THE ABSORPTION LAW *OR* HOW TO KREISEL A HILBERT-BERNAYS-LÖB

ALBERT VISSER

ABSTRACT. In this paper, we show how to construct for a given consistent theory U a Σ^0_1-predicate that both satisfies the Löb Conditions and the Kreisel Condition —even if U is unsound. We do this in such a way that U itself can verify satisfaction of an internal version of the Kreisel Condition.

1. Introduction

When does a predicate P in a theory U count as a provability predicate for U? There are various ideas on the market to explicate this notion. These ideas provide conditions for being a provability predicate that cater to various intuitions.[1] In the present paper, three classes of conditions will be considered: the Hilbert-Bernays-Löb Conditions, the Kreisel Condition and the Feferman Condition. We will introduce the various conditions with some care in Section 3. In the present paper, we will not go into the philosophical discussion about the meaning of the conditions and their relative pro's and con's. However, in Appendix A, we will give examples that illustrate that all three classes of conditions are independent of one another. These examples can help the reader to form her own impression of what the conditions involve and, possibly, help her to get more grip on the issues surrounding the choice between the various classes of conditions.

The aim of our paper is to study the interplay of the Hilbert-Bernays-Löb Conditions and the Kreisel Condition for the case of Σ^0_1-predicates. The Kreisel condition for a provability predicate $\triangle$ for a theory U demands that $U \vdash \triangle A$ iff $U \vdash A$. A first question is whether we can have the Kreisel Condition for a predicate that satisfies the Löb Conditions in case our theory U is unsound. A second question is as follows. If $\triangle$ satisfies the Löb Conditions, the theory U, when consistent, cannot verify both the Kreisel Condition and the internal Kreisel Condition $\triangle\triangle A \leftrightarrow \triangle A$. However, can we have the next best thing, to wit: given an appropriately good provability predicate $\Box$ for U, can we find a predicate $\triangle$, that satisfies the Löb Conditions and for which we have both $U \vdash \triangle A$ iff $U \vdash A$, and $U \vdash \Box\triangle A \leftrightarrow \Box A$? As we will see the answer to the last question is *yes*. We can find, in many cases, a predicate $\triangle$

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[1]We refer the reader to [HV14] and [Vis16a] for some philosophical discussion of these ideas.

that satisfies the Löb Conditions, the Kreisel Condition and the internal Kreisel Condition $U \vdash \Box\triangle A \leftrightarrow \Box A$.

We develop a general construction of a Σ^0_1-predicate $\triangle$ that satisfies both the Löb Conditions and the external and internal Kreisel Conditions from suitable data. The internal Kreisel principle $\vdash \Box\triangle A \leftrightarrow \Box A$ splits in two sub-principles, to wit, *the absorption law* $\vdash \Box\triangle A \to \Box A$ and *the emission law* $\vdash \Box A \to \Box\triangle A$. Our main focus will be on the absorption law.

1.1. **Historical Note.** The following fact is due to Orey. Suppose U is an extension of PA. Then, we can find an elementary α such that α represents the axiom set of U over PA and $U \nvdash \Box_\alpha\bot$. See [Fef60]. See also [Lin03, Chapter 2].

A construction of a Fefermanian predicate with Σ^0_1-axiomatization α for a theory U that extends PA, such that none of the iterated $\Box_\alpha$-inconsistency statements $\Box^n_\alpha\bot$ is provable in U is given in [Bek90]. See also [Kur17].

Clearly our result extends the results of Orey and Beklemishev.

The predicates constructed in the present paper can be viewed as slow provability predicates. The absorption law holds for slow provability predicates over PA. Slow provability over EA was introduced and studied in [Vis12]. Slow provability over PA was introduced and studied in [FRW13]. This notion was further studied in [HP16], [Fre17], [FP17] and [RatXX].

The disadvantage of the present approach to slow provability is that the connection to proof theory and ordinal analysis is not visible. The advantage of the present approach to slow provability compared to the one of [FRW13] is its wider scope. Moreover, as we discuss below, it is not known whether the approach of [FRW13] works for Heyting's Arithmetic HA, the constructive counterpart of PA, where our approach works without problems.

An alternative approach to obtain a provability style predicate that satisfies both the Löb Conditions and the Kreisel Condition can be found in Section 5 of [Vis16b]. The approach in the present paper has a number of advantages. First, it is somewhat more perspicuous. Secondly, the constructed predicates also satisfy the Hilbert-Bernays Condition. Thirdly, the construction of the predicates is fixed-point-free. Fourthly, using the present approach we can also, in a number of cases, construct predicates $\triangle$ with the desired properties that are Fefermanian.

The basic idea for the predicate constructed in this paper is due to Fedor Pakhomov. He suggested considering this predicate when I asked him whether there was a non-model theoretic proof of the absorption law for slow provability. However, the proof of absorption given in this paper is quite different from the one Fedor had in mind.

1.2. **Prerequisites.** The reader should be familiar with basic materials from [HP93]. For certain local results there may be further prerequisites but we will make these clear *in situ*.

2. Basic Conventions, Notations, Definitions

In this section we introduce basic conventions and fix some notations and give some definitions.

2.1. **Theories.** A theory U in this paper is a theory in the signature of arithmetic.[2] A theory is given by a set X of axioms. We will generally assume that X is a recursively enumerable set. However, X is just given as a set and it is not intrinsically connected with a presentation. We will assume as a default that U extends Elementary Arithmetic EA.

Two salient theories of the paper are Elementary Arithmetic EA and Peano Arithmetic PA. The theory EA is $\mathrm{I}\Delta_0 + \mathsf{exp}$. It is finitely axiomatizable by a single axiom B. See [HP93]. The predicate $x = \underline{\ulcorner B \urcorner}$ will be called β. The theory PA has a standard elementary presentation π of the axiom set.

We will also consider the extension of EA with the Σ^0_1-collection principle $\mathrm{B}\Sigma^0_1$. This principle is given by:

$$\vdash \forall x \leq a\, \exists y\, S_0(x,y) \to \exists b\, \forall x \leq a\, \exists y \leq b\, S_0(x,y).$$

Here S_0 is Σ^0_1.

2.2. **Arithmetization.** We will sometimes use implementation properties of the arithmetization like monotonicity and the efficiency of syntactical operations. For this reason, we outline a few features of the Gödel coding we intend to use. We use a style of Gödel numbering that is due to Smullyan (see [Smu61]). Our Gödel numbering is based on the length-first ordering. We enumerate the strings of our finite alphabet according to length and the strings of the same length alphabetically. The Gödel number of a string s will be the number of occurrence in this enumeration. In this ordering the arithmetical function tracing concatenation is of the order of multiplication. We can use our bijective coding of strings to implement sequences of numbers. This has the bonus that also concatenation of sequences of numbers will be of the order of multiplication.[3]

We will in many cases employ modal notations. E.g., let prov_α be the arithmetization of provability from the axioms in α. We write $\Box_\alpha A$ for $\mathsf{prov}_\alpha(\underline{\ulcorner A \urcorner})$. Here $\underline{\ulcorner A \urcorner}$ is the numeral of the Gödel number of A. We will sometimes quantify the sentence-variables inside a modal operator. For example, we write things like:

$$(\dagger) \quad \forall A, B\, ((\Box_\alpha A \wedge \Box_\alpha(A \to B)) \to \Box_\alpha B).$$

This stands for:

$$(\ddagger) \quad \forall x\, \forall y\, \forall z\, ((\mathsf{prov}_\alpha(x) \wedge \mathsf{imp}(x,y,z) \wedge \mathsf{prov}_\alpha(z)) \to \mathsf{prov}_\alpha(y)).$$

Admittedly, such notations are somewhat sloppy, but I think in practice they are very convenient. E.g., (†) is more pleasant to read than (‡).

We employ the usual conventions for quantifiying numerical variables into modal contexts. E.g. $\Box_\alpha A(x)$ means $\exists z\, (\mathsf{sub}(x, \underline{\ulcorner v_0 \urcorner}, \underline{\ulcorner A(v_0) \urcorner}, z) \wedge \mathsf{prov}_\alpha(z))$.

We will employ the witness comparison notation. Suppose $A = \exists x\, A_0(x)$ and $B = \exists x\, B_0(x)$. We write:

- $A \leq B$ for $\exists x\, (A_0(x) \wedge \forall y < x\, \neg B_0(y))$.
- $A < B$ for $\exists x\, (A_0(x) \wedge \forall y \leq x\, \neg B_0(y))$.

[2] Everything in the paper lifts to the more general case where a theory of arithmetic is interpretable in the given theory. However, it is pleasant to avoid the extra notational burden of the more general case. The notational burdens of the present paper seem to be sufficiently heavy.

[3] Usually there is some overhead in defining sequences since we want to add some materials to make the definition of the projection function easy. However the uses of sequences to define syntax and proofs usually only require that we can determine whether something occurs in a sequence before something else. For this one does not need the extra material.

2.3. **Ordering of Predicates for Axioms.** Let $\gamma(x)$ and $\delta(x)$ be formulas with only x free that EA-verifiably represent classes of arithmetical sentences. We write $\gamma \preceq \delta$ for $\mathsf{EA} \vdash \forall A\, (\mathsf{prov}_\gamma(A) \to \mathsf{prov}_\delta(A))$. Here prov_α is a standard arithmetization of provability from α.

3. Conditions for Provability Predicates

In this section, we introduce three (classes of) Conditions that aim to explicate when a predicate is a provability predicate.

3.1. **The Löb Conditions.** To state the Löb conditions we write $\triangle A$ for $P(\underline{\ulcorner A \urcorner})$ and $\vdash$ for provability in U. The Löb conditions (introduced in [Löb55]) are as follows.

