

POSITIVE REINFORCED GENERALIZED TIME-DEPENDENT PÓLYA URNS VIA STOCHASTIC APPROXIMATION

WIOLETTA M. RUSZEL AND DEBLEENA THACKER

ABSTRACT. Consider a generalized time-dependent Pólya urn process defined as follows. Let $d \in \mathbb{N}$ be the number of urns/colors. At each time n , we distribute σ_n balls randomly to the d urns, proportionally to f , where f is a valid reinforcement function. We consider a general class of positive reinforcement functions \mathcal{R} assuming some monotonicity and growth condition. The class \mathcal{R} includes convex functions and the classical case $f(x) = x^\alpha$, $\alpha > 1$. The novelty of the paper lies in extending stochastic approximation techniques to the d -dimensional case and proving that eventually the process will fixate at some random urn and the other urns will not receive any balls any more.

1. INTRODUCTION

The classical Pólya urn model with two urns (colors) to which balls are added randomly was introduced by Eggenberger and Pólya in 1923, [14]. Since then, many generalizations and extensions of the classical model have been studied, see e.g. [26] for a survey. One of the fundamental questions is how the composition of the urns will look like and how it depends on the way balls are added as time goes to infinity. There are numerous applications in economics, computer science and biology where the model is better known as *balls and bins model*, [1, 4, 12, 13, 27, 29] to mention a few.

A popular generalization is the *non-linear Pólya urn model* or *balls and bin model with feedback*. The probability of a new ball choosing a bin with x existing balls is proportional to $f(x)$ where f will be referred to as the feedback function, [13]. A common choice of the feedback function is $f(x) = x^\alpha$, $\alpha > 0$.

For the original case $\alpha = 1$, it was proven in [14] that the proportion of balls in each bin converges to a beta-distributed random variable. In the positive feedback regime $\alpha > 1$, which is also referred to as preferential attachment, the authors in [13] proved *dominance*, i.e. almost surely the proportion of each bin converges to a $\{0, 1\}$ -random variable. A stronger result which we call *fixation* or *monopoly* was proven by [20]. Fixation refers to the event that eventually one bin receives all but a finite number of balls. The onset of the time of fixation (speed of convergence towards the stationary distribution) in the positive feedback regime was studied in [24]. In fact, the author in [24] studies the onset of fixation for more general feedback functions satisfying some growth conditions and being perturbations of the canonical case x^α . In particular for that class of feedback function we have that $\sum_{x=1}^{\infty} \frac{1}{f(x)} < \infty$.

Date: February 1, 2022.

2010 Mathematics Subject Classification. Primary: 60F05, 60F10; Secondary: 60G50.

Key words and phrases. generalized Pólya urn models, time-dependent Pólya urn models, positive reinforcement, stochastic approximation, dominance, fixation.

The *negative feedback regime* is characterized by $\alpha < 1$, and in this case the proportion of balls in the urns converges towards the uniform distribution on $\{1, 2, \dots, d\}$ where d is the number of bins, [13]. A time-dependent version of positively reinforced 2-urn models was studied in [25, 30] where at each time n , σ_n balls (satisfying some growth condition) were randomly added to either one urn [25] or independently to both [30]. In both cases there will be dominance. Depending on some growth conditions of σ_n , [30] demonstrates there might be no fixation.

Other generalizations include considering d urns or colors, $d > 2$ and more general addition rules like addition generating matrices and convex resp. concave feedback functions, [2, 3, 21, 22]. The replacement matrix $H = ((H_{ij}))_{ij}$ models the placement of H_{ij} balls to urn j when urn i is chosen. In [22] the authors prove that if the feedback function is strictly concave and H bi-stochastic, then the urn composition converges towards the uniform distribution $(1/d, \dots, 1/d)$. The limiting distribution will not be uniform with probability one if f is convex. Finally they prove in the same paper that for $d = 2$, f convex and H irreducible, the limiting proportion converges towards the equilibrium points of the corresponding mean-field function resulting from the stochastic approximation approach. A concave feedback function, which includes the negative reinforcement regime, tends to equalize the asymptotic distribution of the proportion of different colors, whereas a convex f which includes the positive reinforcement regime tends to amplify the effect of the generating matrix H . In [19], the author proves CLT type results for the proportion vector of colors around the uniform distribution in the negative reinforcement setting, when H is double-stochastic and f Lipschitz.

Urn models with infinitely many colors were treated in [5, 6, 18]. In [6] the authors introduce a class of balanced urn schemes with infinitely many colors indexed by \mathbb{Z}^d where the replacement schemes are given by the transition matrices associated with bounded increment random walks. They show that the urn composition of the n -th selected ball follows a Gaussian distribution. The authors in [23] generalized the possibly infinite space of urns to general Polish spaces and study the asymptotic behaviour of these measure-valued Pólya urn processes. The author in [18] generalizes results obtained in [6] and [23] and studies measure-valued Pólya urn processes under stochastic replacement matrices H .

A simplistic model for the reinforcement of neural connections in the brain using positive reinforced *interacting Pólya urns* was introduced in [16]. The urns/colors represent the edges of a graph. Roughly speaking, one first chooses a random subset of colours (independent of the past) from n colours of balls, and then positively reinforce a colour from this subset. In [16] the stability of equilibria and examples of different graphs were studied. Interesting follow-up research on percolation questions on different positively reinforced tree-like graphs and its application for neuronal connections were studied in [15, 17].

In this article we consider a generalized time-dependent Pólya urn model with d urns, $d \in \mathbb{N}$. More precisely, at each time step, σ_n many balls are added randomly to the d urns, f -proportionally to their weight with instantaneous replacement. We will assume that

$$(i) \sum_{n=1}^{\infty} \frac{\sigma_n}{\sum_{j=1}^n \sigma_j} = \infty \quad \text{and} \quad (ii) \sum_{n=1}^{\infty} \left(\frac{\sigma_n}{\sum_{j=1}^n \sigma_j} \right)^2 < \infty.$$

Condition (i) ensures that $\sigma_n \geq 1$ for all $n \in \mathbb{N}$ hence we keep on adding balls throughout the whole time-evolution, whereas condition (ii) restricts the growth of σ_n . Polynomial growth is allowed but not exponential growth, since it contradicts (ii). This condition is necessary to ensure that the process, seen as a stochastic approximation (SA) scheme is not subjected to a large noise term which will hinder the process to converge. Both conditions appear naturally when using SA techniques from the dynamical systems viewpoint, see e.g. [28] or [11, Section 1]. The class of reinforcement functions \mathcal{R} (see Section 2.1.1 for its definition) is very general and satisfies some natural continuity and monotony conditions. W.l.o.g. we will evaluate f not on the number of balls in an urn but on the proportions. Additionally, we assume that

$$\alpha := \inf_{x \in (0,1)} \frac{x f'(x)}{f(x)} > 1. \quad (1)$$

Note that if f is convex or of the type $f(x) = x^\alpha$, $\alpha > 1$ then (1) is necessarily satisfied but the converse is not necessarily true. A similar condition can be found in [22, 24], in [22] the authors assume additionally that f is convex and H irreducible. Our case is not covered in [22], since taking H to be the identity matrix is not irreducible. We will prove in Theorems 1 and 2 that there is dominance and fixation for this class of general reinforcement functions \mathcal{R} and generalized urn model. Extending ideas from [30] we will prove that infinitely often the process will move away from non-trivial equilibrium points. Using SA techniques, [7, 9, 10] and coupling the process to an appropriate ODE we will prove that the only possible stationary points of the process are the extremal points of the simplex $[0, 1]^d$. In fact, condition (1) ensures that the Jacobian of the ODE at all non-trivial points will have positive eigenvalues. This will imply that the dynamical system is not stable around non-trivial points. The novelty of the paper lies in proving dominance in this general setting where we consider d urns, $d > 2$, f not necessarily convex, and the addition of balls σ_n is variable. We extend SA techniques to this setting. To the best of our knowledge, SA techniques in the positive reinforcement regime were only applied to $d = 2$ case in the literature, e.g. [22].

1.1. Outline of the paper. The structure of the manuscript is as follows. In Section 2 we introduce the model and assumptions on the class reinforcement functions f . The results are presented in Section 3 whereas Section 4 is devoted to their proofs. Finally, in the Appendix A we introduce stochastic approximation techniques and relevant results.

2. MODEL AND DEFINITIONS

2.1. Model. We assume that all random variables are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We consider the following generalization of Pólya urn scheme where colors are indexed by a non-empty finite set $S := \{1, 2, \dots, d\}$, $d \in \mathbb{N}$. For $n \in \mathbb{N}$ we denote the composition of the urn at time n by $U_n := \{U_{n,j}\}_{j \in S}$, where $U_{n,j}$ is the "weight" of the j -th color at time n .

We start with a non-trivial initial composition U_0 , a given non-negative reinforcement function f and a sequence of positive integers $(\sigma_n)_{n \geq 1}$. At every discrete time point $(n + 1)$, σ_{n+1} balls or colors are added f -proportional to their weight,

with instantaneous replacement. That is, given $U_0, U_1, \dots, U_n, (X_i^{(n+1)})_{1 \leq i \leq \sigma_{n+1}}$ are i.i.d. random vectors, such that

$$\mathbb{P}\left(X_i^{(n+1)} = e_j \mid U_0, U_1, \dots, U_n\right) = \frac{f(\theta_{n,j})}{\sum_{k=1}^d f(\theta_{n,k})}, \quad (2)$$

where $e_j, 1 \leq j \leq d$ is the unit vector corresponding to the canonical basis in \mathbb{R}^d and $\theta_n = (\theta_{n,k})_{1 \leq k \leq d}$ is defined by $\theta_{n,k} = \frac{U_{n,k}}{\sum_{j=1}^d U_{n,j}}$. In other words, $\theta_{n,k}$ is the proportion of balls of color k at time n . The urn composition is then updated according to the following rule:

$$U_{n+1} = U_n + \sum_{i=1}^{\sigma_{n+1}} X_i^{(n+1)}. \quad (3)$$

In words, at time $(n+1)$ if the j -th color is selected at the i -th trial for $1 \leq i \leq \sigma_{n+1}$, then we add a single ball of the same color to the urn.

Let us denote by τ_n the total number of balls at time n . Observe that

$$\tau_n = \|U_n\|_1 = \tau_0 + \sum_{j=1}^{n+1} \sigma_j = \sum_{j=1}^d U_{0,j} + \sum_{j=0}^{n-1} \sigma_j,$$

where $\|\cdot\|_1$ is the $\ell^1(\mathbb{N}^d)$ -norm. Using this notation, $\theta_n = \frac{U_n}{\tau_n}$ and $\theta_n \in \{y \in \mathbb{R}_+^d \setminus \{0\} : \sum_{i=1}^d y_i = 1\}$.

