A Deterministic Polynomial Kernel for Odd Cycle Transversal and Vertex Multiway Cut in Planar Graphs

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Abstract

We show that Odd Cycle Transversal and Vertex Multiway Cut admit deterministic polynomial kernels when restricted to planar graphs and parameterized by the solution size. This answers a question of Saurabh. On the way to these results, we provide an efficient sparsification routine in the flavor of the sparsification routine used for the Steiner Tree problem in planar graphs (FOCS 2014). It differs from the previous work because it preserves the existence of low-cost subgraphs that are not necessarily Steiner trees in the original plane graph, but structures that turn into (supergraphs of) Steiner trees after adding all edges between pairs of vertices that lie on a common face. We also show connections between Vertex Multiway Cut and the Vertex Planarization problem, where the existence of a polynomial kernel remains an important open problem.

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1 Introduction

Kernelization provides a rigorous framework within the paradigm of parameterized complexity to analyze preprocessing routines for various combinatorial problems. A *kernel* of size g for a parameterized problem Π and a computable function g is a polynomial-time algorithm that reduces an input instance x with parameter k of problem Π to an equivalent one with size and parameter value bounded by $g(k)$. Of particular importance are *polynomial kernels*, where the function g is required to be a polynomial, that are interpreted as theoretical tractability of preprocessing for the considered problem Π. Since a kernel (of any size) for a decidable problem implies fixed-parameter tractability (FPT) of the problem at hand, the question whether a *polynomial* kernel exists became a "standard" tractability question one asks about a problem already known to be FPT, and serves as a further finer-grained distinction criterion between FPT problems.

In the recent years, a number of kernelization techniques emerged, including the bidimensionality framework for sparse graph classes [10] and the use of representative sets for graph separation problems [20]. On the hardness side, a lower bound framework against polynomial kernels has been developed and successfully applied to a multitude of problems [1, 5, 7, 11]. For more on kernelization, we refer to the survey [22] for background and to the appropriate chapters of the textbook [3] for basic definitions and examples.

For this work, of particular importance are polynomial kernels for graph separation problems. The framework for such kernels developed by Kratsch and Wahlström in [20, 21], relies on the notion of *representative sets* in linear matroids, especially in gammoids. Among other results, the framework provided a polynomial kernel for Odd Cycle Transversal and for Multiway Cut with a constant number of terminals. However, all kernels for graph separation problems based on representative sets are randomized, due to the randomized nature of all known polynomial-time algorithms that obtain a linear representation of a gammoid. As a corollary, all such kernels have exponentially small probability of turning an input yes-instance into a no-instance.

The question of *deterministic* polynomial kernels for the cut problems that have *randomized* kernels due to the representative sets framework remains widely open. Saket Saurabh, at the open problem session during the Recent Advances in Parameterized Complexity school (Dec 2017, Tel Aviv) [27], asked whether a deterministic polynomial kernel for Odd Cycle Transversal exists when the input graph is planar. In this paper, we answer this question affirmatively, and prove an analogous result for the Multiway Cut problem.

▶ **Theorem 1.1.** Odd Cycle Transversal *and* Vertex Multiway Cut*, when restricted to planar graphs and parameterized by the solution size, admit deterministic polynomial kernels.*

Recall that the Odd Cycle Transversal problem, given a graph G and an integer k, asks for a set $X \subseteq V(G)$ of size at most k such that $G \setminus X$ is bipartite. For the Multiway Cut problem, we consider the Vertex Multiway Cut variant where, given a graph G, a set of terminals $T \subseteq V(G)$, and an integer k, we ask for a set $X \subseteq V(G) \setminus T$ of size at most k such that every connected component of $G \setminus X$ contains at most one terminal. In other words, we focus on the vertex-deletion variant of Multiway Cut with undeletable terminals. In both cases, the allowed deletion budget, k, is our parameter. (A deterministic polynomial kernel for Edge Multiway Cut in planar graphs is known [25, Theorem 1.4].)

Note that in general graphs, Vertex Multiway Cut admits a randomized polynomial kernel with $\mathcal{O}(k^{|T|+1})$ terminals [20], and whether one can remove the dependency on $|T|$ from the exponent is a major open question in the area. Theorem 1.1 answers this question affirmatively in the special case of planar graphs.

Our motivation stems not only from the aforementioned question of Saurabh [27], but also from a second, more challenging question of a polynomial kernel for the Vertex Planarization problem. Here, given a graph G and an integer k, one asks for a set $X \subseteq V(G)$ of size at most k such that $G \setminus X$ is planar. For this problem, an involved $2^{\mathcal{O}(k \log k)} \cdot n$-time fixed-parameter algorithm is known [17], culminating a longer line of research [17, 19, 23]. The question of a polynomial kernel for the problem has not only been posed by Saurabh during the same open problem session [27], but also comes out naturally in another line of research concerning vertex-deletion problems to minor-closed graph classes.

Consider a minor-closed graph class $\mathcal{G}$. By the celebrated Robertson-Seymour theorem, the list of minimal forbidden minors $\mathcal{F}$ of $\mathcal{G}$ is finite, i.e., there is a finite set $\mathcal{F}$ of graphs such that a graph G belongs to $\mathcal{G}$ if and only if G does not contain any graph from $\mathcal{F}$ as a minor. The $\mathcal{F}$-Deletion problem, given a graph G and an integer k, asks to find a set

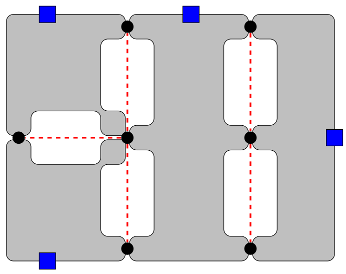

Figure 1 When all terminals (blue squares) lie on the infinite face, a solution to Vertex Multiway Cut (black circles) is a Steiner forest (red dashed connections) in the overlay graph.

$X \subseteq V(G)$ of size at most k such that $G \setminus X$ has no minor belonging to $\mathcal{F}$, i.e., $G \setminus X \in \mathcal{G}$. If $\mathcal{F}$ contains a planar graph or, equivalently, $\mathcal{G}$ has bounded treewidth, then the parameterized and kernelization complexity of the $\mathcal{F}$-Deletion problem is well understood [9]. However, our knowledge is very partial in the other case, when $\mathcal{G}$ contains all planar graphs. The understanding of this general problem has been laid out as one of the future research directions in a monograph of Downey and Fellows [6]. The simplest not fully understood case is when $\mathcal{G}$ is exactly the set of planar graphs, that is, $\mathcal{F} = \{K_{3,3}, K_5\}$, and the $\mathcal{F}$-Deletion becomes the Vertex Planarization problem. The question of a polynomial kernel or a $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$-time FPT algorithm for Vertex Planarization remains open [13, 27].

In Section 6, we observe that there is a simple polynomial-time reduction from Planar Vertex Multiway Cut to Vertex Planarization that keeps the parameter k unchanged. If Vertex Planarization would admit a polynomial kernel, then our reduction would transfer the polynomial kernel back to Planar Vertex Multiway Cut. In the presence of Theorem 1.1, such an implication is trivial, but the reduction itself serves as a motivation: a polynomial kernel for Planar Vertex Multiway Cut should be easier than for Vertex Planarization, and one should begin with the first before proceeding to the latter. Furthermore, we believe the techniques developed in this work can be of use for the more general Vertex Planarization case.

Techniques On the technical side, our starting point is the toolbox of [25] that provides a polynomial kernel for Steiner Tree in planar graphs, parameterized by the number of edges of the solution. The main technical result of [25] is a sparsification routine that, given a connected plane graph G with infinite face surrounded by a simple cycle ∂G, provides a subgraph of G of size polynomial in the length of ∂G that, for every $A \subseteq V(\partial G)$, preserves an optimal Steiner tree connecting A.