L1. If $\vdash A$, then $\vdash \triangle A$
L2. $\vdash (\triangle A \wedge \triangle(A \to B)) \to \triangle B$
L3. $\vdash \triangle A \to \triangle\triangle A$

We obtain the *Hilbert-Bernays Conditions* in case we replace L3 by:

HB. $\vdash S \to \triangle S$, for Σ^0_1-sentences S,

The usual assumption connected to the Hilbert-Bernays conditions is that P be Σ^0_1, so that L3 is a special case of HB. It is easy to see that if P is not Σ^0_1, we can have L1, 2 and HB but not L4. E.g. we may take P to be Feferman provability over PA.

We note that, in case P is Σ^0_1, the Löb conditions are more general than the Hilbert-Bernays Conditions. For example, in a weak theory like S^1_2 we do have the Löb Conditions for a standard provability predicate, but it is unknown whether we have the Hilbert-Bernays Conditions.

Technically, the Löb Conditions constitute a superior analysis of the proof of the Second Incompleteness Theorem. The philosophical use of the Conditions is independent of their technical interest. The philosophical idea is that the Löb Conditions explicate *the theoretical role* that a provability predicate plays in a theory.

We note that the Löb conditions do depend on the choice of Gödel numbering and hence are still not entirely 'coordinatefree'. For a study of this dependence and a proposal to abstract away from it, see [Gra18].

The Löb Conditions also have a uniform and a global version. In the uniform version we allow parameters in the formulas inside the operator. For example, L2 becomes: $\vdash \forall \vec{x}\, (\triangle A(\vec{x}\,) \wedge \triangle(A(\vec{x}\,) \to B(\vec{x}\,))) \to \triangle B(\vec{x}\,)$. In the global version, the quantifiers over sentences are not outside but inside the theory. For example, L2 becomes: $\vdash \forall A, B \in \mathsf{sent}\, ((\triangle A \wedge \triangle(A \to B)) \to \triangle B)$. We note that the global version is stronger than the uniform one. We will not consider the strengthened conditions in the present paper.

3.2. **The Kreisel Condition.** The *Kreisel Condition* was first formulated in [Kre53]. Its statement is as follows:

K. $U \vdash \triangle A$ iff $U \vdash A$.

We note that the Kreisel Condition is of a quite different nature than the Löb Conditions. It just asks that the theory *numerates* its own provability.

One could imagine a variant of the Kreisel Condition where we just ask numerability in a base theory U_0 that is a sub-theory of U.

Finally, we observe that, like the Löb Conditions, the Kreisel Condition does depend on the chosen Gödel numbering.

3.3. **The Feferman Condition.** We explain the idea that a provability-predicate is *Fefermanian.* See [Fef60]. The main ingredient of the idea is simply to fix a preferred arithmetization of provability and allow the choice of the predicate α representing the axiom-set to be free, given that it satisfies certain adequacy conditions.

The best way to present a Fefermanian predicate is to view it as a tuple $\langle U_0, U, \alpha\rangle$. Here U_0 is the *base theory* and U is the *lead theory.* We ask that U extends the base U_0. We demand that α numerates the axiom set in the base theory U_0. So A is an axiom of U iff $U_0 \vdash \alpha(\underline{\ulcorner A \urcorner})$.

We note that the demands on a Fefermanian predicate treat the axioms of the lead theory via a condition similar to the Kreisel Condition.

In the present paper, we will consider Fefermanian predicate modulo provability in the base theory. Thus, we will say that P is Fefermanian for U over U_0 in the relaxed sense iff, there is an α such that $\langle U_0, U, \alpha\rangle$ is Fefermanian in the strict sense and $U_0 \vdash \forall x\,(P(x) \leftrightarrow \mathsf{prov}_\alpha(x))$.

The reader may object that the Feferman Condition does not count as a real condition since it employs an unspecified specification of the arithmetization.[4] Of course, the reader is correct here. Feferman, in his paper, does specify a choice for a proof system and an arithmetization. However, in Feferman's arithmetization the Gödel number of a formula is superexponential in its length, so it is not a convenient Gödel numbering to work with within EA. Moreover, if Feferman's specific Gödel numbering would really be the golden standard, it would be reasonable that everybody would know its specification, but, of course, that is not the case. I see the use of the Feferman idea more as dialogical. The reader is asked to take her favored good arithmetization in mind and read for prov provability according to that arithmetization. So, prov becomes context dependent like the word 'you'. I will employ the Feferman idea in this way.

3.4. **Properties of Fefermanian Predicates.** In this subsection we briefly consider some basic insights on Fefermanian predicates.

Let A_U be the class of all α in Σ^0_1 such that $\langle \mathsf{EA}, U, \alpha\rangle$ is Fefermanian.

Theorem 3.1. *Let U be a theory. Then A_U has a minimum w.r.t. $\preceq$ iff U is finitely axiomatizable.*

Proof. Suppose U is finitely axiomatizable, say by $A_0, \ldots, A_{n-1}$. Let $\alpha_0(x) := \bigvee_{i<n} x = \underline{\ulcorner A_i \urcorner}$. Consider any α in A_U. We find for $i < n$ that $U \vdash A_i$, and, hence $\mathsf{EA} \vdash \Box_\alpha A_i$.

We reason in EA. Suppose p witnesses $\Box_{\alpha_0} A$ and p_i, for $i < n$, witnesses $\Box_\alpha A_i$. We obtain an α-proof q of A by adding the p_i 'above' A_i to p. (Note that we do not need Σ^0_1-collection since n is standard.)

Suppose U is not finitely axiomatizable. Consider any $\alpha \in \mathsf{A}_U$. Clearly, for any n there is a B such that $U \vdash B$ but the axioms in α that are $\leq n$ do not prove B.

[4] As remarked above the other conditions suffer, admittedly to a lesser degree, from the same defect.

Hence, $C := \forall x\, \exists B\, (\Box_\alpha B \wedge \neg\, \Box_{\alpha_x} B)$ is true, where $\alpha_x(y) :\leftrightarrow \alpha(y) \wedge y \leq x$. Thus, $\mathsf{EA} + C$ is consistent. Let $\gamma(x) :\leftrightarrow \beta(x) \vee x = \underline{\ulcorner C \urcorner}$. We define:

$$\alpha'(x) :\leftrightarrow \alpha(x) \wedge \forall y \leq x\, \neg\, \mathsf{proof}_\gamma(x, \underline{\ulcorner \bot \urcorner}).$$

It is evident that $\alpha' \preceq \alpha$. Suppose (†) $\alpha \preceq \alpha'$. We reason inside $\mathsf{EA} + C$. By (†), we have $\forall B\, (\Box_\alpha B \to \Box_{\alpha'} B)$. Suppose p is a γ-proof of $\bot$. It follows that the α' axioms are below p. Consider B such that $\Box_\alpha B$ but not $\Box_{\alpha_p} B$. It follows that $\neg\, \Box_{\alpha'} B$. A contradiction. It follows that there is no γ-proof of $\bot$, in other words, $\Diamond_\gamma \top$. We leave $\mathsf{EA} + C$.

We have shown $\mathsf{EA} + C \vdash \Diamond_\gamma \top$. But this contradics the Second Incompleteness Theorem. Hence (†) must fail. ❑

Theorem 3.2. *Consider theories U_0 and U where* EA *is a sub-theory of U_0 and U_0 is a sub-theory of U. Suppose:*

a. P numerates U in U_0.
b. P contains U_0-provably all predicate-logical tautologies.
c. P is U_0-provably closed under finite conjunctions.
d. P is U_0-provably closed under modus ponens.

Then, P is Fefermanian for U over U_0 with P itself as representation of the axiom set.

Proof. Clearly, we have $U_0 \vdash \forall B \in P\, \mathsf{prov}_P(B)$. Conversely, reason in U_0. Suppose p is a P-proof of B. Let X be the finite set of P-axioms used in p. Then, $(\bigwedge X \to B)$ is a predicate logical tautology, so $(\bigwedge X \to B) \in P$. By closure under conjunction, we have $\bigwedge X \in P$. Hence, by closure under modus ponens, we find $B \in P$. ❑

Theorem 3.3. *Consider theories U_0 and U where $\mathrm{I}\Sigma^0_1$ is a sub-theory of U_0 and U_0 is a sub-theory of U. Suppose:*

a. P numerates U in U_0.
b. P contains U_0-provably all predicate-logical tautologies.
c. P is U_0-provably closed under modus ponens.

Then P is Fefermanian for U over U_0 with P itself as representation of the axiom set.

