical approximation is reliable for the leading and subleading terms in the entropy. The consistency of this approach has been verified in many cases, but mainly for large black holes in $N = 4$ supersymmetric string compactifications based on an $N = 2$ supersymmetric description [16, 19, 9, 20, 11]. Obviously this result is not compatible with the so-called entropy enigma, found in [15]. In the case of small black holes, where the leading contribution is absent, the above arguments do not quite apply and the semiclassical approximation breaks down, although the next-to-leading part in the entropy can still be calculated reliably [16, 6, 27].

3. Constraints on Ω due to exact duality symmetries

In this section we consider specific $N = 2$ models with exact duality symmetry groups. Two such models are the FHSV [28] and the STU model [29, 30]. Their symmetries constrain the form of the real homogeneous function Ω in (2.18), because the corresponding monodromies imply specific transformation rules for the derivatives of Ω. We begin by discussing exact duality symmetries in the context of a larger class of models, which will enable us to make contact with previous work on BPS black hole entropy applied to various string compactifications invariant under 8 or 16 supersymmetries. The results of this section will then be used in later sections.

The FHSV model [28] is a model with 8 supersymmetries. Its type-II realization corresponds to the compactification on the Enriques Calabi-Yau three-fold, which is described as an orbifold $(\mathrm{T}^2 \times \mathrm{K3})/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is a freely acting involution. Its holonomy group equals $\mathrm{SU}(2) \times \mathbb{Z}_2$, which implies that the type-II string compactification is described by an effective four-dimensional theory with $N = 2$ supersymmetry. The Enriques Calabi-Yau is self-mirror with Hodge numbers $h^{(2,1)} = h^{(1,1)} = 11$, so that its Euler number χ vanishes, and the massless sector of the four-dimensional theory comprises 11 vector supermultiplets, 12 hypermultiplets and the $N = 2$ graviton supermultiplet. In what follows we concentrate on the vector multiplet sector, whose classical moduli space, which is not affected by quantum corrections, equals the special-Kähler space,

$$\mathcal{M}_{\text{vector}} = \frac{\text{SL}(2)}{\text{SO}(2)} \times \frac{\text{O}(10,2)}{\text{O}(10) \times \text{O}(2)}\,. \tag{3.1}$$

Its two factors are associated with $\text{T}^2/\mathbb{Z}_2$ and the K3 fiber, and the special coordinates for these two spaces will be denoted by S and T^a, respectively.[4] In the limit $(S+\bar{S}) \to \infty$ one recovers the perturbative result of the dual realization on the corresponding heterotic string orbifold.

Obviously the classical moduli space (3.1) is invariant under the continuous group $\text{SL}(2) \times \text{O}(10,2)$. However, at the quantum level the model is invariant under the product of two discrete groups, namely the $\Gamma(2)$ subgroup of $\text{SL}(2;\mathbb{Z})$, and the group $\text{O}(10,2;\mathbb{Z})$. These groups must be realized as the invariance group of a more complete effective field theory description. We will call those the S- and T-duality groups, respectively, although this nomenclature is not quite appropriate in the type-II context.

Another model with 8 supersymmetries is the STU model [29, 30], which may be regarded as a truncation of the FHSV model, based on 3 vector multiplets and 4 hypermultiplets. Note that the STU model is also self-mirror and has $\chi = 0$. Its corresponding special-Kähler space equals,

$$\mathcal{M}_{\text{vector}} = \frac{\text{SL}(2)}{\text{SO}(2)} \times \frac{\text{SL}(2)}{\text{SO}(2)} \times \frac{\text{SL}(2)}{\text{SO}(2)}\,. \tag{3.2}$$

The duality group of this model is the product of the discrete $\Gamma(2)$ subgroups of each of the three $\text{SL}(2)$ groups.

For reasons of comparison we will also consider the so-called CHL models [59], which are invariant under 16 supersymmetries and whose S-dualities belong to the $\Gamma_1(\tilde{N})$ subgroup of $\text{SL}(2;\mathbb{Z})$. Here $\tilde{N}$ is an integer parameter and the models with $\tilde{N} = 1, 2, 3, 5, 7$, have been studied in the literature [20]. The case $\tilde{N} = 1$ corresponds to the toroidal compactification of heterotic string theory. The rank of the gauge group (corresponding to the number of abelian gauge fields in the effective supergravity action) is then equal to $r = 28, 20, 16, 12$ or 10, respectively, and the corresponding number of $N = 2$ matter vector supermultiplets equals $n = 48/(\tilde{N}+1) - 1$. Many of the studies of BPS black holes in CHL models have been carried out based on an effective $N = 2$ supergravity description.

Let us now consider the underlying holomorphic function $F(Y, \Upsilon)$ in terms of which the Wilsonian action for the vector multiplet sector is encoded. As explained in the previous section the dependence on the field Υ induces the presence of certain higher-order derivative interactions, which, among others, involve the square of the Weyl tensor. The definition of the $n+1$ complex fields Y^I was also discussed in the previous section.[5] The number n will depend on the particular model that one is considering. For example, the FHSV model and the STU model have $n = 11$ and $n = 3$, respectively. Usually one assumes that

[4] The hypermultiplet moduli space contains the type-II dilaton and is of no concern to us. Its classical moduli space is given by the quaternion-Kähler space $\text{O}(12,4)/[\text{O}(12)\times\text{O}(4)]$, as follows from the c-map [28].

[5] See footnote 3. Note that Υ has been subject to a similar rescaling.

the function can be expanded in positive powers of Υ. For type-II compactifications on Calabi-Yau three-folds that are K3 fibrations, the expansion takes the form

$$F(Y,\Upsilon) = -\frac{Y^1 Y^a \eta_{ab} Y^b}{Y^0} + \sum_{g=1}^{\infty} \Upsilon^g\, F^{(g)}(Y)\,, \tag{3.3}$$

where $a, b = 2, \ldots, n$, and the symmetric matrix η_{ab} is an $\mathrm{SO}(n-2,1)$ invariant metric of indefinite signature. Obviously this expression can be parametrized by

$$F(Y,\Upsilon) = \mathrm{i}(Y^0)^2\, S\, T^a \eta_{ab} T^b + \Upsilon\, F^{(1)}(S,T) + \sum_{g=2}^{\infty} \frac{\Upsilon^g}{(Y^0)^{2g-2}}\, F^{(g)}(S,T)\,, \tag{3.4}$$

where

$$S = -\mathrm{i}\, Y^1/Y^0\,, \qquad T^a = -\mathrm{i}\, Y^a/Y^0\,, \tag{3.5}$$

denote the special coordinates that parametrize the moduli space of the Calabi-Yau three-folds. We stress that the classical moduli space described by the first term of (3.4) is exact for the models that we discuss in this paper. The function $F(Y,\Upsilon)$ takes the form of a loop expansion with Y^0 as a loop-counting parameter. This is the form that is used for the topological string where Y^0 is regarded as the inverse topological string coupling constant and the functions $F^{(g)}(S,T)$ are the genus-g twisted partition functions.[6] The latter acquire non-holomorphic corrections encoded by the holomorphic anomaly equation, whose structure is such that the holomorphic dependence on the topological string coupling constant is preserved [24].

On the other hand it is well known that non-holomorphic corrections are also required to realize the relevant symmetries of the effective action [48]. In this context the holomorphic contributions encode the Wilsonian effective action which is supposed to arise from integrating out massive degrees of freedom. The Wilsonian action does not necessarily reflect all the symmetries of the theory and those are recovered upon including the contributions from the massless fields. These contributions contain non-holomorphic terms. As we mentioned already in section 1, it turns out that the non-holomorphic corrections to the effective action are not quite identical to the non-holomorphic contributions to the genus-g partition functions of the topological string, at least for $g > 1$. This will be further discussed in section 5.

Equivalence classes of the holomorphic function $F(Y,\Upsilon)$ are governed by $\mathrm{Sp}(2n+2,\mathbb{Z})$ rotations of the $2(n+1)$-component period vector of the underlying Calabi-Yau holomorphic three-form, corresponding to (Y^I, F_I), where $F_I = (F_0, F_1, F_a)$ denotes the derivatives of F with respect to Y^0, Y^1 and Y^a, respectively. For the models based on (3.4) the invariance group is embedded into an $\mathrm{SL}(2;\mathbb{Z})\times \mathrm{O}(n-1,2;\mathbb{Z})$ subgroup of these monodromy transformations. In this section it is not necessary to precisely specify this embedding. At the classical level, where one retains only the first term in (3.4), the continuous version of these monodromy transformations generate the isometries of the moduli spaces. At the

[6] Hence $F^{(g)}(Y) = (Y^0)^{-2g+2}\, F^{(g)}(S,T)$; when refering to the genus-g partition functions in the text, we usually do not make a distinction between $F^{(g)}(Y)$ and $F^{(g)}(S,T)$.

level of the four-dimensional effective action these transformations are accompanied by electric/magnetic duality transformations.

The period vector (Y^I, F_I) plays a central role in the so-called attractor equations for BPS black holes, which express their imaginary parts (taken at the black hole horizon) in terms of the black hole charges (c.f. (2.10)). Rather than concentrating on the properties of the function (3.4), we will therefore focus attention on the properties of this period vector. On the period vector, invariance transformations are characterized by the fact that the variations of the Y^I induce the action on the $F_I(Y,\Upsilon)$ according to the monodromy matrix that also acts on the black hole charges. Because the BPS attractor equations require Υ to take a specific value at the horizon (namely $\Upsilon = -64$), it is possible that the invariance arguments are not valid for arbitrary Υ. Based on previous work, it seems at least necessary to restrict Υ to a real number. However, in this section this aspect does not yet play a role. Furthermore, because the action of the monodromies on the charges is not subject to corrections, the action of the symmetry on the period vector must remain unchanged upon introducing non-holomorphic corrections.

In view of the above it is of interest to define the monodromies associated with the group $\mathrm{SL}(2;\mathbb{Z})\times\mathrm{O}(n-1,2;\mathbb{Z})$, a subgroup of which is expected to leave the model invariant. The action of the S-duality group is defined as follows,

$$\begin{aligned} Y^0 &\to d\,Y^0 + c\,Y^1\,, & F_0 &\to a\,F_0 - b\,F_1\,,\\ Y^1 &\to a\,Y^1 + b\,Y^0\,, & F_1 &\to d\,F_1 - c\,F_0\,,\\ Y^a &\to d\,Y^a - \tfrac{1}{2}c\,\eta^{ab}\,F_b\,, & F_a &\to a\,F_a - 2b\,\eta_{ab}\,Y^b\,, \end{aligned} \tag{3.6}$$

where a,b,c,d are integer-valued parameters that satisfy $ad-bc=1$ which parametrize (a subgroup of) $\mathrm{SL}(2;\mathbb{Z})$.

For the T-duality group, general transformations are most easily generated by products of a number of specific finite transformations. Those transformations that belong to the $\mathrm{O}(n-2,1;\mathbb{Z})$ subgroup are manifest in the above description and do not have to be considered. Then there are $n-1$ abelian transformations generated by

$$\begin{aligned} Y^0 &\to Y^0\,, & F_0 &\to F_0 + \lambda^a F_a + \lambda^a\eta_{ab}\lambda^b\,Y^1\,,\\ Y^1 &\to Y^1\,, & F_1 &\to F_1 + 2\,\lambda^a\eta_{ab}Y^b - \lambda^a\eta_{ab}\lambda^b\,Y^0\,,\\ Y^a &\to Y^a - \lambda^a\,Y^0\,, & F_a &\to F_a + 2\,\eta_{ab}\lambda^b\,Y^1\,, \end{aligned} \tag{3.7}$$

where the λ^a are integers. Finally the full $\mathrm{O}(n-1,2;\mathbb{Z})$ group is generated provided one also includes the following transformation,

$$\begin{aligned} Y^0 &\to F_1\,, & F_0 &\to -Y^1\,,\\ Y^1 &\to -F_0\,, & F_1 &\to Y^0\,,\\ Y^a &\to Y^a\,, & F_a &\to F_a\,. \end{aligned} \tag{3.8}$$

Observe that the square of this transformation equals the identity.

In the case that the higher-order genus terms in (3.4) are suppressed, it is straightforward to evaluate the behaviour of these transformations on the special coordinates S and T^a. Under S-duality we find the well-known results,

$$S \to \frac{a\,S - \mathrm{i}b}{\mathrm{i}c\,S + d}\,, \qquad T^a \to T^a\,. \tag{3.9}$$

The T-duality transformations (3.7) and (3.8) lead to, respectively,

$$S \to S\,, \qquad T^a \to T^a + \mathrm{i}\,\lambda^a\,, \qquad T^a \to \frac{T^a}{T^b\eta_{bc}T^c}\,. \tag{3.10}$$

However, these S- and T-duality transformations become much more complicated in the presence of higher-genus contributions in (3.4). Insisting on the same symmetry (i.e., characterized by the same monodromy matrix) will restrict these higher-genus contributions. This was demonstrated, for instance in [16], in a simpler situation.

In what follows we concentrate on the periods and thus consider holomorphic derivatives of the function F, which is itself not holomorphic,

$$F = -\frac{Y^1\,Y^a\eta_{ab}Y^b}{Y^0} + 2\mathrm{i}\,\Omega(Y,\bar{Y},\Upsilon,\bar{\Upsilon})\,, \tag{3.11}$$

where Ω encodes the non-classical contributions in accordance with (2.18). We will still be assuming that Ω depends only on positive powers of Υ and $\bar{\Upsilon}$ compensated by negative even powers of Y^0 and/or $\bar{Y}^0$ so as to make (3.11) homogeneous of second degree (but not necessarily holomorphic). Furthermore we expect that Ω vanishes for $\Upsilon = 0$. In the case studied before [19], where the $F^{(g)}$ vanish for $g > 1$, it turns out that Ω could be written as a real function. As long as Ω is harmonic, which implies that it can be written as the difference of a holomorphic and an anti-holomorphic function, this modification has no consequences when considering the periods, as the latter will remain holomorphic. Irrespective of these precise properties the F_I can be written as follows,

$$\begin{aligned} F_0 &= \frac{Y^1}{(Y^0)^2}\,Y^a\eta_{ab}Y^b - \frac{2\mathrm{i}}{Y^0}\left[-Y^0\frac{\partial}{\partial Y^0} + S\frac{\partial}{\partial S} + T^a\frac{\partial}{\partial T^a}\right]\Omega\,,\\ F_1 &= -\frac{1}{Y^0}\,Y^a\eta_{ab}Y^b + \frac{2}{Y^0}\,\frac{\partial\Omega}{\partial S}\,,\\ F_a &= -2\frac{Y^1}{Y^0}\,\eta_{ab}Y^b + \frac{2}{Y^0}\,\frac{\partial\Omega}{\partial T^a}\,, \end{aligned} \tag{3.12}$$

where we regard Ω as a function of Y^0, S and T^a (and possibly their complex conjugates).

