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Polar degree of hypersurfaces with 1-dimensional singularities [☆]

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A B S T R A C T

We prove a formula for the polar degree of projective hypersurfaces in terms of the Milnor data of the singularities, extending to 1-dimensional singularities the Dimca-Papadima result for isolated singularities. We discuss the semi-continuity of the polar degree in deformations, and we classify the homaloidal cubic surfaces with 1-dimensional singular locus. Some open questions are pointed out along the way.

1. Introduction

For any projective hypersurface $V \subset \mathbb{P}^n$, defined by a homogeneous polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$ of degree d, the *polar degree* is defined as the topological degree of the gradient map, also known as the Gauss map:

$$\operatorname{grad} f : \mathbb{P}^n \setminus \operatorname{Sing}(V) \to \mathbb{P}^n. \tag{1}$$

The polar degree depends only on V and not on the defining polynomial f, as conjectured by Dolgachev [6] and proved by Dimca and Papadima in [5]. One therefore denotes it by $\operatorname{pol}(V)$.

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The concept of polar degree goes back to 1851 when Hesse studied hypersurfaces with vanishing Hessian [9,10], which is equivalent to $\mathrm{pol}(V) = 0$, and to Gordon and Noether [8] (see also §5.3 for more comments).

The gradient maps (1) of polar degree equal to 1 are examples of *Cremona transformations* since they are birational maps. The corresponding hypersurfaces V were called *homaloidal*, and Dolgachev [6] found the list of all (reduced) projective plane curves which are homaloidal.

In the beginning of the 2000's, whereas the algebraic approach was dominant before that date, Dimca and Papadima [5] gave the following topological interpretation: *For any projective hypersurface V, if H is a general hyperplane with respect to V, then the reduced homology $\tilde{H}_*(V \setminus H)$ of the affine part is concentrated in dimension $n-1$, and:*

$$\mathrm{pol}(V) = \mathrm{rank}\, H_{n-1}(V \setminus H). \tag{2}$$

The classification of all homaloidal hypersurfaces with *isolated singularities* was carried out by Huh [11] and confirmed a conjecture stated by Dimca and Papadima [5,4] that there are no homaloidal hypersurfaces with isolated singularities besides the smooth quadric and the plane curves found by Dolgachev. Huh [11] proves and uses the bound:

$$\mathrm{pol}(V) \geq \mu_p^{\langle n-2\rangle}(V) := \mu_p(V \cap H), \tag{3}$$

where the Milnor number $\mu_p(V \cap H)$ is > 0 as soon as $p \in \mathrm{Sing}\, V$. This holds at any p such that V is not a cone of apex p.

More recently, the authors together with Steenbrink classified in [20] the hypersurfaces with isolated singularities and polar degree 2, confirming Huh's conjectural list [11]. The finiteness of the range of (n, d) in which there may exist hypersurfaces with isolated singularities and polar degree $k > 2$ has been also proved in [20].

Still for isolated singularities, Dimca and Papadima [5] had shown the formula:

$$\mathrm{pol}(V) = (d-1)^n - \sum_{p \in \mathrm{Sing}\, V} \mu_p(V), \tag{4}$$

which allows to compute the polar degree in terms of the Milnor data of the singular points.

In this paper we consider hypersurfaces with 1-dimensional singular locus. We extend the formula (4) and compute the polar degree from the Milnor data of the singularities (Theorem 2.1):

$$\mathrm{pol}(V) = (d-1)^n - \sum_{p \in \Sigma^{\mathrm{is}}} \mu_p - \sum_{i=1}^{r} c_i \mu_i^{\pitchfork} + (-1)^n \sum_{q \in Q} (\chi(\mathcal{A}_q) - 1)$$

by using the study of the hypersurfaces with 1-dimensional singular locus in [23] and in earlier papers, see [19]. All the notations are explained in §2. Moreover, we observe in Proposition 2.3 that even in the general case of an arbitrary singular locus we may still write a formula of "Dimca-Papadima type":

$$\mathrm{pol}(V) = (d-1)^n - (-1)^{n-1}[\chi^{\curlyvee}(V) - \chi^{\curlyvee}(V \cap H)].$$

Next, we use the semi-continuity of the polar degree in deformations (Proposition 4.1) in order to compare the polar degree of V with 1-dimensional singularity with its deformation V_t (of Yomdin type) to a hypersurface with isolated singularities (Corollary 4.2). Then $\mathrm{pol}(V_t)$ may serve as upper bound for the polar degree of V. As a consequence, we derive a Lefschetz type inequality for the slicing with a generic hyperplane H_{gen}:

$$\mathrm{pol}(V) \leq (d-1)\mathrm{pol}(V \cap H_{gen}).$$

We highlight the concept of *special point* of a hypersurface introduced in [24]. These are points p where the complex link of (V, p) is non-trivial, they are finitely many, and we will show their contribution to lower bounds formulas for the polar degree. We show that, in the 1-dimensional singularity case, one can detect them by the Milnor number jump of the transversal singularity type.