Both Odd Cycle Transversal and Vertex Multiway Cut in a plane graph G translate into Steiner forest-like questions in the *overlay graph* $\mathcal{L}(G)$ of G: a supergraph of G that has a vertex v_f for every face of G, adjacent to every vertex of G incident with f. To see this, consider a special case of Planar Vertex Multiway Cut where all terminals lie on the infinite face of the input embedded graph. Then, an optimal solution is a Steiner forest between some tuples of vertices on the outer face lying between the terminals, cf. Figure 1. Following [25], this suggest the following approach to kernelization of vertex-deletion cut problems in planar graphs:

1. By problem-specific reductions, reduce to the case of a graph of bounded radial diameter.
2. Using the diameter assumption, find a tree in the overlay graph that has size bounded polynomially in the solution size, and that spans all "important" objects in the graph (e.g., neighbors of the terminals in the case of Multiway Cut or odd faces in the case of Odd Cycle Transversal).

3. Cut the graph open along the tree. Using the Steiner forest-like structure of the problem at hand, argue that an optimal solution becomes an optimal Steiner forest for some choice of tuples of terminals on the outer face of the cut-open graph.
4. Sparsify the cut-open graph with a generic sparsification routine that preserves optimal Steiner forests, glue the resulting graph back, and return it as a kernel.

However, contrary to the Steiner tree problem [25], these Steiner forest-like questions optimize a different cost function than merely number of edges, namely *the number of vertices of* G, with the "face" vertices $v_f \in V(\mathcal{L}(G)) \setminus V(G)$ being for free. This cost function is closely related to (half of) the number of edges in case of paths and trees with constant number of leaves, but may diverge significantly in case of trees with high-degree vertices.

For this reason, we need an analog of the main technical sparsification routine of [25] suited for our cost function. To this end, we re-use most of the intermediate results of [25], changing significantly only the final divide&conquer argument. We provide a statement and an overview of the proof of such routine in Section 3. A full proof can be found on arXiv [18].

The application of the obtained sparsification routine to the case of Odd Cycle Transversal, presented in Section 4, follows the phrasing of the problem as a T-join-like problem in the overlay graph due to Fiorini et al [8]. For the sake of reducing the number of odd faces, we adapt the arguments of Suchý [28] for Steiner tree.

The arguments for Vertex Multiway Cut are somewhat more involved and sketched in Section 5. A full version can be found on arXiv [18]. Here, we first use known LP-based rules [4, 12, 14, 26] to reduce the number of terminals and neighbors of terminals to $\mathcal{O}(k)$ and then use an argument based on outerplanarity layers to reduce the diameter.

2 Preliminaries

A finite undirected graph G consists of a vertex set $V(G)$ and edge set $E(G) \subseteq \binom{V(G)}{2}$. We denote the open neighborhood of a vertex v in G by $N_G(v)$. For a vertex set $S \subseteq V(G)$ we define its open neighborhood as $N_G(S) := \bigcup_{v \in S} N_G(v) \setminus S$. For all standard but undefined here terms related to planar graph we refer to [25].

For vertex subsets X, Y of a graph G, we define an (X, Y)-cut as a vertex set $Z \subseteq V(G) \setminus (X \cup Y)$ such that no connected component of $G \setminus Z$ contains both a vertex of X and a vertex of Y. An (X, Y) cut Z is *minimal* if no proper subset of Z is an (X, Y)-cut, and *minimum* if it has minimum possible size.

2.1 Planar graphs

In a connected embedded planar (i.e. plane) graph G, the *boundary walk* of a face f is the unique closed walk in G obtained by going along the face in counter-clockwise direction. Note that a single vertex can appear multiple times on the boundary walk of f and an edge can appear twice if it is a bridge. We denote the number of edges of this walk by $|f|$; note that bridges are counted twice in this definition. The *parity* of a face f is the parity of $|f|$. Then a face is *odd (even)* if its parity is odd (even). The boundary walk of the outer face of G is called the *outer face walk* and denoted ∂G.

We define the *radial distance* in plane graphs, based on a measure that allows to hop between vertices incident on a common face in a single step. Formally speaking, a *radial path* between vertices p and q in a plane graph G is a sequence of vertices $(p = v_0, v_1, \ldots, v_\ell = q)$ such that for each $i \in [\ell]$, the vertices v_{i-1} and v_i are incident on a common face. The *length* of the radial path equals ℓ, so that a trivial radial path from v to itself has length 0. The *radial distance* in plane graph G between p and q, denoted $d^{\mathcal{R}}_G(u, v)$, is defined as the minimum length of a radial pq-path.

For a plane graph G, let $F(G)$ denote the set of faces of G. For a plane (multi)graph G, an *overlay graph* G' of G is a graph with vertex set $V(G) \cup F(G)$ obtained from G as follows. For each face $f \in F(G)$, draw a vertex with identity f in the interior of f. For each connected component C of edges incident on the face f, traverse the boundary walk of C starting at an arbitrary vertex. Every time a vertex v is visited by the boundary walk, draw a new edge between v and the vertex representing f, without crossing previously drawn edges. Doing this independently for all faces of G yields an overlay graph G'. Observe that an overlay graph may have multiple edges between some $f \in F(G)$ and $v \in V(G)$, which occurs for example when v is incident on a bridge that lies on f. The resulting plane multigraph G' is in general not unique, due to different homotopies for how edge bundles may be routed around different connected components inside a face. For our purposes, these distinctions are never important. We therefore write $\mathcal{L}(G)$ to denote an arbitrary fixed overlay graph of G. Observe that $F(G)$ forms an independent set in $\mathcal{L}(G)$.

Apart from the overlay graph, we will also use the related notion of *radial graph* (also known as face-vertex incidence graph). A *radial graph* of a connected plane graph G is a plane multigraph $\mathcal{R}(G)$ obtained from $\mathcal{L}(G)$ by removing all edges with both endpoints in $V(G)$. Hence a radial graph of G is bipartite with vertex set $V(G) \cup F(G)$, where vertices are connected to the representations of their incident faces. From these definitions it follows that $\mathcal{L}(G)$ is the union of G and $\mathcal{R}(G)$, which explains the terminology.

We need also the following simple but useful lemma.

▶ **Lemma 2.1.** *Let G be a connected graph, let $T \subseteq V(G)$ and assume that for each vertex $v \in V(G)$, there is a terminal $t \in T$ that can reach v by a path of at most K edges. Then G contains a Steiner tree of at most $(2K+1)(|T|-1)$ edges on terminal set T, which can be computed in linear time.*

Proof. Observe that there exists a spanning forest in G where each tree is rooted at a vertex of T, and each tree has depth at most K. Such a spanning forest can be computed in linear time by a breadth-first search in G, initializing the BFS-queue to contain all vertices of T with a distance label of 0. Consider the graph H obtained from G by contracting each tree into the terminal forming its root. Since G is connected, H is connected as well. An edge t_1t_2 between two terminals in H implies that in G there is a vertex in the tree of t_1 adjacent to a vertex of the tree of t_2. So for each edge in H, there is a path between the corresponding terminals in G consisting of at most $2K+1$ edges.

Compute an arbitrary spanning tree of the graph H, which has $|T|-1$ edges since H has $|T|$ vertices. As each edge of the tree expands to a path in G between the corresponding terminals of length at most $2K+1$, it follows that G has a connected subgraph F of at most $(2K+1)(|T|-1)$ edges that spans all terminals T. To eliminate potential cycles in F, take a spanning subtree of F as the desired Steiner tree. ◀

▶ **Lemma 2.2** ([16, Lemma 1]). *Let G be a planar bipartite graph with bipartition $V(G) = X \uplus Y$ for $X \neq \emptyset$. If all distinct $u, v \in Y$ satisfy $N_G(u) \not\subseteq N_G(v)$, then $|Y| \leq 5|X|$.*

3 Sparsification

A *plane partitioned graph* is an undirected multigraph G, together with a fixed embedding in the plane and a fixed partition $V(G) = \mathbf{A}(G) \uplus \mathbf{B}(G)$ where $\mathbf{A}(G)$ is an independent set. Consider a subgraph H of a plane partitioned graph G. The *cost* of H is defined as $\mathrm{cost}(H) := |V(H) \cap \mathbf{B}(G)|$, that is, we pay for each vertex of H in the part $\mathbf{B}(G)$. We say that H *connects* a subset $A \subseteq V(G)$ if $A \subseteq V(H)$ and A is contained in a single connected component of H.