With these results the S-duality transformations (3.6) take the form,

$$\begin{aligned} Y^0 &\to \Delta_{\rm S}\,Y^0\,,\\ Y^1 &\to a\,Y^1 + b\,Y^0\,,\\ Y^a &\to \Delta_{\rm S}\,Y^a - \frac{c}{Y^0}\,\eta^{ab}\,\frac{\partial\Omega}{\partial T^b}\,, \end{aligned} \tag{3.13}$$

with

$$\Delta_{\rm S} = d + \mathrm{i}c\,S\,. \tag{3.14}$$

On the special coordinates S and T^a these transformations extend the previous result (3.9),

$$S \to \frac{a\,S - \mathrm{i}b}{\mathrm{i}c\,S + d}\,, \qquad T^a \to T^a + \frac{\mathrm{i}c}{\Delta_{\rm S}\,(Y^0)^2}\,\eta^{ab}\,\frac{\partial\Omega}{\partial T^b}\,, \tag{3.15}$$

and we note the useful relations

$$\frac{\partial S'}{\partial S} = \Delta_{\rm S}{}^{-2}\,, \qquad \frac{1}{S+\bar{S}} \to \frac{|\Delta_{\rm S}|^2}{S+\bar{S}} = \frac{\Delta_{\rm S}{}^2}{S+\bar{S}} - \mathrm{i}c\,\Delta_{\rm S}\,. \tag{3.16}$$

Assuming that the above transformations constitute an invariance of the model, we require that the S-duality transformations of the Y^I induce the expected transformations of the F_I upon substitution. This leads to the following result,[7]

$$\begin{aligned}
\left(\frac{\partial\Omega}{\partial T^a}\right)'_{\mathrm{S}} &= \frac{\partial\Omega}{\partial T^a}\,, \\
\left(\frac{\partial\Omega}{\partial S}\right)'_{\mathrm{S}} - \Delta_{\mathrm{S}}{}^2\,\frac{\partial\Omega}{\partial S} &= \frac{\partial(\Delta_{\mathrm{S}}{}^2)}{\partial S}\left[-\frac{1}{2}Y^0\frac{\partial\Omega}{\partial Y^0} - \frac{\mathrm{i}c}{4\,\Delta_{\mathrm{S}}\,(Y^0)^2}\,\frac{\partial\Omega}{\partial T^a}\eta^{ab}\frac{\partial\Omega}{\partial T^b}\right]\,, \\
\left(Y^0\frac{\partial\Omega}{\partial Y^0}\right)'_{\mathrm{S}} &= Y^0\frac{\partial\Omega}{\partial Y^0} + \frac{\mathrm{i}c}{\Delta_{\mathrm{S}}\,(Y^0)^2}\,\frac{\partial\Omega}{\partial T^a}\eta^{ab}\frac{\partial\Omega}{\partial T^b}\,.
\end{aligned} \tag{3.17}$$

It is instructive to consider the consequences of these equations in case that the dependence on the T-moduli is suppressed (i.e., $\partial\Omega/\partial T^a = 0$) and non-holomorphic terms are absent (so that we may use the decomposition (3.4)). The result is that the functions $F^{(g)}(S,T)$ are modular forms of weight $2g-2$, as the above equations take the form,

$$\begin{aligned}
\partial_S F^{(1)}(S,T) &\longrightarrow \Delta_{\mathrm{S}}^2\,\partial_S F^{(1)}(S,T)\,, \\
F^{(g)}(S,T) &\longrightarrow \Delta_{\mathrm{S}}^{2g-2}F^{(g)}(S,T)\,, \quad (g>1) \\
D_S F^{(g)}(S,T) &\longrightarrow \Delta_{\mathrm{S}}^{2g}D_S F^{(g)}(S,T)\,, \quad (g>1)
\end{aligned} \tag{3.18}$$

where $D_S F^{(g)}(S,T) \equiv [\partial_S - 2(g-1)\partial_S \ln\eta^2]F^{(g)}(S,T)$ with $\eta(S)$ the Dedekind function. Here $\partial_S \ln\eta^2$ acts as a connection, in view of its transformation law,

$$\partial_S \ln\eta^2 \to \Delta_{\mathrm{S}}^2\,\partial_S \ln\eta^2 + \frac{1}{2}\partial_S\Delta_{\mathrm{S}}^2\,, \tag{3.19}$$

but alternative connections exist that will lead to identical results. In the holomorphic case the first derivative with respect to Υ of Ω is known to be an invariant function [52], and this is consistent with the second equation of (3.18).

The same reasoning applies to T-duality. Under the transformation (3.7) it follows from (3.12) that all the derivatives $\partial\Omega/\partial Y^0$, $\partial\Omega/\partial S$ and $\partial\Omega/\partial T^a$ must be invariant under integer shifts $T^a \to T^a + \mathrm{i}\,\lambda^a$. For the T-duality transformation (3.8) the analysis is more subtle. Using (3.12) we derive,

$$\begin{aligned}
Y^0 &\to \Delta_{\mathrm{T}}\,Y^0\,, \\
Y^1 &\to \Delta_{\mathrm{T}}\,Y^1 + \frac{2\mathrm{i}}{Y^0}\left[-Y^0\frac{\partial\Omega}{\partial Y^0} + T^a\frac{\partial\Omega}{\partial T^a}\right]\,, \\
Y^a &\to Y^a\,,
\end{aligned} \tag{3.20}$$

with

$$\Delta_{\mathrm{T}} = T^a\eta_{ab}T^b + \frac{2}{(Y^0)^2}\frac{\partial\Omega}{\partial S}\,. \tag{3.21}$$

[7] $(O)'_{\mathrm{S,T}}$ denotes the change of O under S- or T-duality induced by the transformation of all the arguments on which O depends.

On the special coordinates the transformation (3.20) extends the previous result (3.10),

$$\begin{aligned} S &\to S+\frac{2}{\Delta_{\rm T}(Y^0)^2}\left[-Y^0\frac{\partial\Omega}{\partial Y^0}+T^a\frac{\partial\Omega}{\partial T^a}\right] \,, \\ T^a &\to \frac{T^a}{\Delta_{\rm T}} \,. \end{aligned} \tag{3.22}$$

Again we assume that the above transformations constitute an invariance of the model, and require that the T-duality transformation (3.20) of the Y^I induces the expected transformations of the F_I upon substitution. This leads to

$$\begin{aligned} \left(\frac{\partial\Omega}{\partial S}\right)'_{\rm T} &= \frac{\partial\Omega}{\partial S} \,, \\ \left(\frac{\partial\Omega}{\partial T^a}\right)'_{\rm T} &= \Big(\Delta_{\rm T}\,\delta_a{}^b-2\,\eta_{ac}T^cT^b\Big)\,\frac{\partial\Omega}{\partial T^b}+2\,\eta_{ab}T^b\,Y^0\frac{\partial\Omega}{\partial Y^0} \,, \\ \left(Y^0\frac{\partial\Omega}{\partial Y^0}\right)'_{\rm T} &= Y^0\frac{\partial\Omega}{\partial Y^0}+\frac{4}{\Delta_{\rm T}\,(Y^0)^2}\,\frac{\partial\Omega}{\partial S}\left[-Y^0\frac{\partial\Omega}{\partial Y^0}+T^a\frac{\partial\Omega}{\partial T^a}\right] \,. \end{aligned} \tag{3.23}$$

To appreciate the first term on the right-hand side of the second equation we note

$$\frac{\partial T'^a}{\partial T^b}=\frac{1}{\Delta_{\rm T}}\left[\delta^a{}_b-\frac{2\,T^a\,\eta_{bc}T^c}{\Delta_{\rm T}}-\frac{2\,T^a}{\Delta_{\rm T}(Y^0)^2}\,\frac{\partial^2\Omega}{\partial T^b\partial S}\right] \,. \tag{3.24}$$

In case that the S-dependence is suppressed so that we can drop the terms proportional to $\partial_S\Omega$, (3.24) is precisely the inverse of the term appearing in the second equation (3.23).

As before it is instructive to consider the consequences of these equations in case that non-holomorphic terms are absent (so that we use the decomposition (3.4)), assuming this time that the dependence on the S modulus can be ignored, so that $\partial_S\Omega=0$. The result is that the $F^{(g)}(S,T)$ are holomorphic automorphic forms of weight $2g-2$, as the above equations reduce to (note that $\Delta_{\rm T}=T^a\eta_{ab}T^b$ in this case),

$$\begin{aligned} \partial_{T^a}F^{(1)}(S,T) &\longrightarrow \Big(\Delta_{\rm T}\,\delta_a{}^b-2\,\eta_{ac}T^cT^b\Big)\;\partial_{T^b}F^{(1)}(S,T) \,, \\ F^{(g)}(S,T) &\longrightarrow \Delta_{\rm T}^{2g-2}\,F^{(g)}(S,T) \,, \qquad (g>1) \\ D_{T^a}F^{(g)}(S,T) &\longrightarrow \Big(\Delta_{\rm T}\,\delta_a{}^b-2\,\eta_{ac}T^cT^b\Big)\;\Delta_{\rm T}^{2g-2}D_{T^b}F^{(g)}(S,T) \,, \quad (g>1) \end{aligned} \tag{3.25}$$

where $D_{T^a}F^{(g)}(S,T)\equiv[\partial_{T^a}+(g-1)\partial_{T^a}\ln\Delta_{\rm T}]F^{(g)}(S,T)$. Again this result is consistent with the fact that the first derivative with respect to Υ must be an invariant function in the holomorphic case. Here we made use of a connection $-\frac{1}{2}\partial_T\ln\Delta_{\rm T}$, as

$$-\frac{1}{2}\partial_{T^a}\ln\Delta_{\rm T}\to\Big(\Delta_{\rm T}\,\delta_a{}^b-2\,\eta_{ac}T^cT^b\Big)\left[-\frac{1}{2}\partial_{T^b}\ln\Delta_{\rm T}+\partial_{T^b}\ln\Delta_{\rm T}\right] \,. \tag{3.26}$$

However, other (less trivial) connections are possible. For instance, in the FHSV model one may use $\frac{1}{4}\partial_T\ln\Phi(T)$, where $\Phi(T)$ is the holomorphic automorphic form of weight 4 (c.f.(4.5)). A non-holomorphic connection is given by $-\partial_T\ln[(T+\bar T)^a\eta_{ab}(T+\bar T)^b]$, which is invariant under imaginary shifts of the T^a. Note that, in the same approximation as above, the T-duality transformation (3.20) acts as

$$(T+\bar{T})^a\,\eta_{ab}\,(T+\bar{T})^b \to \frac{1}{|\Delta_{\rm T}|^2}\,(T+\bar{T})^a\,\eta_{ab}\,(T+\bar{T})^b\,. \tag{3.27}$$

We refer to [60] for further discussion.

Returning to the more general case it follows that both $\partial_S\Omega$ and $Y^0\frac{\partial\Omega}{\partial Y^0} - 2\,S\frac{\partial\Omega}{\partial S}$ are T-duality invariant, whereas $\partial_{T^a}\Omega$ is S-duality invariant. Furthermore, the combination $Y^0\frac{\partial\Omega}{\partial Y^0} - T^a\frac{\partial\Omega}{\partial T^a}$ turns out to be invariant under S-duality, while, under the T-duality (3.20), it is invariant up to a sign change. We also note the relations,

$$\begin{aligned}
\Delta_{\rm T} &\xrightarrow{\rm T} \frac{1}{\Delta_{\rm T}}\,,\\
\Delta_{\rm T} &\xrightarrow{\rm S} \Delta_{\rm T} + \frac{2\,{\rm i}\,c}{\Delta_{\rm S}\,(Y^0)^2}\left[-Y^0\frac{\partial\Omega}{\partial Y^0} + T^a\frac{\partial\Omega}{\partial T^a}\right],\\
\Delta_{\rm S} &\xrightarrow{\rm T} \Delta_{\rm S} + \frac{2\,{\rm i}\,c}{\Delta_{\rm T}\,(Y^0)^2}\left[-Y^0\frac{\partial\Omega}{\partial Y^0} + T^a\frac{\partial\Omega}{\partial T^a}\right].
\end{aligned} \tag{3.28}$$

This completes the review of S- and T-duality transformations in the FHSV model and in similar models, such as the STU model. We stress once more that the central results, (3.17) and (3.23), hold in the presence of non-holomorphic modifications. Furthermore, it should be clear that Ω is not an invariant function. While the fields Υ and $\bar{\Upsilon}$ do not enter explicitly into the monodromies (3.6), (3.7) and (3.8), the corresponding transformations induced on Y^0, S, and T^a depend in a complicated way on Υ and $\bar{\Upsilon}$. In the next two sections we will discuss how to solve these equations iteratively in $\Upsilon = \bar{\Upsilon}$. In section 4, we restrict ourselves to terms linear in $\Upsilon = \bar{\Upsilon}$ with the aim of studying the subleading corrections to the mixed black hole partition function. These terms coincide with the genus-1 partition functions of the topological string. Then, in subsection 5.1, we analyse higher-order terms in $\Upsilon = \bar{\Upsilon}$, related to the genus-2 partition function of the topological string. As we intend to demonstrate the result no longer agrees directly with the topological string. The underlying reason for this different result resides in the fact that the transformation rules depend on $\Upsilon, \bar{\Upsilon}$, unlike in the case of the topological string.

4. The measure factor for the mixed partition function

The consequences of the duality symmetry, which are expressed by the equations (3.17) and (3.23) for the function Ω defined in (3.11), can be studied by iteration in powers of Υ and $\bar{\Upsilon}$. Therefore it is convenient to expand Ω as follows,

$$\Omega(Y,\bar{Y},\Upsilon,\bar{\Upsilon}) = \sum_{g=1}^{\infty}\,\Omega^{(g)}(Y,\bar{Y},\Upsilon,\bar{\Upsilon})\,, \tag{4.1}$$

where $\Omega^{(g)}$ may in general contain various monomials in Υ and $\bar{\Upsilon}$ of degree g. As $\Omega^{(g)}$ must be a real function that is homogeneous of degree two, the coefficients of these monomials take the form of functions of S and T^a, as well as of their complex conjugates, divided by homogeneous polynomials of Y^0 and $\bar{Y}^0$ of degree $2(g-1)$. In particular $\Omega^{(1)}(S,T)$ is known for a large variety of models.