Proof. Under the assumptions of the theorem, we can prove that P is closed under finite conjunctions by Σ^0_1-induction. ❑

Example 3.4. We take as base and lead theory PA. The predicate $\Box_\pi \Box_\pi$ is Fefermanian. Similarly, for $\exists x\, \Box^{x+1}_\pi(\cdot)$. The last predicate is, modulo PA-provable equivalence, *Parikh provability* or *fast provability.* Parikh provability can be obtained by adding to an axiomatization based on π the Reflection Rule: $\vdash \Box_\pi A \Rightarrow \vdash A$. See [Par71]. See also [Hen16]. ❏

Theorem 3.5. *Suppose U extends* EA *and P is Fefermanian w.r.t. a $\Delta_0(\mathsf{exp})$-presentation α of the axiom set. Then, P satisfies the Löb Conditions.*

Theorem 3.6. *Suppose U extends $\mathsf{EA} + \mathrm{B}\Sigma^0_1$ and P is Fefermanian w.r.t. a Σ_1-presentation α of the axiom set. Then, P satisfies the Löb Conditions.*

Theorem 3.7. *Suppose $\langle U_0, U, \alpha \rangle$ is a strict Fefermanian representation and suppose U and U_0 are sound. Then, α satisfies the Kreisel Condition for U.*

Proof. Since U_0 is sound, we have $\alpha(\ulcorner A \urcorner)$ iff $U_0 \vdash \alpha(\underline{\ulcorner A \urcorner})$. So, α truly represents the axioms of U. It follows that $U \vdash A$ iff $\Box_\alpha A$. Since U is sound, we find $\Box_\alpha A$ iff $U \vdash \Box_\alpha A$. So, we may conclude $U \vdash A$ iff $U \vdash \Box_\alpha A$. ❑

3.5. **Examples.** We provide a list of examples for coincidence and separation of the conditions. As before β is the standard representation of the axiom of EA and π is the standard representation of the axioms set of Peano Arithmetic. We will, in our examples, prefer EA over PA, Σ^0_1-predicates over more complex ones, and sound theories over unsound ones. Only in the first examples of Example A.5 and Example A.7, perhaps, improvement is possible by finding an example that works for and over EA.

	base	lead	P	Löb	Kreisel	Feferman
Example A.1	EA	EA	Σ^0_1	$+$	$+$	$+$
Example A.2	EA	EA	Σ^0_1	$+$	$+$	$-$
Example A.3	EA	$\mathsf{EA} + \Box_\beta \bot$	Σ^0_1	$+$	$-$	$+$
	EA	EA	Σ^0_2			
Example A.4	EA	EA	Σ^0_1	$+$	$-$	$-$
Example A.5	PA	PA	Σ^0_2	$-$	$+$	$+$
	EA	EA	$\Sigma^0_{1,1}$			
Example A.6	EA	EA	Σ^0_1	$-$	$+$	$-$
Example A.7	PA	PA	Σ^0_2	$-$	$-$	$+$
	EA	$\mathsf{EA} + \Box_\beta \Box_\pi \bot$	$\Sigma^0_{1,1}$			
Example A.8	EA	EA	Σ^0_1	$-$	$-$	$-$

We will give the promised examples in Appendix A.

4. Combining Hilbert-Bernays-Löb and Kreisel

We will first present the basic form of our argument *in abstracto* and then construct concrete implementations.

4.1. **The Basic Argument.** In this subsection we present our main technical argument. The center of the subsection is the proof of Lemma 4.1.

Let U be a theory. Suppose $\alpha(x)$ is an elementary predicate that numerates the axioms of U in U. Let $\theta(y, z)$ be a Σ^0_1 binary predicate. We demand that θ is EA-verifiably, upwards persistent in y, i.e., we assume that

$$\mathsf{EA} \vdash (\theta(y, z) \wedge y < y') \to \theta(y', z).$$

Let $\boxbar_{\theta,y} A$ be $\theta(y, \underline{\ulcorner A \urcorner})$. We write $\boxbar_y A$ as long as θ is given in the context. We define:

- true is the Σ^0_1-truth predicate, which is of the form $\exists y\, \mathsf{true}_0(y, x)$, where true_0 is $\Delta_0(\mathsf{exp})$. We write $\mathsf{true}^z(x)$ for $\exists y \leq z\, \mathsf{true}_0(y, x)$.
- $\Box_{\alpha,(x)} A :\leftrightarrow \exists p \leq x\, \mathsf{proof}_\alpha(p, \underline{\ulcorner A \urcorner})$, where proof is the standard arithmetization of the proof predicate.[5]
- $\mathcal{S}(x) :\leftrightarrow \exists z\, \forall S \leq x\, (\Box_{\alpha,(x)} S \to \mathsf{true}^z(S))$. Here the variable '$S$' ranges over Σ^0_1-sentences.

[5] I use the round brackets to distinguish the intended notion from $\Box_{\alpha,x}$ which is used in some of the literature for $\Box_{\alpha_x}$, where $\alpha_x(y) :\leftrightarrow \alpha(x) \wedge y \leq x$.

- $\triangle_\theta A :\leftrightarrow \exists x(\boxbar_{\theta,x} A \wedge \mathcal{S}(x))$. We will usually write $\triangle$ for $\triangle_\theta$ suppressing the contextually given θ. We note that modulo some rewriting $\triangle_\theta$ is Σ^0_1.

The definition of $\triangle_\theta$ is in essence due to Fedor Pakhomov.

As explained in Subsection 2.2, we assume that we have a reasonable coding of proofs in which the code of the proof is larger than the code of the conclusion. We fix, for the moment θ in the background. We have:

$$
\begin{array}{rcl}
(\dagger)\quad \mathsf{EA} \vdash \triangle A & \rightarrow & \exists x\,(\boxbar_x A \wedge \forall S\,(\Box_{\alpha,(x)} S \rightarrow \mathsf{true}(S))) \\
\mathsf{EA} + \mathrm{B}\Sigma_1 \vdash \triangle A & \leftrightarrow & \exists x\,(\boxbar_x A \wedge \forall S\,(\Box_{\alpha,(x)} S \rightarrow \mathsf{true}(S)))
\end{array}
$$

We note that we can write the right-hand-side of ($\dagger$) as:

$$(\exists x\ \boxbar_x\ A) < (\exists x\,\exists S\,(\Box_{\alpha,(x)} S \wedge \neg\,\mathsf{true}(S))).$$

Here the witness comparison is only concerned with the outer quantifiers.

Lemma 4.1. $\mathsf{EA} + \forall x\,(\Box_\alpha\ \boxbar_x\ A \rightarrow \Box_\alpha A) \vdash \Box_\alpha \triangle A \rightarrow \Box_\alpha A$.

Proof. We find R such that $\mathsf{EA} \vdash R \leftrightarrow (\exists x\ \boxbar_x\ A) < \Box_\alpha R$. We note that R is Σ^0_1.

We reason in $\mathsf{EA} + \forall x\,(\Box_\alpha\ \boxbar_x\ A \rightarrow \Box_\alpha A)$. Suppose $\Box_\alpha \triangle A$. We prove $\Box_\alpha A$.

We reason inside $\Box_\alpha$. Since, by assumption, $\triangle A$, we have, for some x, (i) $\boxbar_x A$ and (ii) $\forall S \leq x\,(\Box_{\alpha,(x)} S \rightarrow \mathsf{true}(S))$. In case not $\Box_{\alpha,(x)} R$, by (i), we find R. If we do have $\Box_{\alpha,(x)} R$, we find R by (ii). We leave the $\Box_\alpha$-environment.

We have shown $\Box_\alpha R$. It follows, (a) that for some p, we have $\Box_\alpha \Box_{\alpha,(p)} R$ and, by the fixed point equation for R, (b) $\Box_\alpha((\exists x\ \boxbar_x\ A) < \Box_\alpha R)$. Combining (a) and (b) and the upward persistence of $\boxbar_x$, we find $\Box_\alpha\ \boxbar_p\ A$, and, thus, $\Box_\alpha A$, as desired. We leave $\mathsf{EA} + \forall x\,(\Box_\alpha\ \boxbar_x\ A \rightarrow \Box_\alpha A)$.

We have shown $\mathsf{EA} + \forall x\,(\Box_\alpha\ \boxbar_x\ A \rightarrow \Box_\alpha A) \vdash \Box_\alpha \triangle A \rightarrow \Box_\alpha A$. ❑

The proof of Lemma 4.1 deserves a few comments.

Remark 4.2. We note that the argument also works when we define $\triangle A$ as $\exists x\,(\boxbar_x A \wedge \forall S\,(\Box_{\alpha,(x)} S \rightarrow \mathsf{true}(S)))$. The argument does not use that $\triangle$ is Σ^0_1. ❑

Remark 4.3. In all applications of Lemma 4.1, EA verifies not just the principle $\forall x\,(\Box_\alpha\ \boxbar_x\ A \rightarrow \Box_\alpha A)$ for the concrete choice of $\boxbar$ of the application, but the stronger $\forall x\,\Box_\alpha(\boxbar_x A \rightarrow A)$. We note that using this last condition, we may obtain the theorem without the demand that $\boxbar_y$ is upward persistent in y. In $\mathsf{EA} + \forall x\,\Box_\alpha(\boxbar_x A \rightarrow A)$, we can go from $\Box_\alpha \Box_{\alpha,(p)} R$ and $\Box_\alpha((\exists x\ \boxbar_x\ A) < \Box_\alpha R)$ to $\Box_\alpha \bigvee_{z<p} \boxbar_z A$, and, hence, $\Box_\alpha A$. ❑

Remark 4.4. The proof of Lemma 4.1 does not use exponentiation and would work in S^1_2. The reason is that we only use $\mathsf{true}(R) \rightarrow R$, which is the direction of $\mathsf{true}(R) \leftrightarrow R$ that does not require exponentiation. ❑

Remark 4.5. Let i-EA be the constructive version of EA. Let U be a constructive theory that extends i-EA. Suppose i-$\mathsf{EA} \vdash (\theta(y,z) \wedge y < y') \rightarrow \theta(y',z)$.

Then, inspection shows that the entire proof of Lemma 4.1 also works when we substitute i-EA for EA. This uses the basic insight that $\Box_{\alpha,(x)} R$ is decidable in i-EA. So the case-splitting in the proof can be constructively justified.

Thus, we find i-$\mathsf{EA} + \forall x\,(\Box_\alpha\ \boxbar_x\ A \rightarrow \Box_\alpha A) \vdash \Box_\alpha \triangle A \rightarrow \Box_\alpha A$. ❑

For the next theorem, we need an important lemma. The lemma, or at least the basic proof idea, is well known. However, just for the record, it is good to have an explicit proof.

