Quantum Black Holes

Some static and dynamical aspects

Navaneeth Krishna Gaddam

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Quantum Black Holes

Some static and dynamical aspects

Kwantum Zwarte Gaten

Sommige statische en dynamische aspecten

(met een samenvatting in het Nederlands)

Proefschrift

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To people of all genders, the genderless and queer,
to those who can but mostly to those who cannot,
to those who believe and those who do not,
to the downtrodden and the hopeless,
to the powerless and the oppressed,
to the mentally ill,
to the layman.

But most of all, to you ...

Contents

Preface

For as long as we can remember, humans have asked questions. Often about the universe(s?) we find ourselves in. We want to know the 'why', the 'what' and the 'how'. This thesis is not ambitious enough to answer any of these questions.

As disappointing as that may be for you, my reader, it is the aim of this thesis to set these questions in a certain context. Using the 'why' as guiding direction, we will answer aspects of the 'what' and the 'how'. We will concern ourselves with the mathematical and physical descriptions of the most fundamental of interactions that take place in nature, as I write. And in fact, even as you read.

These interactions range from those occurring between the smallest building blocks we know, of nature, to those between the largest astrophysical objects we have come to learn of. Quantum Field Theory has been remarkably successful in describing the small. Quantum Electrodynamics and the theories of Weak and Strong forces have all been put together in the Standard Model of Particle Physics, based on the pillars of Quantum Field Theory. These theories have been extremely well tested and experimental evidence continues to pour in, in support of their validity as good descriptions of the fundamental interactions between the smallest of particles we are aware of, in nature. The massive objects, on the other hand, are extremely well described by Einstein's theory of General Relativity [1]. Putting the small and the large together, however, has turned out to be a considerable challenge. Most glaringly when it came to the study of black holes; these are solutions to Einstein's theory [2–9] which have been confirmed to exist in nature [10–12], albeit through indirect detection.

Black holes Bardeen, Bekenstein, Carter and Hawking proposed a remarkable analogy between the laws of black hole mechanics (for stationary black holes with charge and angular momentum) and the laws of thermodynamics [13–15]. The analogy had been formalized with further work by Bekenstein [16] and Hawking [17, 18]. It is these laws that were first indicatives of a quantum structure within black holes, for thermodynamic systems exhibit a quantum statistical structure.

These indicatives spurred extensive research on the so-called 'information paradox'. The premise of which is that in a collapse of matter forming a black hole, the intermediate state post-collapse is a black hole that can be characterized by a small number of physical parameters (mass, charge, angular momentum, etc.). A semi-classical calculation as the one Hawking originally did, however, suggests that black holes radiate as black bodies. That is, with a thermal spectrum. This seems to suggest a gross violation of unitary evolution as all information about the exact in-state that went into forming the black hole appears to have been lost after its evaporation.

For this reason, since the seminal work of [19, 20], the study of black hole microscopics has received significant attention. A quantum understanding of black holes had been plagued with several problems for decades since Bekenstein and Hawking's work. Of them, the apparent infiniteness of the Hilbert space of states associated to the horizons was particularly striking [21]. The microscopic explanation of black hole entropy elegantly solved this problem in a naturally UV complete setting; that of string theory. What is more, the microscopic theory is manifestly unitary.

At this juncture, one had two obvious paths to deliberate between. One, to take the finiteness of the space of states as a final result from string theory and seek an understanding of the special nature of interaction between these degrees of freedom that endow black holes with their exceedingly mysterious dynamics; this is perhaps an obvious path leading towards a quantum understanding of gravitational dynamics. The other, perhaps more modest path, would have been to first seek a refined understanding of the static; more than a mere count of states, that is. Both paths have been travelled, even extensively if one might add, and yet it is fair to say that much is left to be understood.

Static aspects of black hole physics are often easiest to study when there is sufficient symmetry in the game. Think of a balloon. Imagine we said a balloon was approximately spherical. That would leave a child the freedom to poke it a little, see how it responds and play around with it. Imagine we said that the balloon must be exactly spherical all the time. Any touch is going to distort its shape (even if only ever so slightly) and so that does not leave the child any room for play. Nevertheless, it is when nobody touches the balloon that it is easiest to study! That is when we know exactly how to tell its shape; it is spherical. Knowing just the radius of the sphere, we would know its precise surface area and even the volume of air contained inside the balloon. In fact, demanding spherical symmetry and fixing one additional parameter (say the size of the balloon) allows us to completely determine how it behaves with time: exactly nothing would change and the balloon would be a thousand years later just as it is now. Of course, the most fun might be to pierce the balloon with a pin and see it explode! But that is arbitrarily far from any symmetric process—unfortunately rendering it too difficult to write down, say, an evolution equation for.

On account of similar logic, the more symmetric they are, the easier the black holes are to study. The completely static ones are often supersymmetric. They are stable; in fact more so than the balloon. A little poke (or more formally, a small perturbation) doesn't disturb a supersymmetric black hole. The ones that may be disturbed a little while still allowing us some control are often non-supersymmetric. But they tend to retain most of their character upon perturbation by, say, throwing a particle into them; these are called 'large semi-classical black holes' in technical jargon. To burst the balloon is to watch a black hole evaporate into 'nothingness'[1]. In fact, the reverse process of creating a small, non-perturbative black hole is rather exciting too. But again, such fun does not come easy. And in this thesis, we will either let the black holes be or poke them a little, ever so slightly. For good or for bad, we will neither create nor destroy them.

In the case where black holes are static and supersymmetric, we will attempt to obtain a complete and precise understanding of the exact microstates that render them with a horizon and consequently an entropy. Further on, with a view towards what happens when we throw particles into them, when the black holes are near-extremal but still arising in supergravity theories, we will still aim at understanding their entropy, but admittedly with less precision. As patience catches up with us on how unwieldy studying dynamics is, within a controlled string theoretic setting, we will then turn the plate upside down to move to a bottom up approach; after all, our interest is in understanding black holes that are closer to those found in nature. These black holes, being far from supersymmetric or extremal, have horizons with rich dynamics which we will study with the help of quantum mechanics. The top-down and bottom-up approaches may neatly converge to help us understand the complete underlying story in the long-run. Conveniently enough for the impatient, however, an intermediate probe has emerged in recent decades, via a study of strongly coupled gauge theories in the limit of a large number of degrees of freedom.

