

Pokrovsky-Talapov model at finite temperature: A renormalization-group analysis

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We calculate the finite-temperature shift of the critical wave vector Q_c of the Pokrovsky-Talapov model using a renormalization-group analysis. Separating the Hamiltonian into a part that is renormalized and one that is not, we obtain the flow equations for the stiffness and an arbitrary potential. We then specialize to the case of a cosine potential, and compare our results to well-known results for the sine-Gordon model, to which our model reduces in the limit of vanishing driving wave vector $Q=0$. Our results may be applied to describe the commensurate-incommensurate phase transition in several physical systems and allow for a more realistic comparison with experiments, which are always carried out at a finite temperature.

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I. INTRODUCTION

The Pokrovsky-Talapov^{1,2} (PT) model describes a large variety of systems displaying a commensurate/incommensurate (C/IC) transition, ranging from vortex depinning in type-II superconductors³ to adsorbate layers on crystal surfaces,⁴ bromide molecules intercalated onto graphite,⁵ superconductors with modulated thickness,⁶ and quantum-Hall bilayer systems under a tilted magnetic field.⁷⁻⁹ This model is closely related to the sine-Gordon model,¹⁰ with the extra feature of a characteristic driving wave vector Q imposed through the cosine term. It has been studied previously using the Bethe ansatz and the replica trick in the presence of quenched disorder.¹¹ In this work we present a functional renormalization-group calculation of the finite-temperature corrections to the mean-field results.

The Hamiltonian for the PT model is

$$H_{\text{PT}} = \int d^2r \left[\frac{1}{2} \rho_s (\nabla \phi)^2 - t \cos(\phi - Qx) \right], \quad (1)$$

where $\mathbf{r}=(x,y)$ and $d^2r=dx dy$. At mean-field level, one approximates the thermodynamic free energy by the Hamiltonian itself, evaluated at the field configuration that minimizes the energy for given boundary conditions. This neglects the contributions of all fluctuations away from the minimum, therefore becoming less accurate as the temperature is raised and the entropic contribution of fluctuations increases. In this approximation, it is straightforward to compare the free energy of a configuration with the field following the driving wave vector (commensurate phase) to that of one in which the field no longer follows the imposed Q . As Q increases, the presence of the stiffness term, suppressing deviations of the field from uniformity, makes it more and more costly to remain in the commensurate phase, until a critical Q_c is reached at which the incommensurate phase becomes energetically favorable. The aim of this paper is to investigate the effects of thermal fluctuations on this critical Q_c .

We shall use a functional renormalization-group (RG) scheme to study the model at finite temperatures. Our approach is as follows: we perform a simple transformation which maps the PT model to a sine-Gordon model with additional terms depending only on the total topological

“charge” of the system and on the driving wave vector Q . The RG transformation does not couple the sine-Gordon part to the Q -dependent part. Taking advantage of this, we renormalize the sine-Gordon part of the Hamiltonian and obtain a long-wavelength effective action, which we subsequently use to obtain the new value of the critical Q_c .

The main technical complication one faces in constructing a systematic renormalization-group transformation for the sine-Gordon model is the inability to expand the cosine term in Eq. (1) in powers of the field and keep only a finite number of these. There are three main reasons for this complication: first, the periodicity of the cosine is crucial and is destroyed by any finite-order Taylor expansion. Second, we are interested in the two-dimensional case; as follows from simple power counting,^{10,12} polynomial interactions involving any power of the field are relevant in two dimensions. Finally, we are not interested in the fixed point of the RG transformation but in the actual values of the various parameters after integrating out the degrees of freedom that we are not interested in.

Not expanding the cosine means that, in diagrammatic language, we ought to keep track of an infinite number of diagrams to one loop. This problem has already been solved for the case of the Wilsonian RG by Wegner and Houghton in Ref. 13, where these authors derive the eponymous exact renormalization-group equation. This equation is also the limit of an approximate recursion relation first given by Wilson.¹⁴ Wilson’s approximate recursion relation has been applied to the problem of critical wetting in Refs. 15 and 16 because the effective-field theories used in studying critical wetting share with our problem its dimensionality, preventing the use of the more usual perturbation methods. In general, functional renormalization-group approaches are useful for cases where there is an effective potential with a non-trivial functional dependence on the field, such as the cosine term in Eq. (1).