Definition 2.1. We call the event \mathcal{D} *dominance* if

$$\mathcal{D} = \{\exists i \in \{1, \dots, d\} \text{ s.t. } \lim_{n \rightarrow \infty} (\theta_{n,1}, \dots, \theta_{n,d}) = e_i\} \quad (4)$$

where e_i is the i -th coordinate vector in \mathbb{R}^d .

Dominance means that eventually the proportion of different colors in the urn becomes trivial, except for a single color.

Definition 2.2. We say the urn model $(U_n)_{n \geq 0}$ *fixates* if almost surely,

$$\mathcal{F} = \{\exists N \geq 1 \text{ and } i \in \{1, \dots, d\} \text{ s.t. for all } n \geq N, U_{n+1,i} = U_{n,i} + \sigma_{n+1}\}. \quad (5)$$

It is important to note here that if the process fixates at some color J out of $\{1, \dots, d\}$ (which is random), then all other colors stop growing after the random time of fixation. It is also clear that if the process fixates, it implies that there is almost surely dominance. The converse is not always true, (see [30] for examples).

2.1.1. *Class of reinforcement functions \mathcal{R} .* For the purpose of this paper, we will assume that $f : [0, 1] \rightarrow \mathbb{R}_+$ satisfies

- (A) f is a strictly non-decreasing and continuous, such that $f(0) = 0$ and $f(1) = 1$.
- (B) $f \in C^1((0, 1))$ and the semi-derivatives $\lim_{x \rightarrow 0^+} f'(x)$ and $\lim_{x \rightarrow 1^-} f'(x)$ exist.
- (C)

$$\alpha := \inf_{x \in (0,1)} \frac{x f'(x)}{f(x)} > 1. \quad (6)$$

The assumption **(A)** that f is non-decreasing, and $f(0) = 0$ and $f(1) = 1$ is a natural assumption that ensures that "the higher the proportion of a color, more likely it is to be chosen". The assumption **(B)** is for technical purposes. Let \mathcal{R} be the class of all functions that satisfy **(A)**, **(B)** and **(C)**.

Examples are:

- (i) $f(x) = x^\alpha$, $\alpha > 1$
- (ii) $f(x) = x^{2+\epsilon}e^{-x+1}$, $\epsilon > 0$

A classical example of such a function is example (i). From assumption **(C)**, it may seem that for all $f \in \mathcal{R}$, f has to be convex. However, this is not the case as we will show in the following counterexample. Let $\epsilon > 0$ and consider example (ii). Then f satisfies trivially assumptions **(A)** and **(B)**. For the third assumption **(C)**, note that

$$f'(x) = (2 + \epsilon - x)x^{1+\epsilon}e^{-x+1}$$

so that $\inf_{x \in (0,1)} \frac{xf'(x)}{f(x)} = 1 + \epsilon > 1$. This function is not convex, indeed

$$f''(x) = (x^2 - 2(2 + \epsilon)x + (1 + \epsilon)(\epsilon))x^\epsilon e^{-x+1}$$

and we see easily see that for ϵ small, e.g. $\epsilon = 0.1$, the second derivative is changing sign for $x \in (0, 1)$.

Lemma 2.3. *Let $f \in \mathcal{R}$. Then*

- (i) f is Lipschitz on $(0, 1)$.
- (ii)

$$\lim_{x \rightarrow 0^+} \frac{f(x)}{x} = f'(0) \geq 0,$$

is well-defined.

- (iii) The map $x \mapsto \frac{f(x)}{x}$ is increasing for all $x \in (0, 1)$.
- (iv) For all $x \in [0, 1]$ we have that $f(x) \leq x^\alpha$, where α is defined in (6).

Proof. (i) follows immediately from assumption **(B)**. For (ii), observe that $f'(0)$ exists finitely is a part of the assumption **(B)**, and $f'(0) \geq 0$ follows from assumption **(A)**. The statement (iii) follows from the observation that derivative of $\frac{f(x)}{x}$ is given by

$$\frac{xf'(x) - f(x)}{x^2} \geq 0,$$

from assumption **(C)**. Finally (iv) follows from **(C)** and

$$\log \left(\frac{f(1)}{f(x)} \right) = \int_x^1 d(\log f(t)) \geq \alpha \log \left(\frac{1}{x} \right)$$

for $x \in (0, 1)$ and α was defined in (6). □

A major disadvantage of the class of functions \mathcal{R} is that it does not include functions that decay exponentially, for example $f(x) = \frac{1}{a-1}(a^x - 1)$ for $a > 0$, as it fails to satisfy **(C)** ($\alpha = 1$ instead of $\alpha > 1$). It is clear that Theorem 1 should hold even for exponentially decaying functions, since similar result is shown in [20], where $\sigma_n \equiv 1$. However, we could not apply the general stochastic approximation techniques for exponentially growing functions and for a general sequence $(\sigma_n)_{n \geq 0}$.

2.1.2. *Connections to Stochastic Approximation Theory.* In this section, we will connect the proportion vector $(\theta_n)_{n \geq 0}$ to a standard form of recursive equations of the stochastic approximations (SA) method, see e.g. [9–11]. One of the most general forms of the recursion equation associated with SA is of the form

$$Y_{n+1} = Y_n + \gamma_{n+1} [H(Y_n, Z_n) + r_{n+1}], \quad Y_0 \in \mathbb{R}^d, \quad n \geq 0, \quad (7)$$

where $(\gamma_n)_{n \geq 0}$ is the sequence of step sizes, $(Z_n)_{n \geq 0}$ is a sequence of i.i.d. random vectors and $(r_n)_{n \geq 0}$ is the sequence of "error" or remainder terms and $H : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is measurable function.

A general practice is to re-write the above equation (7) in the following form

$$Y_{n+1} = Y_n + \gamma_{n+1} [\mathbb{E}[H(Y_n, Z_n) | Y_0, Z_1, \dots, Z_n] + \Delta M_{n+1} + r_{n+1}], \quad (8)$$

where $\Delta M_{n+1} = H(Y_n, Z_n) - \mathbb{E}[H(Y_n, Z_n) | Y_0, Z_1, \dots, Z_n]$.

The advantage of such a representation is that under suitable conditions on H and $(Z_n)_n$, one can relate the asymptotic properties of Y_n to the zeros of the *mean field function*, which we will define and discuss in details for our model.

Recall our basic recursive equation (3). We can re-write this equation as follows

$$\theta_{n+1} = \theta_n + \frac{1}{\tau_{n+1}} \left[\mathbb{E} \left[\sum_{i=1}^{\sigma_{n+1}} X_i^{(n+1)} \mid \mathcal{F}_n \right] - \sigma_{n+1} \theta_n + \Delta M_{n+1} \right], \quad (9)$$

where \mathcal{F}_n is the sigma algebra generated by U_0, U_1, \dots, U_n , and

$$\Delta M_{n+1} = \sum_{i=1}^{\sigma_{n+1}} X_i^{(n+1)} - \mathbb{E} \left[\sum_{i=1}^{\sigma_{n+1}} X_i^{(n+1)} \mid \mathcal{F}_n \right]. \quad (10)$$

Let us define the function $f_d : \mathbb{R}^d \rightarrow \mathbb{R}^d$ as follows $y \mapsto (f(y_1), f(y_2), \dots, f(y_d))$ for any vector $y = (y_1, y_2, \dots, y_d) \in \mathbb{R}^d$ and $f \in \mathcal{R}$. It is easy to see that

$$\mathbb{E} \left[\sum_{i=1}^{\sigma_{n+1}} X_i^{(n+1)} \mid \mathcal{F}_n \right] = \sigma_{n+1} \frac{f_d(\theta_n)}{\|f_d(\theta_n)\|_1}.$$

An immediate consequence of the construction is the following corollary.

Corollary 2.4. *Given \mathcal{F}_n , the d -dimensional random vector $\sum_{i=1}^{\sigma_{n+1}} X_i^{(n+1)}$ follows a Multinomial distribution with parameters $(\sigma_{n+1}, \frac{f_d(\theta_n)}{\|f_d(\theta_n)\|_1})$, i.e. for $x = (x_1, \dots, x_d) \in \mathbb{N}^d$ such that $\sum_{i=1}^d x_i = \sigma_{n+1}$,*

$$\mathbb{P} \left(\sum_{i=1}^{\sigma_{n+1}} X_i^{(n+1)} = x \right) = \frac{\sigma_{n+1}!}{x_1! \dots x_d!} \frac{f(\theta_{n,1})^{x_1}}{\|f_d(\theta_n)\|_1} \dots \frac{f(\theta_{n,d})^{x_d}}{\|f_d(\theta_n)\|_1}.$$

Re-writing (9), we have

$$\begin{aligned} \theta_{n+1} &= \theta_n + \frac{1}{\tau_{n+1}} \left[\sigma_{n+1} \frac{f_d(\theta_n)}{\|f_d(\theta_n)\|_1} - \sigma_{n+1} \theta_n + \Delta M_{n+1} \right] \\ &= \theta_n + \frac{\sigma_{n+1}}{\tau_{n+1}} \left[h(\theta_n) + \frac{1}{\sigma_{n+1}} \Delta M_{n+1} \right], \end{aligned} \quad (11)$$

where we define $h : \mathbb{R}_+^d \setminus \{0\} \rightarrow \mathbb{R}_+^d$ by $y \mapsto \left(\frac{f_d(y)}{\|f_d(y)\|_1} - y \right)$. This function h is well-defined as f is strictly non-decreasing and $f(0) = 0$. h will be the *mean field*

function, and we later see the relations of h to the limit points of the sequence $(\theta_n)_{n \geq 0}$. Comparing (11) with (8), we observe that it is not exactly in the form of the standard recursive equation as in practice in SA, due to the presence of the coefficient $\frac{1}{\sigma_{n+1}}$ which is multiplied with the martingale difference ΔM_{n+1} . We present all necessary results related to SA required for this paper in the Appendix A.

3. RESULTS

In this section we will present our main results.

Theorem 1. *Suppose that $(\theta_n)_{n \geq 0}$ is as in (11). Assume the following conditions:*

- (i) *Let $(\sigma_n)_{n \geq 1}$ be such that $\sum_{n \geq 1} \frac{\sigma_n}{\tau_n} = \infty$ and $\sum_{n \geq 1} \left(\frac{\sigma_n}{\tau_n}\right)^2 < \infty$,*
- (ii) *$f \in \mathcal{R}$,*

then we have that

$$\mathbb{P}(\mathcal{D}) = 1,$$

where \mathcal{D} was defined in Definition 2.1.

Theorem 2. *Under the assumptions of Theorem 1 we have that*

$$\mathbb{P}(\mathcal{F}) = 1, \tag{12}$$

where \mathcal{F} was defined in Definition 2.2.

Remark 3.1. Ideally, Theorem 1 should be stated as a corollary of Theorem 2. However, we state the theorems in reverse order. This is because, we will prove Theorem 2 using Theorem 1.