Strongly coupled gauge theories Some of the smallest particles that we know nature is built out of, are quarks. They interact with each other via the Strong force. Of the many remarkable successes of 20^{th} century theoretical physics, the theory of Quantum Chromodynamics [22–30] explaining these interactions is among the finest. This theory was discovered to be asymptotically free in the ultra-violet (UV) and known to be confining in the infra-red (IR). A perturbative description for the former and an effective hadronic description for the latter have since been successfully developed and experimentally tested. However, an understanding of this theory at all intermediate energy scales has proven to be a big challenge. Studying the limit of large number of degrees of freedom (called the large N limit) has opened up an unprecedented set of tools to study QCD-like theories, even if in a simplified setting. It paved way for a concrete realization of the holographic principle [21, 31, 32] in the form of gauge-gravity duality [33–35]. Which in turn allowed for an entirely new way of studying gravitational dynamics via field theoretic tools and vice-versa. In fact, some of the most incisive contributions to

[1] What black holes evaporate into, is not decisively known yet.

our modern day understanding of the microscopics of black hole physics are owed to the study of strongly coupled gauge theories in the large N limit.

While static aspects of black holes often entail an understanding of the microscopic origin of their entropy, Wald has shown [36] that any theory with diffeomorphism invariance necessitates black hole entropy as a corresponding Nöther charge. This begs the question of how a microscopic theory may conceivably accommodate an emergent diffeomorphism invariance in the classical limit; often this is asked as 'how or where does gravity emerge from?' An understanding of microscopic theories that allow for such emergent general coordinate invariance naturally allow for a study of gravitational dynamics via the microscopic field theories. Gauge-Gravity duality has already taught us interesting lessons in this regard: we have come to learn that radial evolution in the $d + 1$ dimensional gravitational theory is governed by renormalization group flow in the field theory. Consequently, this allows us to study strongly coupled gauge theories at intermediate scales via their gravitational duals. Conversely, aspects of gravitational dynamics and space-time emergence may be addressed by the field theoretic degrees of freedom. We will embark upon both of these issues in this thesis, using semi-holography.

Semi-holography is a proposal for an effective framework in which one can include both perturbative and non-perturbative effects consistently in a wide range of energy scales. Its present formulation is targeted towards an effective description of asymptotically free theories like QCD which are weakly coupled in the ultraviolet but strongly interacting in the infrared. It is assumed that in the large N limit (i) the infrared non-perturbative effects such as confinement and chiral symmetry breaking can be obtained from a holographic dual description in the form of an appropriate classical theory of gravity[2], and (ii) the perturbative degrees of freedom determine the effective background metric, relevant and marginal couplings, and background gauge-fields (coupling to conserved currents) in which the emergent infrared holographic degrees of freedom live. The second assertion then implies that the perturbative degrees of freedom determine the leading asymptotic behaviour of the classical gravity fields forming the holographic dual of the non-perturbative sector. As we will argue, such a set-up allows for only a few effective parameters[3] in a wide range of energy scales. Concrete phenomenological semi-holographic models with a small number of effective parameters have been proposed for some non-Fermi liquid systems [40, 45–47, 49, 50], and for the quark-gluon plasma (QGP) formed in heavy-ion collisions [48, 55]. In such instances, indeed both perturbative and non-perturbative effects are phenomenologically relevant.

[2]This was demonstrated in the Witten-Sakai-Sugimoto top-down model [37–39] obtained from string theory. We do not assume here that the holographic description of the non-perturbative sector can be embedded in string theory.

[3]See [40–54] for relevant literature.

In this thesis, we will take first steps towards a derivation of the general semi-holographic framework from first principles, i.e. from the fundamental theory describing the microscopic dynamics. This amounts to answering the following questions:

- Which principles tell us how the perturbative degrees of freedom determine the leading asymptotic behaviours of the gravitational fields forming the holographic dual of the non-perturbative sector?
- How do we find the appropriate classical gravity theory which provides the dual holographic description of the non-perturbative sector?

We will arrive at partial answers to both these questions, and also illustrate the full construction of semi-holography with a toy example.

Previously, semi-holography has been conceived of as an effective simplified method for solving low energy holographic dynamics where the asymptotic geometry determining model-dependent features is replaced by simple boundary dynamics which couples to the near-horizon geometry controlling universal scaling exponents [40, 42]. It has also been argued that decoding holography as a form of non-Wilsonian RG flow which preserves Ward identities for single-trace operators (like the energy-momentum tensor) and can self-determine microscopic data via appropriate infrared endpoint conditions naturally gives rise to a more general semi-holographic framework in which the ultraviolet can be asymptotically free so that it is described by perturbative quantum field dynamics rather than by a classical gravity theory [53, 54, 56]. In this thesis, we will deal with the fundamental aspects of construction of the general semi-holographic framework (which may not be embeddable in string theory as we know of it today) by understanding what constrains it structurally and illustrate it with a toy example.

Organization of this thesis This thesis is divided into two parts. The first part concerns static, supersymmetric aspects of black hole physics; the primary focus is on a detailed understanding of the microscopic degrees of freedom contributing towards supersymmetric black hole entropy in $\mathcal{N} = 2$ supergravity theories in four space-time dimensions. Chapter I is a review of literature that is to set Chapter II in context. Familiarity with supersymmetry, supergravity and string theory aside from minimal algebraic geometry, elementary aspects of modular forums, general relativity and (quantum) field theory aids the reading of this part of the thesis. The second part moves towards some dynamical aspects of black holes. In Chapter III, familiarity with string theory is still very useful and references to the static aspects discussed in Part One are frequent. While the aim of this chapter is to move towards dynamical aspects, the results still merely concern black hole entropy. Chapter IV may be read independently and only familiarity with general relativity and field theory should suffice despite references to non-critical string theory that may be overlooked without any interruption to the flow of the thesis.

The results of this chapter concern the dynamical nature of (Schwarzschild) black hole horizons. The last segment of this document is Chapter V where familiarity with bottom-up gauge-gravity duality is useful. Gravitational consequences of this chapter insofar as black hole physics is concerned, is still very much work in progress. The results of this chapter may be viewed as field theoretic in nature.