In the context of large black holes, only $\Omega^{(1)}(S,T)$ is expected to contribute to the mixed partition function (2.26) in the semi-classical approximation, as discussed at the end of section 2. Therefore we restrict ourselves here to the case $g=1$. This result will enable us to evaluate the effective measure factor for the mixed partition function at the end of this section.

We study the constraints imposed by S- and T-duality invariance for the terms linear in Υ and/or $\bar{\Upsilon}$, and their non-holomorphic corrections, proceeding by iteration and assuming that the duality invariance will be realized order-by-order in Υ (subject to $\bar{\Upsilon}=\Upsilon$). We consider both the FHSV and STU models, which have $N=2$ supersymmetry, as well as the $N=4$ supersymmetic CHL models. Considering this variety of models will be helpful in callibrating the normalization of Ω. All these models share the property that the first term in (3.4) is not modified by quantum corrections. In this iterative procedure the term $\Omega^{(1)}$, which is linear in Υ or $\bar{\Upsilon}$, is subject to relatively simple equations,

$$\begin{aligned}
&\frac{\partial\Omega^{(1)}}{\partial T^a} \xrightarrow{\mathrm{S}} \frac{\partial\Omega^{(1)}}{\partial T^a}\,,\\
&\frac{\partial\Omega^{(1)}}{\partial S} \xrightarrow{\mathrm{S}} \Delta_{\mathrm{S}}^{2}\,\frac{\partial\Omega^{(1)}}{\partial S}\,,\\
&\frac{\partial\Omega^{(1)}}{\partial S} \xrightarrow{\mathrm{T}} \frac{\partial\Omega^{(1)}}{\partial S}\,,\\
&\frac{\partial\Omega^{(1)}}{\partial T^a} \xrightarrow{\mathrm{T}} \Big(\eta_{cd}T^cT^d\,\delta_a{}^b - 2\,\eta_{ac}T^cT^b\Big)\,\frac{\partial\Omega^{(1)}}{\partial T^b}\,.
\end{aligned} \tag{4.2}$$

These equations are obviously satisfied by assuming that $\Omega^{(1)}$ is the sum of an S-duality invariant function of S, and a T-duality invariant function of T^a. Such invariant modular and automorphic functions are usually quite rare, so that invariance under the duality group will pose strong restrictions.

The solutions of the above equations are known for the FHSV model, where the contribution linear in Υ or $\bar{\Upsilon}$ takes the following form [22, 61],

$$\begin{aligned}
\Omega^{(1)}_{\mathrm{FHSV}}(S,\bar{S},T,\bar{T},\Upsilon,\bar{\Upsilon}) \;=\; \frac{1}{256\,\pi}\Big[&\frac{1}{2}\Upsilon\ln[\eta^{24}(2S)\,\Phi(T)] + \frac{1}{2}\bar{\Upsilon}\ln[\eta^{24}(2\bar{S})\,\Phi(\bar{T})]\\
&+(\Upsilon+\bar{\Upsilon})\ln[(S+\bar{S})^3(T+\bar{T})^a\eta_{ab}(T+\bar{T})^b]\Big]\,.
\end{aligned} \tag{4.3}$$

For real values of Υ, this result is indeed invariant under S-duality.[8] The S-duality transformations of this model constitute the $\Gamma(2)$ subgroup of $\mathrm{SL}(2;\mathbb{Z})$, defined by $a,d=1$

[8]Here and in the following we make use of the modular transformation rule and the asymptotic expansion of the Dedekind eta function,

$$\begin{aligned}
\ln\eta^{24}(S) \;&\to\; \ln\eta^{24}(S) + 12\,\ln\Delta_{\mathrm{S}}\,,\\
\ln\eta(S) \;&\approx\; -\frac{1}{12}\pi S - \mathrm{e}^{-2\pi S} + \mathcal{O}(\mathrm{e}^{-4\pi S})\,.
\end{aligned} \tag{4.4}$$

mod 2 and $b,c=0 \mod 2$ in (3.9). The result is also T-duality invariant in view of the fact that $\Phi(T)$ is a holomorphic automorphic form of weight 4 [62],

$$\Phi(T)=\prod_{r>0}\left(\frac{1-\mathrm{e}^{-2\pi\, r\cdot T}}{1+\mathrm{e}^{-2\pi\, r\cdot T}}\right)^{2c_1(r^2)}\,, \tag{4.5}$$

transforming under the T-duality transformation (3.20) (suppressing the S-dependence) as

$$\Phi(T)\to\Delta_{\rm T}^{\,4}\,\Phi(T)\,. \tag{4.6}$$

Indeed, (4.3) can be written as the sum of two invariant functions, one of S and $\bar S$ and one of T^a and $\bar T^a$, respectively, which for large real values of S and T^a satisfies,

$$\Omega^{(1)}_{\rm FHSV}\approx-\frac{\Upsilon\, S+\bar\Upsilon\,\bar S}{128}\,. \tag{4.7}$$

It contains non-holomorphic terms, which are crucial for the duality invariance, equal to

$$\Omega^{(1)\,\rm nonholo}_{\rm FHSV}=\frac{\Upsilon+\bar\Upsilon}{256\,\pi}\ln\Big[\big(S+\bar S\big)^3\,\big(T+\bar T\big)^a\,\eta_{ab}\,\big(T+\bar T\big)^b\Big]\,. \tag{4.8}$$

Observe that the duality invariance of $\Omega^{(1)}_{\rm FHSV}$ is only realized for real values of Υ. Therefore we do not know a priori whether to write Υ or its complex conjugate. The way in which this potential ambiguity has been resolved, is by assuming that purely holomorphic terms are always accompanied by a power of Υ and purely anti-holomorphic terms by a power of $\bar\Upsilon$, whereas for the mixed terms we assign Υ and $\bar\Upsilon$ such as to preserve the reality properties of Ω for complex Υ. At this point, it is not quite clear how this procedure will work out at higher orders in Υ and $\bar\Upsilon$, but we know from the explicit evaluation of $\Omega^{(2)}$ for the FHSV model, which we will present in the next section, that no problems are encountered.

Subsequently we turn to the STU model, based on the function

$$F^{(0)}(Y)=-\frac{Y^1Y^2Y^3}{Y^0}=\mathrm{i}\,\big(Y^0\big)^2\,STU\,, \tag{4.9}$$

corresponding to $\eta_{12}=\eta_{21}=\frac{1}{2}$ and $\eta_{11}=\eta_{22}=0$. In this case, we have [30],

$$\begin{aligned}\Omega^{(1)}_{\rm STU}(S,\bar S,T,\bar T,U,\bar U,\Upsilon,\bar\Upsilon)=&\\ \frac{1}{256\,\pi}\Big[&4\,\Upsilon\ln\left[\vartheta_2(S)\,\vartheta_2(T)\,\vartheta_2(U)\right]+4\,\bar\Upsilon\ln\left[\vartheta_2\left(\bar S\right)\,\vartheta_2\left(\bar T\right)\,\vartheta_2(\bar U)\right]\\ &+(\Upsilon+\bar\Upsilon)\ln[(S+\bar S)(T+\bar T)(U+\bar U)]\Big]\,,\end{aligned} \tag{4.10}$$

where

$$\vartheta_2(S)=\frac{2\,\eta^2(2S)}{\eta(S)}\,. \tag{4.11}$$

For large real values of S, T and U, this result yields

$$\Omega^{(1)}_{\rm STU}\approx-\frac{\Upsilon(S+T+U)+\bar\Upsilon(\bar S+\bar T+\bar U)}{256}\,, \tag{4.12}$$

and its non-holomorphic contribution equals,

$$\Omega^{(1)\,\mathrm{nonholo}}_{\mathrm{STU}} = \frac{\Upsilon+\bar\Upsilon}{256\,\pi}\ln\left[(S+\bar S)\,(T+\bar T)\,(U+\bar U)\right]\,. \tag{4.13}$$

Assuming that the real part of S is much larger than that of T and U, the two results (4.7) and (4.12) coincide up to a factor 2. This is related to the fact that the STU model has been defined on the type-II side. The relation between the field S and the heterotic dilaton must involve a factor 2. When this is taken into account the two results are in fact equal, in agreement with [63].

It is instructive to confront some of the previous results with the solution of the holomorphic anomaly equation for $\Omega^{(1)}$ for generic Calabi-Yau compactifications,

$$4\pi\,\Omega^{(1)\,\mathrm{nonholo}}\Big|_{\Upsilon=-64} = -\frac{1}{2}\ln\left|\det[\mathrm{Im}\,2\,F^{(0)}_{KL}]\right| + \Big(\frac{1}{24}\chi-1\Big)\ln\frac{K^{(0)}}{|Y^0|^2}\,, \tag{4.14}$$

where we adjusted the proportionality constant to have agreement with previous results. The quantity K is generally defined by

$$K = \mathrm{i}(\bar Y^I F_I - Y^I\bar F_I)\,. \tag{4.15}$$

Here F_I and F_{IJ} refer to the derivatives of the general function F and may thus contain non-holomorphic contributions. However, $F^{(0)}_{IJ}$ and $K^{(0)}$ refer only to the corresponding expressions for $\Upsilon=0$, so that non-holomorphic terms are absent. Then the Kähler potential $\mathcal{K}$ and the determinant of the special-Kähler metric in the standard representation [50] (see also [64]), are given by

$$\mathcal{K} = -\ln\Big[K^{(0)}/|Y^0|^2\Big]\,,\qquad g = -\,\mathrm{e}^{(n+1)\mathcal{K}}\,\det\Big[\mathrm{Im}\,2\,F^{(0)}_{KL}\Big]\,. \tag{4.16}$$

In the case at hand, where the function $F^{(0)}$ coincides with (3.11) in the $\Upsilon=0$ limit, the expression for the Kähler potential and $\Omega^{(1)}$ are given by

$$\begin{aligned}\mathcal{K} = -\ln\Big[K^{(0)}/|Y^0|^2\Big] &= -\ln\Big[(S+\bar S)(T+\bar T)^a\eta_{ab}(T+\bar T)^b\Big]\,,\\ 4\pi\,\Omega^{(1)\,\mathrm{nonholo}}\Big|_{\Upsilon=-64} &= \Big(\frac{\chi}{24}-2-\frac{n-3}{2}\Big)\ln(S+\bar S)\\ &\quad+\Big(\frac{\chi}{24}-2\Big)\ln\Big[(T+\bar T)^a\eta_{ab}(T+\bar T)^b\Big]\,,\end{aligned} \tag{4.17}$$

where we used the relation,

$$\det\Big[\mathrm{Im}\,2\,F^{(0)}_{KL}\Big] = 2^{n-1}(S+\bar S)^{n-3}\,\det[-\eta_{ab}]\,\left[\frac{K^{(0)}}{|Y^0|^2}\right]^2\,, \tag{4.18}$$

which holds for the same class of functions. For the Enriques Calabi-Yau three-fold, $n=11$ and $\chi=0$, so that (4.14) coincides with (4.8), provided we set $\Upsilon=-64$. Similarly for the STU model, where one has $\chi=0$ and $n=3$, the result coincides with (4.13).

One may also consider the class of CHL models which have $N = 4$ supersymmetry [59] and which we already mentioned in section 3. These models are invariant under the S-duality group $\Gamma_1(\tilde N) \subset \mathrm{SL}(2;\mathbb{Z})$, which is generated by (3.15) with the transformation parameters restricted to $c = 0 \mod \tilde N$ and $a, d = 1 \mod \tilde N$. They contain no higher-genus contributions beyond genus-1. As discussed in [20] the function Ω_k can be expressed in terms of the unique cusp forms of weight $k + 2$ associated with the S-duality group $\Gamma_1(\tilde N) \subset \mathrm{SL}(2;\mathbb{Z})$, defined by $f^{(k)}(S) = \eta^{k+2}(S)\,\eta^{k+2}(\tilde N S)$ where,

$$f^{(k)}(S') = \Delta_{\mathrm{S}}^{k+2}\, f^{(k)}(S)\,. \tag{4.19}$$

The result for Ω_k then takes the following form [11],

$$\Omega_k(S,\bar S,\Upsilon,\bar\Upsilon) = \frac{1}{256\,\pi}\Big[\Upsilon \ln f^{(k)}(S) + \bar\Upsilon \ln f^{(k)}(\bar S) + \frac{1}{2}(\Upsilon+\bar\Upsilon)\ln(S+\bar S)^{k+2}\Big]\,. \tag{4.20}$$

Note that this result agrees with the terms obtained for the corresponding effective actions (see, for instance, [21, 23]).

For large real value of S we obtain the same result (4.7) as for the FHSV model. The non-holomorphic terms in (4.20) can also be confronted with (4.14) and one finds agreement (again, modulo a factor 4π) provided $n = 2(k + 2) + 3$ (here we have included the four gauge fields associated with the extra $N = 2$ gravitino multiplets) and $\chi = 48$. However, this seems a numerical coincidence and we stress that (4.14) is strictly speaking only applicable to $N = 2$ supersymmetric models.