4. PROOFS

The proof of Theorem 1 will be divided into four parts. The first Lemma 4.1 will identify equilibrium points for the mean-field function h . The *equilibrium set* of a function h is the set

$$\mathcal{E}(h) = \{y \in [0, 1]^d : h(y) = 0\}.$$

We call *trivial equilibrium points* the standard orthonormal basis of \mathbb{R}^d , $\{e_1, \dots, e_d\}$. Lemma 4.2 will ensure that the process $(\theta_n)_{n \geq 0}$ converges and the possible limit points are given by the equilibrium points of the mean-field function h . Proposition 4.3 will show that almost surely the process will infinitely often move away by a small enough distance from the non-trivial equilibrium points. Finally Lemma 4.5 will conclude that the only possible limit points are the trivial equilibrium points.

Lemma 4.1. *Let $f \in \mathcal{R}$. Then the equilibrium set is equal to*

$$\mathcal{E}(h) = \left\{ y \in V : y_j = \frac{1}{d - |I|} \text{ for } j \in I^c \text{ and } y_i = 0 \text{ for } i \in I \right\}, \tag{13}$$

where $I = \{i \in \{1, \dots, d\} : y_i = 0\}$.

Proof. To find the equilibrium points of h , we need to find all solutions $y \in \{y \in \mathbb{R}_+^d \setminus \{0\} : \sum_{i=1}^d y_i = 1\}$ for the set of self-consistency equations given by

$$y_1 = \frac{f(y_1)}{\|f_d(y)\|_1}, \dots, y_d = \frac{f(y_d)}{\|f_d(y)\|_1}. \tag{14}$$

Since $y \in \{y \in \mathbb{R}_+^d \setminus \{0\} : \sum_{i=1}^d y_i = 1\}$, it follows immediately that there exists some i , such that $y_i > 0$, and thus $\|f_d(y)\|_1 > 0$. It is clear that the unit coordinate vectors e_i for $1 \leq i \leq d$ are equilibrium points by assumption **(A)**.

So let us assume that $y_1, \dots, y_k \neq 0$ for some $2 \leq k \leq d$. Observe that to prove (13), it is enough to show $y_1 = y_2 = \dots = y_k = \frac{1}{k}$. The previous set of equations (14) can be written as

$$\|f_d(y)\|_1 = \frac{f(y_1)}{y_1}, \dots, \|f_d(y)\|_1 = \frac{f(y_k)}{y_k}, \quad (15)$$

and $f(y_i) = 0$ for all $(k+1) \leq i \leq d$ by assumption **(A)**. W.l.o.g., we may assume that $y_1 > y_2$. Then by Lemma 2.3 (iii), we know that $\frac{f(y_1)}{y_1} > \frac{f(y_2)}{y_2}$, which is a contradiction to (15). This shows that $y_1 = y_2$, and in particular, $y_1 = y_2 = \dots = y_k$ and also $y_1 = y_2 = \dots = y_k = \frac{1}{k}$, since $\sum_{i=1}^k y_i = 1$. \square

Note that the statement of Lemma 4.1 is similar to Proposition 2.4 from [22]. They prove the statement under the assumption that f is concave or convex, which we do not need here. The following lemma is a consequence of Corollary A.6 from the appendix and Lemma 4.1.

Lemma 4.2. *Let $(\theta_n)_{n \geq 0}$ be a stochastic process defined by the recursion*

$$\theta_{n+1} = \theta_n + \frac{\sigma_{n+1}}{\tau_{n+1}} \left[h(\theta_n) + \frac{1}{\sigma_{n+1}} \Delta M_{n+1} \right].$$

Then

$$\theta_n \xrightarrow{a.s.} \theta^*$$

as $n \rightarrow \infty$, where $\theta^* \in \mathcal{E}(h)$.

The following proposition is actually a stronger version of Proposition 4.1 of [30], where we show that for any $1 < k \leq d$, $e = \frac{1}{k}(e_1 + e_2 + \dots + e_k)$ is likely to be unstable. The case $d = 2$ is covered in Proposition 4.1 of [30], so we extend this to $d \geq 3$.

Proposition 4.3. Let $(\sigma_n)_n$ and $(\tau_n)_n$ be as in the assumptions of Theorem 1. For any $e = \frac{1}{k}(e_1 + e_2 + \dots + e_k)$, where $1 < k \leq d$, there exists $\delta_n > 0$, such that

$$(i) \quad \lim_{n \rightarrow \infty} \delta_n = 0, \quad (16)$$

$$(ii) \quad \sum_{n \geq 1} \frac{\delta_n \sigma_{n+1}}{\tau_{n+1}} < \infty. \quad (17)$$

(iii) For any such sequence $(\delta_n)_{n \geq 0}$

$$\mathbb{P}((\theta_{n,1}, \theta_{n,2}, \dots, \theta_{n,d-1}, \theta_{n,d}) \in B_d^c(e, \delta_n) \text{ i.o.}) = 1, \quad (18)$$

where for $a \in \mathbb{R}^d$ and $r > 0$, $B_d(a, r) := \{x \in \mathbb{R}^d : \|x - a\|_2 \leq r\}$.

Remark 4.4. By symmetry, it is clear from the above Proposition 4.3 that any $e \neq e_i$, for $1 \leq i \leq d$ is likely to be unstable as an equilibrium point and therefore, the only possibility is that $\theta_n \rightarrow e_i$ almost surely as $n \rightarrow \infty$, for some $1 \leq i \leq d$.

Proof. (i)+(ii): Recall that by our choice, we have chosen $(\sigma_n)_n$ and $(\tau_n)_n$, such that, $\sum_n \frac{\sigma_n}{\tau_n} = \infty$ and $\sum_n \frac{\sigma_n^2}{\tau_n^2} < \infty$, therefore there always exists δ_n that satisfies $\sum_{n \geq 1} \frac{\delta_n \sigma_{n+1}}{\tau_n + \frac{1}{2} \sigma_{n+1}} < \infty$, which implies that $\sum_{n \geq 1} \frac{\delta_n \sigma_{n+1}}{\tau_{n+1}} < \infty$ as $\tau_{n+1} = \tau_n + \sigma_{n+1}$. We can simply choose $0 \leq \delta_n \leq \frac{\sigma_{n+1}}{\tau_n + 2\sigma_{n+1}}$. It is immediately clear that $\lim_{n \rightarrow \infty} \delta_n = 0$.

(iii): For $e = \frac{1}{k}(e_1 + e_2 + \dots + e_k)$, where $1 < k \leq d$, define

$$\mathcal{H}_m(e) := \{(\theta_{n,1}, \theta_{n,2}, \dots, \theta_{n,d-1}, \theta_{n,d}) \in B_d(e, \delta_n) \text{ for all } n \geq m\}.$$

Since $\mathcal{H}_m(e)$ is non-increasing, therefore it is enough to show that

$$\lim_{m \rightarrow \infty} \mathbb{P}(\mathcal{H}_m(e)) = 0$$

to show (18).

Recall that from (3), we have

$$\theta_{n+1,1} = \frac{\tau_n \theta_{n,1} + B_{n+1,1}}{\tau_{n+1}},$$

where given \mathcal{F}_n , $B_{n+1,1}$ follows $\text{Bin}(\sigma_{n+1}, \Psi(\theta_{n,1}))$, Ψ is short for

$$\Psi(x_i) := \frac{f(x_i)}{\|f_d(x)\|_1}, \quad (19)$$

and $x = (x_1, \dots, x_d) \in \mathbb{R}_+^d$, $f \in \mathcal{R}$. Re-write the above equation as

$$\theta_{n+1,1} = \frac{\tau_n \theta_{n,1} + \sigma_{n+1} \Psi(\theta_{n,1}) + \epsilon_n \sqrt{\sigma_{n+1} \Psi(\theta_{n,1})(1 - \Psi(\theta_{n,1}))}}{\tau_{n+1}}, \quad (20)$$

where $\epsilon_n = \frac{B_{n+1,1} - \sigma_{n+1} \Psi(\theta_{n,1})}{\sqrt{\sigma_{n+1} \Psi(\theta_{n,1})(1 - \Psi(\theta_{n,1}))}}$. Let $\Psi_d : \mathbb{R}^d \rightarrow \mathbb{R}^d$ by $y \mapsto (\Psi(y_1), \dots, \Psi(y_d))$.

By Taylor's theorem, we have for any $x \in \{y \in \mathbb{R}_+^d \setminus \{0\} : \sum_{i=1}^d y_i = 1\}$

$$\Psi_d(x) - e = \sum_{i=1}^{d-1} \frac{\partial \Psi_d(\xi_x)}{\partial x_i} (x_i - \tilde{e}_i), \quad (21)$$

where ξ_x lies on the straight line joining x and $e = (\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_d) \in \mathcal{E}(h)$, since $\Psi(\theta_{n,1}) = \tilde{e}_1$. Re-writing (20), using the Taylor expansion, we get

$$\begin{aligned} \theta_{n+1,1} - \tilde{e}_1 = & \frac{\tau_n}{\tau_{n+1}} (\theta_{n,1} - \tilde{e}_1) + \frac{\sigma_{n+1}}{\tau_{n+1}} \sum_{i=1}^{d-1} \frac{\partial \Psi_d(\xi_{\theta_n})}{\partial x_i} (\theta_{n,i} - \tilde{e}_i) \\ & + \frac{1}{\tau_{n+1}} \epsilon_n \sqrt{\sigma_{n+1} \Psi(\theta_{n,1})(1 - \Psi(\theta_{n,1}))}. \end{aligned} \quad (22)$$

$$\theta_{n+1,1} - \tilde{e}_1 = \kappa_n(\theta_n) (\theta_{n,1} - \tilde{e}_1) + Q_n(\theta_n) + \frac{1}{\tau_{n+1}} \epsilon_n \sqrt{\sigma_{n+1} \Psi(\theta_{n,1})(1 - \Psi(\theta_{n,1}))}, \quad (23)$$

where $\kappa_n(\theta_n) = \left(\frac{\tau_n}{\tau_{n+1}} + \frac{\sigma_{n+1}}{\tau_{n+1}} \frac{\partial \Psi_d(\xi_{\theta_n})}{\partial x_1} \right)$, and

$$Q_n(\theta_n) = \frac{\sigma_{n+1}}{\tau_{n+1}} \sum_{i=2}^{d-1} \frac{\partial \Psi_d(\xi_{\theta_n})}{\partial x_i} (\theta_{n,i} - \tilde{e}_i). \quad (24)$$

Re-iterating the above equation, we obtain

$$\begin{aligned} \theta_{n,1} - \tilde{e}_1 &= \prod_{j=m}^{n-1} \kappa_j(\theta_j) \left[\theta_{m,1} - \tilde{e}_1 + \sum_{j=m}^{n-1} \frac{1}{\left(\prod_{l=m}^j k_l(\theta_l)\right)} Q_j(\theta_j) \right] \\ &\quad + \prod_{j=m}^{n-1} \kappa_j(\theta_j) \sum_{j=m}^{n-1} \frac{1}{\left(\prod_{l=m}^j k_l(\theta_l)\right)} \epsilon_j \frac{\sqrt{\sigma_{j+1} \Psi(\theta_{j,1})(1 - \Psi(\theta_{j,1}))}}{\tau_{j+1}}, \end{aligned}$$

By continuity of the partial derivatives of Ψ_d , for $x \in B_{d-1}(e, \delta_n)$ for all n (large enough), we have $\frac{\partial \Psi_d(\xi_x)}{\partial x_i} = \frac{\partial \Psi_d(e)}{\partial x_i} + \mathcal{O}(\delta_n)$.