We will see, in this thesis, that asking why black holes have entropy guides us in identifying theories where such a question is tractable. Asking why it has been difficult to probe non-extremal black holes within string theory allows us to pin-point what it is that we may be able to concretely learn about near-extremal black holes. Asking how the Schwarzschild black hole could possibly be compatible with quantum mechanics allows us to identify how the corresponding degrees of freedom of the former are in fact re-arranged versions of the latter. And finally, asking why gravitational physics can emerge out of intrinsic intermediate field theoretic energy scales enables us to at least address how this happens in an illustrative toy model. This thesis—an account of my research pursuits in recent years—is based on the following work [57–60]:

- N. Gaddam, *Elliptic genera from multi-centers*, JHEP 1605, 076 (2016), arxiv: 1603.01724.
- N. Gaddam, A. Gnecchi, S. Vandoren and O. Varela, *Rholography, Black Holes and Scherk-Schwarz*, JHEP 1506, 058 (2015), arxiv: 1412.7325.
- P. Betzios, N. Gaddam and O. Papadoulaki, *The Black Hole S-Matrix from Quantum Mechanics*, JHEP 1611, 131 (2016), arxiv: 1607.07885.
- S. Banerjee, N. Gaddam and A. Mukhopadhyay, *Semi-holography illustrated with bi-holography*, IN REVIEW WITH PRD, arxiv: 1701.01229.

For the most part, this document is an identical reproduction of the above papers. A publication of mine that does not find place in this thesis is [61]

- S. Demulder, N. Gaddam and B. Zwiebach, *Doubled geometry and α' corrections*, FORTSCH. PHYS. 64, 279 (2016), in: *Proceedings, The String Theory Universe, 21st European String Workshop and 3rd COST MP1210 Meeting: Leuven, Belgium, September 7-11, 2015*, 279-291p.

A note on referencing All references within a chapter to sections, equations, figures and tables in the same chapter come without the chapter number. All references to those in other chapters are appended with the corresponding chapter number. A reference—in Chapter II—to equation 8 of the third section of Chapter II appears as (3.8) while the same equation is written as II.3.8 when referred to in other chapters.

Part One

The static and the supersymmetric

Chapter I

Supersymmetric black hole entropy

The best known examples where black hole entropy is seen to be originating from a microscopic theory arise in String Theory. The first seminal contributions towards this understanding were made in [19, 20]. In this thesis, we will mostly confine ourselves to four-dimensional black holes. The black holes are often descendants of p-brane solutions to supergravity equations of motion while the microscopic degrees of freedom are those of the world-volume field theory living on D_p-branes in String Theory. The validity of this correspondence between D_p-branes and p-brane solutions is, at least in part, owed to Polchinski [62]. In this chapter, we will present an invitation to how the said entropy matching works. Only to soon point out where the shortcomings lie, as we allow ourselves lesser supersymmetry.

1 Preliminaries

For the purposes of studying microscopic black hole entropy in four dimensions, it has proven significantly useful to use the description of $\mathcal{N} = 2$ supergravity, even in cases where more supersymmetry is available. Since the aim of the first part of this thesis is to understand black hole entropy in $\mathcal{N} = 2$ theories, this is rather convenient. Let us start with a p-brane action that breaks 10-dimensional Lorentz symmetry of Type IIA supergravity down to $SO(1,p) \times SO(9-p)$. The truncated supergravity action now contains only the appropriate Ramond-Ramond gauge fields, the metric and the dilaton appended with appropriate Dirac-Born-Infeld (DBI) and Chern-Simons terms:

$$S = \frac{1}{16\pi G_N} \int \mathrm{d}^{10}x \sqrt{-G} \left(\mathcal{R} - \frac{1}{2}\mathrm{d}\phi^2 - \frac{1}{2} e^{\frac{(3-p)\phi}{2}} \frac{1}{(p+2)!} \left(\mathrm{d}C_{p+1}\right)^2 \right) + T_p (S_{DBI} + S_{CS}) \,. \tag{1.1}$$

Here, G_N is Newton's gravitational coupling constant, $G = \det G_{MN}$ the determinant of the metric tensor, ϕ is the dilaton, C_{p+1} the gauge field associated to the p-brane and T_p the tension associated to it. To find solutions to the equations of motion arising from this action, we could start with a spherically symmetric ansatz of the form [63]

$$\mathrm{d}s^2 = e^{2A(r)}\mathrm{d}\vec{x}^2 + e^{2B(r)}\mathrm{d}\vec{y}^2, \tag{1.2}$$

where $\vec{x}$ and $\vec{y}$ are coordinates longitudinal and transverse to the brane respectively and $A(r)$ and $B(r)$ are some functions of the radius of the black hole solution. One may hope to solve for the equations of motion coming from (1.1) using this ansatz. However, this is a rather cumbersome task. Supersymmetry, though, comes to our rescue. Demanding that the solution preserve half the available supersymmetry, one finds that with

$$\phi = \phi_0 + \frac{p-3}{4}(C - C_0), \quad A(r) = \frac{7-p}{16}C \quad \text{and} \quad B(r) = -\frac{p+1}{16}, \tag{1.3}$$

there exists a general p-brane solution to the equations of motion iff

$$e^{-C} \equiv H_p(r) = e^{-C_0} + \frac{Q_p}{r^7 - p} \tag{1.4}$$

is a harmonic function in the transverse coordinates, with some constant Q_p that can be interpreted as the charge of the brane. The solution in the string frame is

$$\mathrm{d}s^2 = \frac{1}{\sqrt{H_p}}\mathrm{d}\vec{x}^2 + \sqrt{H_p}\mathrm{d}\vec{y}^2, \tag{1.5}$$

where H_p is given by

$$H_p(r) = 1 + N_p\, g_s\, l_s^{p-7}\, \frac{\Gamma\left(\frac{1}{2}(7-p)\right)}{\left(2\sqrt{\pi}\right)^{p-5}\left(r^7 - p\right)}, \tag{1.6}$$

with Γ being the Gamma function, g_S and l_S being the string coupling and length, respectively and N_p the number of p-branes in consideration [64]. The ubiquitous appearance of such harmonic functions is evident in most literature on solutions to supergravity equations of motion. Our interest, though, is in four dimensional black hole solutions. These arise in compactifications of string theory down to four-dimensions, on a Calabi-Yau[1] threefold. Topology of the threefold dictates the field content of the four-dimensional supergravity theory and the (bosonic) action is entirely specified by the scalar manifold. We will label the number of vector multiplets by n_V and the number of hypermultiplets by n_H henceforth. For Type IIA compactifications, these are determined uniquely by the Hodge numbers $h^{1,1}(\mathrm{CY}_3)$ and $h^{2,1}(\mathrm{CY}_3)$ respectively. Certain black hole solutions (which are BPS or contain an AdS_2 horizon, for instance) in these theories exhibit the *attractor mechanism*; where the values of the scalar fields are fixed by the charges of the black hole at the horizon. Regardless of their values at infinity, the scalar fields get *attracted* to specific