The exact functional renormalization equation of Wegner and Houghton relies on a sharp, moving cutoff in momentum space. A sharp cutoff induces long-range interactions in real space and complicates the calculation of the flow of the stiffness ρ_s in Eq. (1). This may be overcome by employing a smooth cutoff function;^{17,18} however, the resulting trajectories depend on the precise form of the cutoff. It has been shown¹⁹ that the Wilsonian RG approach (of which the

Polchinski RG is an example) suffers from strong scheme dependence even in the asymptotic regime. Various alternative formulations of RG transformations exist that do not suffer from this problem; one that has recently been applied to the sine-Gordon model is the functional renormalization of the effective average action^{20,21} in which a transformation is obtained, not for the Hamiltonian itself, but for the generating function for the one-particle irreducible (1PI) Green's functions. This is the RG scheme that we use in this paper.

The outline of this paper is as follows: in Sec. II we describe the model and give a qualitative description of its behavior. In Sec. III we explain the basic idea behind our approach before proceeding directly to the derivation (Sec. IV) and application to the PT model (Sec. V) of the appropriate RG flow equations. Finally, we give a brief discussion of our results in Sec. VI.

II. MODEL

To qualitatively understand the features exhibited by a system described by the PT Hamiltonian given in Eq. (1), consider first the case $Q=0$. It is then clear that, at mean-field level, ϕ will simply remain at one of the minima of the potential $V(\phi)=-t \cos(\phi)$ at $\phi_n=2\pi n$ with $n=0, \pm 1, \pm 2, \dots$

On the other hand, consider the quantity $\langle \phi(\mathbf{r})\phi(\mathbf{r}') \rangle_c = \langle \phi(\mathbf{r})\phi(\mathbf{r}') \rangle - \langle \phi(\mathbf{r}) \rangle \langle \phi(\mathbf{r}') \rangle$; for low enough temperatures, it is given by the one-loop result

$$\langle \phi(\mathbf{r})\phi(\mathbf{r}') \rangle_c = \frac{k_B T}{\rho_s} K_0\left(\frac{|\mathbf{r}-\mathbf{r}'|}{\xi}\right) \quad (2)$$

in which $\xi = \sqrt{\rho_s/t}$, $K_0(x)$ is a modified Bessel function, and $\langle \dots \rangle_c$ indicates a cumulant. Since $\xi \rightarrow \infty$ if $t \rightarrow 0$ and $K_0(x) = \ln(2/x) + \text{const} + O[x^2 \ln(x)]$, the right-hand side of Eq. (2) diverges as $t \rightarrow 0$, i.e., there is no long-range order in the system. This is merely an example of the Mermin-Wagner theorem.^{10,22} In addition, the $Q=0$ system exhibits a Kosterlitz-Thouless (KT)-type transition²² on some line $t(\rho_s)$, which is again completely missed by a mean-field analysis.

This transition is analogous to the roughening transition in interface problems.^{18,23} In this analogy, the phase ϕ corresponds to the height of an interface above a reference level.²⁴ The phase in which $\langle \phi(\mathbf{r})\phi(\mathbf{r}') \rangle_c$ remains finite as $|\mathbf{r}-\mathbf{r}'| \rightarrow \infty$ is called the “smooth” phase while the one in which $\langle \phi(\mathbf{r})\phi(\mathbf{r}') \rangle_c$ diverges is called the “rough” phase. Given the value of $\phi(0) = \phi_0$ at some arbitrary point which we take to be the origin, the value of $\phi(\mathbf{r})$ at some other point \mathbf{r}' arbitrarily far from it either remains within a finite distance from ϕ_0 (smooth phase) or it does not but rather crosses over the maxima of $V(\phi)$. Clearly, the system with $t=0$ is in the rough phase;²⁵ the usual RG analysis of the sine-Gordon model, Eq. (1) with $Q=0$ shows that, for given temperature T and $\rho_s/T < 1/8\pi$, there exists a t_c below which the system is rough and above which it is smooth. For $\rho_s/T > 1/8\pi$ it is always smooth.

Consider now the case of finite Q . In the roughening picture, this corresponds to a potential V that depends on the position x ; as x increases, the minima of the potential move to larger values of $\phi_n = 2\pi n + Qx$. In other words, the poten-

tial is effectively “tilted.” Thus, the potential part of the Hamiltonian tends to favor a ϕ that increases with position and follows the potential, $\phi = \phi_n$ (commensurate phase) while the gradient part favors a spatially constant ϕ (incommensurate phase). The competition between ρ_s and t leads to a transition between the two states as, for example, t is varied.