Step I: In this step we analyse the asymptotic behavior of $\prod_{j=1}^n k_l(\theta_j)$. Observe that at an equilibrium point $e = (\tilde{e}_1, \dots, \tilde{e}_d)$

$$\frac{\partial \Psi_d(e)}{\partial x_1} = \frac{f'(\tilde{e}_1)}{f(\tilde{e}_1) + \dots + f(\tilde{e}_d)} - \frac{f(\tilde{e}_1)(f'(\tilde{e}_1) - f'(1 - \tilde{e}_1 - \tilde{e}_2 - \dots - \tilde{e}_{d-1}))}{(f(\tilde{e}_1) + \dots + f(\tilde{e}_d))^2}.$$

Therefore,

$$\frac{\partial \Psi_d(e)}{\partial x_1} = \begin{cases} \alpha(\tilde{e}_1), & \text{if } \tilde{e}_1 = \tilde{e}_d, \\ \alpha(\tilde{e}_1) - \frac{f(\tilde{e}_1)(f'(\tilde{e}_1) - f'(0))}{(f(\tilde{e}_1) + \dots + f(\tilde{e}_d))^2}, & \text{if } \tilde{e}_d = 0. \end{cases} \quad (25)$$

From Assumption **(C)**, (15) and the observation that $f'(0) \geq 0$, we have

$$\frac{\partial \Psi_d(e)}{\partial x_1} \begin{cases} = \alpha(\tilde{e}_1) > 1, & \text{if } \tilde{e}_1 = \tilde{e}_d, \\ \geq (1 - \tilde{e}_1) \alpha(\tilde{e}_1), & \text{if } \tilde{e}_d = 0, \end{cases} \quad (26)$$

where $\alpha(x) := \frac{x f'(x)}{f(x)}$ and $f \in \mathcal{R}$. Therefore, on \mathcal{H}_m we obtain for $e = \frac{1}{k}(e_1 + e_2 + \dots + e_k)$

$$\kappa_n(\theta_n) = \begin{cases} \frac{\tau_n}{\tau_{n+1}} + \frac{\sigma_{n+1}}{\tau_{n+1}} \left(\frac{k-1}{k} \alpha(\tilde{e}_1) + \mathcal{O}(\delta_n) \right), & \text{if } 1 < k \leq (d-1), \\ \frac{\tau_n}{\tau_{n+1}} + \frac{\sigma_{n+1}}{\tau_{n+1}} (\alpha(\tilde{e}_1) + \mathcal{O}(\delta_n)), & \text{if } k = d. \end{cases}$$

Writing $\pi_{m,n-1}(k) := \prod_{j=m}^{n-1} \left(\frac{\tau_j + \beta_k \sigma_{j+1}}{\tau_{j+1}} \right)$

$$\prod_{j=m}^{n-1} \kappa_j(\theta_j) = \pi_{m,n-1}(k) \prod_{j=m}^{n-1} \left(1 + \frac{\mathcal{O}(\delta_j) \sigma_{j+1}}{\tau_j + \beta_k \sigma_{j+1}} \right),$$

where $\beta_k = \frac{k-1}{k} \alpha(\tilde{e}_1)$, if $1 < k \leq d-1$, and $\beta_k = \alpha(\tilde{e}_1)$ for $k = d$.

$$\prod_{j=m}^{n-1} \left(1 + \frac{\mathcal{O}(\delta_j) \sigma_{j+1}}{\tau_j + \beta_k \sigma_{j+1}} \right) = \exp \left(\sum_{j=m}^{n-1} \frac{\mathcal{O}(\delta_j) \sigma_{j+1}}{\tau_j + \beta_k \sigma_{j+1}} \right).$$

Since $1 \geq (1 - \frac{1}{k}) \geq \frac{1}{2}$ and $\alpha(\tilde{e}_1) > 1$, by our choice of δ_j , in (17), $\sum_j \frac{\delta_j}{\tau_j + \beta_k \sigma_{j+1}} < \infty$. Hence, for all m large enough

$$\prod_{j=m}^{n-1} \left(1 + \frac{\mathcal{O}(\delta_j) \sigma_{j+1}}{\tau_j + \beta_k \sigma_{j+1}} \right) = (1 + o(1)).$$

Therefore, for all m large enough and on \mathcal{H}_m , we have

$$\prod_{j=m}^{n-1} \kappa_j(\theta_j) = \pi_{m,n-1}(k)(1 + o(1)). \quad (27)$$

Step II: In this step we analyse the sum over $Q_j(\theta_j)$, where Q_j was defined in (24).

From our Assumption **(B)**, for $2 \leq i \leq k$, we have $\frac{\partial \Psi_d(\xi_x)}{\partial x_i} = \frac{\partial \Psi_d(\epsilon)}{\partial x_i} + \mathcal{O}(\delta_n)$ is bounded for all $n \geq m$.

On \mathcal{H}_m ,

$$Q_j(\theta_j) = \mathcal{O}(\delta_j) \frac{\sigma_{j+1}}{\tau_{j+1}}. \quad (28)$$

Recall that in **Step I** we proved that $\prod_{j=m}^l \kappa_j(\theta_j) = \pi_{m,l}(k)(1 + o(1))$, where $\pi_{m,l}(k) = \prod_{j=m}^l \left(\frac{\tau_j + \beta_k \sigma_{j+1}}{\tau_{j+1}} \right)$.

Case i: When $k = d$,

$$\pi_{m,l}(k) = \exp \left(\sum_{j=m+1}^l \log \left(1 + (\alpha(\tilde{\epsilon}_1) - 1) \frac{\sigma_{j+1}}{\tau_{j+1}} \right) \right) \rightarrow \infty,$$

as $l \rightarrow \infty$ since $\alpha(\tilde{\epsilon}_1) > 1$ and $\sum_j \frac{\sigma_j}{\tau_j} = \infty$. Therefore,

$$\sum_{j=m}^{n-1} \frac{1}{\left(\prod_{l=m}^j k_l(\theta_l) \right)} Q_j(\theta_j) = (1 + o(1)) \sum_{j=m}^{n-1} \exp \left(- \sum_{j=m+1}^l (\alpha(\tilde{\epsilon}_1) - 1) \frac{\sigma_{j+1}}{\tau_{j+1}} \right) \frac{\sigma_{j+1}}{\tau_{j+1}} \mathcal{O}(\delta_j) < \infty,$$

by our choice of δ_j .

Case ii: When $1 < k \leq d-1$. $\pi_{m,l}(k) = \exp \left(\sum_{j=m+1}^l \log \left(1 + \left(\frac{k-1}{k} \alpha(\tilde{\epsilon}_1) - 1 \right) \frac{\sigma_{j+1}}{\tau_{j+1}} \right) \right) = \exp \left(\sum_{j=m+1}^l \left(\frac{k-1}{k} \alpha(\tilde{\epsilon}_1) - 1 \right) \frac{\sigma_{j+1}}{\tau_{j+1}} \right) + o(1)$ by the assumptions of Theorem 1.

If $\alpha(\tilde{\epsilon}_1) \geq \frac{k}{k-1}$, then it follows similar to Case i, that

$$\begin{aligned} & \sum_{j=m}^{n-1} \frac{1}{\left(\prod_{l=m}^j k_l(\theta_l) \right)} Q_j(\theta_j) \\ &= (1 + o(1)) \sum_{j=m}^{n-1} \exp \left(- \sum_{j=m+1}^l \left(\frac{k-1}{k} \alpha(\tilde{\epsilon}_1) - 1 \right) \frac{\sigma_{j+1}}{\tau_{j+1}} \right) \frac{\sigma_{j+1}}{\tau_{j+1}} \mathcal{O}(\delta_j) < \infty. \end{aligned}$$

If $\alpha(\tilde{\epsilon}_1) < \frac{k}{k-1}$, then because of our choice of δ_j for all m, n (large enough)

$$\begin{aligned} & \left(\prod_{j=m}^{n-1} \kappa_j(\theta_j) \right) \sum_{j=m}^{n-1} \frac{1}{\left(\prod_{l=m}^j k_l(\theta_l) \right)} Q_j(\theta_j) \\ &= \sum_{j=m}^{n-1} \left(\prod_{l=j+1}^{n-1} \kappa_l(\theta_l) \right) Q_j(\theta_j) \\ &= (1 + o(1)) \sum_{j=m}^{n-1} \exp \left(\sum_{l=j+1}^{n-1} \left(\frac{k-1}{k} \alpha(\tilde{\epsilon}_1) - 1 \right) \frac{\sigma_{l+1}}{\tau_{l+1}} \right) \mathcal{O}(\delta_j) \frac{\sigma_{j+1}}{\tau_{j+1}} < \epsilon, \end{aligned} \quad (29)$$

since $\frac{k-1}{k} \alpha(\tilde{\epsilon}_1) < 1$, $\sum_j \frac{\sigma_{j+1}}{\tau_{j+1}} = \infty$ and $\sum_j \delta_j \frac{\sigma_{j+1}}{\tau_{j+1}} < \infty$.