[1]Various inequivalent definitions of a Calabi-Yau manifold exist in the literature; in this thesis, we will call a manifold a Calabi-Yau n-fold iff it has $SU(n)$ holonomy. Furthermore, we will label a Calabi-Yau threefold by CY_3.

values determined by the charges of the black holes. Consider the following spherically symmetric ansatz for the four-dimensional black hole

$$ds^2 = -e^{2U(\tau)}dt^2 + e^{-2U(\tau)}d\vec{x}^2 \quad \text{with} \quad \tau = \frac{1}{r}\,. \tag{1.7}$$

Although first derived in [65], the attractor equations were later written down—in [66]—as first order flow equations on the space of moduli as

$$\partial_\tau U = -e^U|Z| \quad \text{and} \quad \partial_\tau t^A = -2e^U g^{A\bar{B}}\bar{\partial}_{A\bar{B}}|Z|\,, \tag{1.8}$$

where the capital Latin indices run over the number of vector multiplets, n_v, and Z is the central charge in the theory. Along with the additional relation

$$\partial_\tau|Z| = -4e^U\, g^{A\bar{B}}\, \partial_A|Z|\bar{\partial}_{\bar{B}}|Z|, \tag{1.9}$$

the flow equations ensure that $\partial_\tau|Z| \leq 0$. This implies that the gradient of the flow of central charge radially inwards to the black hole, is negative. Therefore, the central charge converges to a fixed value at the horizon. This gives us $2n_V + 1$ equations to uniquely determine the point in moduli space where the central charge is fixed, in terms of the electric and magnetic charges of the black hole; in fact, these determine the $2n_V + 1$ real quantities $U(\tau)$, $\mathrm{Re}(t^A)$ and $\mathrm{Im}(t^A)$. A simple example [67] is rather illustrative and we now turn to one. An explicit version of the attractor equations derived in [66] is the following—

$$\mathrm{Re}(CX^\Lambda) = p^\Lambda \quad \text{and} \quad \mathrm{Re}(CF^\Lambda) = q_\Lambda\,, \tag{1.10}$$

where $X^\Lambda = (X^0, X^A)$ and C is a complex, space-time parameter[2]. Furthermore, p^Λ are magnetic charges while q_Λ are the electric ones carried by the black hole. The charges p^A arise from integrals of the appropriate field strength tensors arising from the n_v vectors in the vector multiplets while q_A arise from their Hodge duals. Finally, p^0 and q_0 arise from the vector in the gravity multiplet; one that is often called the graviphoton. Clearly, the black hole under consideration may be charged under any or all of these gauge fields. As is usual, we will denote the set of all charges that any BPS state (including the black hole) carries in the theory by a charge vector $\alpha = (p^0, p^A, q_A, q_0)$; and the lattice of all possible charges, we will call Γ. For simplicity, let us pick a simple supergravity theory with the tree level prepotential

$$F\left(X^0, X^A\right) = D_{ABC}\frac{X^A X^B X^C}{X^0}, \tag{1.11}$$

[2]For simplicity, we stick to this rather sloppy formulation in this chapter, without specifying C. This parameter can in fact be solved for and the equations written more accurately as can be found in [68, 69], for instance.

in the large volume limit, where D_{ABC} encode the intersection numbers of the compact threefold. This results in the following sections—defined as derivatives of the prepotential $F_\Lambda\left(X^0, X^A\right) := \partial_\Lambda F\left(X^0, X^A\right)$:

$$F_0\left(X^0, X^A\right) = -D_{ABC}\frac{X^A X^B X^C}{\left(X^0\right)^2}\,,\; F_A\left(X^0, X^A\right) = \frac{3D_{ABC}X^B X^C}{X^0}. \tag{1.12}$$

Assuming $p^0 = 0 = q_A$, for simplicity, the attractor equations (1.10) now reduce to

$$\mathrm{Re}(CX^A) = p^A\,, \quad \mathrm{Re}(CX^0) = 0\,, \quad \mathrm{Re}(CF_A) = 0 \quad \text{and} \quad \mathrm{Re}(CF_0) = q_0\,.$$

Writing $X^A = p^A + ib^A$ and plugging these into the expression for F_A we see that $\mathrm{Im}(CX^A) = 0$. Along with a similar constraint from plugging the expressions for X^A into F_0, this gives[3]

$$X^A = p^A \qquad \text{and} \qquad X^0 = i\sqrt{\frac{D}{q_0}}, \tag{1.13}$$

with $D = D_{ABC}p^A p^B p^C$. Therefore, the values of the moduli are fixed exclusively by the charges that the BPS state carries, with no dependence on their values at infinity. In projective coordinates, writing the moduli as $t^A := \frac{X^A}{X^0}$, we will denote the values of the moduli at infinity by t^A_∞. Another form of the BPS equations, also discussed in [66], is

$$2e^{-U}\;\mathrm{Im}(e^{-i\theta}\Omega) = -\alpha\,\tau + 2\,\mathrm{Im}(e^{-i\theta}\Omega)|_{\tau=0}, \tag{1.14}$$

where Ω is the holomorphic section associated to the Special Kähler manifold of the scalars X^Λ and $\theta = \arg(Z)$. The insight we gain with this rewriting is via the identification of the harmonic function describing the solution

$$H(r) = \frac{\alpha}{r} - 2\,\mathrm{Im}(e^{-i\theta})|_{r=\infty}. \tag{1.15}$$

This identification of the harmonic function $H(r)$ allows us to interpret the solution as a charge at $r = 0$ with background moduli chosen as $-2\,\mathrm{Im}(e^{-i\alpha}\Omega)$. Moreover, it allows for a convenient generalization to multi-center black holes.