We treat cubic surfaces in §5. We compute all polar degrees in a topological way and prove that there are 3 homaloidal cubic surfaces with non-isolated singularities.

We discuss homaloidal hypersurfaces with 1-dimensional singularities and transversal type A_1, and we state and discuss the question of the existence of hypersurfaces with 1-dimensional singularities and polar degree equal to zero, in this manner coming back to Hesse's problem cited in the beginning.

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2. Formula for $\mathrm{pol}(V)$ in case of 1-dimensional singularities

Let $V \subset \mathbb{P}^n$ be a projective hypersurface with singular locus $\mathrm{Sing}\,(V)$ of dimension ≤ 1 and $\mathrm{Sing}\,(V) = \Sigma^c \cup \Sigma^{\mathrm{is}}$, where Σ^c is a non-degenerate curve with irreducible components Σ_i^c, $i = 1, \ldots, r$, and Σ^{is} is the finite set of isolated singularities.

Each curve branch Σ_i^c of $\mathrm{Sing}\,(V)$ has a generic transversal type, of transversal Milnor fibre $F_i^{\pitchfork}$ and Milnor number denoted by $\mu_i^{\pitchfork}$.

Each branch Σ_i^c contains a finite set Q_i of points where the transversal type is not the generic one, and which we have called *special points* (see also §3.2 for a more general definition). We denote by $\mathcal{A}_q$ the local Milnor fibre of the hypersurface germ (V, q) for $q \in Q := \cup_{i=1}^r Q_i$, and by $\tilde{\Sigma}_i^c$ the normalisation of Σ_i^c.

At each point $q \in Q_i$ there are finitely many locally irreducible branches of the germ (Σ_i^c, q), we denote by $\gamma_{i,q}$ their number and let $\gamma_i := \sum_{q \in Q_i} \gamma_{i,q}$. In other words, γ_i is equal to the number of "punctures" in $\tilde{\Sigma}_i^c \setminus \tilde{Q}_i$, where $\tilde{Q}_i$ is the inverse image of Q_i by the normalisation map.

This setting has been studied in [23], in particular a formula for $\chi(V)$ has been given in terms of the isolated singularities of V, the special nonisolated singularities, the topology of the curve components of $\mathrm{Sing}\,(V)$, and the transversal singularity type of each such curve component.[1]

Under the above notations, our following formula generalises the Dimca-Papadima formula (4) for isolated singularities to the case of a 1-dimensional singular set.

Theorem 2.1. *Let $V \subset \mathbb{P}^n$ be a hypersurface of degree d with a 1-dimensional singular set. Then:*

$$\mathrm{pol}(V) = (d-1)^n - \sum_{p \in \Sigma^{\mathrm{is}}} \mu_p - \sum_{i=1}^{r} c_i \mu_i^{\pitchfork} + (-1)^n \sum_{q \in Q} (\chi(\mathcal{A}_q) - 1) \tag{5}$$

where $c_i = 2g_i + \gamma_i + (d+1)\deg \Sigma_i^c - 2$, where g_i is the genus of the normalization $\tilde{\Sigma}_i^c$ of Σ_i^c, and where $\deg \Sigma_i^c$ denotes the degree of Σ_i^c as a reduced curve.

Proof. The Euler characteristic $\chi(V)$ has been computed in [23] by using a local pencil of hypersurfaces $V_\varepsilon := \{f_\varepsilon = f + \varepsilon h_d = 0\}$ of degree $d = \deg f$, where h_d is a general homogeneous polynomial Let $A := \{f = h_d = 0\}$ denote the axis of the pencil.

[1] See also Remark 2.2.