Step 3: In this step we will explore the last part of the summand in

$\sum_{j=m}^{n-1} \frac{1}{\left(\prod_{l=m}^j k_l(\theta_l) \right)} \epsilon_j \frac{\sqrt{\sigma_{j+1} \Psi(\theta_{j,1})(1 - \Psi(\theta_{j,1}))}}{\tau_{j+1}}$. The argument is similar to that of Proposition 4.1 in [30], so we present only those details that are crucial and slightly

different from those in [30]. From Step 1, and $\Psi(\theta_{j,1}) = \frac{1}{k} + \mathcal{O}(\delta_j)$, we get for appropriate constants c_k

$$\frac{1}{\left(\prod_{l=m}^j k_l(\theta_l)\right)} \epsilon_j \frac{\sqrt{\sigma_{j+1} \Psi(\theta_{j,1})(1 - \Psi(\theta_{j,1}))}}{\tau_{j+1}} = (c_k + o(1)) \epsilon_j \frac{\sqrt{\sigma_{j+1}}}{\pi_{m,j} \tau_{j+1}}.$$

We know that if $X \sim \text{Bin}(n, p)$, then

$$\mathbb{E} \left[\exp \left(it \frac{X - np}{\sqrt{np(1-p)}} \right) \right] = \left(1 - \frac{t^2}{2n} + \mathcal{O} \left(\frac{t^3}{n^{\frac{3}{2}}} \right) \right)^n = \exp \left(-\frac{t^2}{2} + \mathcal{O} \left(\frac{t^3}{\sqrt{n}} \right) \right), \text{ when } |t| < \delta \text{ for some } \delta > 0. \text{ Therefore,}$$

$$\mathbb{E} [\exp(it\epsilon_j) \mid \mathcal{F}_{j-1}] = \exp \left(-\frac{t^2}{2} + \mathcal{O} \left(\frac{t^3}{\sqrt{\sigma_{j+1}}} \right) \right),$$

since given \mathcal{F}_{j-1} , $\epsilon_j \sim \text{Bin}(\sigma_j, \Psi(\theta_{j-1,1}))$. And hence, we have by tower-property

$$\mathbb{E} \left[\exp \left(it \sum_{j=m}^{n-1} \epsilon_j \right) \middle| \mathcal{F}_{m-1} \right] = \exp \left(\sum_{j=m}^{n-1} \frac{-t^2}{2} + \mathcal{O} \left(\frac{t^3}{\sqrt{\sigma_{j+1}}} \right) \right).$$

Hence, we can write

$$\mathbb{E} \left[\exp \left(\frac{it}{\mu_{m,n}} \sum_{j=m}^{n-1} \epsilon_j \right) \middle| \mathcal{F}_{m-1} \right] = \exp \left(-\frac{t^2}{2} + \mathcal{O} \left(\frac{t^3}{\mu_{m,n}^3} \sum_{j=m}^{n-1} \frac{\sigma_{j+1}}{\pi_{m,j}^3 \tau_{j+1}^3} \right) \right), \quad (30)$$

where $\mu_{m,n} := \sum_{j=m}^{n-1} \frac{\sigma_{j+1}}{\pi_{m,j}^2 \tau_{j+1}^2}$.

For $e = \frac{1}{d}(e_1 + e_2 + \dots + e_d)$, it follows that $\frac{1}{\mu_{m,n}^3} \sum_{j=m}^{n-1} \frac{\sigma_{j+1}}{\pi_{m,j}^3 \tau_{j+1}^3} \rightarrow 0$ as $m, n \rightarrow \infty$.

When $e = \frac{1}{k}(e_1 + e_2 + \dots + e_k)$ for $1 < k \leq d-1$, and $\sup_n \sigma_n = \infty$, exactly same argument as in [30] gives us $\frac{1}{\mu_{m,n}^3} \sum_{j=m}^{n-1} \frac{\sigma_{j+1}}{\pi_{m,j}^3 \tau_{j+1}^3} \rightarrow 0$ as $m, n \rightarrow \infty$.

For $e = \frac{1}{k}(e_1 + e_2 + \dots + e_k)$ for $1 < k \leq d-1$, and $\sup_n \sigma_n < \infty$, we observe that if $\alpha(\tilde{e}_1) > \frac{k}{k-1}$ then similar argument as in [30] works.

Therefore, the only case we need to discuss in details is for $e = \frac{1}{k}(e_1 + e_2 + \dots + e_k)$ for $1 < k \leq d-1$, and $\sup_n \sigma_n < \infty$, and $\alpha \leq \frac{k}{k-1}$.

It is easy to see that

$$\prod_{j=m}^{n-1} \kappa_j(\theta_j) = \exp \left(- \sum_{j=m+1}^{n-1} \left(\frac{k-1}{k} \alpha - 1 + \mathcal{O}(\delta_j) \right) \frac{\sigma_{j+1}}{\tau_{j+1}} \right) \rightarrow 0, \text{ as } m, n \rightarrow \infty, \quad (31)$$

since $\lim_{n \rightarrow \infty} \delta_n = 0$, and $\sum_j \frac{\sigma_j}{\tau_j} = \infty$.

$$\begin{aligned} & \mathbb{E} \left[\exp \left(\frac{it}{\Gamma_{m,n-1}} \sum_{j=m}^{n-1} \left(\prod_{l=j+1}^{n-1} k_l(\theta_l) \right) \epsilon_j \frac{\sqrt{\sigma_{j+1}}}{\tau_{j+1}} \right) \middle| \mathcal{F}_{m-1} \right] \\ &= \exp \left(-\frac{t^2}{2} + \mathcal{O}(1) \frac{t^3}{\Gamma_{m,n-1}^3} \sum_{j=m}^{n-1} \left(\prod_{l=j+1}^n \kappa_l(\theta_l) \right)^3 \frac{\sigma_{j+1}}{\tau_{j+1}^3} \right), \end{aligned} \quad (32)$$

where $\Gamma_{m,n-1} := \left[\sum_{j=m}^{n-1} \left(\prod_{l=j+1}^{n-1} \kappa_l(\theta_l) \right)^2 \frac{\sigma_{j+1}}{\tau_{j+1}^2} \right]^{\frac{1}{2}}$.

Since $\sup_n \sigma_n < \infty$, $j \leq \tau_j \leq Cj$ for suitable constant $C > 0$. From (31), we get

$$\sum_{j=m}^{n-1} \left(\prod_{l=j+1}^{n-1} \kappa_l(\theta_l) \right)^2 \frac{\sigma_{j+1}}{\tau_{j+1}^2} = \mathcal{O}(1) \sum_{j=m}^{n-1} \frac{1}{j^2} = \mathcal{O}(1) \left(\frac{1}{m} - \frac{1}{n} \right) \quad (33)$$

By similar arguments,

$$\sum_{j=m}^{n-1} \left(\prod_{l=j+1}^{n-1} \kappa_l(\theta_l) \right)^3 \frac{\sigma_{j+1}}{\tau_{j+1}^3} = \mathcal{O}(1) \sum_{j=m}^{n-1} \frac{1}{j^3} = \mathcal{O}(1) \left(\frac{1}{m^2} - \frac{1}{n^2} \right) \quad (34)$$

Therefore, as $m \rightarrow \infty$,

$$\frac{1}{\Gamma_{m,n-1}^3} \sum_{j=m}^{n-1} \left(\prod_{l=j+1}^n \kappa_l(\theta_l) \right)^3 \frac{\sigma_{j+1}}{\tau_{j+1}^3} = \mathcal{O}(1) \frac{1}{\sqrt{m}} \rightarrow 0. \quad (35)$$

For this case, choose the subsequence $n_m = 2m$ and choose $\delta_n = \mathcal{O}(1) \frac{1}{n^{\frac{1}{2}+\delta}}$, for some $\delta > 0$. Finally from all previous observations,

$$\mathbb{P}(\mathcal{H}_m(e)) \leq \mathbb{P}(\Gamma_{m,n_m} N \in [-\delta_{n_m}, \delta_{n_m}]) \rightarrow 0 \text{ as } m \rightarrow \infty, \quad (36)$$

since $\frac{\delta_{n_m}}{\Gamma_{m,n_m}} = \mathcal{O}(1) \frac{1}{m^\delta} \rightarrow 0$ as $m \rightarrow \infty$. □

Proposition 4.5. Let $(\theta_n)_{n \geq 0}$ be the process defined recursively in (11). Then

$$\theta_n \xrightarrow{a.s.} e^*,$$

as $n \rightarrow \infty$ where $e^* \in \{e_1, \dots, e_d\}$ are the trivial equilibrium points.

The proof of this proposition is similar to parts of Theorem 1.2. from [30] adapted appropriately for the class of reinforcement functions \mathcal{R} , the main difference is in the final argument leading upto the conclusion. We present all details for the sake of completeness.

Proof. For each $n \in \mathbb{N}$ and $i \in \{1, \dots, d\}$ we have that

$$\theta_{n+1,i} = \frac{\tau_n \theta_{n,i} + B_{n+1,i}}{\tau_{n+1}} \quad (37)$$

where $B_{n,i} \sim \text{Bin}(\sigma_n, \Psi(\theta_{n,i}))$ given \mathcal{F}_n , where Ψ was defined in (19). We can express the system of equations for $i \in \{1, \dots, d\}$ by

$$\begin{aligned} \theta_{n+1,i} &= \theta_{n,i} + \frac{\sigma_{n+1}}{\tau_{n+1}} \left((\Psi(\theta_{n,i}) - \theta_{n,i}) + \frac{1}{\sigma_{n+1}} (B_{n+1,i} - \sigma_{n+1} \Psi(\theta_{n,i})) \right) \\ &= \theta_{n,i} + \frac{B_{n+1,i} - \sigma_{n+1} \Psi(\theta_{n,i})}{\tau_{n+1}} - \frac{\sigma_{n+1}}{\tau_{n+1}} (\theta_{n,i} - \Psi(\theta_{n,i})). \end{aligned} \quad (38)$$

For every $\eta \geq 0$, we have by the above recursive relations that

$$\theta_{n+\eta,i} = \theta_{n,i} + M_{n,i}(\eta) - R_{n,i}(\eta) \quad (39)$$

where

$$M_{n,i}(\eta) = \sum_{j=\eta+1}^{n+\eta} \frac{B_{j,i} - \sigma_j \Psi(\theta_{j-1,i})}{\tau_j}$$

for $i = 1, \dots, d$ and

$$R_{n,i}(\eta) = \sum_{j=\eta+1}^{\eta+n} \frac{\sigma_j}{\tau_j} (\theta_{j-1,i} - \Psi(\theta_{j-1,i})).$$

Then the vector $M_n(\eta) = (M_{n,1}(\eta), \dots, M_{n,d}(\eta))$ is a martingale w.r.t. $(\mathcal{F}_{n+\eta})_n$, since $\mathbb{E}[B_{n+1,i} | \mathcal{F}_n] = \sigma_{n+1} \Psi(\theta_{n,i})$. Note that up to now, we did not use the specific definition of η , which we will define later. For simplicity of notation, we write $M_{n,i}$ instead of $M_{n,i}(\eta)$. Let us show that the martingale is bounded in $L^2(\mathbb{R}^d)$. Note that we can write

$$M_{n,i} = M_{n-1,i} + \frac{B_{n+\eta,i} - \sigma_{n+\eta} \Psi(\theta_{n+\eta-1,i})}{\tau_{n+\eta}}$$

and further estimate

$$\begin{aligned} \mathbb{E}(\langle M_n, M_n \rangle | \mathcal{F}_\eta) &= \sum_{i=1}^d \mathbb{E}((M_{n,i})^2 | \mathcal{F}_\eta) \\ &= \sum_{i=1}^d \mathbb{E}((M_{n-1,i})^2 | \mathcal{F}_\eta) + \mathbb{E} \left(\left(\frac{B_{n+\eta,i} - \sigma_{n+\eta} \Psi(\theta_{n+\eta-1,i})}{\tau_{n+\eta}} \right)^2 \middle| \mathcal{F}_\eta \right) \\ &\quad + 2 \sum_{i=1}^d \mathbb{E} \left(\left[\frac{B_{n+\eta,i} - \sigma_{n+\eta} \Psi(\theta_{n+\eta-1,i})}{\tau_{n+\eta}} \right] M_{n-1,i} \middle| \mathcal{F}_\eta \right) \\ &= \sum_{i=1}^d \mathbb{E}((M_{n-1,i})^2 | \mathcal{F}_\eta) + \frac{1}{\tau_{n+\eta}^2} \mathbb{E} \left(\sigma_{n+\eta} \Psi(\theta_{n+\eta-1,i}) (1 - \Psi(\theta_{n+\eta-1,i})) \middle| \mathcal{F}_\eta \right) \end{aligned} \quad (40)$$