Multi-center black holes Given two BPS charge vectors α_1 and α_2, there is a natural symplectic inner product defined[4] between them $\langle\alpha_1, \alpha_2\rangle$. This can be thought of as purely electric and magnetic 'interaction' between the charge vectors. However, if the repulsion due to this interaction is balanced by the attraction between the two BPS

[3] Up to the complex space-time parameter C

[4] We will return to the exact definition of the inner product in the next chapter.

states (now thought of as black holes), we could well imagine a bound state of them. In supergravity, these are realized as multi-center black hole solutions. These are not just superpositions of single center solutions but are finitely separated and satisfy some conditions to balance the gravitational and electric–magnetic forces. Those solutions that are merely superpositions satisfy $\langle\alpha_1, \alpha_2\rangle =: \alpha_{12} = 0$.

Since these multi center black holes are interacting black holes bound together, a good intuitive way to think about the general case is to imagine a non-trivial Poynting vector at each point on the space spanned by a given solution. This renders the bound state with a non-trivial angular momentum to modify the general ansatz for the metric [70]

$$\mathrm{d}s^2 = -e^{2U}(\mathrm{d}t + \omega)^2 + e^{-2U}\mathrm{d}\vec{x}^2, \tag{1.16}$$

where ω is a one-form to keep track of the above-mentioned angular momentum. The BPS equations now conveniently generalize to

$$\begin{aligned} 2e^{-U}\ \mathrm{Im}(e^{-i\alpha}\Omega) &= -H\,, \\ \star\omega &= \langle dH, H\rangle\,, \end{aligned} \tag{1.17}$$

where the harmonic function now generalizes to

$$H(\vec{r}) = \sum_{i=1}^{N} \frac{\alpha_i}{|\vec{r} - \vec{r}_i|} - 2\,\mathrm{Im}(e^{-i\theta}\Omega)|_{\tau=0}, \tag{1.18}$$

for an N centred black hole bound state in asymptotically flat space. Much like the intrinsic angular momentum in electron-monopole bound states, these bound states also carry an intrinsic angular momentum given by

$$\vec{J} = \frac{1}{2}\sum_{i<j} \alpha_{ij}\vec{r}_{ij}, \tag{1.19}$$

where $\vec{r}_{ij}$ is a vector pointing from $\vec{r}_j$ to $\vec{r}_i$. The equations that govern the distances between the centres, derived from (1.17), are called the *Bubble equations* and are given by

$$\sum_{j=1}^{N} \frac{\alpha_{ij}}{|\vec{r} - \vec{r}_i|} = 2\,\mathrm{Im}(e^{-i\alpha}Z_i)|_{\tau=0}, \tag{1.20}$$

where the Z_i are the central charges for each individual black hole center and all the constants on the R.H.S (for all j) sum to zero. This latter constraint can be used to fix one of the centres to be at the origin—essentially breaking translational invariance—for this does not change the configuration degrees of freedom of our interest. To put things together, a multi center black hole bound state can exist provided there is a solution to the above Bubble equations, the entropy is real and positive and finally if the moduli fields remain physical throughout. This final condition implies that the t^A must remain in one Kähler cone. In a single center solution, the entropy is proportional to the square root of a function called the discriminant function:

$$S \sim \sqrt{\mathcal{D}}. \tag{1.21}$$

For instance, in the Schwarzschild black hole, the discriminant is defined as $\left(1 - \frac{2GM}{r}\right)$ which is always positive outside the horizon. However, for a multi-center solution, since several individual black holes contribute, this regularity of the bound state solution must be independently checked in supergravity. Physically speaking, this condition corresponds to checking that there are no closed time-like curves inside the full solution. This condition, when explicitly spelled out, is [63]

$$\mathcal{D}\left(\beta + \sum_{i=1}^{N} \frac{\alpha_i}{|\vec{r} - \vec{r}_i|}\right) \geq 0, \tag{1.22}$$

where, $\beta = -2\,\mathrm{Im}(e^{-i\alpha}\Omega)$; this is to ensure that

$$S \sim \sqrt{\mathcal{D}} \qquad \in \qquad \mathbb{R}. \tag{1.23}$$

1.1 An invitation to microscopic entropy

The entropy of the single center black holes we discussed so far, has been shown to arise in M-Theory [71]. M-Theory compactified on a compact Calabi-Yau threefold, of volume $\mathcal{V}$ and proper $SU(3)$ holonomy, results in a five-dimensional theory which we further compactify on a circle of radius R. This results in the $\mathcal{N} = 2$, $d = 4$ supergravity. This is merely a T-dual picture of what we considered above. First reducing M-Theory on the circle gives us the IIA theory we picked earlier; so long as our interest is restricted to the supersymmetric sector, the order of compactification is not of much significance. We will often shift between the IIA and M-Theoretic perspectives for circumstantial convenience. For simplicity, let us restrict to compactifications on threefolds with $h^{1,1} = 1$ yielding the t^3 model in four dimensional supergravity theory.

Macroscopics

Prepotentials for compactifications with $h^{1,1} = 1$ are of the simple form

$$F(X) = \kappa \frac{(X^1)^3}{X^0} \tag{1.24}$$

and the corresponding Kähler potential is obtained from the equation

$$ie^{-K^{vec}} = X^\Lambda \bar{F}_\Lambda - \bar{X}^\Lambda F_\Lambda \,. \tag{1.25}$$

Consequently, the charge vector is four-dimensional and again, for simplicity, we pick $\alpha = (0, p, 0, q_0)$. The attractor equations are now solved by

$$CX^0 = i\sqrt{\frac{D}{q_0}} \qquad \text{and} \qquad CX^1 = p \tag{1.26}$$

where $D = \kappa p^3$. The black hole entropy is now given by

$$S_{BH} = \frac{\pi i C\bar{C}}{4}\left(X^\Lambda \bar{F}_\Lambda - \bar{X}^\Lambda F_\Lambda\right) \tag{1.27}$$

$$= 2\pi\sqrt{Dq_0}\,. \tag{1.28}$$

Interestingly, the quantity under the square-root in the last equality above, is quartic in the charges and is uniquely associated to the U-duality group of the supergravity theory in question. Clearly, this invariant depends on the compact manifold under consideration.