Notice that there are two separate effects here: one is the roughening transition (belonging to the Kosterlitz-Thouless universality class), which is already present when $Q=0$ and the other is the commensurate-incommensurate transition, which appears only for finite Q . These two effects may be conveniently separated out as described in the next section.

III. DETERMINATION OF Q_c AT MEAN-FIELD LEVEL

A. Separation of the Hamiltonian

We begin by shifting to the new variable $\theta = \phi - Qx$, whereupon the Hamiltonian becomes

$$H_{\text{PT}}[\theta] = H_{sG} + H_Q, \quad (3a)$$

with

$$H_{sG}[\theta] = \int d^2r \left[\frac{1}{2} \rho_s (\nabla \theta)^2 - t \cos(\theta) \right], \quad (3b)$$

$$H_Q(n_T^\theta) = \frac{1}{2} \rho_s Q^2 + 2\pi \rho_s Q n_T^\theta \quad (3c)$$

and

$$n_T^\theta = \frac{1}{2\pi} \int dx \partial_x \theta = \sum_i n_i^\theta, \quad (3d)$$

that is, the sum of the “charges” of all solitons present (per unit length); the integer i simply indexes the soliton.

This form of the Hamiltonian is advantageous in that it consists of a sine-Gordon part, H_{sG} , which is independent of Q , plus the two terms in H_Q , which do depend on Q . This last term is the essential difference from a simple sine-Gordon model. As we shall see in Sec. IV, H_Q is unaffected by an RG transformation. This will form the basis of our treatment of finite-temperature effects.

B. Determination of critical Q_c

We shall next compute the critical Q_c in the mean-field approximation; this will be straightforwardly extended to the renormalized case in Sec. V.

To obtain the critical Q_c , we notice that the transition from the commensurate to the incommensurate phase is signaled by the appearance of a finite soliton density with charge $-|n_T^\theta|$. We therefore calculate the excess energy per unit area of a configuration with a single soliton (with charge $n^\theta = -1$) over that of one with no solitons, $\theta=0$; we denote this by ΔE . The part of the energy cost of a solitonic configuration due to H_{sG} is calculated in a standard way to be $E_{\text{sol}} = 8\sqrt{\rho_s t}$, hence $\Delta E = E_{\text{sol}} - 2\pi \rho_s Q$. This quantity vanishes

at the transition point, yielding a mean-field critical Q_c of⁸

$$Q_c = \frac{4}{\pi} \sqrt{\frac{t}{\rho_s}}. \quad (4)$$

Note that Q_c diverges as $\rho_s \rightarrow 0$, implying that the system remains in the commensurate state for all momenta; this is in agreement with the discussion at the end of Sec. II, according to which the creation of solitons (hence the transition to the incommensurate state) is caused by the stiffness overcoming the tendency of the phase field ϕ to follow the minimum of the tilted potential.

IV. EXACT RENORMALIZATION-GROUP EQUATIONS

Various schemes have been developed to study renormalization-group transformations of two-dimensional field theories: in the theory of critical wetting, there have been several studies using this formulation of the RG, initially to first order in the potential²⁶ and then to all orders.¹⁵ All these approaches have essentially used a local-potential approximation (LPA), in which the potential is allowed to change under coarse graining, while the stiffness is not. The same method was later extended to a nonlocal model.²⁷ In the same context, there has also been work in which the gradient term is renormalized in an approximate way.¹⁶ The LPA is generally thought to be applicable to wetting problems because the anomalous dimension is expected to be zero.

In the present problem, on the other hand, it is clear that for vanishing Q one should obtain Kosterlitz-Thouless behavior; furthermore, as discussed above, the quantity Q only couples to the number of solitons, which is conserved under the RG flow. Thus, the appropriate RG must capture the Kosterlitz-Thouless type of behavior, which requires ρ_s to flow under the transformation.

As mentioned above, we shall use the effective average functional RG scheme introduced by Wetterich²⁰ and applied recently to the sine-Gordon model.²¹ For completeness, we outline the derivation of the exact flow equation for the effective action before applying it to the sine-Gordon model; for more details, see Refs. 20, 21, and 28.

From this point onwards, we will subsume the temperature into the parameters ρ_s and t , that is, we use units in which the temperature $k_B T = 1$.