Our main sparsification routine is the following.

▶ **Theorem 3.1.** *Given a connected plane partitioned graph G, one can in time $|\partial G|^{\mathcal{O}(1)} \cdot \mathcal{O}(|G|)$ find a subgraph $\widehat{G}$ in G, with the following properties:*
1. *$\widehat{G}$ contains all edges and vertices of ∂G,*
2. *$\widehat{G}$ contains $\mathcal{O}(|\partial G|^{212})$ edges,*
3. *for every set $A \subseteq V(\partial G)$ there exists a subgraph H of $\widehat{G}$ that connects A and has minimum possible cost among all subgraphs of G that connect A.*

In the subsequent sections, given a connected plane graph G, we will apply Theorem 3.1 to a graph G' that is either the overlay graph of G without the vertex corresponding to the outer face, or the radial graph of G. In either case, $\mathbf{A}(G') = V(G') \setminus V(G)$ is the set of face vertices and $\mathbf{B}(G') = V(G)$, i.e., we pay for each "real" vertex, not a face one. If the studied vertex-deletion graph separation problem in G turns into some Steiner problem in G', then we may hope to apply the sparsification routine of Theorem 3.1.

After this brief explanation of the motivation of the statement of Theorem 3.1, we proceed with an overview of its proof. We closely follow the divide&conquer approach of the polynomial kernel for Steiner Tree in planar graphs [25].

We adopt the notation of (strictly) enclosing from [25]. For a closed curve γ on a plane, a point $p \notin \gamma$ is strictly enclosed by γ if γ is not continuously retractable to a single point in the plane punctured at p. A point p is *enclosed* by γ if it is strictly enclosed or lies on γ. The notion of (strict) enclosure naturally extends to vertices, edges, and faces of a plane graph G being (strictly) enclosed by γ; here a face (an edge) is strictly enclosed by γ if every interior point of a face (every point on an edge except for the endpoints, respectively) is strictly enclosed. We also extend this notion to (strict) enclosure by a closed walk W in a plane graph G in a natural manner. Note that this corresponds to the natural notion of (strict) enclosure if W is a cycle or, more generally, a closed walk without self-intersections.

We start with restricting the setting to G being bipartite and ∂G being a simple cycle. Theorem 3.1 follows from Lemma 3.2 by simple manipulations, and its proof can be found on arXiv [18].

▶ **Lemma 3.2.** *The statement of Theorem 3.1 is true in the restricted setting with G being a connected bipartite simple graph with ∂G being a simple cycle and $\mathbf{A}(G)$ being one of the bipartite color classes (so that $\mathbf{B}(G)$ is an independent set as well).*

We now sketch the proof of Lemma 3.2. The full proof can be found on arXiv [18].

First observe that the statement of Lemma 3.2 is well suited for a recursive divide&conquer algorithm. As long as $|\partial G|$ is large enough, we can identify a subgraph S of G such that:
1. The number of edges of S is $\mathcal{O}(|\partial G|)$;
2. For every set $A \subseteq V(\partial G)$ there exists a subgraph H of G that connects A, has minimum possible cost among all subgraphs of G that connect A, and for every finite face f of $S \cup \partial G$, if G_f is the subgraph of G consisting of the edges and vertices embedded within the closure of f, then one of the following holds:
   a. $|\partial G_f| \leq (1-\delta)|\partial G|$ for some universal constant $\delta > 0$;
   b. H does not contain any vertex of degree more than 2 that is strictly inside f.

Similarly as in the case of [25], we show that such a subgraph S is good for recursion. First, we insert S into the constructed sparsifier $\widehat{G}$. Second, we recurse on G_f for every finite face f of $S \cup \partial G$ that satisfies Point 2a. Third, for every other finite face f (i.e., one satisfying Point 2b), we insert into $\widehat{G}$ a naive shortest-paths sparsifier: for every two vertices $u_1, u_2 \in V(\partial G_f)$, we insert into $\widehat{G}$ a minimum-cost path between u_1 and u_2 in G_f.

Property 1 together with the multiplicative progress on $|\partial G|$ in Point 2a ensure that the final size of $\widehat{G}$ is polynomial in $|\partial G|$, with the exponent of the polynomial bound depending on δ and the constant hidden in the big-$\mathcal{O}$ notation in Property 1.

The main steps of constructing S are the same as in [25]. First, we try minimum-size (i.e., with minimum number of edges, as opposed to minimum-cost) Steiner trees for a constant number of terminals on ∂G. If no such trees are found, the main technical result of [25] shows that one can identify a cycle C in G of length $\mathcal{O}(|\partial G|)$ with the guarantee that for any choice of $A \subseteq V(\partial G)$, there exists a minimum-size Steiner tree connecting A that does not contain any Steiner point strictly inside C. In [25] such a cycle is used to construct a desired subgraph S with the inside of C being a face satisfying the Steiner tree analog of Point 2b. In the case of Lemma 3.2, we need to perform some extra work here to show that – by some shortcutting tricks and adding some slack to the constants – one can construct such a cycle C' with the guarantee that the face f inside C' satisfies exactly the statement of Point 2b: that is, no "Steiner points" with regards to minimum-*cost* trees, not minimum-*size* ones.

In other words, the extra work is needed to at some point switch from "minimum-size" subgraphs (treated by [25]) to "minimum-cost" ones (being the main focus of Lemma 3.2). In our proof, we do it as late as possible, trying to re-use as much of the technical details of [25] as possible. Observe that for a path H in G, the cost of H equals $|E(H)|/2$ up to an additive $\pm\frac{1}{2}$ error. Similarly, for a tree H with a constant number of leaves, the cost of H is $|E(H)|/2$ up to an additive error bounded by a constant. Hence, as long as we focus on paths and trees with bounded number of leaves, the "size" and "cost" measures are roughly equivalent. However, if a tree H in G contains a high-degree vertex $v \in \mathbf{B}(G)$, the cost of H may be much smaller than half of the number of edges of H: a star with a center in $\mathbf{B}(G)$ has cost one and arbitrary number of edges. For this reason, the final argument of the proof of Lemma 3.2 that constructs the aforementioned cycle C' using the toolbox of [25] needs to be performed with extra care (and some sacrifice on the constants, as compared to [25]).

4 Odd Cycle Transversal

To understand the Odd Cycle Transversal problem, we rely on the correspondence between odd cycle transversals and T-joins. This correspondence was originally developed by Hadlock [15] for the edge version of Odd Cycle Transversal on planar graphs; for the vertex version discussed here, we build on the work of Fiorini et al. [8]. Given a graph H and set $T \subseteq V(H)$, a *T-join* in H is a set $J \subseteq E(H)$ such that T equals the set of odd-degree vertices in the subgraph of H induced by J. It is known that a connected graph contains a T-join if and only if $|T|$ is even.

▶ **Lemma 4.1** ([8, Lemma 1.1]). *Let T be the set of odd faces of a connected plane graph G. Then $C \subseteq V(G)$ is an odd cycle transversal of G if and only if $\mathcal{R}(G)[C \cup F(G)]$ contains a T-join, that is, each connected component of $\mathcal{R}(G)[C \cup F(G)]$ contains an even number of vertices of T.*

This leads to the following problem :

> Bipartite Steiner T-join **Parameter:** k
>
> **Input:** A connected bipartite graph G, a fixed partition $V(G) = \mathbf{A}(G) \uplus \mathbf{B}(G)$, $T \subseteq \mathbf{A}(G)$, and an integer k.
>
> **Question:** Does there exist a set $C \subseteq \mathbf{B}(G)$ of size at most k such that $G[C \cup \mathbf{A}(G)]$ contains a T-join, that is, each connected component of $G[C \cup \mathbf{A}(G)]$ contains an even number of vertices of T?