Since $\sigma_{n+\eta} \Psi(\theta_{n+\eta-1,i}) (1 - \Psi(\theta_{n+\eta-1,i})) \leq \sigma_{n+\eta}$ and by recursivity we can estimate

$$\mathbb{E}(\langle M_n, M_n \rangle | \mathcal{F}_\eta) \leq d \sum_{j=\eta+1}^{\eta+n} \frac{\sigma_j}{\tau_j^2}.$$

$$\sum_{j=\eta+1}^{\eta+n} \frac{\sigma_j}{\tau_j^2} = \sum_{j=\eta+1}^{\eta+n} \int_{\tau_{j-1}}^{\tau_j} \frac{1}{\tau_j^2} dx \leq \sum_{j=\eta+1}^{\infty} \int_{\tau_{j-1}}^{\tau_j} \frac{1}{x^2} dx = \int_{\tau_\eta}^{\infty} \frac{1}{x^2} dx = \frac{1}{\tau_\eta}, \quad (41)$$

so that M_n is bounded in $L^2(\mathbb{R}^d)$:

$$\mathbb{E}(\mathbb{E}(\langle M_n, M_n \rangle | \mathcal{F}_\eta)) = \mathbb{E}(\langle M_n, M_n \rangle) \leq \frac{d}{\tau_\eta} \quad (42)$$

which implies that M_n converges almost surely and in $L^2(\mathbb{R}^d)$ to some $M_\infty \in L^2(\mathbb{R}^d)$.

Observe that there exists a sequence $(\delta_n)_{n \geq 0}$ (see Lemma 5.1 from [30]), such that, it satisfies (i) and (ii) from Proposition 4.3, and

$$\lim_{n \rightarrow \infty} \frac{1}{\delta_n^2 \tau_n} = 0.$$

For any $\epsilon > 0$, choose $N_1 > 0$, such that, for all $n \geq N_1$

$$\mathbb{P} \left(M_{n,1}(N_1) \geq \frac{\delta_{N_1}}{2} \right) \leq \frac{4d}{\delta_{N_1}^2 \tau_{N_1}} \leq \epsilon. \quad (43)$$

By Proposition 4.3, for δ_n as in (43) W.l.o.g., we may assume that $\mathbb{P}(N_2 < \infty) > 0$, where $N_2 := \inf \{n : \theta_{n,1} < \frac{1}{k} - \delta_n\}$. Choose $\eta = \max\{N_1, N_2\}$. Henceforth, we will work with $M_{n,1}(\eta)$ and $R_{n,1}(\eta)$ for this choice of η . As before, we write $M_{n,1}$ and $R_{n,1}$ for notational simplicity.

Lemma 4.2 yields that $\theta_n \rightarrow \theta^*$ as $n \rightarrow \infty$, hence together with $M_n \rightarrow M_\infty$ a.s., we deduce that R_n has to converge as well. It remains to show that $\theta^* \neq e$ where $e = \frac{1}{k}(e_1 + \dots + e_k)$ and $2 \leq k \leq d$. Assume that $\theta^* = e$. Recall that $\Psi(\theta_i^*) = \frac{1}{k}$ for $1 \leq i \leq k$, and $\Psi(\theta_i^*) = 0$ for all other i . Since $\Psi(\cdot)$ is continuous, and $\mathcal{E}(h)$ consists of isolated points, either $\Psi(x) < x$ for all $x \in (\frac{1}{k+1}, \frac{1}{k})$ and $x < \Psi(x)$ for all $x \in (\frac{1}{k}, \frac{1}{k-1})$ or $\Psi(x) > x$ for all $x \in (\frac{1}{k+1}, \frac{1}{k})$ and $x > \Psi(x)$ for all $x \in (\frac{1}{k}, \frac{1}{k-1})$. W.l.o.g., we may assume that $\Psi(x) < x$ for all $x \in (\frac{1}{k+1}, \frac{1}{k})$.

On the event $E_\eta := \{\theta_{\eta,1} < \frac{1}{k} - \delta_\eta\} \cap \{M_{n,1}(\eta) \leq \frac{\delta_\eta}{2}\}$, we have $\theta_{\eta,1} - \Psi(\theta_{\eta,1}) > 0$ and from (43) $\mathbb{P}(E_\eta) > 0$. Now let show by induction that on E_η , $\theta_{n,1} < \frac{1}{k} - \frac{\delta_\eta}{2}$ for $n \geq \eta$. The base step of the induction for $n = \eta$ is obvious. Let us assume that $\theta_{n-1,1} < \frac{1}{k} - \frac{\delta_\eta}{2}$, which implies that $R_{n,1} > 0$ and $\theta_{j,1} - \Psi(\theta_{j,1}) > 0$ for $\eta \leq j \leq n-1$. Therefore,

$$\theta_{n,1} < \theta_{n-1,1} + M_{n,1} \leq \frac{1}{k} - \delta_\eta + \frac{\delta_\eta}{2} = \frac{1}{k} - \frac{\delta_\eta}{2}.$$

Therefore, on E_η , we have $\theta_{n,1} - \Psi(\theta_{n,1}) > 0$ for all $n \geq \eta$ and $0 < \lim_{n \rightarrow \infty} R_{n,1} < \infty$. But this is a contradiction, since on E_η

$$\lim_{n \rightarrow \infty} R_{n,1} = \lim_{n \rightarrow \infty} (\theta_{n,1} + M_{n,1} - \theta_{n,1}) \leq -\frac{\delta_\eta}{2}. \quad (44)$$

□

Our proof for Theorem 2 is simpler than the proof of Proposition 7.1 from [30]. This is due to the assumption $\sum_n \left(\frac{\sigma_n}{\tau_n}\right)^2 < \infty$.

Proof of Theorem 2. From Theorem 1, we know that

$$\mathbb{P}(\mathcal{D}) = 1.$$

By symmetry, W.l.o.g. we may assume $\theta_{n,1} \rightarrow 0$ as $n \rightarrow \infty$. To prove (12), it is enough to show that conditioned on the event $\theta_{n,1} \rightarrow 0$ as $n \rightarrow \infty$, if $\sup_{n \geq 1} U_{n,1} = \infty$, then we get a contradiction.

Let us assume that $\sup_{n \geq 1} U_{n,1} = \infty$. Since $U_{n,1}$ is non-decreasing, it follows $U_{n,1} \rightarrow \infty$, as $n \rightarrow \infty$.

W.l.o.g. we may assume that for some m, n (large enough), $U_{i+1,1} - U_{i,1} > 0$ for $n \leq i \leq m$. For some $C > 0$ by assumption **(C)**,

$$\sum_{i=n}^m \frac{U_{i+1,1} - U_{i,1}}{U_i^\alpha} \geq \int_{U_{n,1}}^{U_{m,1}} \frac{1}{x^\alpha} dx \sim \frac{C}{U_{n,1}^{\alpha-1}}. \quad (45)$$

Recall that we know from the (3)

$$U_{n+1,1} = U_{n,1} + B_{n+1,1},$$

where $B_{n+1,1}$ follows $\text{Bin}(\sigma_{n+1}, \Psi(\theta_{n,1}))$.

Let us re-write the above equation as

$$U_{n+1,1} = U_{n,1} + \sigma_{n+1} \Psi(\theta_{n,1}) + \epsilon_n \sqrt{\sigma_{n+1} \Psi(\theta_{n,1})(1 - \Psi(\theta_{n,1}))}, \quad (46)$$

where $\epsilon_n = \frac{B_{n+1,1} - \sigma_{n+1}\Psi(\theta_{n,1})}{\sqrt{\sigma_{n+1}\Psi(\theta_{n,1})(1-\Psi(\theta_{n,1}))}}$.

Since $x_1 + x_2 + \dots + x_d = 1$, and f is non-decreasing, we have $\sum_{i=1}^d f(x_i) \geq f(\frac{1}{d}) =: C_d^{-1}$, which implies that

$$\Psi(x) \leq C_d f(x). \quad (47)$$

By the estimate from Lemma 2.3 (iv) applied to (46), we get

$$\begin{aligned} U_{n+1,1} &\leq U_{n,1} + C_d \sigma_{n+1} \theta_{n,1}^\alpha + \epsilon_n \sqrt{\sigma_{n+1} \Psi(\theta_{n,1})(1 - \Psi(\theta_{n,1}))} \\ \frac{U_{n+1,1} - U_{n,1}}{U_{n,1}^\alpha} &\leq C_d \frac{\sigma_{n+1}}{\tau_n^\alpha} + \frac{1}{U_{n,1}^\alpha} \epsilon_n \sqrt{\sigma_{n+1} \Psi(\theta_{n,1})(1 - \Psi(\theta_{n,1}))}. \end{aligned} \quad (48)$$

Thus,

$$\sum_{i=n}^{\infty} \frac{U_{i+1,1} - U_{i,1}}{U_{i,1}^\alpha} \leq C_d \sum_{i \geq n} \frac{\sigma_{i+1}}{\tau_i^\alpha} + \sum_{i \geq n} \frac{1}{U_{i,1}^\alpha} \epsilon_i \sqrt{\sigma_{i+1} \Psi(\theta_{i,1})(1 - \Psi(\theta_{i,1}))}. \quad (49)$$

Combining this with (45), we get

$$\frac{C}{U_{n,1}^{\alpha-1}} \leq C_d \sum_{i \geq n} \frac{\sigma_{i+1}}{\tau_i^\alpha} + \sum_{i \geq n} \frac{1}{U_{i,1}^\alpha} \epsilon_i \sqrt{\sigma_{i+1} \Psi(\theta_{i,1})(1 - \Psi(\theta_{i,1}))}. \quad (50)$$

We obtain using arguments similar to (41), for some suitable constant $C' > 0$,

$$\sum_{i \geq n} \frac{\sigma_{i+1}}{\tau_i^\alpha} \leq C' \int_{\tau_n}^{\infty} \frac{1}{x^\alpha} dx = C' \frac{1}{\tau_n^{\alpha-1}}.$$