As an application we can now give the expressions of the measure for the mixed partition function as it appears in (2.26). Because the mixed partition function usually refers to the holomorphic part of $\mathcal{F}_{\mathrm{E}}$, we extract the non-holomorphic contribution from (2.24) and absorb it into measure, so that the factor $\sqrt{\Delta^-}$ is replaced by $\sqrt{\Delta^-}\,\exp[4\pi\,\Omega^{(1)\mathrm{nonholo}}]$. Evaluating the expression based on (4.14), we find the following universal result,

$$\sqrt{\Delta^-}\,\mathrm{e}^{4\pi\,\Omega^{(1)\mathrm{nonholo}}} \propto \left[\frac{K^{(0)}}{|Y^0|^2}\right]^{\chi/24-1}\,, \tag{4.21}$$

where we only kept the leading terms which scale with zero weight in the large-charge limit, and we dropped an irrelevant proportionality constant. This result applies to $N = 2$ only. For the CHL models one can perform the same calculation, employing an $N = 2$ description. Provided that one chooses $n = 2(k + 2) + 3$, accounting again for the extra four gauge fields belonging to the $N = 2$ gravitino multiplets, one obtains,

$$\sqrt{\Delta^-}\,\mathrm{e}^{4\pi\,\Omega^{(1)\mathrm{nonholo}}} \propto \left[\frac{K^{(0)}}{|Y^0|^2}\right]\,. \tag{4.22}$$

This latter result has been confirmed for the CHL models [9, 11] based on the corresponding microscopic degeneracy formulae [53 − 55, 20]. Observe that for the FHSV and STU models, $\chi = 0$, so that the semiclassical measure factors for these models and for the CHL models are inversely proportional. In contrast with the $N = 4$ models the semiclassical prediction for the $N = 2$ and $N = 8$ models does not agree with other results in the literature. The

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$N = 2$ results of [15] for compact Calabi-Yau manifolds are qualitatively different as they apply to large topological string coupling, whereas the semiclassical results refer to small coupling. Hence these two results apply to different regimes. Actually the measure factor of [15] will diverge when uniformly taking the charges and the Y^I large, which reflects the so-called entropy enigma. We expect the semiclassical results to apply to singlecentered solutions, which are insensitive to the entropy enigma. For the $N = 8$ result of [9] the situation is rather different, because here the measure factor is subleading as compared to semiclassical arguments. This seems to indicate that the semiclassical contribution will actually vanish in this particular case, presumably as the result of the high degree of symmetry of the $N = 8$ model.

We evaluate $\Omega^{(2)}$ for the FHSV model in the next section. Obviously these results will only be determined up to invariant functions, just as the non-holomorphic anomaly equation of the topological string enables the determination of the genus-g partition functions up to holomorphic terms. We will demonstrate that the results for $\Omega^{(2)}$ do not coincide with the corresponding expressions found for the topological string in [60].

5. Non-holomorphic corrections and the topological string

In this section we solve the constraints posed on $\Omega^{(2)}$ for the FHSV model, and compare the result with that for the genus-2 partition function of the topological string. As we already indicated previously, the two results do not agree. Obviously the discrepancy raises a variety of questions. First of all, it is important to realize that the transformations depend on Υ, so that we are dealing with an iteration in Υ, both in the function Ω as well as in the transformation rules. This situation is crucially different from the setting in which the non-holomorphic terms arise for the topological string, and this explains why the two results are different. As is well known, $\Omega^{(2)}$ encodes certain terms in the full effective action that are not necessarily local, which arise upon integrating out the massless modes. These terms affect the holomorphicity that underlies the Wilsonian effective action. The full effective Lagrangian must reproduce the physically relevant invariances, and for that the presence of the non-holomorphic corrections can be crucial. Indeed we will demonstrate that the free energy (2.15) is invariant up to second order in Υ in the presence of the non-holomorphic corrections. This will be discussed for the FHSV model in subsection 5.1.

A second, even more subtle, issue concerns the electric/magnetic duality transformations. Electric/magnetic duality is defined at the level of the effective action and its consequences are not a priori restricted to the Wilsonian action. This duality is not necessarily a statement about invariances, but about equivalence classes: the same physics can be described in the context of different electric/magnetic duality frames with different corresponding Lagrangians. These equivalence classes are well understood for $N = 2$ supersymmetric theories at the level of the Wilsonian action, based exclusively on the holomorphic contributions. It is reasonable to expect that the full effective action that includes the effect of the non-holomorphic terms remains subjected to electric/magnetic duality, so that the functions in terms of which the full effective Lagrangian can be encoded, should still fall into similar equivalence classes. This requires that one can establish the existence of a different function encoding a different Lagrangian which is related to the former by an electric/magnetic duality transformation induced by symplectic rotations of the period vector. In subsection 5.2 we show that this situation is indeed realized in certain cases. We prove that upon electric/magnetic duality, there are indeed equivalence classes of functions. Furthermore, for the class of functions that we consider in this paper, the free energy transforms as a function under duality. In this way the results of subsection 5.1 can be understood in a more general context.

5.1 Duality constraints on $\Omega^{(2)}(S, \bar{S}, T, \bar{T}, Y^0, \bar{Y}^0)$

In this subsection we consider the duality constraints at second order in Υ and $\bar{\Upsilon}$. We concentrate on the FHSV model, but the corresponding result for the STU model can be derived along the same lines. For the CHL models the $\Omega^{(g)}$ vanish for $g > 1$ so that (4.20) represents the complete result.

We start by solving the constraints imposed by S-duality, which are given in (3.17). Because the $\partial\Omega/\partial T^a$ are S-duality invariant, we can solve the second and third equation and write $\partial\Omega/\partial S$ and $Y^0\partial\Omega/\partial Y^0$ in terms of two functions transforming homogeneously under S-duality. To this end we employ the holomorphic function $G_2(2S) = \frac{1}{2}\partial_S \ln \eta^2(2S)$, which transforms under S-duality as,

$$G_2(2S) \to \Delta_{\mathrm{S}}^2\, G_2(2S) + \frac{1}{2}\mathrm{i}c\, \Delta_{\mathrm{S}}\,. \tag{5.1}$$

Observe that to $G_2(2S)$ one can always add a modular form of weight two but this ambiguity will be absorbed in the various functions that we will introduce shortly. We stress that we cannot assume holomorphicity for these functions in view of the non-holomorphic corrections noted previously. The choice for the argument $2S$ in (5.1) is made in view of the S-duality transformations which constitute the group $\Gamma(2)$. We now solve the third equation (3.17) by writing,

$$Y^0\,\frac{\partial\Omega}{\partial Y^0} = w^{(0)} + \frac{2\,G_2(2S)}{(Y^0)^2}\,\frac{\partial\Omega}{\partial T^a}\,\eta^{ab}\,\frac{\partial\Omega}{\partial T^b}\,, \tag{5.2}$$

where $w^{(0)}$ is invariant under S-duality. Substituting this result into the second equation (3.17), we obtain the following expression for $\partial\Omega/\partial S$,

$$\frac{\partial\Omega}{\partial S} = w^{(2)} - 2\,G_2(2S)\,w^{(0)} - \frac{2\,[G_2(2S)]^2}{(Y^0)^2}\,\frac{\partial\Omega}{\partial T^a}\,\eta^{ab}\,\frac{\partial\Omega}{\partial T^b}\,, \tag{5.3}$$

where $w^{(2)}$ is now a function transforming under S-duality as $w^{(2)} \to \Delta_{\mathrm{S}}^2 w^{(2)}$.

The above two equations should be integrated to yield a solution for Ω. In order to do so we first note the identity

$$[G_2(2S)]^2 = \frac{1}{2}\frac{\partial G_2(2S)}{\partial S} + G_4(2S)\,, \tag{5.4}$$

where G_4 is a modular form of weight four, which is proportional to the corresponding Eisenstein function $G_4(S) = (\pi/6)^2 E_4(S)$. This identity enables one to write the square of

G_2 in the last term of (5.3) as an S-derivative of G_2, because the term proportional to G_4 transforms under S-duality in such a way that it can be absorbed into the function $w^{(2)}$. Furthermore the second term proportional to $w^{(0)}$ can also be related to an S-derivative, as can be seen by writing it as a power series in Y^0,

$$w^{(0)}(S,\bar S,T,\bar T,Y^0,\bar Y^0) = \sum_{m\neq 0} \frac{v^m(S,\bar S,T,\bar T,\bar Y^0)}{(Y^0)^m}\,, \tag{5.5}$$

where the functions v^m transform under S-duality as modular forms,

$$v^m \to \Delta_{\rm S}^m\, v^m\,. \tag{5.6}$$

The reason that the contribution with $m=0$ is not included, is related to the fact that such a term can not show up in (5.2) in the context of a power expansion in Y^0. Using the definition of the covariant holomorphic derivative $D_S v^m = (\partial_S - 2mG_2(2S))v^m$, we can write $2\,G_2(2S)\,w^{(0)}$ as

$$2\,G_2(2S)\,w^{(0)} = \sum_{m\neq 0} \frac{(\partial_S - D_S)v^m}{m\,(Y^0)^m}\,. \tag{5.7}$$

The terms proportional to $D_S v^m$ transform under S-duality exactly as $w^{(2)}$, and can thus be absorbed into it. Hence we are left with,

$$\frac{\partial\Omega}{\partial S} = w^{(2)} - \sum_{m\neq 0}\frac{1}{m\,(Y^0)^m}\frac{\partial v^m}{\partial S} - \frac{1}{(Y^0)^2}\frac{\partial G_2(2S)}{\partial S}\,\frac{\partial\Omega}{\partial T^a}\,\eta^{ab}\,\frac{\partial\Omega}{\partial T^b}\,. \tag{5.8}$$

The two equations (5.2) and (5.8) can be integrated provided the following condition holds,

$$\frac{\partial\Omega}{\partial T^a}\,\eta^{ab}\left[4\,\frac{G_2(2S)}{Y^0}\,\frac{\partial^2\Omega}{\partial S\partial T^b} + 2\,\frac{\partial G_2(2S)}{\partial S}\,\frac{\partial^2\Omega}{\partial Y^0\partial T^b}\right] = (Y^0)^2\,\frac{\partial w^{(2)}}{\partial Y^0}\,. \tag{5.9}$$

Now we concentrate on the terms $\Omega^{(g)}$ with $g=1,2$, which depend at most quadratically on Υ and/or $\bar\Upsilon$. In that case the T-derivatives of Ω in the above formulae can be restricted to the corresponding derivatives of $\Omega^{(1)}$ and thus follow from the results of the previous subsection. In particular, these T-derivatives depend only on T^a and $\bar T^a$. According to (5.9) it then follows that $w^{(2)}$ does not depend on Y^0.

We are thus left with the first equation (3.17), which implies that the T-derivatives of Ω are S-duality invariant. Since the derivative of $\Omega^{(1)}$ was invariant under the first term in the S-duality variation of the T^a specified in (3.15), this equation leads to,

$$\begin{aligned}\left(\frac{\partial\Omega^{(2)}}{\partial T^a}\right)'_{\rm S} &+ \frac{{\rm i}c}{\Delta_{\rm S}(Y^0)^2}\,\frac{\partial^2\Omega^{(1)}}{\partial T^a\partial T^b}\eta^{bc}\frac{\partial\Omega^{(1)}}{\partial T^c}\\ &- \frac{{\rm i}c}{\bar\Delta_{\rm S}(\bar Y^0)^2}\,\frac{\partial^2\Omega^{(1)}}{\partial T^a\partial \bar T^b}\eta^{bc}\frac{\partial\bar\Omega^{(1)}}{\partial \bar T^c} = \frac{\partial\Omega^{(2)}}{\partial T^a}\,.\end{aligned} \tag{5.10}$$

Note that, in the approximation that we are working, the S-duality transformation on the left-hand side will not involve any variations of the T^a as those would be of even higher order in Υ or $\bar{\Upsilon}$. Furthermore we make use of the fact that $\Omega^{(1)}$ is real, so that we extract an overall T^a-derivative and establish that,

$$\Omega^{(2)}(S,\bar{S},T,\bar{T},Y^0,\bar{Y}^0) = -\frac{G_2(2S)}{(Y^0)^2}\,\frac{\partial\Omega^{(1)}}{\partial T^a}\eta^{ab}\frac{\partial\Omega^{(1)}}{\partial T^b} - \frac{G_2(2\bar{S})}{(\bar{Y}^0)^2}\,\frac{\partial\Omega^{(1)}}{\partial \bar{T}^a}\eta^{ab}\frac{\partial\Omega^{(1)}}{\partial \bar{T}^b} + u^{(0)}\,, \tag{5.11}$$

where $u^{(0)}$ is an S-duality invariant function quadratic in $\Upsilon,\bar{\Upsilon}$. Its S-derivative must obviously coincide with the first two terms on the right-hand side of (5.8) as far as they are of the same order in $\Upsilon,\bar{\Upsilon}$.

Further constraints follow from imposing the T-duality equations (3.23), where we will now deal exclusively with contributions of second order in $\Upsilon,\bar{\Upsilon}$. We first consider the third equation of (3.23) and note that the term proportional to $Y^0\partial\Omega/\partial Y^0$ on the right hand side of the third equation can be dropped in this order. Using that $\partial\Omega/\partial S$ is invariant under T-duality we find that third equation is solved by

$$Y^0\frac{\partial\Omega^{(2)}}{\partial Y^0} = r^{(0)} + \frac{1}{2(Y^0)^2}\,\frac{\partial\log\Phi(T)}{\partial T^a}\,\eta^{ab}\,\frac{\partial\Omega^{(1)}}{\partial T^b}\,\frac{\partial\Omega^{(1)}}{\partial S}\,, \tag{5.12}$$

where $\frac{1}{4}\partial_T\log\Phi(T)$ acts as a connection for T-duality, as discussed below (3.26). Here $r^{(0)}$ denotes a T-duality invariant function. The first equation of (3.23), on the other hand, results in

$$\left(\frac{\partial\Omega^{(2)}}{\partial S}\right)'_{\mathrm{T}} + \frac{2}{\Delta_{\mathrm{T}}(Y^0)^2}\,\frac{\partial^2\Omega^{(1)}}{\partial S^2}\,T^a\,\frac{\partial\Omega^{(1)}}{\partial T^a} + \frac{2}{\bar{\Delta}_{\mathrm{T}}(\bar{Y}^0)^2}\,\frac{\partial^2\Omega^{(1)}}{\partial S\partial\bar{S}}\,\bar{T}^a\,\frac{\partial\Omega^{(1)}}{\partial\bar{T}^a} = \frac{\partial\Omega^{(2)}}{\partial S}\,, \tag{5.13}$$

where we used again that $\Omega^{(1)}$ is real. Following the same steps as before, this equation is solved by

$$\begin{aligned}\frac{\partial\Omega^{(2)}}{\partial S} = s^{(0)} &- \frac{1}{4(Y^0)^2}\,\frac{\partial^2\Omega^{(1)}}{\partial S^2}\,\frac{\partial\ln\Phi(T)}{\partial T^a}\,\eta^{ab}\,\frac{\partial\Omega^{(1)}}{\partial T^b}\\ &- \frac{1}{4(\bar{Y}^0)^2}\,\frac{\partial^2\Omega^{(1)}}{\partial S\partial\bar{S}}\,\frac{\partial\ln\bar{\Phi}(\bar{T})}{\partial\bar{T}^a}\,\eta^{ab}\,\frac{\partial\Omega^{(1)}}{\partial\bar{T}^b}\,,\end{aligned} \tag{5.14}$$

where $s^{(0)}$ denotes a T-duality invariant function. Observe that (5.14) is consistent with the expression (5.8) for $\partial\Omega/\partial S$ following from S-duality invariance. Namely, the last term in (5.8) is of the type $s^{(0)}$, while the second and third term in (5.14) are of the type v^2 and $w^{(2)}$, respectively.