In particular, we are interested in the problem when G is a plane graph, which we call Plane Bipartite Steiner T-join. We call T the set of *terminals* of the instance; $\mathbf{A}(G) \setminus T$ is the set of *non-terminals*. We call $C \subseteq \mathbf{B}(G)$ a *solution* to an instance of Bipartite Steiner T-join if $|C| \leq k$ and $G[C \cup \mathbf{A}(G)]$ contains a T-join.

▶ **Lemma 4.2.** *If* Plane Bipartite Steiner T-join *has a polynomial kernel, then* Plane Odd Cycle Transversal *has a polynomial kernel.*

Proof. By Lemma 4.1, the answer to a plane instance (G, T, k) of Odd Cycle Transversal is equivalent to the answer of the Plane Bipartite Steiner T-join instance on the graph $\mathcal{R}(G)$, with the face vertices $F(G)$ taking the role of $\mathbf{A}$, $V(G)$ taking the role of $\mathbf{B}$, and $T \subseteq F(G)$ being the odd faces. So if Plane Bipartite Steiner T-join has a polynomial kernel, then an instance of Plane Odd Cycle Transversal can be compressed to size polynomial in k by transforming it into an instance of Plane Bipartite Steiner T-join and applying the kernel to it. Since Plane Bipartite Steiner T-join is in NP and Plane Odd Cycle Transversal is NP-hard, by standard arguments (cf. [2]) the T-join instance can be reduced back to an instance of the original problem of size polynomial in k, which forms the kernel. ◀

Below, we will give a polynomial kernel for Plane Bipartite Steiner T-join. Combined with Lemma 4.2, this implies a polynomial kernel for Plane Odd Cycle Transversal.

4.1 Reducing the number of terminals

Let $(G, \mathbf{A}(G), \mathbf{B}(G), T, k)$ be an instance of Plane Bipartite Steiner T-join. As a first step, we show that the graph can be reduced so that there remain at most $6k^2$ terminals. To this end, we adapt the rules that Suchý [28] developed for Plane Steiner Tree parameterized by the number of Steiner vertices of the solution tree. Each of the rules is applied exhaustively before a next rule will be applied.

▶ **Observation 4.3.** *Let C be a solution for the instance. Then each vertex of T has a neighbor in C.*

This is the analogue of [28, Lemma 2] and is immediate from the bipartiteness of G.

▶ **Observation 4.4.** *If $k < 0$ or there is a connected component containing exactly one terminal $t \in T$, then we can safely answer NO.*

▶ **Lemma 4.5.** *Let $X \subseteq T$ be a maximal set such that $N_G(x) = N_G(y)$ for all $x, y \in X$. Remove all but $2 - (|X| \bmod 2)$ vertices of X from the graph and T. The resulting instance $(G', \mathbf{A}(G'), \mathbf{B}(G'), T', k)$ has a solution if and only if $(G, \mathbf{A}(G), \mathbf{B}(G), T, k)$ has a solution.*

Proof. Let $Y \subseteq X$ be the set of remaining vertices of X. Observe that $|X| \equiv |Y| \pmod 2$ and that $|Y| \geq 1$. The equivalence is now immediate. ◀

▶ **Lemma 4.6.** *Let $u, v \in \mathbf{B}(G)$ and let $L = N_G(u) \cap N_G(v) \cap T$ with $L \neq \emptyset$. If a connected component X of $G \setminus (L \cup \{u, v\})$ exists that contains no terminals, then remove X from the graph. The resulting instance $(G', \mathbf{A}(G'), \mathbf{B}(G'), T, k)$ has a solution if and only if $(G, \mathbf{A}(G), \mathbf{B}(G), T, k)$ has a solution.*

Proof. If $(G', \mathbf{A}(G'), \mathbf{B}(G'), T, k)$ has a solution C, then C is a solution for the instance $(G, \mathbf{A}(G), \mathbf{B}(G), T, k)$, as G' is an induced subgraph of G with the same terminal set.

Suppose that C is a minimal solution for $(G, \mathbf{A}(G), \mathbf{B}(G), T, k)$. We construct a solution for $(G', \mathbf{A}(G'), \mathbf{B}(G'), T, k)$ such that $C' \cap X = \emptyset$. Suppose that $C \cap X \neq \emptyset$; otherwise, let $C' = C$. Let $C' = (C \setminus X) \cup \{v\}$. In either case, $|C'| \leq |C| \leq k$. We claim that C' is still a solution for $(G, \mathbf{A}(G), \mathbf{B}(G), T, k)$. To this end, first consider $C \cup \{v\}$. All connected components of $G[C \cup \mathbf{A}(G)]$ that neighbor v will then be unified into a single connected component Z of $G[C \cup \{v\} \cup \mathbf{A}(G)]$. The parity of $|Z \cap T|$ is equal to the sum (mod 2) of the parities of $|Z' \cap T|$ of the connected components Z' of $G[C \cup \mathbf{A}(G)]$ neighboring v. Since these parities are 0, their sum is 0, and $|Z \cap T|$ is even. Now consider the connected component ZZ of $G[C' \cup \mathbf{A}(G)]$ that contains v. Clearly, $ZZ \cap T = Z \cap T$, because $X \cap T = \emptyset$ and any path in $G[C \cup \{v\} \cup \mathbf{A}(G)]$ that intersects X can be re-routed through v and the vertices of $L \subseteq \mathbf{A}(G)$. The claim follows, and thus the lemma as well. ◀

We now present the final two reduction rules. Each relies on the following operation.

▶ **Lemma 4.7.** *Let $v \in \mathbf{B}(G)$. Let G' be obtained from G by contracting all edges between v and its neighbors in G. Let v' be the resulting vertex, and let $\mathbf{A}(G')$ and $\mathbf{B}(G')$ be the resulting color classes, where $v' \in \mathbf{A}(G')$. Let T' be obtained from T by removing $N_G(v) \cap T$, and adding v' to T' if and only if $|N_G(v) \cap T| \equiv 1 \pmod 2$.*

- *If $(G, \mathbf{A}(G), \mathbf{B}(G), T, k)$ has a solution C with $v \in C$, then $(G', \mathbf{A}(G'), \mathbf{B}(G'), T', k-1)$ has a solution;*
- *if $(G', \mathbf{A}(G'), \mathbf{B}(G'), T', k-1)$ has a solution, then $(G, \mathbf{A}(G), \mathbf{B}(G), T, k)$ has a solution.*

Proof. Suppose there is a solution C to $(G, \mathbf{A}(G), \mathbf{B}(G), T, k)$ such that $v \in C$. Then the vertices of $T \cap N_G(v)$ are in the same connected component Z of $G[C \cup \mathbf{A}(G)]$. Let $C' = C \setminus \{v\}$ and let Z' be obtained from Z by contracting all edges between v and $N_G(v)$. Then Z' is a connected component of $G'[C' \cup \mathbf{A}(G')]$. By the construction of T', Z' contains an even number of vertices of T'. Moreover, $|C'| = |C| - 1 \leq k - 1$. Hence, C' is a solution to $(G', \mathbf{A}(G'), \mathbf{B}(G'), T', k-1)$.