One begins by defining the bare action $\mathcal{S}[\theta] = H[\theta]$ and adding to it a piece

$$\Delta S_R[\theta] = \frac{1}{2} \sum_{\mathbf{q}} R_\Lambda(\mathbf{q}) \theta(\mathbf{q}) \theta^*(-\mathbf{q}),$$

where $R_\Lambda(\mathbf{q})$ is called a regulator function (see below). One also adds a source term $\Delta S_S[j, \theta] = \sum_{\mathbf{q}} [j(\mathbf{q}) \theta^*(-\mathbf{q}) + j^*(-\mathbf{q}) \theta(\mathbf{q})]$ and writes $\mathcal{S}_\Lambda[j, \theta] = \mathcal{S}[\theta] + \Delta S_R[\theta] + \Delta S_S[j, \theta]$. The quantity

$$W_\Lambda[j] = \log \int \mathcal{D}\theta \exp(-S_\Lambda[j, \theta])$$

is then the generator of connected correlation functions¹² for the action S_Λ . Its Legendre transform is

$$\tilde{\Gamma}_\Lambda[\bar{\theta}] + W_\Lambda[j] = \sum_{\mathbf{q}} [j^*(-\mathbf{q}) \bar{\theta}(\mathbf{q}) + j(\mathbf{q}) \bar{\theta}^*(-\mathbf{q})],$$

where $\bar{\theta}(\mathbf{q}) = \delta W_\Lambda[j] / \delta j^*(-\mathbf{q})$ is the average of the field. We also define a new, related function

$$\Gamma_\Lambda[\bar{\theta}] = \tilde{\Gamma}_\Lambda[\bar{\theta}] - \Delta S_R[\bar{\theta}]. \quad (5)$$

Using well-known¹² properties of W and Γ , and writing $\epsilon = \ln(\Lambda_0/\Lambda)$ where Λ_0 is the initial value of the cutoff Λ , one finds

$$\partial_\epsilon \Gamma_\Lambda[\bar{\theta}]|_{\bar{\theta}} = \frac{1}{2} \text{Tr} \{ \partial_\epsilon R_\Lambda(\mathbf{q}) [\Gamma_\Lambda^{(2)}[\bar{\theta}] + R_\Lambda(\mathbf{q})]^{-1} \} \quad (6)$$

with $\Gamma_\Lambda^{(n)}(\bar{\theta})$ indicating the n th functional derivative of Γ_Λ . This is an exact result.²⁰ It can be shown²⁰ that if $R_\Lambda(q) \rightarrow \infty$ as $\Lambda \rightarrow \infty$ then $\Gamma_\Lambda[\bar{\theta}] \rightarrow S_\Lambda[\bar{\theta}]$: fluctuations about the mean-field solution are completely suppressed. Conversely, if $R_\Lambda(q) \rightarrow 0$ as $\Lambda \rightarrow 0$ then $\Gamma_\Lambda[\bar{\theta}] \rightarrow \Gamma[\bar{\theta}]$, so that the full generator of 1PI vertices is obtained.

A full solution of Eq. (6) for Γ_Λ would amount to computing all 1PI functions of the system at some length scale $\sim 1/\Lambda$, including the full effects of fluctuations. This is not a simple problem, and one must resort to approximations. We will take the form of $\Gamma_\Lambda[\bar{\theta}]$ to be

$$\Gamma_\Lambda[\bar{\theta}] = \int d^2r \left[\frac{1}{2} \rho_s(\epsilon) (\nabla \bar{\theta})^2 + V(\epsilon, \bar{\theta}) \right].$$

A tedious but straightforward computation leads to²⁸⁻³⁰

$$\partial_\epsilon V = 2V - \frac{1}{2} \int \frac{d^2q}{(2\pi)^2} \mathcal{A}(\mathbf{q}) \partial_\epsilon R_\Lambda(\mathbf{q}), \quad (7a)$$

$$\begin{aligned} \partial_\epsilon \rho_s = & \frac{1}{2} \mathcal{P}(V^{(3)})^2 \int \frac{d^2q}{(2\pi)^2} \mathcal{A}^4(\mathbf{q}) \partial_\epsilon R_\Lambda(\mathbf{q}) [-\rho_s(\epsilon) \\ & + 2\rho_s^2(\epsilon) \mathcal{A}(\mathbf{q}) \mathbf{q}^2], \end{aligned} \quad (7b)$$