All results obtained so far give rise to the following expression for $\Omega^{(2)}$, up to an S- and T-duality invariant function,

$$\Omega^{(2)} = -\frac{G_2(2S)}{(Y^0)^2}\,\frac{\partial\Omega^{(1)}}{\partial T^a}\,\eta^{ab}\,\frac{\partial\Omega^{(1)}}{\partial T^b} - \frac{1}{4(Y^0)^2}\,\frac{\partial\ln\Phi(T)}{\partial T^a}\,\eta^{ab}\,\frac{\partial\Omega^{(1)}}{\partial T^b}\,\frac{\partial\Omega^{(1)}}{\partial S} + \text{c.c}\,. \tag{5.15}$$

The reader may verify that all previous results (5.2), (5.11), (5.12) and (5.14) are reproduced. Furthermore, the result is consistent with the assumption that $\Omega^{(2)}$ is real.

The result (5.15) can be confronted with the manifestly duality invariant expression,

$$F^{(2)}(Y) \propto \frac{1}{(Y^0)^2}\,\hat{G}_2(2S,2\bar{S})\,\frac{\partial\Omega^{(1)}}{\partial T^a}\,\eta^{ab}\,\frac{\partial\Omega^{(1)}}{\partial T^b}\,, \tag{5.16}$$

where $\hat{G}_2(S,\bar{S}) = G_2(S) + [2\,(S+\bar{S})]^{-1}$. Note that the right hand side of (5.16) is non-holomorphic. This latter expression is the one obtained for the topological string [60], which is clearly invariant under the lowest order S- and T-duality transformation by virtue of the non-holomorphic terms in $\hat{G}_2$ and $\Omega^{(1)}$. It is clear that the real part of (5.16) and (5.15) are quite different. Indeed, $\Omega^{(2)}$ is *not* duality invariant in leading order of Υ and $\bar{\Upsilon}$. It varies as follows under S- and T-duality,

$$\begin{aligned}
\big(\Omega^{(2)}\big)'_{\rm S} &= \Omega^{(2)} - \left(\frac{\mathrm{i}c}{2\,\Delta_{\rm S}(Y^0)^2}\,\frac{\partial\Omega^{(1)}}{\partial T^a}\eta^{ab}\frac{\partial\Omega^{(1)}}{\partial T^b} + \text{c.c.}\right)\,,\\
\big(\Omega^{(2)}\big)'_{\rm T} &= \Omega^{(2)} - \left(\frac{2}{\Delta_{\rm T}(Y^0)^2}\,T^a\frac{\partial\Omega^{(1)}}{\partial T^a}\,\frac{\partial\Omega^{(1)}}{\partial S} + \text{c.c.}\right)\,.
\end{aligned}\tag{5.17}$$

The lack of invariance poses no problem as the function $\Omega^{(1)}$ is invariant in lowest order of Υ or $\bar{\Upsilon}$, but still receives corrections from variations of S and T and their complex conjugates that are themselves linear in Υ or $\bar{\Upsilon}$. This leads to the following variations, quadratic in $\Upsilon,\bar{\Upsilon}$,

$$\begin{aligned}
\big(\Omega^{(1)}\big)'_{\rm S} &= \Omega^{(1)} + \left(\frac{\mathrm{i}c}{\Delta_{\rm S}(Y^0)^2}\,\frac{\partial\Omega^{(1)}}{\partial T^a}\eta^{ab}\frac{\partial\Omega^{(1)}}{\partial T^b} + \text{c.c.}\right)\,,\\
\big(\Omega^{(1)}\big)'_{\rm T} &= \Omega^{(1)} + \left(\frac{4}{\Delta_{\rm T}(Y^0)^2}\,T^a\frac{\partial\Omega^{(1)}}{\partial T^a}\,\frac{\partial\Omega^{(1)}}{\partial S} + \text{c.c.}\right)\,.
\end{aligned}\tag{5.18}$$

Observe that $\Delta_{\rm T}$ can be replaced by its lowest-order value $T^a\eta_{ab}T^b$ in the second equation of (5.17) and of (5.18). With these results one can verify that (5.15) also satisfies the second equation in (3.23). This follows directly from the second equations in (5.17) and (5.18), taking into account that all fields T^a, S and Y^0, as well as their complex conjugates, transform under T-duality. Hence we have established that Ω satisfies the restrictions posed by the dualities to second order in (real) Υ.

While $\Omega^{(1)}+\Omega^{(2)}$ is not invariant, the quantity $\mathrm{Im}[\Upsilon\partial_\Upsilon F] \propto [\Upsilon\partial_\Upsilon + \bar{\Upsilon}\partial_{\bar{\Upsilon}}]\Omega$ is invariant for real values of Υ at this level of approximation. Therefore it follows that the free energy defined in (2.15) is indeed invariant under S- and T-duality to second order in real Υ!

For genus $g>2$ the deviations between the functions that encode the full effective action and the topological string twisted partition functions will persist. The reason is that both the function Ω and the duality transformation rules depend on Υ, which is in striking contrast to the situation in the topological string, where the duality transformations are independent of Υ and determined, once and for all, by the classical contribution of the function F. Therefore the twisted partition functions, $F^{(g)}$, of the topological string must be different from the contributions appearing in Ω. The former are invariant under the dualities whereas the latter are not invariant, but they are determined by the requirement that the corresponding periods transform according to the correct monodromy transformations.

In section 1 we have already pointed out how this discrepancy can possibly be resolved. The topological string partition functions correspond to certain string amplitudes [32, 24], which are also encoded in the full effective action that describes all the irreducible graphs.

On the other hand the latter is not invariant under duality, unlike the partition functions of the topological string. Therefore the information contained in the topological string and in the relevant terms of the effective action can certainly be in agreement, although the corresponding mathematical expressions are different. It is suggestive that the connected and irreducible graphs are related by a Legendre transform, whereas the action (or its underlying function) can also be converted to an invariant expression (e.g. an Hamiltonian or a Hesse potential) by a Legendre transform. Obviously resolving these subtleties is a challenge.

5.2 Non-holomorphic deformations of special geometry?

Motivated by the results of the preceding section we consider some of the more conceptual issues related to the presence of non-holomorphic corrections. Let us consider electric/magnetic dualities on the periods (X^I, F_I), which take the form of $\mathrm{Sp}(2n)$ rotations. Here we do not assume that the F_I are holomorphic functions or sections. Hence we have holomorphic and anti-holomorphic coordinates X^I and $\bar X^{\bar I}$, while the F_I may depend on both X^I and $\bar X^I$. To avoid ambiguous notation we will use anti-holomorphic indices $\bar I$ wherever necessary. In this subsection homogeneity properties do not play a role.

Electric/magnetic dualities are defined by monodromy transformations of the periods, defined in the usual way,

$$\begin{aligned} X^I &\to \tilde X^I = U^I{}_J X^J + Z^{IJ} F_J\,, \\ F_I &\to \tilde F_I = V_I{}^J F_J + W_{IJ} X^J\,, \end{aligned} \tag{5.19}$$

where U, V, Z and W are the $(n+1)\times(n+1)$ submatrices that constitute an element of $\mathrm{Sp}(2n+2,\mathbb{R})$. As a result the relation between the old and the new fields, X^I and $\tilde X^I$, will no longer define a holomorphic map, and we note,

$$\frac{\partial \tilde X^I}{\partial X^J} \equiv \mathcal{S}^I{}_J = U^I{}_J + Z^{IK} F_{KJ}\,, \qquad \frac{\partial \tilde X^I}{\partial \bar X^J} = Z^{IK} F_{K\bar J}\,, \tag{5.20}$$

where $F_{IJ} = \partial F_I/\partial X^J$ and $F_{I\bar J} = \partial F_I/\partial \bar X^J$. Subsequently we consider the transformation behaviour of the derivatives F_{IJ} and $F_{I\bar J}$ induced by electric/magnetic duality (5.19). Straightforward use of the chain rule yields the relation,

$$F_{IJ} \to \tilde F_{IJ} = (V_I{}^L \hat F_{LK} + W_{IK})\,[\hat{\mathcal{S}}^{-1}]^K{}_J\,, \tag{5.21}$$

where

$$\begin{aligned} \hat F_{IJ} &= F_{IJ} - F_{I\bar K}\,\bar{\mathcal{Z}}^{\bar K\bar L}\,\bar F_{\bar L J}\,, \\ \hat{\mathcal{S}}^I{}_J &= U^I{}_J + Z^{IK}\hat F_{KJ}\,, \\ \mathcal{Z}^{IJ} &= [\mathcal{S}^{-1}]^I{}_K\, Z^{KJ}\,. \end{aligned} \tag{5.22}$$

As was shown in [34], $\mathcal{Z}^{IJ}$ is a symmetric matrix by virtue of the fact that the duality matrix belongs to $\mathrm{Sp}(2n+2,\mathbb{R})$. For the same reason $[\hat{\mathcal{S}}^{-1}]^I{}_K\, Z^{KJ}$ is also symmetric in (I,J). Observe that $\mathcal{Z}^{IJ}$ satisfies the equation,

$$\delta\mathcal{Z}^{IJ} = -\mathcal{Z}^{IK}\delta F_{KL}\mathcal{Z}^{LJ}\,. \tag{5.23}$$

Let us now assume that F_{IJ} is symmetric in I and J. This symmetry implies that the F_I can be written as the holomorphic derivatives of some function $F(X,\bar{X})$. It is of interest to determine whether this symmetry is preserved under duality. In general this is not the case. However, when we assume that

$$F_{I\bar{J}} = \pm \bar{F}_{\bar{J}I}\,, \tag{5.24}$$

then $\hat{F}_{IJ}$ will also be symmetric. In that case one can derive from (5.21) that $\tilde{F}_{IJ}$ must be symmetric as well, so that the $\tilde{F}_I$ can be expressed as the holomorphic derivatives of some function $\tilde{F}(\tilde{X},\bar{\tilde{X}})$ with respect to $\tilde{X}^I$. This is a first indication that non-holomorphic deformations satisfying (5.24) can be consistent with the special geometry transformations of the periods. Henceforth we will assume that (5.24) holds. Observe that terms in F that depend exclusively on $\bar{X}^{\bar{I}}$ are not determined by the above arguments.

Furthermore one can show that

$$F_{I\bar{J}} \to \tilde{F}_{I\bar{J}} = [\hat{\mathcal{S}}^{-1}]^K{}_I\,[\bar{\mathcal{S}}^{-1}]^{\bar{L}}{}_{\bar{J}}\,F_{K\bar{L}}\,. \tag{5.25}$$

It seems that the holomorphic and anti-holomorphic indices are treated somewhat asymmetrically in this transformation rule. However, noting the relation

$$\left(\mathcal{S}^{-1}\hat{\mathcal{S}}\right)^I{}_J = \delta^I{}_J - \mathcal{Z}^{IK}F_{K\bar{L}}\bar{\mathcal{Z}}^{\bar{L}\bar{M}}\bar{F}_{\bar{M}J}\,, \tag{5.26}$$

which follows from (5.22), and upon inverting the above expression and writing it as a power series, one observes that $\mathcal{S}^K{}_I\,\bar{\mathcal{S}}^{\bar{L}}{}_{\bar{J}}\,\tilde{F}_{K\bar{L}}$ takes a more symmetric form. This enables one to show that (5.25) can be expressed in two ways,

$$F_{I\bar{J}} \to \tilde{F}_{I\bar{J}} = \left[\hat{\mathcal{S}}^{-1}\right]^K{}_I\,\left[\bar{\mathcal{S}}^{-1}\right]^{\bar{L}}{}_{\bar{J}}\,F_{K\bar{L}} = \left[\mathcal{S}^{-1}\right]^K{}_I\,\left[\bar{\hat{\mathcal{S}}}^{-1}\right]^{\bar{L}}{}_{\bar{J}}\,F_{K\bar{L}}\,. \tag{5.27}$$

Let us now assume that the function F depends on some auxiliary real parameter η and consider partial derivatives with respect to it. A little calculation shows that $\partial_\eta F_I$ transforms in the following way,

$$\partial_\eta \tilde{F}_I = \left[\hat{S}^{-1}\right]^J{}_I\left[\partial_\eta F_J - F_{J\bar{K}}\,\bar{\mathcal{Z}}^{\bar{K}\bar{L}}\,\partial_\eta \bar{F}_{\bar{L}}\right]\,, \tag{5.28}$$

where the η-derivative in $\partial_\eta \tilde{F}_I(\tilde{X},\bar{\tilde{X}};\eta)$ is a partial derivative that does not act on the arguments $\tilde{X}^I$ and their complex conjugates, and likewise, in $\partial_\eta F_I(X,\bar{X};\eta)$ the arguments X^I and their complex conjugates are kept fixed. Let us now assume that the function $F(X,\bar{X};\eta)$ decomposes into a holomorphic function of X^I and a purely imaginary function that depends on X^I, its complex conjugates, and on the auxiliary parameter η,

$$F(X,\bar{X};\eta) = F^{(0)}(X) + 2\mathrm{i}\,\Omega(X,\bar{X};\eta)\,, \tag{5.29}$$

where Ω is real, just as the functions we have been considering in this paper. For this class of functions we have the following identities,

$$F_{I\bar{J}} = -\bar{F}_{\bar{J}I},\,, \qquad \partial_\eta F_{\bar{I}} = -\partial_\eta \bar{F}_{\bar{I}}\,, \tag{5.30}$$

so that we must adopt the minus sign in (5.24). With this result we can establish that

$$\partial_\eta \tilde{F}(\tilde{X}, \bar{\tilde{X}}; \eta) = \partial_\eta F(X, \bar{X}; \eta)\,, \tag{5.31}$$

up to terms that no longer depend on X^I and $\bar{X}^{\bar{I}}$. Ignoring such terms on the ground that they are not relevant for the vector multiplet Lagrangian, this implies that the first derivative of the function F with respect to some auxiliary parameter transforms as a function under electric/magnetic duality. Of course, it is crucial that we assumed the decomposition (5.29) so that η appears only in the non-holomorphic component Ω of F.