Lemma 4.6. $\mathsf{EA} \vdash \forall x\, \Box_\alpha \mathcal{S}(x)$.

Proof. We work in EA. We prove the desired result by induction on x. We need a multi-exponential bound for the $\Box_\alpha$-proofs. We will extract the desired bound by inspecting the induction step.

The base case is trivial since there will be no $S \leq 0$. The proof witnessing the base will be given by a standard number $\underline{n}$.

We turn to the induction step. Suppose p_0 witnesses $\Box_\alpha \mathcal{S}(x)$. We have two possibilities: $\mathsf{proof}_\alpha(x+1, S^*)$, for some $S^* \leq x+1$, or $\neg\, \mathsf{proof}_\alpha(x+1, S^*)$, for all $S^* \leq x+1$.

Suppose $\mathsf{proof}_\alpha(x+1, S^*)$. Inspecting the proof of the truth-lemma for true in [HP93, Ch V, Section 5b, pp361–366], we obtain a proof code p_1 such that $\mathsf{proof}_\alpha(p_1, S^* \to \mathsf{true}(S^*))$. The transformation $S^* \mapsto p_1$ is p-time. By [HP93, Ch III, Lemma 3.14, p175], we obtain an α-proof p_2 of $\mathsf{proof}_\alpha(x+1, S^*)$. The transformation $x+1 \mapsto p_2$ is of order $2^{x+1}_{\underline{k}}$, where k is standard and the subscript $\underline{k}$ indicates the number of iterations of exponentiation. Working inside $\Box_\alpha$ we can put these facts together to obtain (a) $\mathcal{S}(x)$, (b) $\mathsf{proof}_\alpha(x+1, S^*)$ and (c) $\mathsf{true}(S^*)$. Let z_0 be the witness of (a), let z_1 be the witness of (c). Then, it is easily seen that $z := \mathsf{max}(z_0, z_1)$ witnesses $\mathcal{S}(x+1)$.

Suppose $\forall S^* \leq x+1\, \neg\, \mathsf{proof}_\alpha(x+1, S^*)$. By [HP93, Ch III, Lemma 3.14, p175], we may find an α-proof p_3 of $\forall S^* \leq x+1\, \neg\, \mathsf{proof}_\alpha(x+1, S^*)$ where the transformation $x+1 \mapsto p_3$ is of order $2^{x+1}_{\underline{k}}$. Using (d) $\forall S^* \leq x+1\, \neg\, \mathsf{proof}_\alpha(x+1, S^*)$ inside $\Box_\alpha$, we easily find the desired proof of $\mathcal{S}(x+1)$.

We note that apart from a bit of overhead we extend p_0 with at most two proofs that are estimated by $2^{x+1}_{\underline{k}}$. So, the resulting proof will be of order $p_0 \times (2^{x+1}_{\underline{k}})^2$. Thus, after all is said and done, the proof we obtain of $\mathcal{S}(x+1)$ will be estimated by $\underline{n} \times (2^{x+1}_{\underline{k}})^{2(x+1)} = 2^{2^{x+1}_{\underline{k}-1} \times 2(x+1)} \leq 2^{2x+1}_{\underline{k}}$, assuming that $k \geq 2$. ❑

Lemma 4.7. $\mathsf{EA} \vdash \forall x\, \Box_\alpha(\boxbar_x A \to \vartriangle A)$. *Hence,*

$$\mathsf{EA} + (\Box_\alpha A \to \exists x\, \Box_\alpha\, \boxbar_x\, A) \vdash \Box_\alpha A \to \Box_\alpha \vartriangle A.$$

Proof. We work in EA. Let x be given. By Lemma 4.6, we find $\Box_\alpha \mathcal{S}(x)$. Thus, $\Box_\alpha(\boxbar_x A \to (\boxbar_x A \wedge \mathcal{S}(x)))$. This gives us $\Box_\alpha(\boxbar_x A \to \vartriangle A)$. ❑

Lemma 4.8. $\mathsf{EA} + \forall x\, ((\boxbar_x A \wedge \boxbar_x(A \to B)) \to \boxbar_x B) \vdash (\vartriangle A \wedge \vartriangle(A \to B)) \to \vartriangle B$.

Proof. We work in $\mathsf{EA} + \forall x\, ((\boxbar_x A \wedge \boxbar_x(A \to B)) \to \boxbar_x B)$. Suppose $\vartriangle A$ and $\vartriangle(A \to B)$. It follows that, for some x, we have $\boxbar_x A$ and $\mathcal{S}(x)$ and that, for some y, we have $\boxbar_y(A \to B)$ and $\mathcal{S}(y)$. Let $z := \mathsf{max}(x, y)$. It is easily seen that $\boxbar_z A$ and $\boxbar_z(A \to B)$ and $\mathcal{S}(z)$. Hence, $\boxbar_z B$ and $\mathcal{S}(z)$, and, thus, $\vartriangle B$. ❑

Lemma 4.9. $\mathsf{EA} + \forall S\, \exists x\, \Box_\alpha(S \to \boxbar_x S) \vdash \Box_\alpha(S \to \vartriangle S)$.

Proof. This is immediate by Lemma 4.7. ❑

We formulate the obvious theorem that follows from the Lemmas. Let $\mathsf{W}_{\alpha,\theta}$ be EA plus the following principles:

a. $\forall x\,(\Box_\alpha \boxbar_x A \to \Box_\alpha A)$
b. $\Box_\alpha A \to \exists x\, \Box_\alpha \boxbar_x A$
c. $\forall x\,((\boxbar_x A \wedge \boxbar_x (A \to B)) \to \boxbar_x B)$
d. $\forall S\, \exists x\, \Box_\alpha (S \to \boxbar_x S)$

Let $\mathsf{W}^+_{\alpha,\theta}$ be EA plus the following principles.

A. $\Box_\alpha \vartriangle A \to \Box_\alpha A$
B. $\Box_\alpha A \to \Box_\alpha \vartriangle A$
C. $(\vartriangle A \wedge \vartriangle (A \to B)) \to \vartriangle B$
D. $\forall S\, \Box_\alpha (S \to \vartriangle S)$

Theorem 4.10. *Let α be a $\Delta_0(\mathsf{exp})$-predicate that numerates the axioms of U in EA, or, equivalently, in true arithmetic. Let θ be a Σ^0_1-predicate that satisfies $\mathsf{EA} \vdash (\theta(y,z) \wedge y < y') \to \theta(y',z)$. Then, $\mathsf{W}_{\alpha,\theta}$ implies $\mathsf{W}^+_{\alpha,\theta}$.*

The logic GLT is the bimodal propositional logic which has GL both for $\Box$ and $\vartriangle$, plus the following principles.

- $\vdash \vartriangle\phi \to \Box\phi$.
- $\vdash \Box\phi \to \vartriangle\Box\phi$.
- $\vdash \Box\phi \to \Box\vartriangle\phi$.
- $\vdash \Box\vartriangle\phi \to \Box\phi$.

By Theorem 4.10, we have:

Theorem 4.11. *Let α be a $\Delta_0(\mathsf{exp})$-predicate that numerates the axioms of U in EA, or, equivalently, in true arithmetic. Let θ be a Σ^0_1-predicate that satisfies $\mathsf{EA} \vdash (\theta(y,z) \wedge y < y') \to \theta(y',z)$.*

Suppose $\mathsf{W}_{\alpha,\theta}$ is a true theory and $U \vdash \mathsf{W}_{\alpha,\theta}$. Then GLT is arithmetically valid in U. In addition, U satisfies HB both for $\Box_\alpha$ and for $\vartriangle_\theta$. Finally, $\vartriangle_\theta$ satisfies the Kreisel Condition in U.

4.2. **Extensions of Peano Arithmetic.** Let U be a consistent extension of PA and let α be an elementary predicate numerating the axioms of U in EA with $\pi \preceq \alpha$. We note that, equivalently, α numerates the axioms of U in true arithmetic. Let $\alpha_z(x) :\leftrightarrow \alpha(x) \wedge x \leq z$. We take $\Theta_\alpha(z,x) := \mathsf{prov}_{\alpha_z}(x)$ in the role of θ. Thus, we have $\Box_{\alpha_z} A = \Theta_\alpha(z, \underline{\ulcorner A \urcorner})$ in the role of $\boxbar_z A$ and we have $\widetilde{\Box}_\alpha A := \exists z\,(\Box_{\alpha_z} A \wedge \mathcal{S}(z))$ the role of $\vartriangle A$.

We define: $\widetilde{\alpha}(a) :\leftrightarrow \alpha(a) \wedge \mathcal{S}(a)$. We have:

Lemma 4.12. $\mathsf{EA} \vdash \forall A\,(\widetilde{\Box}_\alpha A \leftrightarrow \Box_{\widetilde{\alpha}} A)$.

Proof. We reason in EA.

Suppose $\widetilde{\Box}_\alpha A$. Then, for some z, we have $\Box_{\alpha_z} A$ and $\mathcal{S}(z)$. Suppose p witnesses $\Box_{\alpha_z} A$ and B is an axiom used in p. Then, $\alpha(B)$ and $B \leq z$. Since $\mathcal{S}$ is downward persistent w.r.t. $\leq$, we find $\mathcal{S}(B)$, and, hence, $\widetilde{\alpha}(B)$.