where the operator \mathcal{P} projects the function to its right onto the field-independent functional subspace³¹ and $\mathcal{A}(\mathbf{q}) = [\rho_s q^2 + R_\Lambda(\mathbf{q}) + V^{(2)}]^{-1}$. We take the cutoff function to be

$$R_\Lambda(q) = q^2 \left(\frac{\Lambda^2}{q^2} \right)^b. \quad (8)$$

The parameter b controls the sharpness of the regulator function R_Λ in both wave vector and real space: for large b , R_Λ is local in wave-vector space and long range in real space; for $b \rightarrow 1$, it is instead smooth in momentum space but sharp in real space.

If $V(\epsilon, \bar{\theta})$ is restricted to its leading Fourier component $V(\epsilon, \bar{\theta}) = -t(\epsilon) \cos(\bar{\theta})$, the flow equations are, to leading order and after rescale

$$\partial_\epsilon t(\epsilon) = \left(2 - \frac{1}{4\pi\rho_s} \right) t(\epsilon) / \Lambda^2$$

$$\partial_\epsilon \rho_s(\epsilon) = \frac{[t(\epsilon)/\Lambda^2]^2}{[\rho_s(\epsilon)]^{2-2/b}} \tau_b$$

with $\tau_b = b\Gamma(3-2/b)\Gamma(1+1/b)/(48\pi)$, reproducing the well-known leading-order flow equations for the sine-Gordon model.^{10,18,32,33}

Including higher-order terms on the right-hand side of Eqs. (7a) and (7b) is, in principle, straightforward; for example, expanding $\mathcal{A}(\mathbf{q})$ in powers of $V(\epsilon, \bar{\theta})$ and computing the integrals, one obtains

$$\begin{aligned} \partial_\epsilon V(\epsilon, \bar{\theta}) = & -\frac{V^{(2)}}{2b\Lambda\rho} + \frac{(-1+b)\pi\left(\frac{\rho}{\Lambda}\right)^{1+1/b} (V^{(2)})^2}{4b^3\rho^3 \sin\left(\frac{2\pi}{b}\right)} \\ & - \frac{(-2+b)(-1+b)\pi\left(\frac{\rho}{\Lambda}\right)^{(2+b)/b} (V^{(2)})^3}{3b^4\rho^4 \sin\left(\frac{2\pi}{b}\right)} + \dots \end{aligned} \quad (9)$$

A similarly ungainly equation holds for $\partial_\epsilon \rho_s(\epsilon)$. These expressions are greatly simplified if we make the choice³⁴ $b=1$ in the regulator function, Eq. (8). For the potential, the series on the right-hand side of Eq. (9) may be summed to yield

$$\partial_\epsilon V = \frac{\Lambda}{4\pi\rho_s} \ln\left(1 + \frac{V^{(2)}}{\Lambda^2}\right).$$

Restricting the potential to the form $V(\epsilon, \bar{\theta}) = -t(\epsilon)\cos(\bar{\theta})$ we obtain the flow equation²¹ for $t(\epsilon)$

$$\partial_\epsilon t(\epsilon) = 2t(\epsilon) - \frac{1}{2\pi\rho_s(\epsilon)t(\epsilon)/\Lambda^2} \left\{ 1 - \sqrt{1 - \left[\frac{t(\epsilon)}{\Lambda^2}\right]^2} \right\}. \quad (10)$$

In a similar way, we obtain for the flow of $\rho_s(\epsilon)$

$$\partial_\epsilon \rho_s(\epsilon) = \frac{t^2(\epsilon)/\Lambda^2}{24\pi[1 - t^2(\epsilon)/\Lambda^4]^{3/2}}. \quad (11)$$

The flow diagram corresponding to Eqs. (10) and (11) is shown in Fig. 1. We now turn to the application of the flow equations to the PT model.