When the electric/magnetic duality defines a symmetry, then it follows that $\partial_\eta F$ must be invariant under this symmetry. As we explained previously, S- and T-duality requires real values of Υ. The above arguments can now be applied to the free energy for BPS black holes defined in (2.15), with the real Υ playing the role of the auxiliary parameter η. Therefore the second term in the free energy proportional to the Υ-derivative of F is duality invariant while the first term equals the symplectic product of the period vector and its complex conjugate. As a result the free energy is thus duality invariant.

We stress once more that the effective action encoded in a non-holomorphic function F is not fully known. Although the arguments presented above indicate that, indeed, non-holomorphic deformations are possible within the context of special gometry, a lot of work remains to be done in order to establish the full consistency and the implications of this approach.

6. The STU model

The analysis of the last section can be repeated for the STU model, and undoubtedly the results will be rather similar. Nevertheless, we still turn to a detailed analysis of this model to confront our general results with the proposal of [31] for the statistical degeneracies in the STU model. The STU model is based on four fields, Y^0, Y^1, Y^2 and Y^3, of which the latter three appear symmetrically. The fields S, T, and U are defined by $S = -\mathrm{i}Y^1/Y^0$, $T = -\mathrm{i}Y^2/Y^0$ and $U = -\mathrm{i}Y^3/Y^0$. Much of the information has already been given in section 3. The T-duality group is contained in $\mathrm{SO}(2,2) \cong \mathrm{SL}(2) \times \mathrm{SL}(2)$, and the combined S- and T-duality group is the product group $\Gamma(2)_S \times \Gamma(2)_T \times \Gamma(2)_U$, where $\Gamma(2) \subset \mathrm{SL}(2;\mathbb{Z})$ with $a, d \in 2\,\mathbb{Z}+1$ and $b, c \in 2\,\mathbb{Z}$, with $ad - bd = 1$. Furthermore there exists a triality symmetry according to which one can interchange Y^1, Y^2, Y^3 or, equivalently S, T, U. Under this interchange the corresponding $\Gamma(2)$ factors of the duality groups are interchanged accordingly.

The distinction between S- and T-duality disappears for this model and in view of that the set-up adopted in section 3 is not the most convenient one. However, we can simply start from the S-duality as explained there and recover the other $\Gamma(2)$ factors upon interchanging the corresponding moduli. Hence we start from (3.6), which we present on the corresponding charges,

$$
\begin{aligned}
p^0 &\to d\,p^0 + c\,p^1\,, \quad & q_0 &\to a\,q_0 - b\,q_1\,,\\
p^1 &\to a\,p^1 + b\,p^0\,, \quad & q_1 &\to d\,q_1 - c\,q_0\,,\\
p^2 &\to d\,p^2 - c\,q_3\,, \quad & q_2 &\to a\,q_2 - b\,p^3\,,\\
p^3 &\to d\,p^3 - c\,q_2\,, \quad & q_3 &\to a\,q_3 - b\,p^2\,.
\end{aligned}
\tag{6.1}
$$

The T-duality (U-duality) transformations are now obtained upon interchanging the labels $1 \leftrightarrow 2$ $(1 \leftrightarrow 3)$. From these transformation rules it follows that the eight charges transform according to the $(\mathbf{2},\mathbf{2},\mathbf{2})$ representation of $\Gamma(2)_S \times \Gamma(2)_T \times \Gamma(2)_U$. Consequently the charge bilinears transform as $\Gamma(2)$ triplets, $(\mathbf{3},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{3})$, or in the $(\mathbf{3},\mathbf{3},\mathbf{3})$ representation. Only the triplets are relevant for what follows and we start by defining the following three charge bilinears,

$$
\begin{aligned}
\langle Q,Q\rangle_s &= 2\,(q_0p^1 - q_2q_3)\,,\\
\langle P,P\rangle_s &= -2\,(q_1p^0 + p^2p^3)\,,\\
\langle P,Q\rangle_s &= q_0p^0 - q_1p^1 + q_2p^2 + q_3p^3\,,
\end{aligned}
\tag{6.2}
$$

which are invariant under $\Gamma(2)_T \times \Gamma(2)_U$ and transform as a vector under $\Gamma(2)_S$,

$$
\begin{aligned}
\langle Q,Q\rangle_s &\to a^2\,\langle Q,Q\rangle_s + b^2\,\langle P,P\rangle_s + 2\,ab\,\langle P,Q\rangle_s\,,\\
\langle P,P\rangle_s &\to c^2\,\langle Q,Q\rangle_s + d^2\,\langle P,P\rangle_s + 2\,cd\,\langle P,Q\rangle_s\,,\\
\langle P,Q\rangle_s &\to ac\,\langle Q,Q\rangle_s + bd\,\langle P,P\rangle_s + (ad+bc)\,\langle P,Q\rangle_s\,.
\end{aligned}
\tag{6.3}
$$

The $\Gamma(2)_S$ invariant norm of this vector,

$$
D(p,q) \equiv \langle Q,Q\rangle_s\,\langle P,P\rangle_s - \langle P,Q\rangle_s^{\,2}\,,
\tag{6.4}
$$

is also invariant under triality, so that the two triplets of charge bilinears that follow from (6.2) by triality, have the same invariant norm.These two other triplets,$(\langle Q,Q\rangle_t, \langle P,P\rangle_t, \langle P,Q\rangle_t)$ and $(\langle Q,Q\rangle_u, \langle P,P\rangle_u, \langle P,Q\rangle_u)$, transform as a vector under $\Gamma(2)_T$ and $\Gamma_{\rm U}(2)$, respectively, and are singlets under the two remaining $\Gamma(2)$ subgroups.

In the next subsections we discuss the macroscopic determination of the entropy of large and small black holes based on the entropy function (2.8) and the free energy (2.15), which will include the non-holomorphic corrections. Subsequently we consider the statistical degeneracy formula for the STU model proposed in [31].

6.1 Macroscopic evaluation of the BPS entropy

Here we apply the results of the preceding sections and determine the attractor equations and the black hole entropy including the first non-trivial subleading corrections. For convenience we recall the relations (3.12) for the STU model,

$$
\begin{aligned}
F_0 &= \frac{Y^1Y^2Y^3}{(Y^0)^2} - \frac{2\mathrm{i}}{Y^0}\left[-Y^0\frac{\partial}{\partial Y^0} + S\frac{\partial}{\partial S} + T\frac{\partial}{\partial T} + U\frac{\partial}{\partial U}\right]\Omega\,,\\
F_1 &= -\frac{Y^2Y^3}{Y^0} + \frac{2}{Y^0}\frac{\partial\Omega}{\partial S}\,,\\
F_2 &= -\frac{Y^1Y^3}{Y^0} + \frac{2}{Y^0}\frac{\partial\Omega}{\partial T}\,,\\
F_3 &= -\frac{Y^1Y^2}{Y^0} + \frac{2}{Y^0}\frac{\partial\Omega}{\partial U}\,,
\end{aligned} \tag{6.5}
$$

which clearly exhibits the triality symmetry, provided that Ω is triality invariant. Under $\Gamma(2)_S$ the fields transform as follows (c.f. (3.13)),

$$
\begin{aligned}
&Y^0 \to \Delta_{\mathrm{S}}\,Y^0\,, && Y^1 \to a\,Y^1 + b\,Y^0\,,\\
&Y^2 \to \Delta_{\mathrm{S}}\,Y^2 - \frac{2\,c}{Y^0}\frac{\partial\Omega}{\partial U}\,, && Y^3 \to \Delta_{\mathrm{S}}\,Y^3 - \frac{2\,c}{Y^0}\frac{\partial\Omega}{\partial T}\,.
\end{aligned} \tag{6.6}
$$

This result leads to the following transformations of the special coordinates (c.f. (3.15)),

$$
S \to \frac{a\,S - \mathrm{i}b}{\mathrm{i}c\,S + d}\,,\qquad T \to T + \frac{2\mathrm{i}c}{\Delta_{\mathrm{S}}\,(Y^0)^2}\frac{\partial\Omega}{\partial U}\,,\qquad U \to U + \frac{2\mathrm{i}c}{\Delta_{\mathrm{S}}\,(Y^0)^2}\frac{\partial\Omega}{\partial T}\,. \tag{6.7}
$$

Requiring these transformations to induce the corresponding variations on the periods, we obtain (c.f. (3.17)),

$$
\begin{aligned}
\left(\frac{\partial\Omega}{\partial T}\right)'_{\mathrm{S}} &= \frac{\partial\Omega}{\partial T}\,, \qquad \left(\frac{\partial\Omega}{\partial U}\right)'_{\mathrm{S}} = \frac{\partial\Omega}{\partial U}\,,\\
\left(\frac{\partial\Omega}{\partial S}\right)'_{\mathrm{S}} - \Delta_{\mathrm{S}}{}^2\frac{\partial\Omega}{\partial S} &= \frac{\partial(\Delta_{\mathrm{S}}{}^2)}{\partial S}\left[-\frac{1}{2}Y^0\frac{\partial\Omega}{\partial Y^0} - \frac{\mathrm{i}c}{\Delta_{\mathrm{S}}\,(Y^0)^2}\frac{\partial\Omega}{\partial T}\frac{\partial\Omega}{\partial U}\right]\,,\\
\left(Y^0\frac{\partial\Omega}{\partial Y^0}\right)'_{\mathrm{S}} &= Y^0\frac{\partial\Omega}{\partial Y^0} + \frac{4\mathrm{i}c}{\Delta_{\mathrm{S}}\,(Y^0)^2}\frac{\partial\Omega}{\partial T}\frac{\partial\Omega}{\partial U}\,.
\end{aligned} \tag{6.8}
$$

Corresponding results under T- and U-duality follow directly by triality. Subsequently we evaluate the free energy,

$$
\begin{aligned}
\mathcal{F} = {}& -|Y^0|^2(S+\bar S)(T+\bar T)(U+\bar U) + 4\,\Omega^{(1)}\\
& - 2\left\{\frac{\bar Y^0}{Y^0}\Big[(S+\bar S)\frac{\partial\Omega^{(1)}}{\partial S} + (T+\bar T)\frac{\partial\Omega^{(1)}}{\partial T} + (U+\bar U)\frac{\partial\Omega^{(1)}}{\partial U}\Big] + \text{h.c.}\right\}\,,
\end{aligned} \tag{6.9}
$$

where we dropped all the higher-order Υ contributions. Henceforth we will consistently restrict Ω to $\Omega^{(1)}$, but we will nevertheless keep writing Ω for notational clarity. The above free energy is invariant under S-, T- and U-duality, up to terms that are quadratic in $\Omega^{(1)}$, as can be verified by explicit calculation. These higher-order terms will eventually be cancelled by variations of the higher-order $\Omega^{(g)}$.

Expressing Y^2 and Y^3 in terms of the charges and the field S,

$$
\begin{aligned}
Y^2 &= \frac{1}{S+\bar S}\left\{-q_3 + \mathrm{i}\bar S\,p^2 - 2\mathrm{i}\left(\frac{\partial_U\Omega}{Y^0} - \frac{\partial_{\bar U}\Omega}{\bar Y^0}\right)\right\}\,,\\
Y^3 &= \frac{1}{S+\bar S}\left\{-q_2 + \mathrm{i}\bar S\,p^3 - 2\mathrm{i}\left(\frac{\partial_T\Omega}{Y^0} - \frac{\partial_{\bar T}\Omega}{\bar Y^0}\right)\right\}\,,
\end{aligned} \tag{6.10}
$$

and imposing the remaining magnetic attractor equations, $Y^1 - \bar{Y}^1 = \mathrm{i}\, p^1$ and $Y^0 - \bar{Y}^0 = \mathrm{i}\, p^0$, one finds,

$$\Sigma(S,\bar{S},p,q) = -\frac{\langle Q,Q\rangle_s - \mathrm{i}\langle P,Q\rangle_s\,(S-\bar{S}) + \langle P,P\rangle_s\,|S|^2}{S+\bar{S}} + 4\,\Omega(S,\bar{S},T,\bar{T},U,\bar{U}), \tag{6.11}$$

where T and U are no longer independent variables but denote the S-dependent values of the moduli that follow from (6.10) to first order in Ω. To evaluate those we use the definitions,

$$\begin{aligned} Q(S) &= q_0 + \mathrm{i}Sq_1\,, \qquad & Q_2(S) &= q_2 + \mathrm{i}S\,p^3\,,\\ P(S) &= p^1 - \mathrm{i}Sp^0\,, \qquad & Q_3(S) &= q_3 + \mathrm{i}S\,p^2\,. \end{aligned} \tag{6.12}$$

transforming under S-duality as $P(S) \to \Delta_S^{-1}\,P(S)$, and likewise for $Q(S)$, $Q_2(S)$ and $Q_3(S)$. Furthermore we note the expression

$$Y^0 = \frac{\bar{P}(\bar{S})}{S+\bar{S}}\,, \tag{6.13}$$

so that (6.10) leads to the following S-dependent expressions for T and U,

$$\begin{aligned} T &= \mathrm{i}\,\frac{\bar{Q}_3(\bar{S})}{\bar{P}(\bar{S})} - \frac{2\,(S+\bar{S})}{\bar{P}(\bar{S})}\left(\frac{\partial_U\Omega}{\bar{P}(\bar{S})} - \frac{\partial_{\bar{U}}\Omega}{P(S)}\right)\,,\\ U &= \mathrm{i}\,\frac{\bar{Q}_2(\bar{S})}{\bar{P}(\bar{S})} - \frac{2\,(S+\bar{S})}{\bar{P}(\bar{S})}\left(\frac{\partial_T\Omega}{\bar{P}(\bar{S})} - \frac{\partial_{\bar{T}}\Omega}{P(S)}\right)\,. \end{aligned} \tag{6.14}$$

Observe that the S-duality transformation of these equations coincides with the results (6.7). For what follows we need to evaluate the derivatives of $\bar{T}$ and $\bar{U}$ with respect to S,

$$\begin{aligned} \frac{\partial\bar{T}}{\partial S} &= -\frac{1}{2}\langle P,P\rangle_u\,P^{-2}(S) + \cdots\,,\\ \frac{\partial\bar{U}}{\partial S} &= -\frac{1}{2}\langle P,P\rangle_t\,P^{-2}(S) + \cdots\,, \end{aligned} \tag{6.15}$$

where we suppressed terms proportional to the derivatives of Ω.