Let $\mathbb{V}_\Delta := \{(x, \varepsilon) \in \mathbb{P}^{n+1} \times \Delta \mid f + \varepsilon h_d = 0\}$ denote the total space of the pencil, where $V_0 := V \subset \mathbb{P}^n \times \{0\}$ and Δ is a small enough disk centred at $0 \in \mathbb{C}$ such that V_ε is nonsingular for all $\varepsilon \in \Delta^*$. The existence of small enough disks Δ is ensured[2] by the genericity of h_d. Note that $\mathbb{V}_\Delta$ retracts to V.

We have introduced in [23] the *vanishing homology* of projective hypersurfaces V with $\dim \mathrm{Sing}(V) = 1$, defined as:

$$H_*^{\curlyvee}(V) := H_*(\mathbb{V}_\Delta, V_\varepsilon; \mathbb{Z}) \tag{6}$$

and in its study we have established the following Euler characteristic formula for $\chi(\mathbb{V}_\Delta, V_\varepsilon)$ which equals $\chi(V) - \chi(V_\varepsilon)$, since $\mathbb{V}_\Delta$ retracts to V:

$$\chi(V) - \chi(V_\varepsilon) = (-1)^n \sum_{i=1}^{r} (2g_i + \gamma_i + \nu_i - 2)\mu_i^{\pitchfork} - \sum_{q \in Q} (\chi(\mathcal{A}_q) - 1) + (-1)^n \sum_{p \in \Sigma^{\mathrm{is}}} \mu_p$$

where $\nu_i := \#A \cap \Sigma_i^c$ is the number of axis points, namely $\nu_i = \mathrm{int}(\{h_d = 0\}, \Sigma_i^c) = d \deg \Sigma_i^c$.

Since V_ε is a nonsingular hypersurface in $\mathbb{P}^n$, its Euler characteristic is that of the smooth hypersurface of degree d in $\mathbb{P}^n$, namely:

$$\chi^{n,d} := n + 1 - \frac{1}{d}[1 + (-1)^n (d-1)^{n+1}].$$

Let now H be a generic hyperplane with respect to the canonical Whitney stratification of V. Since $V \cap H$ has only isolated singularities, one has the well-known formula:

$$\chi(V \cap H) = \chi^{n-1,d} + (-1)^{n-1} \sum_{a \in \mathrm{Sing}(V \cap H)} \mu_a(V \cap H),$$

where $\mathrm{Sing}(V \cap H) = \Sigma^c \cap H$. The number $\#\Sigma^c \cap H$ of intersection points is then $\sum_{i=1}^r \deg \Sigma_i^c$, and $\mu_a(V \cap H)$, for $a \in \Sigma_i^c$, is precisely the transversal Milnor number $\mu_i^{\pitchfork}$.

To prove our claimed formula for $\mathrm{pol}(V)$ we use the above formulas and the relation (2) of [5] in the form:

$$\chi(V \setminus H) = 1 + (-1)^{n-1} \mathrm{pol}(V)$$

where $\chi(V \setminus H) = \chi(V) - \chi(V \cap H)$ by the additivity of χ. $\square$

Remark 2.2. The vanishing (co)homology (6) has been studied more recently in the general singular setting in [14] by using sheaf cohomology. The Euler characteristic version of (6), let us denote it here by $\chi^{\curlyvee}(V) := \chi(\mathbb{V}_\Delta, V_\varepsilon; \mathbb{Z})$ and call it *vanishing Euler characteristic*, appears to be known for some time under a different name: Parusiński and Pragacz [16] had produced a formula for $\chi^{\curlyvee}(V)$ in terms of Chern classes, later generalised by Maxim, Saito and Schürmann [15]. See also the more elementary presentation of their formula in Maxim's book [13, Section 10.4].

The above proof also shows that the polar degree is related to the vanishing Euler characteristic by the following formula which represents a generalisation of Dimca-Papadima formula (4) to higher dimensional singular locus:

[2] See e.g. [22, Prop. 2.2] for a detailed explanation.

Proposition 2.3. *Let $V \subset \mathbb{P}^n$ be a projective hypersurface with arbitrary singular set, and let H be a generic hyperplane. Then:*

$$\mathrm{pol}(V) = (d-1)^n - (-1)^{n-1}[\chi^{\curlyvee}(V) - \chi^{\curlyvee}(V \cap H)]. \quad \square$$

Formula (5) is useful as soon as the local topological information is available. We show this in some examples taken from [23].