Conversely, suppose $\Box_{\widetilde{\alpha}} A$. let q be a witnessing proof. Let B be the maximal α-axiom used in q. We find $\mathcal{S}(B)$. Thus, $\Box_{\alpha_B} A$ and $\mathcal{S}(B)$, i.e., $\widetilde{\Box}_\alpha A$. ❑

Lemma 4.13. *The predicate $\widetilde{\alpha}$ numerates the axioms of U in U. Hence, $\widetilde{\Box}_\alpha$ is Fefermanian in U over U.*

Proof. Let X be the axioms set of U.

Suppose $n \in X$. Then $\alpha(n)$ and hence $U \vdash \alpha(\underline{n})$. Since also, by Lemma 4.6, we have $U \vdash \mathcal{S}(\underline{n})$, it follows that $U \vdash \widetilde{\alpha}(\underline{n})$.

Suppose $n \notin X$. Then $\neg\,\alpha(n)$. Hence, $U \vdash \neg\,\alpha(\underline{n})$. Hence $U \vdash \neg\,\widetilde{\alpha}(\underline{n})$. ❑

Lemma 4.14. EA *verifies* $\mathsf{W}_{\alpha,\Theta_\alpha}$.

Proof. The principle (a) follows by essential reflexivity. The principles (b) and (c) are trivial. The principle (c) follows since for a sufficiently large n we will have, in EA, that $\Box_{\alpha_{\underline{n}}} B$, where B is a single axiom for EA. ❑

By Theorem 4.11 and Lemma 4.14 we find:

Theorem 4.15. *The logic* GLT *is arithmetically valid for* $\Box_\alpha$ *and for* $\Box_{\widetilde{\alpha}}$ *over* U. *In addition, we have* HB *both for* $\Box_\alpha$ *and for* $\Box_{\widetilde{\alpha}}$ *over* U. *Finally,* $\Box_{\widetilde{\alpha}}$ *satisfies the Kreisel Condition in* U.

In case U is sound, one easily sees that the pair $\Box_\alpha$ and $\Box_{\widetilde{\alpha}}$ satisfies the conditions of Theorem 16 of [HP16]. It follows that GLT is precisely the bi-modal propositional logic of $\Box_\alpha$ and $\Box_{\widetilde{\alpha}}$ in U, for sound U.

We started the basic construction of this subsection with an α such that $\pi \preceq \alpha$. We note that we do not get $\pi \preceq \widetilde{\alpha}$, so the construction cannot be iterated.

Remark 4.16. The soundness proofs of our subsection can be extended to constructive logic. In this case we still have the representations β for the axiom set of i-EA and π for the axiom set of HA. So the whole development remains unchanged. One just has to check that never an essentially classical step was taken. The completeness proof for the cosntructive version of GLT fails radically, since the completeness proof for the the constructive version of GL w.r.t. $\Box_\alpha$ already fails, as is illustrated e.g. in [LV18].

The intuitionistic development has an important point. In their paper [AM14], Mohammad Ardeshir and Mojtaba Mojtahedi characterize the provability logic of HA for Σ^0_1-substitutions. This is the most informative result on the provability logic of HA at the moment of writing. An alternative proof is been developed in [VZ18]. This proof uses slow provability in the style of Friedman, Rathjen & Weiermann for HA. The proof works because only a restricted version of the absorption law is needed. The validity of the full absorption law is plausible but not proved. Replacement by of Friedman-Rathjen-Weiermann slow provability by slow provability in the style of the present paper (as suggested by Fedor Pakhomov) does give us full absorption.

We show that we get a strengthened version of absorption in the case of HA. The proof is intended for readers with some background in the metamathematics of constructive arithmetical theories.

Theorem 4.17. i-EA $\vdash \Box_\pi(A \vee B) \leftrightarrow \Box_\pi(A \vee \Box_{\widetilde{\pi}} B)$.

Proof. We reason in i-EA.

Suppose $\Box_\pi(A \vee B)$. It follows by either q-realizability or the de Jongh translation that, for some x, we have $\Box_\pi(A \vee \Box_{\pi_x} B)$. From this we have $\Box_\pi(A \vee \Box_{\widetilde{\pi}} B)$.

Conversely, suppose $\Box_\pi(A \vee \Box_{\widetilde{\pi}} B)$. By the left-to-right case (with change of variables), we have $\Box_\pi(\Box_{\widetilde{\pi}} A \vee \Box_{\widetilde{\pi}} B)$. Hence, $\Box_\pi \Box_{\widetilde{\pi}}(A \vee B)$. So, by absorption, $\Box_\pi(A \vee B)$. ❑

Thus, the alternative predicates that satisfy the absorption law also have a rich constructive life. ❑

Remark 4.18. Let $\mathsf{EA} + \mathsf{ref}$ be EA plus sentential reflection for predicate logic. Let τ be a standard axiomatization for $\mathsf{EA} + \mathsf{ref}$. Let U be an extension of $\mathsf{EA} + \mathsf{ref}$ and let α be an elementary axiomatization of U such that $\tau \preceq \alpha$. With these basic ingredients we can repeat the development of the present section noting that we are always looking at sentential reflection rather than uniform reflection.

In [Vis14] we introduced the theory Peano Corto, which has many analogies to $\mathsf{EA} + \mathsf{ref}$. It would be interesting to see how much of our development can be repeated for the case of Peano Corto. ❑

5. Extensions of Elementary Arithmetic

We first take a moment to see that, in order to get the desired combinations of properties for extensions of EA, we indeed need to leave the realm of the Fefermanian predicates.

5.1. Two Examples.

Our first example addresses the case that we only demand that our Fefermanian predicate is Kreiselian.

Example 5.1. Consider the theory $U := \mathsf{EA} + \Box_\beta \bot$. Suppose there would be a Fefermanian predicate for U over EA that is Kreiselian. Say the witnessing predicate for the axiom set is α. Let $\gamma(x) :\leftrightarrow \beta(x) \vee x = \ulcorner \Box_\beta \bot \urcorner$. We have $\gamma \preceq \alpha$. Since, $U \vdash \Box_\beta \bot$, it follows that $U \vdash \Box_\gamma \bot$, and, hence, $U \vdash \Box_\alpha \bot$. So, $U \vdash \bot$. *Quod non.* ❑

In the previous example, we needed an unsound theory. In our second example, we consider the case that our example satisfies absorption. Here we can use a sound theory.

Example 5.2. Let $U := \mathsf{EA}$. Suppose there is a Fefermanian predicate P based on α for EA over EA. We write $\vartriangle$ for P. We note that $\beta \preceq \alpha$. Suppose we would have the absorption law for $\vartriangle$ and $\Box_\beta$. Then, it would follow that:

$$\begin{aligned} \mathsf{EA} \vdash \Box_\beta \Box_\beta \bot \ &\to\ \Box_\beta \vartriangle \bot \\ &\to\ \Box_\beta \bot \end{aligned}$$

Quod non. ❑

Open Question 5.3. We note that our examples are of finitely axiomatized theories. The construction of Section 4.2 gives us Fefermanian predicates for theories extending Peano Arithmetic. As pointed out in Remark 4.18, we can improve this to extensions of $\mathsf{EA} + \mathsf{ref}$. Obviously there is a big gap between examples and counter examples. So, there is some further work to be done to narrow the gap. ❑

5.2. Motivating Remarks for Our Construction.

We may construct the desired predicates $\boxbar_x$ in many ways. However, it good to maximize the meaningfulness of the construction. Obviously, this is good for didactic reasons. However, I also think it may help to inspire further work.

As a first step, we note that we have the conditions of Theorems 4.1 and 4.7, for $\Box_{\alpha,(x)}$. So, to obtain absorption and emission, $\boxbar_x := \Box_{\alpha,(x)}$ is already sufficient. The idea of our construction is simply to add closure under modus ponens and closure under HB in a minimal way to $\Box_{\alpha,(x)}$.

The minimal way to obtain the addition of modus ponens is simply to close of the α-theorems with proofs $\leq x$ under modus ponens. However, we can strengthen the analogy with our approach to the case of extensions of PA by working with a Hilbert system that only has modus ponens as a rule. Such deduction systems are described in [Qui96] (first edition 1940) and in [Fef60]. When we have such a system we can, for the definition of $\boxminus_x$, consider the theorems whose proofs contain only axioms *whether logical or non-logical* which are $\leq x$. Thus, the main difference between our approach for the extensions of PA and the new one is that we stop treating logical and non-logical axioms as different.

What to do to obtain the Hilbert-Bernays condition? Simple: we add the true Σ^0_1-sentences to our original axiom set.

There is a small technical complication, due to the lack of Σ^0_1-collection, that necessitates us to stipulate a bound on the witnesses of the truth of the Σ^0_1 sentences involved in a proof, but this complication disappears as soon as we have Σ^0_1-collection in the ambient theory.

5.3. **The Construction.** We fix a Hilbert system $\mathfrak{H}$ with as only rule *modus ponens*. Let $\mathsf{logic}(x)$ be a $\Delta_0(\mathsf{exp})$-formula that numerates the axioms of $\mathfrak{H}$ in EA.

We assume that a Σ^0_1-sentence begins with a, possibly vacuous, existential quantifier.

We give the basic definitions for our approach. Let a theory U be given and a $\Delta_0(\mathsf{exp})$-formula α that numerates the axioms of U in EA (or, equivalently, in true arithmetic).