Suppose there is a solution C' to $(G', \mathbf{A}(G'), \mathbf{B}(G'), T', k-1)$. Let $C = C' \cup \{v\}$. Let Z' be the connected component of $G'[C' \cup \mathbf{A}(G')]$ that contains v', and let Z be obtained from Z' by adding $N_G[v]$ and removing v'. Then Z is a connected component of $G[C \cup \mathbf{A}(G)]$. Moreover, by the construction of T', Z contains an even number of vertices of T. Finally, $|C| = |C'| + 1 \leq k$. Hence, C is a solution to $(G, \mathbf{A}(G), \mathbf{B}(G), T, k)$. ◀

▶ **Lemma 4.8.** *Let $u, v \in \mathbf{B}(G)$ and let $L = N_G(u) \cap N_G(v) \cap T$ with $L \neq \emptyset$. If a connected component X of $G \setminus (L \cup \{u, v\})$ exists for which all terminals in $X \cap T$ neighbor v and there is a solution C to the instance $(G, \mathbf{A}(G), \mathbf{B}(G), T, k)$, then there is a solution that contains v.*

Proof. Assume that $v \notin C$, or the lemma would already follow. Since the rule of Lemma 4.6 is inapplicable, there is a terminal in X. Moreover, no terminal in $X \cap T$ neighbors u, because any such terminal would be in L and thus not in X. Since every terminal has to have a neighbor in C, it follows that $C \cap X \neq \emptyset$. Therefore, $C' = (C \setminus X) \cup \{v\}$ is not larger than C. We claim that C' is still a solution. To this end, first consider $C \cup \{v\}$. All connected components of $G[C \cup \mathbf{A}(G)]$ that neighbor v will then be unified into a single connected component Z of $G[C \cup \{v\} \cup \mathbf{A}(G)]$. In particular, Z contains $X \cap T$. The parity of $|Z \cap T|$ is equal to the sum (mod 2) of the parities of $|Z' \cap T|$ of the connected components Z' of $G[C \cup \mathbf{A}(G)]$ that neighbor v. Since these parities are 0, their sum is 0, and $|Z \cap T|$ is even. Now consider the connected component ZZ of $G[C' \cup \mathbf{A}(G)]$ that contains v. Clearly, $ZZ \cap T = Z \cap T$, because any path in $G[C \cup \{v\} \cup \mathbf{A}(G)]$ that intersects X can be re-routed through v and the vertices of L. The claim follows, and thus the lemma as well. ◀

▶ **Lemma 4.9.** *If there is a vertex $v \in \mathbf{B}(G)$ adjacent to more than $6k$ terminals and there is a solution C to the instance $(G, \mathbf{A}(G), \mathbf{B}(G), T, k)$, then there is a solution that contains v.*

Proof. The proof is completely analogous to the proof of [28, Lemma 11]. If $v \in C$, then we are done. So assume that $v \notin C$. Let $B \subseteq C$ be the set of vertices in C adjacent to at least two terminals in $N_G(v)$. Given $b \in B$, let x, y be any two terminals in $N_G(b) \cap N_G(v)$ and consider the region R that is enclosed by the cycle x, b, y, v and that does not contain the outer face. If R does not contain any other terminal of $N_G(b) \cap N_G(v)$, then R is called the *(internal) eye* of x, b, y, v. The *support* of $b \in B$, denoted $\text{supp}(b)$, is the set of vertices $a \in B$ such that a is contained inside an eye R of b, but not inside an eye of any $b' \in B \setminus \{b\}$ for which b' is inside R. The bound of $6k$ (instead of $5k$) ensures that the proof of [28, Lemma 16] can be modified (straightforwardly) to yield a vertex $b \in \mathbf{B}(G)$ adjacent to more than $2\,|\text{supp}(b)| + 4$ vertices of T. The further arguments then imply the existence of a twin set in T of size at least 3, thus contradicting the exhaustive execution of the rule of Lemma 4.5. ◀

Lemma 4.8 and 4.9, when combined with Lemma 4.7, naturally lead to two reduction rules. After exhaustively applying all the reduction rules in this section, each vertex of $\mathbf{B}(G)$ neighbors at most $6k$ terminals.

▶ **Observation 4.10.** *If $|T| > 6k^2$, then we can safely answer NO.*

This rule is immediate from Observation 4.3 and the fact that any solution contains at most k vertices that are each adjacent to at most $6k$ terminals by Lemma 4.9.

4.2 Reducing the diameter and obtaining the kernel

We now reduce the diameter of the graph. Our arguments here are a generalization of the arguments of Fiorini et al. [8] in their FPT-algorithm for Plane Odd Cycle Transversal.

▶ **Lemma 4.11.** *Suppose there is a solution for $(G, \mathbf{A}(G), \mathbf{B}(G), T, k)$. Let C be a minimal solution. Then each vertex $v \in C$ has distance at most $k+1$ in $G[C \cup \mathbf{A}(G)]$ to a vertex of T.*

Proof. Suppose for sake of contradiction that $v \in C$ has distance at least $k+1$ to each vertex of T in $G[C \cup \mathbf{A}(G)]$. Since C is minimal, there are two connected components X and Y of $G[(C \setminus \{v\}) \cup \mathbf{A}(G)]$ with an odd number of terminals. Let $x \in X \cap T$ and $y \in Y \cap T$. Consider a shortest path in $G[C \cup \mathbf{A}(G)]$ from x to v. This path P is fully contained in $G[V(X) \cup \{v\}]$ and has length at least $k+1$. As P connects vertices on opposite sides of the bipartite graph, $|V(P) \cap C| \geq 1 + k/2$. Hence, $|V(X) \cap C| \geq k/2$. Similarly, $|V(Y) \cap C| \geq k/2$. Since X and Y are vertex disjoint, it follows that $|C| \geq 2k/2 + 1 > k$, a contradiction. ◀

▶ **Corollary 4.12.** *Suppose there is a solution for $(G, \mathbf{A}(G), \mathbf{B}(G), T, k)$. Let C be a minimal solution. Then every vertex of $C \cup T$ has distance at most $k+2$ to T in $G[C \cup \mathbf{A}(G)]$.*

▶ **Lemma 4.13.** *We can safely answer NO, or we can compute, in polynomial time, disjoint subgraphs $G_1, \ldots, G_\ell$ of G for some $\ell \leq k$ such that:*
1. *the graphs G_i jointly contain all terminals;*
2. *for each i and for each vertex $v \in V(G_i)$, there is a terminal $t \in T \cap V(G_i)$ that can reach v by a path of at most $k+2$ edges;*
3. *for any solution C for $(G, \mathbf{A}(G), \mathbf{B}(G), T, k)$, $C \cap V(G_i)$ is a solution for $(G_i, \mathbf{A}(G_i), \mathbf{B}(G_i), T \cap V(G_i), k_i)$ for each i, where $k_i = |C \cap V(G_i)|$;*
4. *if $(G_i, \mathbf{A}(G_i), \mathbf{B}(G_i), T \cap V(G_i), k_i)$ has a solution for each i for some $k_1, \ldots, k_\ell \geq 0$ that sum up to at most k, then $(G, \mathbf{A}(G), \mathbf{B}(G), T, k)$ has a solution.*

Proof. For each terminal $t \in T$, let $B(t)$ be the set of all vertices within distance $k+2$ of t. Let $G_1, \dots, G_\ell$ be the connected components of $G[\bigcup_{t \in T} B(t)]$. If $\ell > k$, then G has more than k terminals with disjoint neighborhoods in $\mathbf{B}(G)$, and we can safely answer NO. We now consider the properties set forth in the lemma statement:

1. True by construction and the definition of the function B.
2. True by construction and the definition of the function B.
3. True by construction, the definition of the function B, and Corollary 4.12.
4. We take the union C of the solutions C_i of the sub-instances. Note that the subgraphs G_i are disjoint and thus contain disjoint sets of terminals. Hence, any connected component of $G[C \cup \mathbf{A}(G)]$ that contains connected components of $G[C_i \cup \mathbf{A}(G)]$ for multiple i, still contains an even number of terminals.

This finishes the proof. ◀

Property 2 of Lemma 4.13 implies that each constructed subgraph G_i has diameter $\mathcal{O}(k \cdot |T \cap V(G_i)|)$, which is $\mathcal{O}(k^3)$ using Observation 4.10. The proof of Theorem 4.14 employs an additional argument to obtain a quadratic-size Steiner tree to cut open.

▶ **Theorem 4.14.** Plane Bipartite Steiner T-join *has a kernel of size* $\mathcal{O}(k^{425})$*.*

Proof. We first exhaustively apply the reduction rules of Subsection 4.1 until each vertex of $\mathbf{B}(G)$ neighbors at most $6k$ terminals. The rules can clearly be executed exhaustively in polynomial time. As per Observation 4.10, we may assume that $|T| \leq 6k^2$. Then we apply Lemma 4.13 and consider each of the ℓ subgraphs G_i separately. Let $T_i = T \cap V(G_i)$; note that $|T_i| \leq 6k^2$. Moreover, we can assume that $|T_i|$ is even, or we can safely answer NO.