[23, Ex 7.1a]: $V := \{x^2z + y^2w = 0\} \subset \mathbb{P}^3$. Then $\mathrm{Sing}\,(V)$ is a projective line with transversal type A_1 and with two special points of type D_∞, the Milnor fibre of which is homotopy equivalent to S^2. Applying formula (5) we get $\mathrm{pol}(V) = 8 - 0 - (0 + 2 + 4 - 2) - 2 = 2$.

[23, Ex 7.1b]: $V := \{x^2z + y^2w + t^3 = 0\} \subset \mathbb{P}^4$. Then $\mathrm{Sing}\,(V) = \mathbb{P}^1$ is also a projective line but here with transversal type A_2 and with two special points having Milnor fibre $S^3 \vee S^3$. From formula (5) we get $\mathrm{pol}(V) = 16 - 0 - (0 + 2 + 4 - 2) \cdot 2 - 4 = 4$.

3. Special points and lower bounds

If one looks for hypersurfaces with small polar degree, for instance homaloidal, it is useful to have lower bounds for the polar degree in terms of the singularities of V or its dual. An example is Huh's bound (3) for isolated singularities, which has been used in [11] and in [20] to determine all hypersurfaces with isolated singularities which have polar degree 1 or 2. A more general bound has been found in [24].

3.1. Quantisation of the polar degree, after [24]

Let us fix a Whitney stratification $\mathcal{W}$ of $V \subset \mathbb{P}^n$. This depends only on the reduced structure of V. We consider hyperplanes H which are stratified transversal to all strata of V, except at finitely many points. One says that the hyperplane $H = \{l = 0\}$ is *admissible* iff its non-transversality locus consists of isolated points only, and if the affine polar locus[3] $\Gamma(l, f)$ has dimension ≤ 1.

If H has an isolated non-transversality at $p \in V$ then the linear function $l : \mathbb{C}^n \to \mathbb{C}$ defining H near p has a stratified isolated singularity at p. Consequently, its local Milnor-Lê fibre $B_\varepsilon \cap V \cap \{l = s\}$, for some s close enough to $l(p)$, has the homotopy type of a bouquet of spheres of dimension $n-2$, cf Lê's results [12]. We denote by $\alpha_p(V, H)$ its Milnor-Lê number. In case $H = H_{\mathrm{gen}}$ is a general hyperplane through p, then $\alpha_p(V, H_{\mathrm{gen}})$ is the Milnor number of the *complex link* of V at p, denoted by $\alpha_p(V)$, and we have $\alpha_p(V, H) \geq \alpha_p(V)$.

Let us denote by $\alpha(V, H)$ the sum $\sum_p \alpha_p(V, H)$ for all points p of the non-transversality locus of the admissible hyperplane H. We have shown in [24]:

Theorem 3.1. *[24, Theorem 5.4] If H is an admissible hyperplane, then there is the following decomposition of the polar degree:*

$$\mathrm{pol}(V) = \alpha(V, H) + \beta(V, H)$$

where $\beta(V, H)$ counts the vanishing cycles of the restriction $f_| : \{l = 1\} \to \mathbb{C}$ which are vanishing outside V. $\square$

Let us consider now the special case of hypersurfaces V with 1-dimensional singular locus. In this case, the numbers $\alpha_p(V, H)$ can be described by sectional Milnor numbers, as follows:

[3] $\Gamma(l, f)$ is the closure in $\mathbb{C}^{n+1}$ of the set $\mathrm{Sing}\,(l, f) \setminus \mathrm{Sing}\, f$.

(i) For isolated singular points $p \in V$ we have $\alpha_p(V, H) = \mu_p(V \cap H)$ by definition.
(ii) For points p on the 1-dimensional singular set, we have the formula proved in [21]:

$$\alpha_p(V, H) = \mu(V \cap H, p) - \sum_j \mu(V \cap H_s, p_j), \tag{7}$$

where $H_s := \{l = s\}$, where $f_p = 0$ is a reduced local equation for V, and where the points p_j are the singular points of the restriction $f_{p|H_s}$ in $B_\varepsilon \cap V \cap H_s$.
The number $\alpha_p(V, H)$ was called the *Milnor number jump* at p for the family of functions $f_{p|H_s}$. Even if H_0 may not be the most generic at p, for $s \neq 0$ the hyperplane H_s is a locally generic slice of any branch Σ_i^c, and so the above equality reads:

$$\alpha_p(V, H) = \mu(V \cap H, p) - \sum_i \mathrm{mult}_p \Sigma_i^c \cdot \mu_i^{\pitchfork}, \tag{8}$$

where $\mu_i^{\pitchfork}$ is the generic transversal Milnor number and does not depend on the choice of the point of the irreducible component Σ_i^c. Here the sum is taken over all local branches at p.