- We define $\mathsf{ass}^\circ(p)$ as the set of assumptions op p, where now a logical axiom also counts as an assumption. In other words, anything not proved from previous items using modus ponens counts as an assumption.
- We write $\mathsf{proof}^\circ_\gamma(p,x)$ for $\mathsf{proof}(p,x) \wedge \forall y \in \mathsf{ass}^\circ(p)\,\gamma(y)$.
- We write $\boxdot_\gamma A$ for $\exists p\,\mathsf{proof}^\circ_\gamma(p, \ulcorner \underline{A} \urcorner)$.
- B is a *direct* $\circ$*-subformula* of A if A is of the form $(C \to B)$ or $(B \to C)$. The $\circ$*-subformulas* of A are the smallest set that contains A and is closed under taking direct $\circ$-subformulas.
- $\alpha^+(a) := \alpha(a) \vee \mathsf{logic}(a) \vee \mathsf{true}(a)$.
- $\widetilde{\alpha}^+(a) := (\alpha(a) \vee \mathsf{logic}(a) \vee \mathsf{true}(a)) \wedge \mathcal{S}(a)$.
- $\alpha^+_{x,z}(a) :\leftrightarrow (\alpha(a) \vee \mathsf{logic}(a) \vee \mathsf{true}^z(a)) \wedge a \leq x$.
- $\alpha^+_x(a) :\leftrightarrow (\alpha(a) \vee \mathsf{logic}(a) \vee \mathsf{true}(a)) \wedge a \leq x$.
- $\Theta^\circ_\alpha(x, A) :\leftrightarrow \boxdot_{\alpha^+_{x,*}} A :\leftrightarrow \exists z\, \boxdot_{\alpha^+_{x,z}} A$. We use Θ°_α in the role of θ. So $\boxdot_{\alpha^+_{x,*}}(\cdot)$ has the role of $\boxminus_x$.
- We define $\widetilde{\boxdot}_{\alpha^+} A :\leftrightarrow \exists x\,(\boxdot_{\alpha^+_{x,*}} A \wedge \mathcal{S}(x))$. So, $\widetilde{\boxdot}_{\alpha^+}$ has the role of $\triangle$.

In case we have Σ^0_1-collection, the situation simplifies. We note that in the absence of Σ^0_1-collection $\boxdot_{\alpha^+_x} A$ is not Σ^0_1 but $\Sigma^0_{1,1}$. We have:

Lemma 5.4. *a.* $\mathsf{EA} \vdash \boxdot_{\alpha^+_{x,*}} A \to \boxdot_{\alpha^+_x} A$.
b. $\mathsf{EA} + \mathrm{B}\Sigma_1 \vdash \boxdot_{\alpha^+_x} A \leftrightarrow \boxdot_{\alpha^+_{x,*}} A$.
c. $\mathsf{EA} \vdash \widetilde{\boxdot}_{\alpha^+} A \to \boxdot_{\widetilde{\alpha}^+} A$.
d. $\mathsf{EA} + \mathrm{B}\Sigma_1 \vdash \boxdot_{\widetilde{\alpha}^+} A \leftrightarrow \widetilde{\boxdot}_{\alpha^+} A$.

Proof. (a) is trivial. (b) is an immediate application of collection. (c) and (d) are analogous to the proof of Lemma 4.12, using respectively (a) and (b). ❑

Suppose U is a consistent theory with axiom set X that extends $\mathsf{EA}+\mathrm{B}\Sigma_1$. Suppose α is a $\Delta_0(\mathsf{exp})$-formula numerating X in U. Let $\widetilde{X}$ be the X plus logic plus the set of Σ^0_1-sentences S such that $U \vdash S$ and let $\widetilde{U}$ be the theory axiomatized by $\widetilde{X}$. Then, $\boxdot_{\widetilde{\alpha}^+} A$ is analogous to a Fefermanian predicate for $\widetilde{U}$ over $\widetilde{U}$. The difference is only that we switch from provability over predicate logic to modus-ponens provability.

We start with a well-known lemma.

Lemma 5.5. *Suppose U extends EA. Let α be a $\Delta_0(\mathsf{exp})$-predicate numerating the axiom set of U over EA. Then,* $\mathsf{EA} \vdash \forall x, A\, \Box_\alpha(\Box_{\alpha,(x)} A \to A)$.

Proof. We reason in EA. Suppose, for some $p \leq x$, we have $\mathsf{proof}_\alpha(p, A)$. It clearly follows that $\Box_\alpha A$ and, hence, *a fortiori*, $\Box_\alpha(\Box_{\alpha,(x)} A \to A)$.

Suppose, for all $q \leq x$, we have $\neg\,\mathsf{proof}_\alpha(q, A)$. If follows, by Σ^0_1-completeness, that $\Box_\alpha \forall q \leq p\, \neg\,\mathsf{proof}_\alpha(q, A)$. In other words, $\Box_\alpha \neg\, \Box_{\alpha,(x)} A$. If follows that $\Box_\alpha(\Box_{\alpha,(x)} A \to A)$. ❑

The next lemma is in the spirit of the previous one, but takes a bit more work.

Lemma 5.6. *Suppose U extends EA. Let α be a $\Delta_0(\mathsf{exp})$-predicate numerating the axiom set of U over EA. We have* $\mathsf{EA} \vdash \forall x, A\, \Box_\alpha(\boxdot_{\alpha^+_{x,*}} A \to A)$.

Proof. We will use a well-known fact, to wit that

$$\mathsf{EA} \vdash \forall x\, \Box_\alpha \forall y\, (y \leq x \leftrightarrow \bigvee_{z \leq x} z = y).$$

This fact means that we do not have to worry that undesirable non-standard elements creep in below elements that are internally standard in EA.

We reason in EA. Let x be given.

We reason inside $\Box_\alpha$. Suppose (\$) $\boxdot_{\alpha^+_{x,*}} A$. Let z and p witness $\boxdot_{\alpha^+_{x,*}} A$. Keeping z fixed, we may, by the $\Delta_0(\mathsf{exp})$-mimimum Principle, find a p_0 that is minimal with this property.

Suppose that p_0 contains a formula B twice. If B is the conclusion A of p_0 we may omit the part after the first occurrence of A, obtaining a shorter proof. This contradicts the minimality of p_0. If B is not the conclusion of p_0, we may omit all occurrences of B after the first one, obtaining a shorter proof. This again contradicts the minimality of p_0. We may conclude that all sentences in p_0 occur only once in p_0.

We claim that every formula that is a (sub)conclusion of p_0 is a $\circ$-subformula of a formula in $\mathsf{ass}^\circ(p_0)$. Suppose not. Let B be the first such formula. Clearly, B cannot be a $\circ$-assumption. So, it must be the conclusion of an application of modus ponens and, thus, a direct $\circ$-subformula of a previous formula of the form $(C \to B)$. But this formula is by assumption a $\circ$-subformula of $\mathsf{ass}^\circ(p_0)$. A contradiction.

So, all sentences occurring in p_0 are in $\circ$-subformulas of $\mathsf{ass}^\circ(p_0)$ and occur only once. It follows that the sentences in p_0 are all $\leq x$ and, hence, the number of these sentences is also $\leq x$. So, by our assumptions on coding, we find $p_0 \approx x^x$. So, certainly p_0 will be estimated by $2^{x^2} + \underline{k}$, for a sufficiently large standard k.

It follows that $\bigvee_{q \leq 2^{x^2}+\underline{k}} \mathsf{proof}^\circ_{\alpha^+}(q, A)$ and, hence, (†) $\bigvee_{q \leq 2^{x^2}+\underline{k}} \mathsf{proof}_{\alpha\cup\mathsf{true}}(q, A)$. (Here the q are standard on the $\Box_\alpha$-external EA-level.)

Now, suppose (‡) $\mathsf{proof}_{\alpha\cup\mathsf{true}}(q, A)$, where $q \leq 2^{x^2} + \underline{k}$. We transform q as follows. Let $\mathscr{S}$ be the set of the Σ^0_1-sentences in $\mathsf{ass}(q)$ that are not in α. It follows that all $S \in \mathscr{S}$ are true. We transform q in two steps. First we form a proof q' from the assumptions $(\mathsf{ass}(q) \setminus \mathscr{S}) \cup \bigwedge \mathscr{S}$ with conclusion A. Then, we transform q' to q'' with assumptions $\mathsf{ass}(q) \setminus \mathscr{S}$ to $\bigwedge \mathscr{S} \to A$. We note that the big conjunction makes sense, since $\bigwedge \mathscr{S} \approx q$ which is $\Box_\alpha$-external.

We easily see that $|q'|$ can be bounded by a linear term in $|q|$. The transformation $q' \mapsto q''$ uses the deduction theorem. Inspection of the proof shows that here also $|q''|$ is linear in $|q'|$. Thus, q'' is bounded by $2^{\underline{m}x^2} + \underline{n}$, for appropriate standard m and n. We conclude that q'' is also $\Box_\alpha$-external. We have found that $\Box_{\alpha_{q''}}(\bigwedge \mathscr{S} \to A)$, where q'' is $\Box_\alpha$-external.

We apply Lemma 5.5 to obtain $\bigwedge \mathscr{S} \to A$. We also have $\bigwedge\{\mathsf{true}(S) \mid S \in \mathscr{S}\}$. Combining these, we find A.

By (†) we find A without assumption (‡). We now cancel ($) to obtain the sentence: $\boxdot_{\alpha^+_{x,*}} A \to A$.

We leave the $\Box_\alpha$-environment. We have shown $\Box_\alpha(\boxdot_{\alpha^+_{x,*}} A \to A)$, as desired. ❑

We insert a quick corollary of Lemma 5.6.

Corollary 5.7. $\mathsf{EA} \vdash \forall A\,(\exists x\ \boxdot_{\alpha^+_{x,*}} A \leftrightarrow \Box_\alpha A)$.