V. APPLICATION OF RG TO THE PT MODEL

A. Calculation of Q_c using the RG results

To apply our formalism to the PT model, we use a Ginzburg criterion.³⁵ Using the fact that mean-field theory applies if³⁶ $\Lambda\xi(\epsilon) \ll 2\pi$, we conclude that the $\epsilon = \epsilon_f$ at which the integration may be stopped may be located by integrating up to the point at which $\xi(\epsilon)$ is a minimum. At this point, the mean-field approach of Sec. III may be used to determine Q_c , with the effects of fluctuations having been taken into account by the renormalization of the parameters. The mean-field treatment of the action at this point is justified *a poste-*

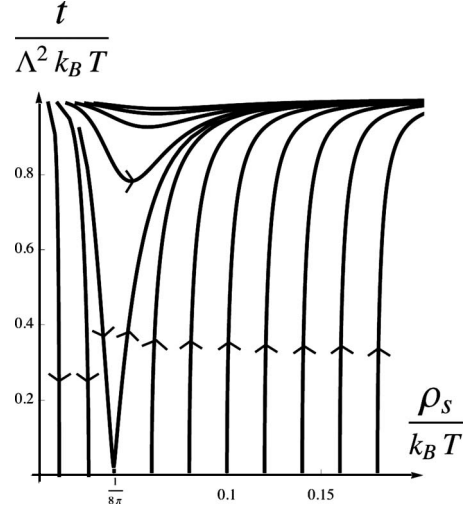


FIG. 1. Flow diagrams for the RG Eqs. (10) and (11).

riori if indeed $\Lambda\xi(\epsilon_f) \ll 2\pi$. This condition is satisfied for all parameter values we have studied (see Fig. 2 for a representative example).

In Fig. 2 we show a representative plot of the evolution of $\Lambda\xi(\epsilon)$ with ϵ . Evidently, the minimum is well below 2π so that mean-field theory is applicable to the renormalized Γ .

Having determined the appropriate ϵ_f , we repeat the procedure of Sec. III to determine the critical Q . How would this calculation be affected by the RG transformation? As we saw in Sec. IV, H_Q is invariant under the RG transformation, while H_{sG} is not, i.e., the parameters in H_{sG} will change to $\rho_s(\epsilon)$ and $t(\epsilon)$, respectively. To obtain the critical Q_c , we notice that while H_Q is unaffected by the RG, the energy of a single soliton now depends on ϵ : we have $E_{\text{sol}}(\epsilon) = 8\sqrt{\rho_s(\epsilon)}t(\epsilon)$. This results in an energy difference between the phase with no solitons and the phase with a single soliton given by $\Delta E = E_{\text{sol}}(\epsilon) - 2\pi\rho_s(0)Q$. Setting this equal to zero and solving for Q , as in Sec. III, yields

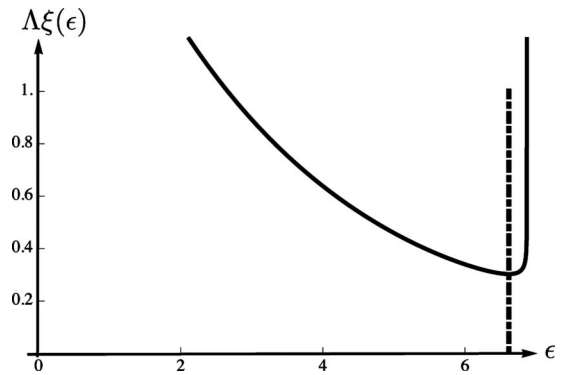


FIG. 2. Evolution of $\Lambda\xi(\epsilon)$ (dark full line) as a function of ϵ . The dot-dashed line indicates the position of the minimum of $\xi(\epsilon)$. The steep increase in $\xi(\epsilon)$ after its minimum is a result of our approximations and therefore unphysical. Notice that the minimum of $\Lambda\xi(\epsilon) < 2\pi$. The initial values for this figure are $\rho_s/k_B T = 0.06$ and $t/\Lambda^2 k_B T = 0.01$.

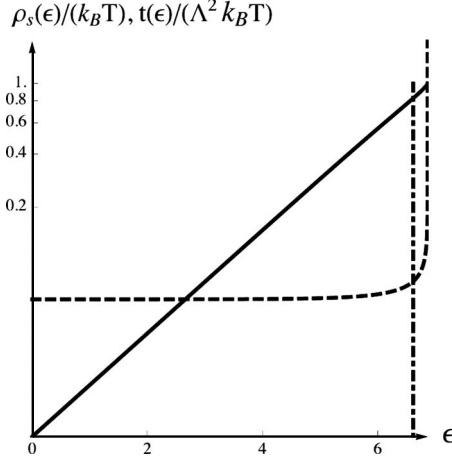


FIG. 3. Demonstration of the asymptotic simple scaling of the trajectories, demonstrating their scheme independence (see text). The full line is $t(\epsilon)/\Lambda^2 k_B T$, the dashed line is $\rho_s(\epsilon)/k_B T$, and the dot-dashed line indicates the position of the minimum of $\xi(\epsilon)$ (see Fig. 2). The steep increase in $\xi(\epsilon)$ after its minimum is a result of our approximations and therefore unphysical (see text). The initial values for this figure are the same as for Fig. 2: $\rho_s/k_B T=0.06$ and $t/\Lambda^2 k_B T=0.01$.