Using the above bound in (50), we obtain where $C, C' > 0$ are suitable constants that may change accordingly

$$\begin{aligned} \frac{C}{U_{n,1}^{\alpha-1}} &\leq C' \frac{1}{\tau_n^{\alpha-1}} + \sum_{i \geq n} \frac{1}{U_{i,1}^\alpha} \epsilon_i \sqrt{\sigma_{i+1} \Psi(\theta_{i,1})(1 - \Psi(\theta_{i,1}))} \\ C &\leq C' \theta_{n,1}^{\alpha-1} + U_{n,1}^{\alpha-1} \sum_{i \geq n} \frac{1}{U_{i,1}^\alpha} \epsilon_i \sqrt{\sigma_{i+1} \Psi(\theta_{i,1})(1 - \Psi(\theta_{i,1}))}. \end{aligned} \quad (51)$$

Let us denote by

$$T_n := U_{n,1}^{\alpha-1} \sum_{i \geq n} \frac{1}{U_{i,1}^\alpha} \epsilon_i \sqrt{\sigma_{i+1} \Psi(\theta_{i,1})(1 - \Psi(\theta_{i,1}))}. \quad (52)$$

Using (47) and $f(x) \leq x^\alpha$, we get for some $C' > 0$

$$T_n \leq C' U_{n,1}^{\alpha-1} \sum_{i \geq n} \frac{1}{U_{i,1}^{\frac{\alpha}{2}}} \epsilon_i \sqrt{\sigma_{i+1} \frac{1}{\tau_i^\alpha} (1 - \Psi(\theta_{i,1}))}.$$

Since $U_{n,1}$ is non-decreasing in n , we have $\frac{U_{n,1}}{U_{i,1}} \leq 1$. Therefore, from the above equation, we get

$$\begin{aligned}
 T_n &\leq C' \frac{1}{U_{n,1}} \sum_{i \geq n} \epsilon_i \sqrt{\sigma_{i+1} \frac{U_{n,1}^\alpha}{\tau_i^\alpha} (1 - \Psi(\theta_{i,1}))} \\
 &= C' \frac{1}{\sqrt{U_{n,1}}} \sum_{i \geq n} \epsilon_i \sqrt{\frac{\sigma_{i+1}}{\tau_i} \left(\frac{U_{n,1}}{\tau_i}\right)^{\alpha-1} (1 - \Psi(\theta_{i,1}))} \\
 &\leq C' \frac{1}{\sqrt{U_{n,1}}} \sum_{i \geq n} \epsilon_i \sqrt{\frac{\sigma_{i+1}}{\tau_i} \left(\frac{\tau_n}{\tau_i}\right)^{\alpha-1}}, \tag{53}
 \end{aligned}$$

where we have used $(1 - \Psi(\theta_{i,1})) \leq 1$, and $U_{n,1} \leq \tau_n$.

It is easy to see that

$$\frac{\sigma_{n+1}}{\tau_n} = \frac{\frac{\sigma_{n+1}}{\tau_{n+1}}}{1 - \frac{\sigma_{n+1}}{\tau_{n+1}}}. \tag{54}$$

Observe that by our assumption $\sum_{n \geq 1} \frac{\sigma_n^2}{\tau_n} < \infty$, we have $\lim_{n \rightarrow \infty} \frac{\sigma_n}{\tau_n} = 0$ and hence, from (54) we have $\frac{\sigma_{n+1}}{\tau_n} \rightarrow 0$, as $n \rightarrow \infty$.

Therefore, for all n (large enough), we obtain

$$\begin{aligned}
 \frac{\tau_i}{\tau_n} &= \prod_{j=n}^{i-1} \frac{\tau_{j+1}}{\tau_j} \\
 &= \prod_{j=n}^{i-1} \left(1 + \frac{\sigma_{j+1}}{\tau_j}\right) \\
 &= (1 + o(1))^{i-n}, \text{ for all } i \geq n. \tag{55}
 \end{aligned}$$

Therefore, using the above inequality and $\alpha > 1$,

$$\sum_{i \geq n} \left(\frac{\tau_n}{\tau_i}\right)^{\alpha-1} = \sum_{i \geq 1} \left(\frac{1}{(1 + o(1))^i}\right)^{\alpha-1} < \infty. \tag{56}$$

Similar to (42), it can be easily shown $\left(N_n = \sum_{i=1}^n \epsilon_i \sqrt{\frac{\sigma_{i+1}}{\tau_i} \left(\frac{\tau_n}{\tau_i}\right)^{\alpha-1}}\right)_{n \geq 1}$ is a $L^2(\mathbb{R})$ bounded martingale, and we have for some $C > 0$

$$\sup_{n \geq 1} \mathbb{E} [N_n^2] \leq C \sum_{i=1}^{\infty} \left(\frac{\tau_n}{\tau_i}\right)^{\alpha-1} < \infty, \tag{57}$$

where we have used (55) and $\frac{\sigma_{n+1}}{\tau_n} \rightarrow 0$, as $n \rightarrow \infty$.

Since N_n is an $L^2(\mathbb{R})$ bounded, N_n converges almost surely and in $L^2(\mathbb{R})$. This implies that $\left(\sum_{i \geq n} \epsilon_i \sqrt{\frac{\sigma_{i+1}}{\tau_i} \left(\frac{\tau_n}{\tau_i}\right)^{\alpha-1}}\right)^2 < \infty$, almost surely.

Therefore, since we assumed that $U_{n,1} \rightarrow \infty$ as $n \rightarrow \infty$, we get

$$T_n \leq \frac{1}{\sqrt{U_{n,1}}} \sum_{i \geq n} \epsilon_i \sqrt{\frac{\sigma_{i+1}}{\tau_i} \left(\frac{\tau_n}{\tau_i}\right)^{\alpha-1}} \rightarrow 0, \text{ as } n \rightarrow \infty. \tag{58}$$

From (51), we get on the event $\{\theta_{n,1} \rightarrow 0\}$,

$$C \leq C' \theta_{n,1}^{\alpha-1} + T_n \rightarrow 0, \text{ as } n \rightarrow \infty,$$

which is a contradiction since $C > 0$. \square

APPENDIX A. STOCHASTIC APPROXIMATION

In this section we will introduce relevant quantities and results for the stochastic approximation techniques which are relevant for our model. We will use notation from [7, 10]. Define the following recursive scheme on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, \mathbb{P})$ with values in a compact set $\Gamma \subset \mathbb{R}^d$:

$$\theta_{n+1} = \theta_n + \frac{\sigma_{n+1}}{\tau_{n+1}} \left[h(\theta_n) + \frac{1}{\sigma_{n+1}} \Delta M_{n+1} \right], \quad (59)$$

where $(\frac{\sigma_n}{\tau_n})_{n \geq 0}$ is a positive sequence of numbers, $h : \Gamma \rightarrow \mathbb{R}^d$ is a continuous function, $(\Delta M_n)_{n \geq 0}$ a $(\mathcal{F}_n)_{n \geq 0}$ -martingale increment, defined in (10). The mean-field function h was defined as

$$h(y) = \frac{f_d(y)}{\|f_d(y)\|_1} - y \quad (60)$$

for $y \in \Gamma$.

The main idea of stochastic approximation techniques is to show that the recursive system defined in (59) behaves like an ODE perturbed by a small noise term. Then the recursive system is shown to converge towards the equilibrium points of the deterministic flow induced by the driving term h .

Call $L(\{w_n\}_{n \geq 1})$ the *limit set* of the sequence $\{w_n\}_{n \geq 1}$ is the set of points $x \in \mathbb{R}^d$ such that there exists a subsequence $(n_k)_{k \geq 1}$ such that $\lim_{k \rightarrow \infty} n_k = \infty$ for which $\lim_{k \rightarrow \infty} w_{n_k} = x$.

Consider furthermore the ODE

$$\dot{y} = h(y) \quad (61)$$

and associated to h look at $(\Phi(t, x))_{t \geq 0, x \in \Gamma}$ the Γ -valued *flow* of the system. The family $\{\Phi_t\}_{t \in \mathbb{R}_+}$, where $\Phi_t(x) = \Phi(t, x)$, satisfies the group property. $\Phi_0 = \text{Id}$ and for all $(t, s) \in \mathbb{R}_+^2$

$$\Phi_{t+s} = \Phi_t \circ \Phi_s.$$

For every $x \in \Gamma$, $(\Phi(t, x))_{t \geq 0}$ is the unique solution to the ODE (61)

$$\frac{d}{dt} \Phi_t(x) = h(\Phi_t(x)). \quad (62)$$

It exists since h is locally Lipschitz which follows from continuity of the function f . Denote by $\mathcal{E}(\Phi) = \{p \in \Gamma : \Phi_t(p) = p \text{ for all } t > 0\}$ the equilibrium set for the flow Φ .

Definition A.1. A compact subset $A \subset \Gamma$ is called internally chain recurrent if and only if for each $x \in A$ is chain recurrent for the flow Φ restricted to A . The point x is called chain recurrent in A if and only if for all $\delta > 0$, and $T > 0$, there exist $k \in \mathbb{N}$ and points $y_0, \dots, y_{k-1} \in A$ and t_1, \dots, t_{k-1} such that for all $i = 0, \dots, k-1$ and $y_k = x$

$$t_i \geq T; \quad d(y_0, x) < \delta; \quad d(\Phi_{t_i}(y_i), y_{i+1}) < \delta$$

where $d(\cdot, \cdot)$ denotes the Euclidean distance in \mathbb{R}^d . The set of all chain recurrent points of the flow Φ will be denoted by $\text{CR}(\Phi)$.

The omega limit set of Φ is defined by $\mathcal{L}(\Phi) = \bigcup_{x \in \Gamma} \omega(x)$, where

$$\omega(x) = \left\{ p \in \Gamma : \exists (t_k)_k \text{ such that } t_k \rightarrow \infty \text{ and } p = \lim_{k \rightarrow \infty} \Phi_{t_k}(x) \right\}.$$

One has that

$$\mathcal{E}(\Phi) \subset \mathcal{L}(\Phi) \subset \text{CR}(\Phi), \quad (63)$$

see page 21 of [10].

We will need the notion of a strict Lyapunov function for the flow Φ , see also Section 3 of [7].

Definition A.2. A strict Lyapunov function for the flow Φ (or h) on Γ is a continuous map $V : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $t \mapsto V(\Phi_t(x))$ is constant for $x \in \Gamma$ and strictly decreasing for all $x \in \mathbb{R}^d \setminus \Gamma$ for all $t > 0$. If such a V exists we call h a gradient-like vector field.

Lemma A.3. *Let the flow Φ be defined in (62) and h in (60) and $F : [0, 1]^d \rightarrow \mathbb{R}$ by*

$$F(y) = \sum_{i=1}^d \int_0^{y_i} \frac{f(z_i)}{z_i} dz_i.$$

Then $V = -F$ is a strict Lyapunov function for Φ on $\mathcal{E}(\Phi)$.