We construct a small set $A_i \subseteq V(G_i)$ such that $G[A_i]$ is connected and contains T_i. Start by adding T_i to A_i. Then, we find a subset T_i' of T_i such that the sets $N_{G_i}(t)$ are pairwise disjoint for $t \in T_i'$ by the following iterative marking procedure: add any unmarked $t \in T_i$ to T_i' and then mark all terminals in $N_{G_i}(N_{G_i}(t))$. It follows from Observation 4.3 that $|T_i'| \leq k$, or we can safely answer NO. Now apply Lemma 2.1 to find a Steiner tree of at most $(2(k+2)+1)\,(|T_i'|-1)$ edges (and vertices) on T_i'. Add these vertices to A_i. Finally, for each $t \in T_i$, let $t' \in T_i'$ be a terminal such that $t \in N_{G_i}(N_{G_i}(t')) \cup \{t'\}$ and add an arbitrary vertex of $N(t) \cap N(t')$ to A_i. Then $|A_i| \leq 6k^2 + 6k^2 + (2k+5)\,|T_i'| = \mathcal{O}(k^2)$. Moreover, by construction, $G_i[A_i]$ is connected and contains T_i.

Let S_i be a spanning tree of $G_i[A_i]$. Note that S_i has size $\mathcal{O}(k^2)$ by the construction of A_i and contains T_i. We cut the plane open along S_i and make the resulting face the outer face. Let $\hat{G}_i$ denote the resulting plane graph. That is, we create a walk W_i on the edges of S_i that visits each edge of S_i exactly twice. This walk has $\mathcal{O}(k^2)$ edges. Then we duplicate the edges of S_i and duplicate each vertex v of S_i exactly $d_{S_i}(v) - 1$ times, where $d_{S_i}(v)$ is the degree of v in S_i. We can then create a face in the embedding that has W_i as boundary. Then we obtain $\hat{G}_i$ by creating an embedding in which this new face is the outer face. See Figure 2. This also yields a natural mapping π from $E(\hat{G}_i)$ to $E(G_i)$ and from $V(\hat{G}_i)$ to $V(G_i)$. Finally, we observe that the terminals T_i are all on the outer face of $\hat{G}_i$ and that $\hat{G}_i$ is a connected plane partitioned graph.

Now apply Theorem 3.1 to $\hat{G}_i$ and let $\tilde{G}_i$ be the resulting graph. Let $F_i = \pi(\tilde{G}_i)$. Note that $\tilde{G}_i$ has $\mathcal{O}(|\partial \hat{G}_i|^{212}) = \mathcal{O}(|W_i|^{212}) = \mathcal{O}(k^{424})$ edges, and thus so has F_i. Finally, let $F = \bigcup_{i=1}^{\ell} F_i$. Clearly, $|F| = \mathcal{O}(k^{425})$, as $\ell < k$. Also note that each of the reduction rules, the above marking procedures, and F itself can be computed in polynomial time.

We claim that $(F, \mathbf{A}(F), \mathbf{B}(F), T, k)$ is a kernel. Since F is a subgraph of G, it follows that if $(F, \mathbf{A}(F), \mathbf{B}(F), T, k)$ has a solution, then so does $(G, \mathbf{A}(G), \mathbf{B}(G), T, k)$. Now let C be a minimum solution for $(G, \mathbf{A}(G), \mathbf{B}(G), T, k)$. Then $C_i = C \cap V(G_i)$ is a solution for

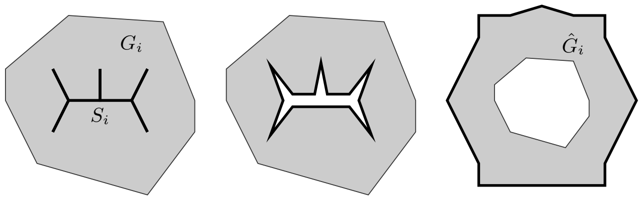

Figure 2 The process of cutting open the graph G_i along the tree S_i. Adapted from [24] with permission.

$(G_i, \mathbf{A}(G_i), \mathbf{B}(G_i), T \cap V(G_i), k_i)$ for each i, where $k_i = |C \cap V(G_i)|$. Consider some i and let J_i be a T-join of $G_i[C_i \cup \mathbf{A}(G_i)]$.

Let Z be a connected component of $G_i[J_i]$. We show how to find a connected subgraph Z' of F_i (and thus of G_i) such that $V(Z') \cap T_i \supseteq V(Z) \cap T_i$ and $|V(Z') \cap \mathbf{B}(G_i)| \leq |V(Z) \cap \mathbf{B}(G_i)|$. Consider the subgraph $\hat{Z}$ of $\hat{G}_i$ formed by $\pi^{-1}(V(Z) \cup E(Z))$. Note that any connected component Y of $\hat{Z}$ connects $A = V(Y) \cap \partial\hat{G}_i$. Then by Theorem 3.1, there is a subgraph $H(Y)$ of $\tilde{G}_i$ that connects A and has minimum possible cost among all subgraphs of G_i that connect A. Hence, $|V(H(Y)) \cap \mathbf{B}(G_i)| \leq |V(Y) \cap \mathbf{B}(G_i)|$. Now let H be the union of $H(Y)$ over all connected components Y of $\hat{Z}$. Observe that H is a subgraph of $\tilde{G}_i$. Let $Z' = (\pi(V(H)), \pi(E(H)))$. Observe that, by construction, Z' is a subgraph of F_i with the claimed properties. In particular, observe that although $\hat{Z}$ can be much larger than Z due to the duplication of vertices of $Z \cap S_i$ when G_i was cut open along S_i, we de-duplicate these vertices when using $\pi(V(H))$.

Consider the union J_i' of all these connected subgraphs Z' over all connected components Z of $G_i[J_i]$. Then $|V(G_i[J_i']) \cap \mathbf{B}(G_i)| \leq |V(G_i[J_i]) \cap \mathbf{B}(G_i)| = |C_i|$. Moreover, by construction, for each connected component ZZ of $G_i[J_i']$ there exists a set $\mathcal{Z}(ZZ)$ of connected components Z of $G_i[J_i]$ such that $ZZ \cap T$ is the union of $Z \cap T$ over all these connected components Z. We note that the sets $\mathcal{Z}(ZZ)$ induce a partition of the connected components of $G_i[J_i]$. Observe that the parity of $|ZZ \cap T|$ is equal to the sum (mod 2) of the parities of the corresponding connected components of $G_i[J_i]$, and thus, equal to 0. It follows that $G_i[J_i']$ contains a T_i-join. Hence, $V(G_i[J_i']) \cap \mathbf{B}(G_i)$ is a solution for $(G_i, \mathbf{A}(G_i), \mathbf{B}(G_i), T_i, k_i)$. By repeating this procedure for all i, it follows from the proof of Lemma 4.13 that the union of these solutions is a solution for $(G, \mathbf{A}(G), \mathbf{B}(G), T, k)$. Moreover, any T-join that is contained in this solution is fully contained in F. Hence, $(F, \mathbf{A}(F), \mathbf{B}(F), T, k)$ has a solution. ◀

▶ **Corollary 4.15.** Plane Odd Cycle Transversal *has a polynomial kernel.*

5 Vertex Multiway Cut

In this section we sketch our polynomial kernel for Planar Vertex Multiway Cut. Many important details are omitted in this presentation, but the full version can be found on arXiv [18]. Now, let (G, T, k) be an input of Planar Vertex Multiway Cut.