Proof. We reason in EA. The left-to-right direction works as follows. We use Lemma 5.6.

$$\begin{aligned} \exists x\ \boxdot_{\alpha^+_{x,*}} A \quad & \to \quad \exists x\, \Box_\alpha\, \boxdot_{\alpha^+_{x,*}} A \\ & \to \quad \Box_\alpha A \end{aligned}$$

The right-to-left direction is immediate since $\Box_\alpha A$ implies $\Box_{\alpha,(x)} A$, for some x, and $\Box_{\alpha,(x)} A$ implies $\boxdot_{\alpha^+_{x,*}} A$. ❑

Lemma 5.8. EA *verifies* $\mathsf{W}_{\alpha,\Theta^\circ_\alpha}$.

Proof. The principle (a) follows by Lemma 5.6. The principle (b) follows by:

$$\begin{aligned} \mathsf{EA} \vdash \Box_\alpha A \quad & \to \quad \exists x\, \Box_\alpha(\Box_{\alpha,(x)} A \wedge \mathcal{S}(x)) \\ & \to \quad \exists x\, \Box_\alpha(\boxdot_{\alpha^+_{x,*}} A \wedge \mathcal{S}(x)) \end{aligned}$$

The principles (c) and (d) are immediate by the construction of $\boxdot_{\alpha^+_{x,*}}$. ❑

By Theorem 4.11 and Lemma 5.8 we find:

Theorem 5.9. *The logic* GLT *is arithmetically valid in* U *for* $\Box_\alpha$ *and* $\widetilde{\boxdot}_{\alpha^+}$. *In addition, we have* HB *over* U *both for* $\Box_\alpha$ *and for* $\Box_{\widetilde{\alpha}}$. *Finally,* $\widetilde{\boxdot}_{\alpha^+}$ *satisfies the Kreisel Condition in* U.

5.4. **An Application.** Consider a theory U. Let α be a $\Delta_0(\mathsf{exp})$-formula that numerates the axioms of U in EA. We write $\triangle_\alpha$ for $\widetilde{\boxdot}_{\alpha^+}$. We have:

Theorem 5.10. *i.* U *is* Π^0_1*-conservative over* $U + \triangle_\alpha \bot$.
ii. U *is* Σ^0_1*-conservative over* $U + \neg \triangle_\alpha \bot$.

Our result is EA*-verifiable w.r.t.* $\Box_\alpha$.

Proof. We prove (i). Let P be a Π^0_1-sentence. Suppose $U + \triangle_\alpha \bot \vdash P$. Then, (a) $U + \neg P \vdash \neg \triangle_\alpha \bot$. Hence, $U \vdash \triangle_\alpha \neg P \to \triangle_\alpha \neg \triangle_\alpha \bot$. It follows by Σ^0_1-completeness and the formalized Second Incompleteness Theorem for $\triangle_\alpha$ that (b) $U + \neg P \vdash \triangle_\alpha \bot$. Combining (a) and (b), we find $U \vdash P$.

We prove (ii). Let S be a Σ^0_1-sentence. Suppose $U + \neg \vartriangle_\alpha \bot \vdash S$. It follows that $U \vdash \vartriangle_\alpha \bot \vee S$, and, hence, by Σ^0_1-completeness, $U \vdash \vartriangle_\alpha S$. By Kreisel/absorption, we find $U \vdash S$.

The EA-verifiability is immediate. ❑

The proof of (i) is ascribed by Per Lindström, in [Lin03, p94], to Georg Kreisel in [Kre62]. For extensions U of Peano Arithmetic, the existence of a Σ^0_1-sentence S, such that U is Π^0_1-conservative over $U + S$ and U is Σ^0_1-conservative over $U + \neg S$ is a special case of a result due to Robert Solovay. See [Gua79]. See also [Lin03, Chapter 5].

We note that our result implies $\vartriangle_\alpha \bot$ is *a fortiori* a Rosser sentence for U. The resulting proof of Rosser's Theorem is like the proof of the Second Incompleteness Theorem in the following sense. The sentence under consideration is self-reference-free, but in the proof of the desired property we use self-reference.

Remark 5.11. Another example of a self-reference-free Σ^0_1 Rosser sentence (for extensions of PA) is due to Fedor Pakhomov. See [Pak17]. We note that Pakhomov's construction is, in a sense, orthogonal to ours. An essential feature of Pakhomov's construction is that, like the ordinary Rosser sentence and its opposite, it produces Σ^0_1-sentences S_0 and S_1 each with the Rosser property over U such that we have $U \vdash \neg (S_0 \wedge S_1)$ and $U \vdash \Box_\alpha \bot \leftrightarrow (S_0 \vee S_1)$. It follows that e.g. $U \vdash S_0 \to \neg S_1$, but $U \nvdash \neg S_1$. So, S_0 is not Π^0_1-conservative. The non-Π^0_1-conservativity of Pakhomov's sentences is an important feature since it allows him to use them for his alternative proof of Solovay's arithmetical completeness theorem for Löb's Logic. ❑

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Appendix A. Examples

For convenience, we repeat our overview of the examples.

	base	lead	P	Löb	Kreisel	Feferman
Example A.1	EA	EA	Σ^0_1	$+$	$+$	$+$
Example A.2	EA	EA	Σ^0_1	$+$	$+$	$-$
Example A.3	EA	$\mathsf{EA} + \Box_\beta\bot$	Σ^0_1	$+$	$-$	$+$
	EA	EA	Σ^0_2			
Example A.4	EA	EA	Σ^0_1	$+$	$-$	$-$
Example A.5	PA	PA	Σ^0_2	$-$	$+$	$+$
	EA	EA	$\Sigma^0_{1,1}$			
Example A.6	EA	EA	Σ^0_1	$-$	$+$	$-$
Example A.7	PA	PA	Σ^0_2	$-$	$-$	$+$
	EA	$\mathsf{EA} + \Box_\beta\Box_\pi\bot$	$\Sigma^0_{1,1}$			
Example A.8	EA	EA	Σ^0_1	$-$	$-$	$-$

Example A.1. $+{+}+$: We take $U_0 := U := \mathsf{EA}$ and $P := \mathsf{prov}_\beta$. Clearly, this P satisfies all three conditions for EA.

We note that our example satisfies the Hilbert-Bernays Condition too. ❑

Example A.2. $+{+}-$: In Section 5, we provide an example of a Σ^0_1-predicate P for $U := \mathsf{EA}$, that satisfies the absorption principle w.r.t. $\Box_\beta$. By Example 5.2, the predicate P cannot be Fefermanian.

Here is a second example of a somewhat simpler nature. Let $U_0 := U := \mathsf{EA}$. We define $\mathsf{prov}_{\beta,x}(y)$ as the arithmetization of 'y is provable from β by a proof that only employs formulas of depth of quantifier alternations at most x'. We define

$$\mathsf{prov}^*_\beta(y) :\leftrightarrow \exists x\, (\mathsf{prov}_{\beta,x}(y) \wedge \forall z \leq x\, \neg\, \mathsf{prov}_\beta(\underline{\ulcorner\bot\urcorner})).$$

We write $\Box^*_\beta$ for prov^*_β.

It is easy to see that $\Box^*_\beta$ satisfies both the Löb Conditions and the Kreisel Condition over EA.

Suppose EA would prove that $\Box^*_\beta$ is co-extensional with a Fefermanian predicate. Let $B_0 := \bot$ and $B_{n+1} := \forall v_0\, \exists v_0\, B_n$.

Reason in EA. Suppose p is a proof of $\bot$. It follows that $\neg\, \Box^*_\beta B_p$, since B_p is too complex to be in the scope of $\Box^*_\beta$. On the other hand, since predicate logic proves B_p, we find, by the Feferman Condition, that $\Box^* B_p$. A contradiction. So EA is consistent. We leave EA.

We have shown, on the assumption that prov^*_β is Fefermanian in and over EA, that EA proves its own consistency. Quod non.

A final example is $\mathsf{cfprov}_\beta(x)$, which stands for cut-free provability in EA. Let's write $\Box^{\mathsf{cf}}_\beta A$ for $\mathsf{cfprov}(\underline{\ulcorner A\urcorner})$. We have Löb's Logic for $\Box^{\mathsf{cf}}_\beta$. See [Vis90] and [Kal91]. Also we easily see that $\Box^{\mathsf{cf}}_\beta$ satisfies the Kreisel condition. However, $\Box^{\mathsf{cf}}_\beta$ cannot be Fefermanian for EA over EA. If it were Fefermanian, we would have $\mathsf{EA} \vdash \Box_\beta\bot \to \Box^{\mathsf{cf}}_\beta\bot$. To prove that this is impossible is outside the scope of the present article. We just give the outline of the proof, so that the reader can see the basic idea.

Suppose $\mathsf{EA} \vdash \Box_\beta\bot \to \Box^{\mathsf{cf}}_\beta\bot$. It follows that $\mathsf{EA} \vdash \Diamond^{\mathsf{cf}}_\beta\top \to \Diamond_\beta\top$. Then, by a meta-theorem from [WP87], it follows that (a) $\mathsf{S}^1_2 + \Diamond^{\mathsf{cf}}_\beta\top \vdash \Diamond^J_\beta\top$, for a definable cut J. We also have that (b) EA interprets $\mathsf{S}^1_2 + \Diamond^{\mathsf{cf}}_\beta\top$. Combining (a) and (b), we find that EA interprets $\mathsf{S}^1_2 + \Diamond_\beta \top$. But this contradicts the Second Incompleteness Theorem.