3.2. The special points of V

For any singular projective hypersurface V, we say after [24] that $p \in V$ is a *special point of* V if $\alpha_p(V) > 0$. It has been shown that the set V_{spec} of special points is finite.

In case $\dim \mathrm{Sing}\,(V) = 1$, the set of special points of V consists of the isolated singularities of V together with the set Q of points p where the generic transversal Milnor number is jumping (see §2), which is equivalent to the inequality $\alpha_p(V, H) > 0$. Indeed, it is well-known that $\alpha_p(V, H) = 0$ implies that $\mathrm{Sing}\,(V)$ is smooth at p and that $V \cap H_s$ is a μ-constant local family of hypersurfaces. This is equivalent to A'Campo's "non-splitting principle" [1].

It is known (see [24, Remark 4.3]) that the set of admissible hyperplanes for f containing a fixed point $p \in \mathrm{Sing}\,V$ is a Zariski-open subset of the set of all hyperplanes through p. The following useful lower bound then holds in general, as a consequence of Theorem 3.1:

Corollary 3.2. *[24, Corollary 6.6] Let $V \subset \mathbb{P}^n$ be a projective hypersurface which is not a cone of apex p. Then:*

$$\mathrm{pol}(V) \geq \alpha_p(V). \tag{9}$$

In particular, if V is not a cone, then:

$$\mathrm{pol}(V) \geq \max_{q \in V_{\mathrm{spec}}} \alpha_q(V). \quad \square \tag{10}$$

We shall apply this lower bound result in case of homaloidal hypersurfaces, see §5.2.

4. Semi-continuity of pol(V) under deformations

The following result is general, it holds for any singular locus and whatever $\mathrm{pol}(V)$ might be, including the case $\mathrm{pol}(V) = 0$. This could be folklore, but we didn't find a precise reference.

Proposition 4.1. *The polar degree is lower semi-continuous in deformations of fixed degree d. More precisely, if f_s is a deformation of $f_0 := f$ of constant degree, then $\mathrm{pol}(V_s) \geq \mathrm{pol}(V)$ for $s \in \mathbb{C}$ close enough to 0, where $V_s := \{f_s = 0\}$.*

Proof. Let $b \in \mathbb{P}^n$ be a regular value for $\operatorname{grad} f : \mathbb{P}^n \setminus \operatorname{Sing} V \to \mathbb{P}^n$ and let $(\operatorname{grad} f)^{-1}(b) = \{a_1, \cdots, a_k\}$, $k \geq 0$. There exist disjoint compact neighbourhoods U_i of a_i and U' of b such that $\operatorname{grad} f : (U_i, a_i) \to (U', b)$ is a diffeomorphism. Next take s so close to 0 such that $\operatorname{grad} f_s | U_i$ are still diffeomorphisms, and that $\operatorname{grad} f_s(U_i)$ still contains b in its interior. Let $W = \cap_{i=1}^k \operatorname{grad} f_s(U_i)$ and $Z_i = (\operatorname{grad} f_s)^{-1}(W) \cap U_i$. The restriction $\operatorname{grad} f_s : \bigsqcup_i Z_i \to W$ is a diffeomorphism on each component Z_i, and has topological degree $\operatorname{pol}(V)$. Moreover b is still a regular value for this restriction, but perhaps not anymore for the full map $\operatorname{grad} f_s : \mathbb{P}^n \setminus \operatorname{Sing} V_s \to \mathbb{P}^n$. Arbitrarily close to b there exist points b' which are regular values for $\operatorname{grad} f_s$. Then the number of counter-images $\#(\operatorname{grad} f_s)^{-1}(b')$ is $\operatorname{pol}(V_s)$ and $(\operatorname{grad} f_s)^{-1}(b')$ contains at least one point in each Z_i. This shows the inequality $\operatorname{pol}(V_s) \geq \operatorname{pol}(V)$. $\square$