Microscopics

The horizon of the above macroscopic black hole preserves eight supercharges. Though the black hole is half-BPS, there is an enhancement at the horizon. From the point of view of the 10d Type IIA theory, these are space-time quarter-BPS states. The black hole can be modelled by considering appropriately charged D-branes extending in the compact dimensions. Naívely, one may think of a D-branes living entirely in the compact dimensions as appearing to be localized charged particles in the non-compact space-time. Since D-branes break half of the supersymmetry, space-time quarter-BPS states correspond to the half-BPS states of the D-brane world-volume theory. Therefore, we are interested in counting these half-BPS states of the D-brane world-volume theory; these are the ones that contribute to the are of the horizon of the black hole. However, the black hole carries more quantum numbers than mere supersymmetry. The D-brane being considered must have the same Ramond-Ramond (R-R) charge as the magnetic charge p of the black hole. Additionally, the black hole being considered carries an electric q_0 charge. Therefore, supersymmetric excitations of the appropriate D-brane world-volume theory carrying precisely this electric charge appear to correspond to the black hole of interest.

In M-Theory, the M5 brane is the only object supported by the R-R charge in 11 dimensions. A stack of p M5-branes would carry exactly the same quantum magnetic charge as the black hole of interest. The M5-brane worldvolume is a 5 + 1 dimensional theory with $(0,2)$ supersymmetry. Wrapping this brane on a holomorphic divisor in the threefold yields a $(0,4)$ CFT (often called the MSW-CFT after the authors of [71]) in two dimensions. BPS states in this theory are those that are annihilated by half the supercharges. Denoting the four supercharges by $G^{\pm}$ and $\bar{G}^{\pm}$, their modes can be written as $G^{\pm}_r$ and $\bar{G}^{\pm}_s$. Since a BPS state is invariant under two of these four supercharges, say[5] $\bar{G}^{\pm}|BPS\rangle = 0$, all their corresponding modes annihilate these states as well: $\bar{G}^{\pm}_r|\mathrm{BPS}\rangle = 0$. In particular, the zero modes kill the BPS states: $\bar{G}^{\pm}_0|\mathrm{BPS}\rangle = 0$. This results in a constraint on the momentum of the right moving states in the CFT via the following expression in the algebra [72]

$$\{\bar{G}^a_r, \bar{G}^b_s\} = 2\delta^{ab}\bar{L}_{r+s} - \delta^{ab}\frac{c_R}{12}, \tag{1.29}$$

where $a, b = \pm$. Since the modes $\bar{G}^a_0$ annihilate the BPS states, we have that

$$2\bar{L}_0 - \frac{c_R}{12} = 0, \tag{1.30}$$

which implies that

$$\bar{L}_0 = \frac{c_R}{24}. \tag{1.31}$$

The shift in the ground state value of $\bar{L}_0$ can be understood in the following way. The dilatation operator in a classical CFT is given by $l_0 + \bar{l}_0$. However, upon radial quantization, the Virasoro modes are given by a Laurent expansion of the stress energy tensor $T(z)$. In such a quantum theory, the Schwarzian derivative of the stress-energy tensor results in a conformal anomaly with exactly this shift of $\frac{c}{12}$. This shift is carried over to the Virasoro modes when the Laurent expansion of $T(z)$ is inverted. As a result, the Hamiltonian is shifted by

$$H = L_0 - \frac{c_L}{24} + \bar{L}_0 - \frac{c_R}{24} = L_0 - \frac{c_L}{24}. \tag{1.32}$$

The BPS states of interest—aside from being the desired supersymmetric excitations of the CFT—carry additional charges.

- Since these states arise from the M5-brane worldvolume, they naturally carry the RR charge p.
- Moreover, they also carry momentum around the M-Theory circle. If P is the momentum operator on the circle, we have

$$P = L_0 - \frac{c_L}{24} - \left(\bar{L}_0 - \frac{c_R}{24}\right) = L_0 - \frac{c_L}{24} = q_0, \tag{1.33}$$

[5] For non-vanishing p^0 and q_1 charges, $\left(\bar{G}^{\pm} - (p^0 q_0 + p^1 q_1)\bar{\psi}^{\pm}_{\alpha}\right)|BPS\rangle = 0$ is the appropriate modification of this condition. The $\bar{\psi}^{\pm}_{\alpha}$ are the fermionic Goldstino modes that sit in the universal hypermultiplet. Since the short $\mathcal{N} = 4$ SCA of the MSW-CFT is extended by the universal hypermultiplet, this modification of the BPS condition is one way to see how the additional symmetry affects the states in the theory.

where the last equality is owed to the knowledge that electric charge q_0 in the supergravity theory arises from quantized Kaluza-Klein momentum along the M-Theory circle.

From (1.32) and (1.33), we see that the momentum of the BPS states is all left moving. And therefore, they are right moving ground states. For a unitary two-dimensional CFT, Cardy's formula shows that the asymptotic growth of states for $L_0 - \frac{c_L}{24} = n$, when $n \gg 1$, is given by [73]

$$d(n, c) = \exp\left(2\pi\sqrt{\frac{n\,c}{6}}\right), \tag{1.34}$$

where c is the central charge of the theory. Modular properties of the partition function of the 2d CFT are extremely powerful in determining such a growth of states. An illustrative example of how an estimate for the growth of states may be made is presented in Appendix A. In the MSW-CFT we have at hand, we want to pick out the left moving excitations with total momentum q_0. Therefore we have that $n = q_0$. It was shown in [71] that the central charges $c_{L,R}$ of the theory can be computed from the topological data of the divisor being wrapped by the M5-brane. This was achieved via an assumption on the very-ample nature of the divisor inside a toric threefold (when Kodaira's embedding theorem in Algebraic Geometry comes to use); the result was found to be that $c_L = 6D$, where D encodes the triple intersection numbers as before. Therefore, the micro-canonical entropy arising from this degeneracy of states with charge q_0 is given by

$$S_{CFT} = 2\pi\sqrt{\frac{c_L q_0}{6}} = 2\pi\sqrt{Dq_0}\,, \tag{1.35}$$

which is exactly the same as the entropy coming from the macroscopic black hole (1.27). It is the degrees of freedom from the non-supersymmetric left moving sector that give the black hole its entropy!