The first step towards the kernel is a preprocessing routine that ostensibly aims to reduce the diameter of G. Recall that for Odd Cycle Transversal, we could reduce the diameter of the radial graph to $\mathcal{O}(k^3)$ by Lemma 4.13, which enabled us to find a small tree connecting the terminals in Theorem 4.14 along which we could cut open the graph. For Planar Vertex Multiway Cut, we use a much more involved argument to find this small tree.

To be more precise, we partition the vertices of the input plane graph G using its outerplanarity layers. A vertex belongs to outerplanarity layer $k \geq 1$ if it is on the outer face after $k - 1$ times simultaneously removing all vertices on the outer face. We then obtain a tree, denoted $\mathbb{T}(G)$, by simultaneously contracting all edges whose endpoints belong to the same layer; note that this operation shrinks each connected component induced by each layer to a node of the tree, and each node u of the tree corresponds to a set $\kappa(u)$ of vertices of G. We call a node u in $\mathbb{T}(G)$ *important* if $\kappa(u)$ contains a terminal or if two distinct children of u have a descendant v in $\mathbb{T}(G)$ for which $\kappa(v)$ contains a terminal. We then argue that if the unique path of P in $\mathbb{T}(G)$ between two important nodes has length more than $2(k+1)$, then any optimal T-multiway cut will only use vertices corresponding to the first $k+1$ nodes (call this set Q), to the last $k+1$ nodes (call this set R), or to a $(\kappa(x), \kappa(y))$-cut of size at most k for some $x \in Q, y \in R$. This means that only $\mathcal{O}(k^3)$ nodes along P are relevant, which combined with the definition of important nodes leads to $\mathcal{O}(k^3|T|)$ relevant nodes in $\mathbb{T}(G)$.

The intuition behind relevant nodes is that we are only interested in the part of the graph induced by relevant nodes, and thus we simultaneously contract all edges of G whose endpoints both belong to a non-relevant node of $\mathbb{T}(G)$. We denote by Z the set of vertices that arise due to this contraction. We must forbid that Z belongs to the solution of the kernel, a detail later dealt with by replacing each vertex of Z by a suitably chosen grid. We now use the LP-based reduction rules of Cygan et al. [4] to reduce the number of terminals to $2k$, so that there are only $\mathcal{O}(k^4)$ relevant nodes. The definition of $\mathbb{T}(G)$ combined with the contraction we described earlier implies that the radial distance of each terminal to the outer face is $\mathcal{O}(k^4)$. This enables us to find a tree H of size $\mathcal{O}(k^5)$ in the overlay graph of G along which we cut it open (cf. Figure 2). Call the resulting graph $\hat{G}$.

The second step of the kernel is to establish a correspondence between vertex cuts X in G and Steiner trees in $\hat{G}$ that connect vertices along $\partial\hat{G}$. To this end, observe that each connected component of $G \setminus X$ can be bounded by a closed curve γ that intersects the drawing of G only in vertices of X. This curve corresponds to a closed curve γ^* in the overlay graph of G that intersects its drawing only in vertices of X or $F(G)$. This set of intersected vertices X' will contain vertices of H such that in $\hat{G}$, X' can be decomposed to induce several Steiner trees that connect vertices along $\partial\hat{G}$. Then it suffices to note that the cost of each Steiner tree only depends on the number of vertices of $V(G)$ it contains and that vertices of $F(G)$ are free. By picking $\mathbf{A}(\hat{G}) = V(G)$ and $\mathbf{B}(\hat{G}) = F(G)$, we can then apply Theorem 3.1.

Note that the kernel contains, for each cut $X \subseteq V(G)$, a set $X' \subseteq V(G)$ that *mimics* the set X in the following way: for each $Y \subseteq X \cap V(\partial\hat{G})$ which is contained in a single connected component of $\hat{G}[X \cup F]$, the set Y is contained in a single connected component of $\hat{G}[X' \cup F]$. Then for every pair of vertices $u, v \in T$, if X is a (u, v)-cut in G then X' is also a (u, v)-cut in G. Hence by preserving minimum connectors for subsets of $V(\partial\hat{G})$, we preserve minimum solutions to Planar Vertex Multiway Cut.

6 Reductions to Vertex Planarization

In this section we show two reductions from Planar Vertex Multiway Cut: one to the disjoint version of Vertex Planarization, and one to the regular one. We start with recalling formal problem definitions.

Vertex Planarization	**Parameter:** k
Input: A graph G and an integer k.	
Question: Does there exist a set $X \subseteq V(G)$ such that $G \setminus X$ is planar?	

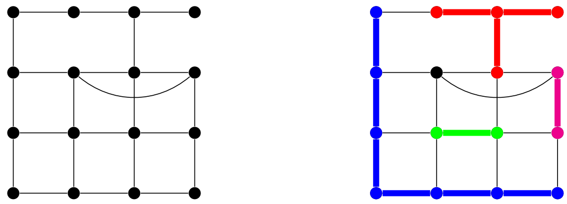

Figure 3 The graph H_0 of Observation 6.1 with the K_5 minor model on the right.

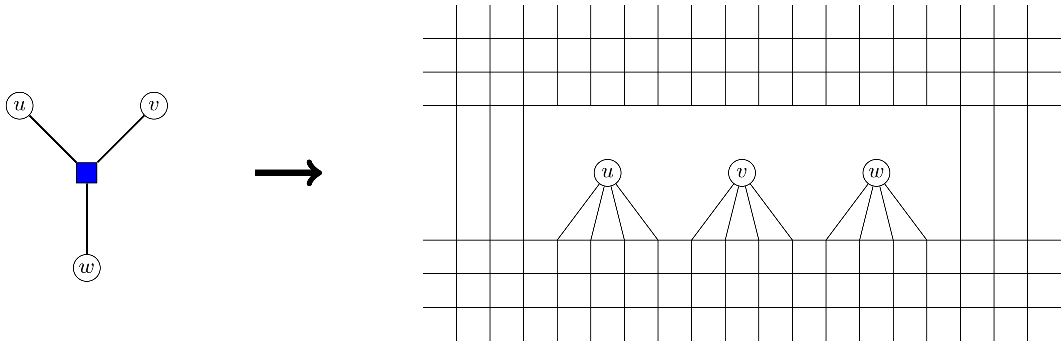

Figure 4 Embedding neighbors of a terminal (blue square) into a hole cut out in a large grid. Every neighbor of a terminal is connected to $k+1$ vertices of the grid ($k+1=4$ in the figure).

DISJOINT VERTEX PLANARIZATION	**Parameter:** $k+\|S\|$

Input: A graph G, a set $S \subseteq V(G)$ such that $G \setminus S$ is planar, and an integer k.
Question: Does there exist a set $X \subseteq V(G) \setminus S$ of size at most k such that $G \setminus X$ is planar?

Lemma 6.2 and 6.3 below give polynomial-parameter transformations from PLANAR VERTEX MULTIWAY CUT to DISJOINT VERTEX PLANARIZATION and VERTEX PLANARIZATION.

Both reductions rely on the same idea: if a VERTEX PLANARIZATION instance contains a large grid, the budget of k deletions is not able to effectively break it, and there is essentially only one way to embed it in the plane. If some parts of the graph are attached to vertices of the grid incident to faces far away from each other, a solution to VERTEX PLANARIZATION needs to separate such parts from each other. This allows to embed a PLANAR VERTEX MULTIWAY CUT instance. Formally, we rely on the following observation (see Figure 3).