Proof. For $i = 1, \dots, d$, write the ODE (61) as

$$\begin{aligned} \dot{y}_i(t) &= \frac{y_i(t)}{\|f_d(y(t))\|_1} \left(\frac{f(y_i(t))}{y_i(t)} - \sum_{j=1}^d y_j(t) \frac{f(y_j(t))}{y_j(t)} \right) \\ &= \frac{y_i(t)}{\|f_d(y(t))\|_1} \left(\frac{\partial}{\partial y_i} F(y(t)) - \sum_{j=1}^d y_j(t) \frac{\partial}{\partial y_j} F(y(t)) \right) \end{aligned} \quad (64)$$

where $F : \mathbb{R}^d \rightarrow \mathbb{R}$ and $F(y) = \sum_{i=1}^d F_i(y_i)$. We have that $F(0) = 0$ and

$$\frac{\partial}{\partial y_i} F(y) = \frac{f(y_i)}{y_i}.$$

For each coordinate $i = 1, \dots, d$ we can write,

$$F_i(y_i) = \int_0^{y_i} \frac{\partial}{\partial z} F_i(z) dz = \int_0^{y_i} \frac{f(z)}{z} dz$$

where $z \in \mathbb{R}$. The function $-F$ is strictly Lyapunov for Φ since

$$\begin{aligned} \frac{d}{dt} F(y(t)) &= \sum_{i=1}^d \dot{y}_i(t) \frac{\partial}{\partial y_i} F(y(t)) \\ &= \frac{1}{\|f_d(y)\|_1} \left\{ \sum_{i=1}^d y_i(t) \left(\frac{\partial}{\partial y_i} F(y(t)) \right)^2 - \left(\sum_{i=1}^d y_i(t) \frac{\partial}{\partial y_i} F(y(t)) \right)^2 \right\} > 0 \end{aligned}$$

where the last step follows from Jensen's inequality. Finally note that it is constant on the equilibrium set $\mathcal{E}(h) = \mathcal{E}(\Phi)$:

$$\begin{aligned} & \sum_{i=1}^d y_i(t) \left(\frac{\partial}{\partial y_i} F(y(t)) \right)^2 - \left(\sum_{i=1}^d y_i(t) \frac{\partial}{\partial y_i} F(y(t)) \right)^2 \\ &= \sum_{i=1}^d \frac{f^2(y_i(t))}{y_i(t)} - \left(\sum_{i=1}^d f(y_i(t)) \right)^2 \\ &= \sum_{i=1}^d \|f_d(y(t))\|_1 f(y_i(t)) - \|f_d(y(t))\|_1^2 = 0, \end{aligned}$$

where in the before last equality we used (15). \square

Proposition A.4. Let Φ be the flow on Γ , where Γ is compact. Furthermore, let $\Lambda \subset \Gamma$ be compact invariant set and $V : \Gamma \rightarrow \mathbb{R}$ a Lyapunov function for Φ on Λ with finite $V(\Lambda)$, then

$$CR(\Phi) \subset \Lambda.$$

The last proposition is same as Proposition 1.2 from [7]. Choosing $V = -F$ and $\Lambda = \mathcal{E}(\Phi)$ in the last proposition together with (63) implies that

$$\mathcal{E}(\Phi) = \mathcal{L}(\Phi) = CR(\Phi).$$

The following theorem is equivalent to Theorem 6 from [19], based on Theorem 5.7 from [10].

Theorem A.5. *Let $\{\theta_n\}_{n \geq 1}$ be a solution to (59) and assume that*

- (i) $\sum_{n=1}^{\infty} \left(\frac{\sigma_n}{\tau_n} \right)^2 < \infty$.
- (ii) h is a local Lipschitz function.
- (iii) $\sup_{n \geq 1} \mathbb{E}(\|\Delta M_n\|_2^2 | \mathcal{F}_n) < \infty$ a.s.

Then almost surely $L(\{\theta_n\}_{n \geq 1})$ is a connected set, internally chain-recurrent for the flow Φ induced by h , $L(\{\theta_n\}_{n \geq 1}) \subset CR(\Phi)$.

Proof. The first assumption is trivially satisfied, the second follows from the fact that f is locally Lipschitz since it is $C^1((0, 1))$ and the third follows trivially from writing the martingale difference as in (10). \square

The next corollary is analogous to Corollary 3.3. from [7], the proof is adapted.

Corollary A.6. *Let $\{\theta_n\}_{n \geq 1}$ be defined in (59). Then $\theta_n \rightarrow \theta^*$ a.s. as $n \rightarrow \infty$ where $\theta^* \in \mathcal{E}(h)$.*

Proof. Let Φ be the flow induced by h defined in (60) and let $\Gamma = L(\{\theta_n\}_{n \geq 1})$ the set of all almost sure limiting points of $\{\theta_n\}_{n \geq 1}$. By Theorem A.5 is a connected set, internally chain-recurrent for the flow Φ induced by h . By Lemma A.3 we know that there exists a Lyapunov function for the flow on Γ and by Proposition A.4 the set $L(\{\theta_n\}_{n \geq 1})$ consists of equilibria, $\mathcal{E}(\Phi) = \mathcal{E}(h)$. Since they are isolated (Lemma 4.1), $L(\{\theta_n\}_{n \geq 1})$ is an equilibrium. \square

ACKNOWLEDGEMENT

The authors are grateful to V.S. Borkar for his help with stochastic approximations. The authors are grateful to Andrew Wade for his valuable insight with the class of reinforcement functions.

REFERENCES

- [1] R. Armenter and M. Koren, *A Balls-and-Bins Model of Trade*, American Economic Review, Vol. 104, No. 7, 2014.
- [2] Z.-D. Bai and F. Hu, *Asymptotic theorems for urn models with nonhomogeneous generating matrices*, Stochastic Process. Appl., 80(1):87–101, 1999.
- [3] Z.-D. Bai and F. Hu, *Asymptotics in randomized urn models*, Ann. Appl. Probab., 15(1B):914–940, 2005.
- [4] Z.-D. Bai, F. Hu, and L. Shen, *An adaptive design for multi-arm clinical trials*, J. Multivariate Anal., 81(1):1–18, 2002.
- [5] A. Bandyopadhyay, S. Janson, D. Thacker, *Strong convergence of infinite color balanced urns under uniform ergodicity*, Jour. of Appl. Probab., 57(3), 853–865, 2020.
- [6] A. Bandyopadhyay, D. Thacker, *Pólya urn schemes with infinitely many colors*, Bernoulli 23 (4B) 3243 – 3267, 2017.
- [7] M. Benaïm, *A dynamical system approach to stochastic approximations*, Siam J. Control Optim. 34, 437–472, 1996.
- [8] M. Benaïm, *Vertex-reinforces random walks and a conjecture of Pemantle*, Ann. of Probab. 25, no. 1, 361–392, 1997.
- [9] M. Benaïm, *Recursive algorithms, urn processes and chaining number of chain recurrent sets*, Ergodic Theory Dynam. Systems, 18, no 1, 53–87, 1998.
- [10] M. Benaïm, *Dynamics of stochastic approximation algorithms*, In Séminaire de Probabilités, XXXIII, volume 1709 of Lecture Notes in Math., pages 1–68. Springer, Berlin, 1999.
- [11] V.S. Borkar, *Stochastic approximation - a dynamical systems viewpoint*, Springer 2008.
- [12] P. Donnelly, *Partition structures, Polya urns, the Ewens sampling formula, and the ages of alleles*, Theo. Pop. Biology, Vol. 30, Issue 2, p. 271–288, 1986.
- [13] E. Drinea, A. Frieze and M. Mitzenmacher, *Balls in bins processes with feedback*, Proceedings of the 11th Annual ACM-SIAM Symposium on Discrete Algorithms, 308–315. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2002
- [14] F. Eggenberger and G. Pólya, *Über die Statistik verketteter Vorgänge*, Z. Angew. Math. Mech. 3, 279–289, 1923.
- [15] C. Hirsch, M. Holmes, M., V. Kleptsyn, *Absence of WARM percolation in the very strong reinforcement regime*, Ann. Appl. Probab. 31 (1), 199–217, 2021.
- [16] R. v.d. Hofstad, M. Holmes, A. Kuznetsov, W.M. Ruszel, *Strongly reinforced Pólya urns with graph-based competition*, The Annals of Applied Probability 26, no. 4 , 2494–2539, 2016.
- [17] M. Holmes, V. Kleptsyn, *Proof of the WARM whisker conjecture for neuronal connections*, Chaos. Apr;27(4):043104, 2017.
- [18] S. Janson, *Random replacements in Pólya urns with infinitely many colours*, Electron. Commun. Probab. 24 1 – 11, 2019.

- [19] G. Kaur, *Negatively reinforced balanced urn schemes*, Adv. Applied Math. 105, 48–82, 2019.
- [20] K. Khanin, R. Khanin, *A probabilistic model for the establishment of neuron polarity*, J. Math. Biol., 42(1):26-40, 2001.
- [21] S. Laruelle and G. Pagès, *Randomized urn models revisited using stochastic approximation*, Ann. Appl. Probab., 23(4):1409–1436, 2013.
- [22] S. Laruelle and G. Pagès, *Nonlinear randomized urn models: a stochastic approximation viewpoint*, Electron. J. Probab. 24, np. 98, p.1–47, 2019.
- [23] C. Mailler and J.-F. Marckert, *Measure-valued Pólya processes*, Electron. J. Probab. 22, no. 26, 1–33, 2017.
- [24] R.I. Oliveira, *The onset of dominance in balls-in-bins processes with feedback*, Random Struct. Algorithms, 34, Issue 4, 454–477, 2009.
- [25] R. Pemantle, *A time-dependent version of Pólya’s urn*, J. Theoret. Probab., 3:627–637, 1990.
- [26] R. Pemantle, *A survey of random processes with reinforcement*, Probab. Surveys 4, 1 – 79, 2007.
- [27] M. Raab and A. Steger, *Balls into Bins - A Simple and Tight Analysis*, Luby M., Rolim J.D.P., Serna M. (eds) Randomization and Approximation Techniques in Computer Science. RANDOM 1998. Lecture Notes in Computer Science, vol 1518. Springer, Berlin, Heidelberg, 1998.
- [28] H. Robbins, S. Monro, *A stochastic approximation method*, Annals of Mathematical Statistics 22, 400–407, 1951.
- [29] H. Shi and Z. Jiang, *The missing trade of China: balls-and-bins model*, Empir Econ 50, 1511–1526, 2016.
- [30] N. Sidorova, *Time-dependent balls and bin model with positive feedback*, arXiv: 1809.02221, 2018.

(Utrecht University) MATHEMATICAL INSTITUTE

BUDAPESTLAAN 6, 3584 CD UTRECHT, THE NETHERLANDS

Email address: `w.m.ruszel@uu.nl`

(Durham University) DEPARTMENT OF MATHEMATICAL SCIENCES

CAMPUS, STOCKTON RD, DURHAM DH1 3LE, UNITED KINGDOM

Email address: `debleena.thacker@durham.ac.uk`