4.1. Yomdin type families and polar degree

Let $V = \{f = 0\}$ have at most 1-dimensional singularities, and let us consider the particular deformation[4] $f_s = f + sl^d$ of degree d, where $H = \{l = 0\}$ is a hyperplane such that $H \cap V$ has isolated singularities only. By direct computation, one can show that for generic $s \neq 0$ (actually except of a finite number of values of $s \in \mathbb{C}$), the singular locus of $V_s = \{f_s = 0\}$ is the set $H \cap \operatorname{Sing} V$, and therefore $\operatorname{Sing} V_s$ consists of isolated singular points. We may then show:

Theorem 4.2. *If V has at most 1-dimensional singularities, and if H_{gen} is a generic hyperplane, then:*

$$\operatorname{pol}(V) \leq \operatorname{pol}(V_s) = (d-1)\operatorname{pol}(V \cap H_{gen}) \tag{11}$$

and

$$\sum_i \tilde{d}_i \mu_i^{\pitchfork} \leq (d-1)^{n-1}, \tag{12}$$

where $\tilde{d}_i := \operatorname{int}(\Sigma_i^c, H_{gen})$ denotes the global intersection number.

Proof. By [25], one has the following formula for the local "Yomdin series" $g_N = g + sl^N$ of a function germ $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with 1-dimensional singular locus $\Sigma = \cup_j \Sigma_j$, where l is a general linear form, see also [19]:

$$\mu(g_N) = b_{n-1}(g) - b_{n-2}(g) + N \sum_j d_j \mu_j^{\pitchfork} \tag{13}$$

where $\mu(g_N)$ is the Milnor number of g_N, where $b_{n-1}(g)$ and $b_{n-2}(g)$ are the respective Betti numbers of the local Milnor fibre of g, and where $\mu_j^{\pitchfork}$ is the transversal Milnor number of the local branch Σ_j. The sum is taken over the branches Σ_j of Σ, and $d_j = \operatorname{int}_0(\Sigma_j, \{l = 0\})$ is the intersection multiplicity of $\{l = 0\}$ and Σ_j with reduced structure.

This formula was proved initially for "high enough N" depending on the polar ratios of the discriminant of the map (l, g). It was shown in [18] that actually it holds for N greater or equal to the highest polar ratio. Moreover, from the proof of this formula in [18, pag 187], one deduces the following statement:

Lemma 4.3. *If the polar locus $\Gamma(l, g)$ is empty, then formula (13) holds for any $N \geq 2$.* $\square$

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[4] Known as the *Yomdin series* in the local context.

Let us remark that if $\Gamma(l, g) = \emptyset$ then $b_{n-1}(g) = 0$ and $b_{n-2}(g) = \mu(g_{|l=0})$. Thus formula (13) becomes:

$$\mu(g_N) = -\mu(g_{|l=0}) + N \sum_j d_j \mu_j^{\pitchfork} \tag{14}$$

We will apply Lemma 4.3 and formula (14) at some point $p \in H_{\mathrm{gen}} \cap \Sigma_i^c$, for the following data: $g := f_p$, where $f_p = 0$ is a local equation of V at p, $N = d$ and $g_d := f_s = f + sl^d$ as defined above.

We remark that the genericity of H_{gen} with respect to $\mathrm{Sing}\, V$ implies that the local polar locus $\Gamma_p(l, f)$ is empty, and therefore Lemma 4.3 holds indeed for our data at p. Moreover, since H_{gen} is generic, Σ_i^c is smooth at $p \in H_{gen} \cap \Sigma_i^c$, thus in formula (14) we have a single term in the sum, a single multiplicity $d_i = 1$, a single transversal Milnor number $\mu_i^{\pitchfork}$, and $b_{n-2}(f_p) = \mu_i^{\pitchfork}$. Therefore formula (14) reduces to:

$$\mu_p(V_s) = (d-1)\mu_i^{\pitchfork}. \tag{15}$$

Taking the sum over all points $p \in \mathrm{Sing}\, V_s$, we get:

$$\sum_{p \in H_{\mathrm{gen}} \cap \mathrm{Sing}\, V} \mu_p(V_s) = (d-1) \sum_i \tilde{d}_i \mu_i^{\pitchfork}.$$

Inserting this in the Dimca-Papadima formula (4) for V_s, we get:

$$\begin{aligned} \mathrm{pol}(V_s) &= (d-1)^n - (d-1) \textstyle\sum_i \tilde{d}_i \mu_i^{\pitchfork} \\ &= (d-1)\big[(d-1)^{n-1} - \textstyle\sum_p \mu_p(V \cap H_{gen})\big] = (d-1)\mathrm{pol}(V \cap H_{gen}). \end{aligned} \tag{16}$$

By applying now the semi-continuity result Proposition 4.1, we obtain (11).