2 Entropy in $\mathcal{N} = 8$ and $\mathcal{N} = 4$ theories

The microscopic origin of black hole entropy we saw so far is of fair significance in that it encompasses all the half-BPS dyonic black holes that solve $\mathcal{N} = 2$, $d = 4$ supergravity equations of motion. Including all those that arise in appropriate truncations of $\mathcal{N} = 4$ and $\mathcal{N} = 8$ supergravity theories; after all, these theories are at least $\mathcal{N} = 2$. However, the microscopic entropy we found in (1.35) is a leading order result. There are logarithmic and polynomial corrections—in the charges—that we have not kept track of in the microscopic counting of states. Moreover, the prepotential of the supergravity theory in (1.24)—upon string compactification—receives perturbative and instanton corrections. Consequently, the black hole entropy in (1.27) also receives corrections that we have ignored. We saw that $\mathcal{N} = 2$ supersymmetry was powerful enough to allow us to solve gravity equations of motion (via the first order flow equations) easily and allow us to identify the microstates too. As one might naturally guess, life is easier with more symmetry. And one can go far beyond this leading order entropy matching and in fact arrive at an exact counting of states on both the microscopic and macroscopic fronts in $\mathcal{N} = 4$ and $\mathcal{N} = 8$ theories.

2.1 $\mathcal{N} = 8$ supergravity and IIB string theory on T^6

Circle compactifications do not break supersymmetry. Therefore, reducing Type IIB string theory on $T^6 = T^4 \times S^1 \times \tilde{S}^1$ preserves all the 32 available supercharges in 10 dimensions. The two circles have been illustratively written separately for convenience that is to be apparent soon; there is no topological difference between any of the circles in consideration and the manifold is an honest six-torus. The resulting supergravity theory is $\mathcal{N} = 8$, in four dimensions. The corresponding U-duality group is $E_{7,7}(\mathbb{Z})$ whose associated invariant of charges is given by $\Delta := q_e^2 q_m^2 - (q_e \cdot q_m)^2$, where q_e and q_m are the electric and magnetic charge vectors, respectively.

Microscopics On the microscopic front, the configuration to consider this time in the Type IIB picture is that of a D5-brane that wraps the T^4, a D1-brane wrapping a circle—say the one we called S^1, momentum n along the circle S^1 and finally, a unit of Kaluza-Klein monopole charge along $\tilde{S}^1$. With the charge vectors given by $\frac{1}{2}q_e^2 = n$, $\frac{1}{2}q_m^2 = 1$ and $q_e \cdot q_m = l$, the U-duality invariant is then $\Delta = 4n - l^2$. This configuration is one instance of the famous D1-D5 system that was the first example of a microscopic counting of states [19]. Half-BPS black holes in $\mathcal{N} = 2$ supergravity are now $\frac{1}{8}$-BPS states in the $\mathcal{N} = 8$ theory. It was shown in [74] that the partition function for the counting of such states of this theory is given by

$$
\begin{aligned}
Z_{\frac{1}{8}-\mathrm{BPS}}(\tau, z) &= \sum_{n,l \in \mathbb{Z}} a(n,l)\, q^n\, y^l \\
&= \frac{\vartheta_1(\tau, z)^2}{\eta(\tau)^6} \\
&= h_0(\tau)\vartheta_{1,0}(\tau, z) + h_1(\tau)\vartheta_{1,1}(\tau, z) \\
&= \left(-\frac{\vartheta_{1,1}(\tau, 0)}{\eta(\tau)^6}\right)\vartheta_{1,0}(\tau, z) + \left(\frac{\vartheta_{1,0}(\tau, 0)}{\eta(\tau)^6}\right)\vartheta_{1,1}(\tau, z) \\
&= \left(-2 - 12q + \dots\right)\vartheta_{1,0}(\tau, z) + \left(q^{-1/4} + 8q^{3/4} + \dots\right)\vartheta_{1,1}(\tau, z),
\end{aligned}
\tag{2.1}
$$

where $q = e^{2\pi i \tau}$ and $y = e^{2\pi i z}$. This is a weight -2 and index 1 Jacobi form. And it was shown that the degeneracies of the $\frac{1}{8}$-BPS states of the D1-D5 CFT we picked are related to the Fourier coefficients of the above partition function via $d(\Delta) = (-1)^{\Delta+1}\, a(n,l)$, with

$l = \Delta \mod 2$. Counting these degeneracies, then, is a rather non-trivial task if it has to be done exactly. This difficult task is rather beautifully achieved [75] by the *Rademacher expansion*[6]. Since there is only one polar term in this theory, and with degeneracy 1, the Rademacher expansion yields the rather simple result that

$$a(n,l) = 2\pi\left(\frac{\pi}{2}\right)^{7/2} \sum_{c=1}^{\infty} c^{-9/2} K_c \tilde{I}_{7/2}\left(\frac{\pi\sqrt{\Delta}}{c}\right), \tag{2.2}$$

where K_c represents a particular combination of the Kloosterman sums[7] with $K_1(\Delta) = 1$.

Macroscopics The large amount of supersymmetry, as we might guess, allows for this expression to be entirely reproduced from a macroscopic calculation. First, we note that since the $\frac{1}{8}$-BPS solutions in $\mathcal{N} = 8$ theories are still $\frac{1}{2}$-BPS solutions in a truncated $\mathcal{N} = 2$ theory, the attractor mechanism we saw is still in play. This ensures that Sen's quantum entropy function is applicable at the AdS_2 horizon and that it indeed computes the corresponding black hole entropy [79]. Moreover, the technique to be used this time for computing the said entropy is *supersymmetric localization* [80]. As it turns out, this technique needs off-shell supersymmetry. For which, a convenient off-shell formulation of the $\mathcal{N} = 2$ supergravity theory can be employed; it is that of conformal supergravity [81–83]. The prepotential[8] in conformal supergravity is not only a function of the scalar fields as before but also of the (rescaled[9]) lowest component, say $\widehat{A}$, of the square of the Weyl multiplet: $F = F\left(X^\Lambda, \widehat{A}\right)$. While supersymmetry freezes the Weyl multiplet to its attractor value [88], localization techniques have been employed to show [89–92] that the quantum entropy arising from the leading saddle-point results in

$$\begin{aligned}
\exp[\mathcal{S}] &\coloneqq \widehat{W} \\
&= \int_{\mathcal{M}_Q} \prod_{A=0}^{n_v} \left[d\phi^A\right] \exp\left(-\pi\, q_A \phi^A + 4\,\mathrm{Im}\, F\left(\frac{\phi^A + i p^A}{2}\right)\right) Z_{1\text{-loop}}\left(\phi^A\right) \\
&= 2\pi\left(\frac{\pi}{2}\right)^{7/2} \tilde{I}_{7/2}\left(\pi\sqrt{\Delta}\right).
\end{aligned} \tag{2.3}$$

[6] See [76] for a brief exposition on the expansions and references therein for more details. We also refer to [77] for all relevant definitions and notions of (mock) modular forms, (weak) Jacobi forms and Siegel forms; all of which repeatedly appear in this thesis.