▶ **Observation 6.1.** *Consider the following graph H_0: we start with H_0 being a 4×4 grid with vertices $x_{a,b}$, $1 \le a,b \le 4$ (i.e., the vertex $x_{a,b}$ lies in a-th row and b-th column of the grid) and then add an edge $x_{2,2}x_{2,4}$ but delete edges $x_{1,2}x_{2,2}$ and $x_{1,4}x_{2,4}$. Then H_0 contains a K_5 minor and is therefore not planar.*

▶ **Lemma 6.2.** *Given a* PLANAR VERTEX MULTIWAY CUT *instance (G,T,k), one can in linear time compute an equivalent* DISJOINT VERTEX PLANARIZATION *instance (G',S,k) with $|S| \le 8|T|$.*

Proof. If $|T| \le 1$, then the input instance is trivial, and we can output $G' = S = \emptyset$. Otherwise, let $T = \{t_1, t_2, \dots, t_{|T|}\}$. We start by constructing a $4 \times 2|T|$ grid H. Denote $S = V(H)$; note that $|S| = 8|T|$ as promised. For $1 \le i \le |T|$, let x_i be the $(2i)$-th vertex in the second row of H. We construct the graph G' from $G \uplus H$ by identifying t_i with x_i for every $1 \leq i \leq |T|$. We claim that the resulting Disjoint Vertex Planarization instance (G', S, k) is equivalent to the input Planar Vertex Multiway Cut instance (G, T, k). Note that $V(G) \setminus T = V(G') \setminus S$.

In one direction, let $X \subseteq V(G) \setminus T$ be a solution to Planar Vertex Multiway Cut on (G, T, k). We show that X is also a solution to Disjoint Vertex Planarization on (G', S, k) by showing a planar embedding of $G' \setminus X$. First, embed H in the natural way. Second, for every connected component C of $G \setminus X$, proceed as follows. If C contains a terminal t_i, then fix a planar embedding of C that keeps t_i incident to the infinite face, and embed C in one of the faces of H incident with x_i. Otherwise, if C does not contain any terminal, embed C in the infinite face of H. Since every connected component C contains at most one terminal, this is a valid planar embedding of $G' \setminus X$.

In the other direction, let $X \subseteq V(G') \setminus S$ be a solution to Disjoint Vertex Planarization on (G', S, k). We claim that X is also a solution to Planar Vertex Multiway Cut on (G, T, k). Assume the contrary; since $|X| \leq k$ and $X \subseteq V(G') \setminus S = V(G) \setminus T$ by assumption, we have two terminals $t_i, t_j \in T$ and a $t_i - t_j$ path P in $G \setminus X$. Consider the subgraph $H \cup P$ of $G' \setminus X$ and contract P to a single edge $t_i t_j$. Then, this minor of $G' \setminus X$ contains H_0 from Observation 6.1 as a minor. By Observation 6.1, $G' \setminus X$ contains K_5 as a minor, contradicting its planarity. ◀

▶ **Lemma 6.3.** *Given a* Planar Vertex Multiway Cut *instance* (G, T, k)*, one can in polynomial time compute an equivalent* Vertex Planarization *instance* (G', k) *with* $|E(G')| + |V(G')| \leq \mathcal{O}(k(|E(G)| + |V(G)|))$.

Proof. We proceed as in the proof of Lemma 6.2, but we need to make H thicker in order not to allow any tampering.

If $|T| \leq 1$, then the input instance is trivial, and we can output $G' = \emptyset$. Similarly, we output a trivial no-instance if two terminals of T are adjacent. Otherwise, fix a planar embedding ϕ of G and let $T = \{t_1, t_2, \ldots, t_{|T|}\}$. For every $1 \leq i \leq |T|$, let d_i be the degree of t_i in G and let $v_i^1, \ldots, v_i^{d_i}$ be the neighbors of t_i in G in clockwise order around t_i in ϕ. Let $D = \sum_{i=1}^{|T|} d_i$.

We define a graph H as follows. We start with H being a $4(k+1) \times (D + |T|)(k+1)$-grid with vertices $x_{a,b}$, $1 \leq a \leq 4(k+1)$, $1 \leq b \leq (D + |T|)(k+1)$ (i.e., the vertex $x_{a,b}$ lies in a-th row and b-th column). For every $1 \leq i \leq |T|$, let $b_i^{\leftarrow} = (i + \sum_{j<i} d_j)(k+1)$ and $b_i^{\rightarrow} = b_i^{\leftarrow} + d_i(k+1)$; additionally, let $b_0^{\rightarrow} = 0$. For every $1 \leq i \leq |T|$ and every $b_i^{\leftarrow} < b \leq b_i^{\rightarrow}$, we delete from H the edge $x_{k+1,b}x_{k+2,b}$; see Figure 4.

We now define the graph G' as follows. We start with $G' = H \uplus (G \setminus T)$. Then, for every $1 \leq i \leq |T|$ and every $1 \leq j \leq d_i$, we make v_i^j adjacent to $x_{k+2,b}$ for every $b_i^{\leftarrow} + (j-1)(k+1) < b \leq b_i^{\leftarrow} + j(k+1)$. This finishes the construction of the Vertex Planarization instance (G', k). We now show that it is equivalent to Planar Vertex Multiway Cut on (G, T, k).

In one direction, let X be a solution to Planar Vertex Multiway Cut on (G, T, k). We show that X is also a solution to Vertex Planarization on (G', k) by constructing a planar embedding of $G' \setminus X$. First, we embed H naturally and for every $1 \leq i \leq |T|$ let f_i be the face of the embedding that is incident with vertices $x_{k+2,b}$ for every $b_i^{\leftarrow} < b \leq b_i^{\rightarrow}$. Then, for every connected component C of $G \setminus X$ we proceed as follows. If C does not contain a terminal, then since $X \cap T = \emptyset$, component C contains no neighbors of terminals either; hence the vertices of C are not adjacent to H in G'. We embed C in the infinite face of H. Otherwise, assume that the only terminal of C is t_i. We take the embedding of

C induced by ϕ, change the infinite face so that t_i is incident with the infinite face, and embed $C \setminus t_i$ with the induced embedding into f_i. The fact that $v_i^1, \ldots, v_i^{d_i}$ are embedded around t_i in ϕ in this order allows us now to draw all edges between vertices of $N_G(t_i)$ and $\{x_{k+2,b} | b_i^{\leftarrow} < b \leq b_i^{\rightarrow}\}$ in a planar fashion.

In the other direction, let X' be a solution to Vertex Planarization on (G', k). We claim that $X := X' \cap (V(G) \setminus T)$ is a solution to Planar Vertex Multiway Cut on (G, T, k). If this is not the case, then there exist two terminals t_{i_1}, t_{i_2}, $1 \leq i_1 < i_2 \leq |T|$ and a path P from t_{i_1} to t_{i_2} in $G \setminus X$. Let $v_{i_1}^{j_1}$ be the neighbor of t_{i_1} on P and $v_{i_2}^{j_2}$ be the neighbor of t_{i_2} on P. Since $|X'| \leq k$, there exist:

- indices $1 \leq a_1 \leq k+1$, $k+2 \leq a_2 \leq 2k+2$, $2k+3 \leq a_3 \leq 3k+3$, $3k+4 \leq a_4 \leq 4k+4$ such that no vertex of X' is in rows numbered a_1, a_2, a_3, nor a_4 of H;
- for every $1 \leq i \leq |T|$, an index $b_{i-1}^{\rightarrow} < b_i \leq b_i^{\leftarrow}$ with no vertex of X' in the b_i-th column of H; and
- for every $1 \leq i \leq |T|$ and every $1 \leq j \leq d_i$ an index $b_i^{\leftarrow} + (j-1)(k+1) < b_i^j \leq b_i^{\leftarrow} + j(k+1)$ with no vertex of X' in the b_i^j-th column of H.

We conclude by observing that the graph H_0 from Observation 6.1 is a minor of a subgraph of $G' \setminus X$ induced by P, the a_1-th, a_2-th, a_3-th, and a_4-th rows of H, and columns of H with numbers b_{i_1}, $b_{i_1}^{j_1}$, b_{i_2}, $b_{i_2}^{j_2}$. ◀

7 Conclusions

We conclude with several open problems. First, the exponents in the polynomial bounds of our kernel sizes are enormous, similarly as for planar Steiner tree [25]. Thus, we reiterate the question of reducing the bound of the main sparsification routine of [25] to quadratic. Second, we hope that our tools can pave the way to a polynomial kernel for Vertex Planarization, which remains an important open problem. Third, nothing is known about the kernelization of Multiway Cut parameterized above the LP lower bound [4], even in the case of planar graphs and edge deletions.

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