From (16) we also get $\mathrm{pol}(V \cap H_{gen}) = (d-1)^{n-1} - \sum_i \tilde{d}_i \mu_i^{\pitchfork} \geq 0$, hence our claimed inequality (12) follows too. □

5. Classification results, and questions

5.1. Cubic surfaces

We list here the polar degrees of all reduced cubic surfaces, based on the classification done by Bruce and Wall [2] with a singularity theory approach. In case of *isolated singularities*, we give the number of singularities and their types. By using the Dimca-Papadima formula (4) one gets:

$\mathrm{pol}(V) = 8$: the smooth cubic.
$\mathrm{pol}(V) = 7 : A_1$.
$\mathrm{pol}(V) = 6 : 2A_1$ or A_2.
$\mathrm{pol}(V) = 5 : 3A_1$ or A_1A_2 or A_3.
$\mathrm{pol}(V) = 4 : 4A_1$ or $A_2 2A_1$ or A_3A_1 or $2A_2$ or A_4 or D_4.
$\mathrm{pol}(V) = 3 : A_3 2A_1$ or $A_1 2A_2$ or A_4A_1 or A_5 or D_5.
$\mathrm{pol}(V) = 2 : 3A_2$ or A_5A_1 or E_6.
$\mathrm{pol}(V) = 1$: no homaloidal surfaces.
$\mathrm{pol}(V) = 0 : \tilde{E}_6$, which is a cone.

Next, the Bruce-Wall classification [2] of *irreducible* cubics with *nonisolated singularities* contains:

(CN) cone over a nodal curve,

(CC) cone over a cuspidal curve

both with $\mathrm{pol}(V) = 0$ because they are cones, and two other cases, for which we use our formula (5) to compute $\mathrm{pol}(V)$:

(E1) $x_0^2x_2 + x_1^2x_3$; $\mathrm{pol}(V) = 2$
(E2) $x_0^2x_2 + x_0x_1x_3 + x_1^3$; $\mathrm{pol}(V) = 1$,

where the singular set is a projective line with two special points of type D_∞ in the first case, and a single special point of type $J_{2,\infty}$ in the second case.[5]

The irreducible surface (E2) with the $J_{2,\infty}$ point is also mentioned in [3, Section 3.2] as $Y(1, 2)$, related to a rational scroll surface and in [3, Example 4.7] as a sub-Hankel surface.

Among the *reducible* cubics there are only the following three cases with non-zero polar degree:

(QP) The union of a smooth quadratic with a general hyperplane: $\mathrm{pol}(V) = 2$.
(QT) The union of a smooth quadratic with a tangent hyperplane: $\mathrm{pol}(V) = 1$.
(CP) The union of a quadratic cone and a general hyperplane: $\mathrm{pol}(V) = 1$.

For these computations one can use the union formula (17) for the union of two hypersurfaces $V = V_1 \cup V_2 \subset \mathbb{P}^n$:

$$\mathrm{pol}(V_1 \cup V_2) = \mathrm{pol}(V_1) + \mathrm{pol}(V_2) + (-1)^n[\chi(V_1 \cap V_2 \setminus H) - 1], \tag{17}$$

where H is a generic hyperplane with respect to V. This is a consequence of (2) and of the inclusion-exclusion principle for the Euler-characteristic. It was observed in several papers, e.g. [7, Cor. 4.3].

All the other reducible cubics are cones and thus have $\mathrm{pol}(V) = 0$. In particular, there are only three homaloidal cubic surfaces, all with nonisolated singularities.

5.2. Homaloidal singularities with transversal type A_1

A necessary condition for V to be homaloidal is that $\alpha_q(V) = 1$ for any special point $p \in V$, cf Corollary 3.2. This restriction tells for instance that all isolated singularities of V must be of type A_k, as observed in [11] in case of isolated singularities only, but the argument holds at any isolated singular point, cf Corollary 3.2, whatever the other singularities of V might be.