Finally the attractor equation for S follows from requiring the S-derivative of (6.11) to vanish,

$$\begin{aligned} &\langle Q,Q\rangle_s + 2\mathrm{i}\,\langle P,Q\rangle_s\bar{S} - \langle P,P\rangle_s\,\bar{S}^2\\ &+ 2(S+\bar{S})^2\left\{2\,\partial_S\Omega - \frac{\langle P,P\rangle_u}{P^2(S)}\,\partial_{\bar{T}}\Omega - \frac{\langle P,P\rangle_t}{P^2(S)}\,\partial_{\bar{U}}\Omega\right\} = 0\,. \end{aligned} \tag{6.16}$$

It is important to check the behaviour of this result under the various dualities. It is covariant under S-duality, because, in this approximation, the term proportional to the derivatives of Ω scale under S-duality with the same factor $\bar{\Delta}_{\mathrm{S}}^{-2}$ as the other terms in (6.16).

In the following, we will consider large black holes, i.e. black holes with charges such that $D(p,q) > 0$, and hence with $\langle P,P\rangle_s \neq 0$. In that case the solution of (6.16) takes the following form,

$$
\begin{aligned}
S = {}& \sqrt{\frac{D}{\langle P,P\rangle_s^2}}\left\{1+\frac{4}{\langle P,P\rangle_s}\left[2\,\partial_{\bar S}\Omega-\frac{\langle P,P\rangle_u}{\bar P^2(\bar S)}\,\partial_T\Omega-\frac{\langle P,P\rangle_t}{\bar P^2(\bar S)}\,\partial_U\Omega\right]\right\}\\
&-\frac{\mathrm{i}\,\langle P,Q\rangle_s}{\langle P,P\rangle_s}\,, \qquad (6.17)
\end{aligned}
$$

where the arguments in Ω are the leading values of $S,\bar T,\bar U$ as our results hold only to first order of in Ω. At this point it is easy to substitute these values for S into (6.14) and we find the same equations for the fixed-point values for T and U as in (6.17) upon triality transformations. These results are the extension of the lowest-order expressions that were obtained long ago [65].

Before considering the behaviour under T- and U-duality of (6.17), we note the following identities, which hold at the attractor point,

$$
\begin{aligned}
2\,\langle P,P\rangle_s\,|P(S)|^2 &= -\langle P,P\rangle_t\,\langle P,P\rangle_u+\cdots\,,\\
2\,\langle P,P\rangle_s\,Q_2(S)\,\bar P(\bar S) &= \langle P,P\rangle_t\,\langle P,Q\rangle_u-\frac{1}{2}\mathrm{i}\,\langle P,P\rangle_s\langle P,P\rangle_t\,(S+\bar S)+\cdots\,,\\
T+\bar T &= -\frac{1}{2}\langle P,P\rangle_u\,\frac{S+\bar S}{|P(S)|^2}+\cdots\,, \qquad (6.18)
\end{aligned}
$$

as well as similar identities obtained by triality. Furthermore we note the transformations,

$$
P(S)\xrightarrow{S}\frac{P(S)}{\Delta_S}\,,\qquad P(S)\xrightarrow{\mathrm{T,U}}\bar\Delta_{\mathrm{T,U}}\,P(S)+\cdots\,. \qquad (6.19)
$$

With these equations one establishes that the expression (6.17) for S transforms under T- and U-duality as,

$$
S\xrightarrow{\mathrm{T,U}}S+\frac{2\mathrm{i}c_{T,U}}{\Delta_{\mathrm{T,U}}\,(Y^0)^2}\,\partial_{U,T}\Omega\,, \qquad (6.20)
$$

which is precisely compatible with (6.7) upon triality.

Now we can introduce a modified field S^{inv} invariant under T- and U-duality by

$$
S^{\mathrm{inv}}=\sqrt{\frac{D}{\langle P,P\rangle_s^2}}\left\{1+\frac{8\,\partial_{\bar S}\Omega}{\langle P,P\rangle_s}\right\}-\frac{\mathrm{i}\,\langle P,Q\rangle_s}{\langle P,P\rangle_s}\,, \qquad (6.21)
$$

which transforms in the usual way under S-duality as it is the solution of an S-duality covariant equation,

$$
\langle Q,Q\rangle_s+2\mathrm{i}\,\langle P,Q\rangle_s\bar S-\langle P,P\rangle_s\,\bar S^2+4(S+\bar S)^2\,\partial_S\Omega=0\,. \qquad (6.22)
$$

This equation results from the condition that (we set $\Upsilon=-64$)

$$
\begin{aligned}
\Sigma^{\mathrm{S}}(S^{\mathrm{inv}},\bar S^{\mathrm{inv}};p,q) = {}& -\frac{\langle Q,Q\rangle_s-\mathrm{i}\langle P,Q\rangle_s\,(S^{\mathrm{inv}}-\bar S^{\mathrm{inv}})+\langle P,P\rangle_s\,|S^{\mathrm{inv}}|^2}{S^{\mathrm{inv}}+\bar S^{\mathrm{inv}}}\\
&-\frac{2}{\pi}\,\ln\left[|\vartheta_2(S^{\mathrm{inv}})|^4(S^{\mathrm{inv}}+\bar S^{\mathrm{inv}})\right]\,, \qquad (6.23)
\end{aligned}
$$

is stationary. Likewise we can introduce similar equations for fields $T^{\rm inv}$ and $U^{\rm inv}$ which transform as usual under T- and U-duality respectively, but are invariant under the other dualities. These fields are the solutions of the equations that follow from (6.22) by triality.

The result for the entropy now follows from substituting the value of S into (6.11). All the Ω-dependent terms in the solutions for S, T, U cancel generically, and one is left with (6.11) with S (and thus T and U in Ω) equal to their classical values. Observe that this is so because we are only considering the first-order corrections to the entropy. In principle there are higher-order terms which will represent next-to-subleading corrections to the entropy. The result for the entropy thus takes the form,

$$\begin{aligned}\mathcal{S}_{\rm STU}(p,q) &= \pi\,\Sigma\Big|_{\rm attractor}\\ &= \pi\,\sqrt{D(p,q)} - 2\,\ln\left[|\vartheta_2(S)|^4\,(S+\bar S)\right]\\ &\quad - 2\,\ln\left[|\vartheta_2(T)|^4\,(T+\bar T)\right] - 2\,\ln\left[|\vartheta_2(U)|^4\,(U+\bar U)\right]\,, \end{aligned} \tag{6.24}$$

where, in the last terms S, T and U are fixed to their lowest-order attractor values. Here we made use of (4.10).

Alternatively, the entropy (6.24) can be obtained from an entropy function $\tilde\Sigma$ that depends on the invariant fields $S^{\rm inv}$, $T^{\rm inv}$ and $U^{\rm inv}$, where these fields are treated as independent. This entropy function is given by

$$\tilde\Sigma(S^{\rm inv},\bar S^{\rm inv},T^{\rm inv},\bar T^{\rm inv},U^{\rm inv},\bar U^{\rm inv};p,q) = \frac{1}{3}\left[\tilde\Sigma^{\rm S}+\tilde\Sigma^{\rm T}+\tilde\Sigma^{\rm U}\right]\,, \tag{6.25}$$

where $\tilde\Sigma^{\rm S}$ is S-, T- and U-duality invariant and equal to,

$$\begin{aligned}\tilde\Sigma^{\rm S}(S^{\rm inv},\bar S^{\rm inv};p,q) = &-\frac{\langle Q,Q\rangle_s - {\rm i}\langle P,Q\rangle_s\,(S^{\rm inv}-\bar S^{\rm inv}) + \langle P,P\rangle_s\,|S^{\rm inv}|^2}{S^{\rm inv}+\bar S^{\rm inv}}\\ &-\frac{6}{\pi}\,\ln\left[|\vartheta_2(S^{\rm inv})|^4(S^{\rm inv}+\bar S^{\rm inv})\right]\,,\end{aligned} \tag{6.26}$$

and $\tilde\Sigma^{\rm T}$ and $\tilde\Sigma^{\rm U}$ follow by triality. Extremizing $\tilde\Sigma$ with respect to $S^{\rm inv}$, $T^{\rm inv}$ and $U^{\rm inv}$ and substituting the resulting values into $\tilde\Sigma$ yields the entropy (6.24), where we work in the same order of approximation as before. Note, however, that $\tilde\Sigma^{\rm S}$ does not equal (6.23) so that the value of the attractor point will be different, although, at this order of approximation, such a deviation has no effect on the entropy.

6.2 Small black holes

To explore some other aspects of the STU model, we now consider possible small black hole solutions. Small black holes satisfy $D(p,q)=0$, with D given in (6.4). The higher-curvature corrections encoded in Ω are then crucial to ensure that the moduli are attracted to finite values at the horizon. For the STU model, the associated $\Omega^{(1)}$, given in (4.10), depends on all three moduli S, T and U, which implies that in order for the three moduli to take finite values at the horizon, the charges carried by the small black hole have to be chosen in such a way as to result in three non-vanishing charge bilinears (out of the nine bilinears introduced earlier). This differs from the situation encountered in $N=4$ models, where the associated $\Omega^{(1)}$ only depends on one modulus, so that only one non-vanishing charge bilinear is required to construct a small black hole [6].

An obvious possibility consists in choosing charges such that only $\langle Q,Q\rangle_s$, $\langle Q,Q\rangle_t$ and $\langle Q,Q\rangle_u$ are different from zero. Such a configuration can be obtained by switching on the charges q_0, q_1, q_2, q_3 while leaving the remaining ones equal to zero, so that $\langle Q,Q\rangle_s = -2\,q_2 q_3$, $\langle Q,Q\rangle_t = -2\,q_1 q_3$ and $\langle Q,Q\rangle_u = -2\,q_1 q_2$. Then, at the horizon, Y^0, Y^1, Y^2, Y^3 are all real, so that S, T, U are purely imaginary, which does not constitute a well-behaved situation (since, for instance, the non-holomorphic terms contained in $\Omega^{(1)}$ are expressed in terms of the real part of the moduli fields). Therefore, we discard this choice of charges and take instead p^0, q_2, q_3 as non-vanishing charges. Then, the non-vanishing charge bilinears are,

$$\langle Q,Q\rangle_s = -2\,q_2 q_3\,, \quad \langle P,P\rangle_t = -2\,p^0 q_2\,, \quad \langle P,P\rangle_u = -2\,p^0 q_3\,. \tag{6.27}$$

In that case Y^1, Y^2, Y^3 are real, but Y^0 is not in view of the fact that $p^0 \neq 0$. Using the definition of S, T and U we establish the following expressions for these quantities,

$$\begin{aligned} Y^0 &= \mathrm{i}\bar S\,\frac{p^0}{S+\bar S}\,, \qquad & Y^2 &= -\bar S T\,\frac{p^0}{S+\bar S}\,, \\ Y^1 &= -\bar S S\,\frac{p^0}{S+\bar S}\,, \qquad & Y^3 &= -\bar S U\,\frac{p^0}{S+\bar S}\,, \end{aligned} \tag{6.28}$$

so that $\bar S U$ and $\bar S T$ are real. Inserting (6.28) into (6.5) and restricting Ω to $\Omega^{(1)}$ gives,

$$\begin{aligned} F_0 &= \frac{1}{\bar S^2}\left\{\frac{p^0\, STU\bar S^3}{S+\bar S} - \frac{2(S+\bar S)\bar S}{p^0}\big[S\partial_S\Omega + T\partial_T\Omega + U\partial_U\Omega\big]\right\}\,, \\ F_1 &= \frac{\mathrm{i}}{\bar S}\left\{\frac{p^0\, TU\bar S^2}{S+\bar S} - \frac{2(S+\bar S)}{p^0}\,\partial_S\Omega\right\}\,, \\ F_2 &= \frac{\mathrm{i}}{\bar S}\left\{\frac{p^0\, US\bar S^2}{S+\bar S} - \frac{2(S+\bar S)}{p^0}\,\partial_T\Omega\right\}\,, \\ F_3 &= \frac{\mathrm{i}}{\bar S}\left\{\frac{p^0\, TS\bar S^2}{S+\bar S} - \frac{2(S+\bar S)}{p^0}\,\partial_U\Omega\right\}\,. \end{aligned} \tag{6.29}$$

Using $\bar T = \bar S T/S$ and $\bar U = \bar S U/S$, we find that the attractor equations $F_0 = \bar F_{\bar 0}$ and $F_1 = \bar F_{\bar 1}$ yield, respectively,

$$\begin{aligned} (S-\bar S)\bar S^2 TU &= \frac{2}{(p^0)^2}(S+\bar S)\left(S^2\partial_S\Omega - \bar S^2\partial_{\bar S}\Omega\right. \\ &\qquad \left. + ST\partial_T\Omega - \bar S\bar T\partial_{\bar T}\Omega + SU\partial_U\Omega - \bar S\bar U\partial_{\bar U}\Omega\right)\,, \\ \bar S^2 TU &= \frac{2}{(p^0)^2}(S+\bar S)\left(S\partial_S\Omega + \bar S\partial_{\bar S}\Omega\right)\,, \end{aligned} \tag{6.30}$$

while the attractor equations $F_2 - \bar F_{\bar 2} = iq_2$ and $F_3 - \bar F_{\bar 3} = iq_3$ read,

$$\begin{aligned} \bar S U &= \frac{q_2}{p^0} + \frac{2}{(p^0)^2}\frac{(S+\bar S)}{|S|^2}\left(S\partial_T\Omega + \bar S\partial_{\bar T}\Omega\right)\,, \\ \bar S T &= \frac{q_3}{p^0} + \frac{2}{(p^0)^2}\frac{(S+\bar S)}{|S|^2}\left(S\partial_U\Omega + \bar S\partial_{\bar U}\Omega\right)\,. \end{aligned} \tag{6.31}$$

In the absence of higher-curvature corrections, inspection of (6.30) and (6.31) shows that there are no solutions with finite values of S, T and U. When including higher-curvature corrections, on the other hand, we deduce from the structure of (6.30) and (6.31) that a likely solution exists with finite, but small values for T and U, and a large, but finite value for S. Therefore, we expand Ω around large values of S and small values of T and U. Using $\Omega^{(1)}$ given in (4.10), we obtain accordingly (with $\Upsilon = -64$),

$$\Omega^{(1)}_{\rm STU} = \frac{1}{4}\,(S+\bar S) - \frac{1}{2\,\pi}\left(\log(S+\bar S) + \log\left(\frac{1}{T}+\frac{1}{\bar T}\right) + \log\left(\frac{1}{U}+\frac{1}{\bar U}\right)\right) . \tag{6.32}$$

Here we used (4.4) in the expansion of ϑ_2. Observe that the non-holomorphic terms in $\Omega^{(1)}$ are crucial for obtaining finite horizon values for T and U.