$$Q_c(\epsilon) = \exp(-\epsilon) \frac{4}{\pi} \frac{1}{\rho_s(0)} \sqrt{\rho_s(\epsilon)t(\epsilon)}. \quad (12)$$

The factor $\exp(-\epsilon)$ ensures that we are using physical (as opposed to rescaled) units. Equation (12) reduces to the correct mean-field expression, Eq. (4), for $\epsilon=0$.

The result given in Eq. (12) is nonuniversal, involving the microscopic parameter $\rho_s(0)$. This is expected: in calculating a nonuniversal quantity such as Q_c , is natural for microscopic, thus nonuniversal, quantities to appear. Here, the dependence on the microscopic physics arises both from the presence of $\rho_s(0)$ and from ϵ_f , which is a function of $\rho_s(0)$ and $t(0)$.

Let us now discuss the scheme dependence of our calculation. In Ref. 19, it is shown that the trajectories resulting from the effective action functional RG scheme we use are scheme independent, provided that the quantities t and ρ_s flow as powers of the parameter $\exp(\epsilon)$, i.e., $\rho_s \sim \exp(d_\rho \epsilon)$ with some d_ρ (and similarly for t). They term the region in which this occurs the “freezing region.” Note that this scheme independence does not hold in general for the case of Wilson-type renormalization.

Figure 3 is a log plot of the evolution of ρ_s and t with ϵ . The full line is $t(\epsilon)$, the dashed line is $\rho_s(\epsilon)$, and the dot-dashed line indicates the position of the minimum of $\xi(\epsilon)$ (see also Fig. 2). Evidently, both $\rho_s(\epsilon)$ and $t(\epsilon)$ are in the freezing region at the values of ϵ that we are interested in. This happens for all initial values of ρ_s and t that we have checked.

B. Results and discussion

In Figs. 4(a) and 4(b) we show the ratio $\gamma=Q_c(\epsilon)/Q_c(0)$ determined by numerically integrating the flow equations and applying the method discussed above; this corresponds