For instance, a hypersurface V with 1-dimensional singular locus can be viewed (by slicing) as a 1-parameter family of hypersurfaces with isolated singularities. Therefore, a natural classification question is: *what are the 1-parameter families of hypersurfaces with isolated singularities with Milnor jumps equal to 1?*

It is known that in certain generic transversal types (e.g. $\tilde{E}_6$, $\tilde{E}_7$, or $\tilde{E}_8$) this jump does not exist. Let us look now to the case of the transversal type A_1, more precisely the subclass of "smooth singular locus and generic transversal type A_1".

Proposition 5.1. *Let $V \subset \mathbb{P}^n$ be a hypersurface with singular locus $\mathrm{Sing}(V) = \Sigma^c \sqcup \Sigma^{\mathrm{is}}$ such that Σ^c is a smooth projective line with transversal type A_1, and Σ^{is} is a finite set of isolated points.*

Then $\alpha_p(V) = 1$ if and only if the hypersurface germ (V, p) is a $J_{k,\infty}$ singularity, i.e. has local equation: $y^2(y - x^k) + z_1^2 + \cdots + z_{n-2}^2 = 0$.

[5] For the type notation, see [17].

In particular, if V is homaloidal, then all its special points on Σ^c must be of type $J_{k,\infty}$, and all its isolated singular points of type A_k.

Proof. The generic transversal type is A_1 by hypothesis, with transversal Milnor number $\mu^{\pitchfork} = 1$. Then by (7) the condition $\alpha_p(V, H) = 1$ is equivalent to $\mu(V \cap H, p) = 2$, which means that the hypersurface germ $(V \cap H, p)$ is an A_2-singularity. One may then apply the local classification of line singularities [17] and obtain, firstly that the local function defining (V, p) is $\mathcal{R}$-equivalent to $y^2 g(x, y) + z_1^2 + \cdots + z_{n-2}^2$, with $g(0, 0) = 0$, and secondly that $y^2 g(x, y)$ is $\mathcal{R}$-equivalent to $y^2(y - x^k)$ for some k. This shows our first claim.

To show our second claim, we apply Corollary 3.2 as in the first paragraph of this subsection, together with the result that we have just proved. □

5.3. On projective cones and other hypersurfaces with polar degree zero

Let $V \subset \mathbb{P}^n$ be defined by $f(x_0, x_1, \cdots, x_n) = g(x_1, \cdots, x_n) = 0$. This is one of the possible definitions of a projective cone V over the hypersurface $V' \subset \mathbb{P}^{n-1}$ given by $g(x_1, \cdots, x_n) = 0$. The point $q = [1 : 0 : \cdots : 0]$ is called the apex of the cone. From the definition (1) it follows that $\mathrm{pol}(V) = 0$.

Let V' have isolated singularities only, with Milnor numbers $\mu_1, \cdots, \mu_r$. Then $\mathrm{Sing}(V) = \cup_i^r \Sigma_i^c$, where each Σ_i^c is a projective line with transversal type μ_i, and such that all these lines intersect only at the apex q.

Let us compute $\mathrm{pol}(V)$ with formula (5). First observe that $g_i = 0$, $\gamma_i = 1$ and $\deg \Sigma_i^c = 1$. There are no isolated critical points. The only special point is the apex q, where a local affine equation is given by $g = 0$. As computed in [19], one has $\chi(\mathcal{A}_q) = (-1)^n((d-1)^n - d\sum_i \mu_i^{\pitchfork})$. We therefore get $\mathrm{pol}(V) = (d-1)^n - d\sum_i \mu_i^{\pitchfork} - (d-1)^n + d\sum_i \mu_i^{\pitchfork} = 0$, which is indeed what was expected since V is a cone.

What can be said about other hypersurfaces with polar degree zero? Historically, Hesse claimed that hypersurfaces with vanishing Hessian are always projective cones. In 1875 Gordan and Noether [8] gave several examples with polar degree zero but not cones. All known examples seem to have a singular locus of dimension at least 2. It follows from the lower bound results [24] that the polar degree zero can not occur if V contains isolated singularities. The following question is open:

Do there exist hypersurfaces with 1-dimensional critical set and polar degree 0?

We have the following partial result, by [24, Corollary 6.9]: *Let $V \subset \mathbb{P}^n$ be a hypersurface which is not a cone and such that* $\mathrm{pol}(V) = 0$. *Then V has no special point. In particular V has no isolated singularity (besides its non-isolated singularities).*

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