Using (6.32) we find that the first equation in (6.30) is identical to the second equation in (6.30) multiplied by $S-\bar S$. This means that $S-\bar S$ does not get determined at the horizon. The second equation yields

$$\bar S^2 T U = \frac{2}{(p^0)^2}(S+\bar S)\left(\frac{1}{4}(S+\bar S) - \frac{1}{2\pi}\right) , \tag{6.33}$$

while from (6.31) we obtain

$$\begin{aligned} \bar S U &= \frac{q_2}{p^0} + \frac{S+\bar S}{\pi\,(p^0)^2\,\bar S T} , \\ \bar S T &= \frac{q_3}{p^0} + \frac{S+\bar S}{\pi\,(p^0)^2\,\bar S U} . \end{aligned} \tag{6.34}$$

Thus we see that the attractor equations determine the values of $S+\bar S, \bar S U$ and $\bar S T$, while the remaining moduli are left undetermined.

In the following, we take p^0, q_2, q_3 to be positive and uniformly large. For large $S+\bar S$, (6.33) can be approximated by $S+\bar S = \sqrt{2} p^0 \sqrt{\bar S^2 T U}$, while (6.34) implies that $\bar S U$ and $\bar S T$ are of order one with approximate values given by

$$\bar S U = \frac{|\langle Q,Q\rangle_s|}{|\langle P,P\rangle_u|} \quad , \quad \bar S T = \frac{|\langle Q,Q\rangle_s|}{|\langle P,P\rangle_t|} , \tag{6.35}$$

where we made use of the charge bilinears (6.27). Reinserting this into $S+\bar S$ gives

$$S+\bar S = \sqrt{|\langle Q,Q\rangle_s|} . \tag{6.36}$$

The entropy of this small black hole can be computed using (6.11) at the attractor point. Its value is entirely determined in terms of (6.35) and (6.36). We obtain, up to an additive constant,

$$\mathcal{S}_{\rm macro} = 2\pi\,\sqrt{|\langle Q,Q\rangle_s|} - 2\log\left(\frac{|\langle P,P\rangle_t\,\langle P,P\rangle_u|}{\sqrt{|\langle Q,Q\rangle_s|}}\right) . \tag{6.37}$$

We note that for large charges, the leading term in the entropy depends only on one of the bilinears (6.27). This is in contrast to what one naively obtains when considering the microstate degeneracy proposal of [31] and evaluating the degeneracy integral on an electric or magnetic divisor. There one expects to obtain a microscopic degeneracy which, to leading order, is given by the sum of three terms, each involving the square root of one of the three charge bilinears (6.27). This, however, is in conflict with (6.37), which indicates the need for a better understanding of the microstate degeneracy proposal of [31].

6.3 Comparison with microstate degeneracies

Recently, a proposal [31] was put forward for the microscopic degeneracies of twisted sector dyons in the STU model in terms of the residues of certain products of Siegel modular forms, and it was shown that the leading and subleading results for the entropy of these dyons agree with the macroscopic analysis that we have presented in subsection 6.1. Here we briefly review the analysis of the asymptotic degeneracies based on the microscopic formula in the notation of [11]. It is based on the procedure used earlier in [19, 20]. The degeneracy of dyons depends on the residues of the inverse of a modular form $\Phi_0(\rho,\sigma,\upsilon)$ of weight zero under a subgroup of $\mathrm{Sp}(2;\mathbb{Z})$. The three modular parameters, ρ,σ,υ, parametrize the period matrix of an auxiliary genus-two Riemann surface which takes the form of a complex, symmetric, two-by-two matrix. For the STU model the proposed degeneracies are given by the product of three of the following integrals over appropriate 3-cycles,

$$I(K,L,M) \propto \oint \mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\;\frac{\mathrm{e}^{\mathrm{i}\pi[\rho\,K+\sigma\,L+(2\upsilon-1)\,M]}}{\Phi_0(\rho,\sigma,\upsilon)}\,. \tag{6.38}$$

The quantities K,L,M are integers proportional to the charge bilinears $\langle P,P\rangle$, $\langle Q,Q\rangle$ and $\langle P,Q\rangle$, and thus transform as triplets under $\Gamma(2)$. The inverse of the modular form Φ_0 takes the form of an infinite Fourier sum with integer powers of $\exp[\pi\mathrm{i}\rho]$, $\exp[\pi\mathrm{i}\sigma]$ and $\exp[2\pi\mathrm{i}\upsilon]$, and the 3-cycle is then defined by choosing integration contours where the real parts of ρ and σ take values in the interval $(0,2)$ and the real part of υ takes values in the interval $(0,1)$. The leading behaviour of the dyonic degeneracy is associated with the rational quadratic divisor $\mathcal{D}=\upsilon+\rho\sigma-\upsilon^2=0$ of Φ_0, near which $1/\Phi_0$ takes the form,

$$\frac{1}{\Phi_0(\rho,\sigma,\upsilon)} \approx \frac{1}{\mathcal{D}^2}\,\frac{\sigma^2}{f^{(0)}(\gamma')\,f^{(0)}(\sigma')} + \mathcal{O}(\mathcal{D}^0)\,, \tag{6.39}$$

where

$$\gamma' = \frac{\rho\sigma-\upsilon^2}{\sigma}\,,\qquad \sigma' = \frac{\rho\sigma-(\upsilon-1)^2}{\sigma}\,, \tag{6.40}$$

and $f^{(0)}(\gamma)=\vartheta_2^4(\gamma)$. The divisor is invariant under the following $\Gamma(2)$ transformations,

$$\begin{aligned}
\rho &\to a^2\,\rho + b^2\,\sigma - 2\,ab\,\upsilon + ab\,,\\
\sigma &\to c^2\rho + d^2\,\sigma - 2\,cd\,\upsilon + cd\,,\\
\upsilon &\to -\,ac\,\rho - bd\,\sigma + (ad+bc)\upsilon - bc\,,
\end{aligned} \tag{6.41}$$

which belong to the invariance group of Φ_0. With this information it can be verified straightforwardly that the function (6.38) is therefore invariant under $\Gamma(2)$ using that K,L,M transform precisely as the charge bilinears in (6.3).

As stated above, the proposal for the dyon degeneracy reads,

$$d_{\rm STU}(p,q) = I(K_s,L_s,M_s)\; I(K_t,L_t,M_t)\; I(K_u,L_u,M_u)\;, \tag{6.42}$$

which is manifestly invariant under triality. When performing an asymptotic evaluation of the integral (6.38), one must specify which limit in the charges is taken. Large black holes correspond to a limit where both electric and magnetic charges are taken to be large. More precisely, one takes $KL - M^2 \gg 1$, and $K + L$ must be large and negative. Under a uniform scaling of the charges the field $S^{\rm inv}$ given in (6.21) will then remain finite; to ensure that it is nevertheless large one must assume that $|K|$ is sufficiently small as compared to $\sqrt{KL - M^2}$. In this way one can recover the non-perturbative string corrections, as was stressed in [19].

Clearly, $\Phi_0(\rho_s,\sigma_\sigma,\upsilon_s)$ has double zeros at $\upsilon_{s\pm} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4\rho_s\sigma_s}$ on the divisor. The evaluation of the integral (6.38) proceeds by first evaluating the contour integral for υ around either one of the poles $\upsilon_{s\,\pm}$, and subsequently evaluating the two remaining integrals over ρ_s and σ_s in saddle-point approximation. The saddle-point values of ρ_s, σ_s, and hence of $\upsilon_{s\pm}$, can be parametrized by

$$\rho_s = \frac{\mathrm{i}|S^{\rm inv}|^2}{S^{\rm inv}+\bar S^{\rm inv}}\;, \qquad \sigma_s = \frac{\mathrm{i}}{S^{\rm inv}+\bar S^{\rm inv}}\;, \qquad \upsilon_{s\pm} = \frac{S^{\rm inv}}{S^{\rm inv}+\bar S^{\rm inv}}\;, \tag{6.43}$$

with $S^{\rm inv}$ given in (6.21).[9] The same considerations apply to the other integrals in (6.42) with identical results. As argued in [19], these values describe the unique solution to the saddle-point equations for which the state degeneracy $d(p,q)$ takes a real value. The resulting expression for $\log d_{\rm STU}(p,q)$ precisely equals the expression for the macroscopic entropy (6.24), with S (and similarly T and U) expressed in terms of the charges through the first term in (6.17). The result is valid up to a constant and up to terms that are suppressed by inverse powers of the charges. Other divisors are expected to give rise to exponentially suppressed corrections to the microscopic entropy $\mathcal{S}_{\rm micro} = \log d_{\rm STU}(p,q)$. This result is in accordance with the generic features of the semiclassical approximation that we have outlined in section 2.

The microstate degeneracy proposal of [31] does, however, raise a few questions which in our mind indicate that a better understanding of the microstate degeneracy is needed. First of all, the saddle-point equation for $S^{\rm inv}$ resulting from the asymptotic evaluation of (6.38), is the one following from (6.26) and therefore it does not agree with the attractor equation (6.22) derived from the macroscopic analysis. This is in contrast to the situation encountered in the $N = 4$ models discussed in [19, 20].

Second, when considering the small black hole discussed in (6.37), it is not clear how the microstate proposal (6.42) can reproduce the leading term of the entropy of this small black hole. In the case of a small black hole, the degeneracy integral (6.38) needs to be evaluated on either an electric or a magnetic divisor, and to leading order this yields a

[9] Observe that ρ, σ, υ constitute the complex two-by-two period matrix, which appears in the exponential factor of the integrand in (6.38) sandwiched between the charge vectors. At the divisor, the imaginary part of this matrix is proportional to the coset representative of $\mathrm{SO}(2,1)/\mathrm{SO}(2)$, parametrized by the invariant dilaton field.

contribution to the microscopic entropy proportional to the square root of the appropriate charge bilinear. Since the microscopic degeneracy proposal (6.42) involves three integrals, with each integral contributing a term of this type, the resulting microscopic entropy consists of a sum of three terms, each involving the square root of one of the three charge bilinears (6.27). This, however, is in conflict with (6.37).

Finally, we have considered the computation of the mixed black hole partition function, as was done in the context of $N = 8$ [9] and $N = 4$ [9, 11] models, in the hope of reproducing (2.26). Hence we start from the definition of the mixed black hole partition function (2.26) with $d_{\rm STU}(p,q)$ expressed by (6.42), and with K, L, M given by the charge bilinears $\langle P,P\rangle$, $\langle Q,Q\rangle$ and $\langle P,Q\rangle$ (here we omit a proportionality factor between these two sets of bilinears, for simplicity). The summation over q_0 leads to a delta function, whereas the sum over q_1, q_2, q_3 can be done by a Poisson resummation. In this way we obtain the following result,

$$
\begin{aligned}
Z_{\rm STU}(p,\phi) = & \sum_{\phi-\text{shifts}} \oint\oint\oint \frac{1}{\sqrt{\sigma_s\sigma_t\sigma_u}\,\Phi_0(\rho_s,\sigma_s,\upsilon_s)\,\Phi_0(\rho_t,\sigma_t,\upsilon_t)\,\Phi_0(\rho_u,\sigma_u,\upsilon_u)} \\
& \times \delta\big(\phi^0 + \mathrm{i}p^0(2\upsilon_s + 2\upsilon_t + 2\upsilon_u - 3) + 2\mathrm{i}(p^1\sigma_s + p^2\sigma_t + p^3\sigma_u)\big) \\
& \times \exp\left(-2\pi\mathrm{i}\left[p^2p^3\rho_s + p^3p^1\rho_t + p^1p^2\rho_u - \frac{\phi^{s2} + \phi^{t2} + \phi^{u2} - 2(\phi^s\phi^t + \phi^t\phi^u + \phi^u\phi^s)}{16\,\sigma_s\sigma_t\sigma_u}\right]\right),
\end{aligned} \tag{6.44}
$$

where the sum over shifts of ϕ are by arbitrary integer steps of 2i. The quantities ϕ^s, ϕ^t and ϕ^u are given by

$$
\phi^s = \sigma_s\phi^1 - 2\mathrm{i}p^0\rho_s\sigma_s - \mathrm{i}p^1\sigma_s(2\upsilon_s - 2\upsilon_t - 2\upsilon_u + 1)\,, \tag{6.45}
$$

with ϕ^t and ϕ^u related by triality. The resulting integral is supposed to be a function of $\phi^0, \phi^1, \phi^2, \phi^3$, and of the charges p^0, p^1, p^2, p^3, but this feature is no longer manifest in the expression (6.44). Unlike in the $N = 4$ models, it is a non-trivial task to explicitly evaluate the integral, although it should, for instance, be possible to use a saddle-point approximation and make contact with semiclassical predictions.

Note added. Meanwhile this problem has been addressed in [66].

7. Conclusion

In this paper we demonstrated that non-holomorphic corrections are crucial for obtaining a BPS black hole free energy that is manifestly invariant under duality transformations. In our approach, these corrections are encoded in a single real homogeneous function Ω, in order to ensure that the attractor equations will still follow by requiring stationarity of the free energy. We presented evidence that these corrections describe a consistent non-holomorphic deformation of special geometry. The precise relationship between the non-holomorphic terms encoded in Ω and the effective supersymmetric action remains to be worked out.

In the context of $N = 2$ models with exact duality symmetries, such as the FHSV and the STU models, an explicit evaluation of the non-holomorphic corrections to Ω reveals that these are related to, but quantitatively different from the non-holomorphic corrections to the topological string. This difference may be related to the Legendre transformation that transforms the holomorphic prepotential of complex special geometry into the real Hesse potential of real special geometry. The latter is related to the BPS black hole free energy and therefore manifestly duality invariant. It would be very interesting to investigate this further.

Duality invariance of the black hole partition function also requires the presence of a non-trivial integration measure when writing the BPS degeneracies in the form of an inverse Laplace transform over a mixed partition function [5]. We gave a prediction for the measure factor for a class of $N = 2$ black holes using semiclassical arguments, which, however, disagrees with the results for string compactifications based on compact Calabi-Yau manifolds at strong topological string coupling [15]. A direct test of our semiclassical prediction for the measure factor requires knowledge of the exact microscopic state degeneracy. When confronting our macroscopic results for large and small black holes in the STU model with the microstate degeneracy proposal of [31], we identify a number of subtle issues that to us indicate the need for a better understanding of the microstate degeneracy of the STU model.

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