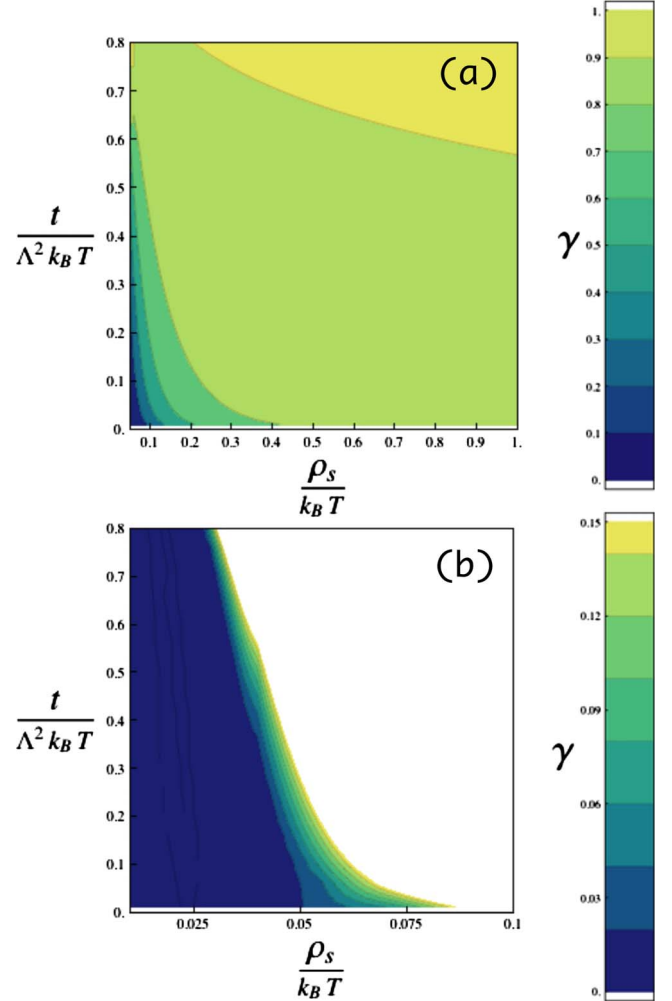


FIG. 4. (Color online) The ratio $\gamma=Q_c^{\text{RG}}/Q_c^{\text{MF}}$ predicted by our RG analysis to its mean-field value. Part (b) corresponds to the lower left corner of part (a). The strips on the right indicate meaning of shade. Notice the difference in scales between the two plots. White corresponds to $\gamma>0.15$.

to the ratio $\gamma=Q_c^{\text{RG}}/Q_c^{\text{MF}}$. A darker color indicates a larger decrease in the critical Q_c due to thermal effects (see the insets and notice the different scales). The plot in Fig. 4(b) is a zoomed-in part of Fig. 4(a) (notice the range of the axes and also the changed color coding).

For the purposes of this section, we will switch back to using natural units by defining $\tilde{\rho}_s/k_B T=\rho_s$ and $\tilde{t}/(k_B T \Lambda^2)=t$, i.e., $\tilde{\rho}_s$ and \tilde{t} are the parameters in physical units (while t and ρ_s are in units in which $k_B T=\Lambda=1$).

First let us fix $\tilde{\rho}_s$ and T and vary \tilde{t} , i.e., fix ρ_s and vary t . For decreasing (increasing) t , γ decreases (increases), vanishing as $t \rightarrow 0$. This occurs because, as described in Sec. II, for vanishing t there is no long-range order even in the absence of Q . This effect is not present at mean-field level, hence γ vanishes with decreasing t .

Next, fix \tilde{t} and T and vary ρ_s . From Fig. 4, γ increases (decreases) for decreasing (increasing) ρ_s or $\tilde{\rho}_s$.

Finally, consider fixing \tilde{t} and $\tilde{\rho}_s$ and varying the temperature T . This corresponds to fixed $t/\Lambda^2 \rho_s$ or, in terms of Fig. 4, to moving on rays emanating from the origin with gradient

t/ρ_s . Increasing (decreasing) the temperature T corresponds to moving inward (outward) from the origin. As T is increased, a T will be reached at which the system will enter the rough phase at the extreme lower left corner of the diagram; thus, at high enough T , Q_c vanishes for all $\tilde{\rho}_s$ and \tilde{t} . Equivalently, this may be viewed as a proliferation of solitons because the RG flow there is such that $t(\epsilon) \rightarrow 0$ monotonically as $\epsilon \rightarrow \infty$, so that we also have $E_{\text{sol}}(\epsilon) \rightarrow 0$. This brings up the intriguing possibility of a purely temperature-driven commensurate-incommensurate transition in suitable systems. For low enough temperatures, $\gamma \rightarrow 1$ as expected.

It would be desirable to compare our results to numerical calculations. Unfortunately, the nonlinear nature of the problem makes simple numerical approaches unreliable. For example, approximately including temperature effects via a noise term in the extremization condition does not work for reasons explained in Ref. 37. Full Monte Carlo numerical results would be a useful check on the accuracy of our results; we hope that our work will motivate such analysis.

VI. CONCLUSIONS

We have studied thermal effects on the commensurate-incommensurate transition point of the PT model using a renormalization-group approach. Our scheme relies on split-

ting the PT Hamiltonian into a sine-Gordon part, H_{SG} , and a part depending only on the number of solitons present, H_Q . We then derive a functional RG transformation which acts on the sine-Gordon part while leaving the soliton part invariant. We are thus able to determine the critical Q_c at which the incommensurate phase eventually becomes stable, taking into account thermal effects. We find a general lowering of Q_c compared to the mean-field result. Furthermore, there exists a regime in the ρ_s - t plane in which $Q_c \rightarrow 0$ even for finite t . This is due to the roughening transition, which is of Kosterlitz-Thouless type.

Since real experiments are performed at finite temperatures, the approach developed here may be applied to describe the C/IC transition in several physical systems. An application to the case of a quantum-Hall bilayer at total filling $\nu_T=1$ in the presence of a tilted magnetic field, motivated by recent experiments,³⁸ will be described in a forthcoming publication.⁷

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