We note that our last two examples also satisfy the Hilbert-Bernays Condition. A disadvantage of these examples is do not work for the global version of the Löb Conditions, where the quantifiers over sentences for $\mathsf{L2}$ and $\mathsf{L3}$ are inside the theory. It would be interesting to have an example for this case. ❏

Example A.3. $+-+$: Here is an example of a Fefermanian predicate that does satisfy the Löb Conditions and does not satisfy the Kreisel Condition. Let $U_0 := \mathsf{EA}$, $U := \mathsf{EA} + \Box_\beta \bot$. Let $P := \mathsf{prov}_\gamma$, where $\gamma(x) :\leftrightarrow \beta(x) \vee x = \underline{\ulcorner \Box_\beta \bot \urcorner}$.

We have $U \vdash \Box_\gamma \bot$, but $U \nvdash \bot$, so the Kreisel condition fails for P and U.

We note that our example also works for $U_0 := U := \mathsf{EA} + \Box_\beta \bot$.

We provide a second example, where the base and the lead theories are sound. By Theorem 3.7, the predicate that represents the axioms cannot be Σ^0_1.

We take $U_0 := U := \mathsf{EA}$. We define: $\delta(x) :\leftrightarrow \beta(x) \vee (\Diamond_\beta \top \wedge x = \underline{\ulcorner \bot \urcorner})$. We note that δ numerates $\{B\}$ in EA, where B is the single axiom for EA. We find:

$$\begin{aligned} \mathsf{EA} \vdash \Box_\delta \bot \quad & \leftrightarrow \quad (\Box_\beta \bot \wedge \Box_\beta \bot) \vee (\Diamond_\beta \top \wedge \Box_\beta (\bot \to \bot)) \\ & \leftrightarrow \quad \top \end{aligned}$$

So $\mathsf{EA} \vdash \Box_\delta \bot$. It follows that prov_δ is not Kreiselian and satisfies the Feferman Conditions and the Löb Conditions.

We note that prov_δ is Σ^0_2. ❏

Example A.4. $+--$: Let $U_0 := U := \mathsf{EA}$ and let P be $x = x$. Clearly, P satisfies the Löb conditions in EA, but P is not Kreiselian. Since EA is sound and P is Σ^0_1, a Fefermanian P must be Kreiselian. So, P is also not Fefermanian. ❏

Example A.5. $-++$: The case of Fefermanian predicates that do not satisfy the Löb Conditions is among the most interesting of our cases. The study of the possibilities for such predicates for the case of extensions of Peano Arithmetic has been taken up by Taishi Kurahashi in great depth. See [Kur17] and [Kur18].

A classical example of such a predicate is *Feferman Provability*. We write π_x for a standard arithmetization of the set of $y \leq x$ that are codes of single axioms of the theories $\mathrm{I}\Sigma_z$. We define $\pi^\star(y) :\leftrightarrow \exists x\, (\pi_x(y) \wedge \Diamond_{\pi_x} \top)$. Let $P := \mathsf{prov}_{\pi^\star}$. This predicate was introduced by Solomon Feferman in his classical paper [Fef60]. By the essential reflexivity of PA, one finds that $\Box_{\pi^\star}$ is Fefermanian for PA over PA. For closely related reasons $\Box^\star$ is Kreiselian. However, $\Box^\star$ does not satisfy the Löb Conditions. The bimodal provability logic of $\Box_\pi$ and $\Box_{\pi^\star}$ has been characterized by Volodya Shavrukov in [Sha94]. For some further work, see [Mon78] and [Vis89]. We note that $\Box_{\pi^\star}$ is Σ^0_2.

An example of quite different flavor uses the fact that EA does not verify Σ^0_1-collection. We refer the reader to [Vis17, Subsection 6.2]. This example provides a Σ^0_1-axiomatization σ. As a consequence prov_σ is $\Sigma^0_{1,1}$. We refer the reader to e.g. [Vis14] for a further explanation of the relevant formula hierarchy. ❏

Example A.6. $-+-$: Here is an example of a P that satisfies the Kreisel Condition but not the Löb Conditions and the Feferman Condition. Let $U_0 := U := \mathsf{EA}$. Let $P(x) := (\mathsf{prov}_\pi(x) \wedge x \neq \underline{\ulcorner \bot \urcorner})$.

We note that $\mathsf{L2}$ fails for P over EA. This shows that P does not satisfy the Löb Conditions and cannot be Fefermanian. ❑

Example A.7. $--+$: The examples are adaptations of the predicates and theories in Example A.5. We just add something to make the examples non-Kreiselian. We use the notations of Example A.5.

We give our first example of a non-Kreiselian Fefermanian predicate that does not satisfy the Löb Conditions. Let $U_0 := U := \mathsf{PA}$. We take:

$$\pi^\circ(x) :\leftrightarrow \pi^\star(x) \vee (\Diamond_\pi \top \wedge x = \underline{\ulcorner \Box_\pi \bot \urcorner}).$$

Let $P := \mathsf{prov}_{\pi^\circ}$.

It is easily seen that π° numerates the axioms of PA in PA. We have, using the fact that we have HB for $\Box_{\pi^\star}$:

$$\begin{aligned} \mathsf{PA} \vdash \Box_{\pi^\circ} \Box_\pi \bot \quad & \leftrightarrow \quad (\Box_\pi \bot \wedge \Box_{\pi^\star} \Box_\pi \bot) \vee (\Diamond_\pi \top \wedge \Box_{\pi^\star}(\Box_\pi \bot \to \Box_\pi \bot)) \\ & \leftrightarrow \quad \Box_\pi \bot \vee \Diamond_\pi \top \\ & \leftrightarrow \quad \top \end{aligned}$$

It follows that $\mathsf{PA} \vdash \Box_{\pi^\circ} \Box_\pi \bot$. However, $\mathsf{PA} \nvdash \Box_\pi \bot$, so $\mathsf{prov}_{\pi^\circ}$ is not Kreiselian. We have:

$$\begin{aligned} \mathsf{PA} \vdash \Box_{\pi^\circ} \bot \quad & \leftrightarrow \quad (\Box_\pi \bot \wedge \Box_{\pi^\star} \bot) \vee (\Diamond_\pi \top \wedge \Box_{\pi^\star} \neg \Box_\pi \bot) \\ & \leftrightarrow \quad (\Box_\pi \bot \wedge \bot) \vee (\Diamond_\pi \top \wedge \Box_\pi \neg \Box_\pi \bot) \\ & \leftrightarrow \quad \Diamond_\pi \top \wedge \Box_\pi \bot \\ & \leftrightarrow \quad \bot \end{aligned}$$

So $\mathsf{PA} \vdash \neg \Box_{\pi^\circ} \bot$. Thus, $\Box_{\pi^\circ}$ cannot satisfy the Löb Conditions.

Finally, $\Box^\circ$ is clearly Σ^0_2.

Here is our second example. The presentation of our example presupposes that the reader has [Vis17, Subsection 6.2] at hand. We take $U_0 := \mathsf{EA}$, $U := \mathsf{EA} + \Box_\beta \Box_\pi \bot$. We define $\sigma^\circ(x) :\leftrightarrow \sigma(x) \vee x = \underline{\ulcorner \Box_\beta \Box_\pi \bot \urcorner}$. Clearly, σ° numerates the axioms of U in EA. We take $P(x) := \mathsf{prov}_{\sigma^\circ}(x)$. Evidently, P is Fefermanian for $\mathsf{EA} + \Box_\beta \Box_\pi \bot$ over EA.

Since, we have $\mathsf{EA} \vdash \Box_\beta C \to \Box_\sigma C$ and $\mathsf{EA} \vdash \Box_\sigma C \to \Box_{\sigma^\circ} C$. We find $U \vdash \Box_U \Box_\pi \bot$. Suppose we would have $U \vdash \Box_\pi \bot$. In would follow that $\mathsf{EA} + \Box_\beta \Box_\pi \bot \vdash \Box_\pi \bot$, and, hence, $\mathsf{EA} \vdash \Box_\pi \bot$. *Quod non.* Thus $U \nvdash \Box_\pi \bot$. So, P is not Kreiselian.

We note that over U we have, by Σ^0_1-completeness, that $\Box_\sigma$ and $\Box_{\sigma^\circ}$ coincide also in iterated $\Box_\sigma$-contexts. Suppose $\Box_{\sigma^\circ}$ satisfies the Löb Conditions over U. It follows that $\Box_\sigma$ also satisfies the Löb Conditions over U. So, *a fortiori*, we find $U \vdash \Box_\sigma \Diamond_\sigma \top \to \Box_\sigma \bot$. By Lemma 6.11 and Lemma 6.12 of [Vis17, Subsection 6.2], we find:

$$\mathsf{EA} + \Box_\beta \Box_\pi \bot \vdash (\mathsf{S}^\star \vee \Box_\beta \bot) \to ((\mathsf{S}^\star \wedge \Box_\beta \Box_\beta \bot) \vee \Box_\beta \bot).$$

It follows that $\mathsf{EA} + \Box_\pi \bot + \mathsf{S}^\star \vdash \Box_\beta \Box_\beta \bot$. However, we can construct a model of $\mathsf{EA} + \Box_\pi \bot + \mathsf{S}^\star + \neg \Box_\beta \Box_\beta \bot$ using the construction described in [Vis17, Subsection 6.2]. ❑

Example A.8. $---$: We take $U_0 := U := \mathsf{EA}$ and $P(x) := \bot$. It is clear that P does not satisfy the Löb Conditions. Nor is it Kreiselian or Fefermanian. ❑

Philosophy, Faculty of Humanities, Utrecht University, Janskerkhof 13, 3512BL Utrecht, The Netherlands

E-mail address: a.visser@uu.nl