[7] See [78] for basic definitions and some details on these number theoretic phases.

[8] There are terms one may add to the conformal supergravity action that are not encoded in the prepotential. These are called the D-terms. However, these do not contribute to the exact quantum entropy [84, 85] of full BPS black holes; they may contribute to the quarter BPS solutions [86]. The terms that are always characterized by the prepotential are called the F-terms.

[9] In [87], the rescaled field was called Υ while the field itself went with $\widehat{A}$.

In the first equality, $\mathcal{M}_Q$ is the manifold of localization; one that is specified by the set of all field configurations satisfying $Q\psi = 0$, for a chosen supercharge Q and all fermions ψ in the theory. The choice of Q was such that $Q^2 = L_0 - J_0$, where L_0 and J_0 are Cartan generators of the enhanced $SL(2)$ and $SU(2)$ bosonic symmetries—respectively—at the $AdS_2 \times S^2$ horizon. The localizing manifold is parametrized by coordinates ϕ^A with $[d\phi^A]$ accounting for the non-trivial measure arising from the curvature of $\mathcal{M}_Q$. In the toroidal compactification we are considering, q_A and p^A label the electric and magnetic charges as before while[10] $\widehat{A} = -64$. Finally, the one-loop determinant $Z_{1-\text{loop}}(\phi^A)$ takes care of the fluctuations of the fields orthogonal to the manifold of localization. Direct inspection shows that the result (2.3) agrees with the $c = 1$ contribution to the microscopic Rademacher expansion in (2.2). In fact, it was further shown in [78] that gravitational saddle-points of the kind $AdS_2 \times S^2/\mathbb{Z}_c$ give rise to $c > 1$ contributions to $\widehat{W}$ that exactly match the corresponding microscopic Rademacher series in (2.2).

The punchline of this analysis is that all the macroscopic entropy of $\frac{1}{8}$-BPS black holes in $\mathcal{N} = 8$ supergravity in four dimensions can be accounted for, by an exact microscopic identification of states in appropriate dual conformal field theories arising in string theory. In fact, corresponding sub-leading gravitational saddles can also be identified, that contribute to the supersymmetric sector of the quantum gravity path integral!

2.2 $\mathcal{N} = 4$ supergravity and IIB string theory on $K3 \times T^2$

One may wonder how much of the exact identification of microstates is owed to the large degree of supersymmetry. The natural step next is to reduce supersymmetry. This is easily achieved by replacing the T^4 part of the toroidal compactification with a $K3$ surface, to reduce IIB string theory on $K3 \times S^1 \times \tilde{S}^1$. The $K3$ surface is a Calabi-Yau twofold and therefore breaks space-time supersymmetry upon compactification to result in $\mathcal{N} = 4$ supergravity. This time, we will be interested in $\frac{1}{4}$-BPS states for these are the $\frac{1}{2}$-BPS solutions to $\mathcal{N} = 2$ supergravity. The U-duality group this time is $SL(2,\mathbb{Z}) \times SO(6,22)$ and the invariant charges that label the states are given by the vector $\left(\frac{1}{2}q_e^2, q_e \cdot q_m, \frac{1}{2}q_m^2\right) =: (n, l, m)$.

Microscopics The brane configuration is almost identical to the toroidal compactification with the only difference being that the D5-brane this time wraps the $K3$ as opposed to the four-torus. In fact, it has been shown that the corresponding partition function can be computed with Q_5 D5-branes on $K3$ and Q_1 D1-branes wrapping the S^1. As in the toroidal case, there are n units of momentum along S^1, l units of momentum along with one unit of Kaluza-Klein monopole charge on the $\tilde{S}^1$. These are related to the charge

[10] Strictly speaking, it is the rescaled field that is given by the number -64 and not $\widehat{A}$ itself [87].

vector via $\frac{1}{2}q_e^2 = n$, $q_e \cdot q_m = l$ and $\frac{1}{2}q_m^2 = Q_5Q_1$.

$$\mathcal{Z}_{\frac{1}{4}-\mathrm{BPS}}(\tau, z, \sigma) = \frac{1}{\Phi_{10}(\tau, z, \sigma)}, \tag{2.4}$$

where the function $\Phi_{10}(\tau, z, \sigma)$ is the famous Igusa cusp form [93–97]; it is the unique weight 10 Siegel cusp form. The generating function of symmetric products of $K3$ surfaces (in fact, a multiplicative lift thereof) is related to this Igusa cusp form via the division of suitable Jacobi form; the interested reader may consult Section 5.2 of [77] for details on this exact relation. For our purposes, it suffices to note the relationship between elliptic genera of the target space of the D5-brane and the supersymmetric partition function of the microscopic theory. We will see this feature repeating itself in several examples of $\mathcal{N} = 2$ compactifications we will consider in Chapter II. The degeneracies for a given charge vector (n, l, m) can be extracted from the above $\frac{1}{4}$-BPS partition function via an integral with a specification of an appropriate contour as shown in [98]

$$d(n, l, m) = (-1)^{l+1} \int_C \mathrm{d}\tau\, \mathrm{d}z\, \mathrm{d}\sigma \, \frac{\exp\left[-i\pi(\tau n + 2zl + \sigma m)\right]}{\Phi_{10}(\tau, z, \sigma)}. \tag{2.5}$$

Unfortunately for computational simplicity and fortunately for richness in structure, this degeneracy does not always return the degeneracies of a given single-center black hole. It receives contributions from two-center configurations with the total charge adding up to the chosen charge vector. One reason for this is that the Igusa cusp form is meromorphic in z, in that it has finitely many poles. Therefore, different choices of contours return different results [99, 100]. However, as was shown in [77], first Fourier expanding the Siegel form in σ

$$\frac{1}{\Phi_{10}(\tau, z, \sigma)} = \sum_{m \geq -1} \psi_m(\tau, z) \exp(2\pi i m \sigma), \tag{2.6}$$

in terms of meromorphic Jacobi forms $\psi_m(\tau, z)$. These allow for the following splitting:

$$\psi_m(\tau, z) = \psi_m^{\mathrm{F}}(\tau, z) + \psi_m^{\mathrm{P}}(\tau, z), \tag{2.7}$$

where $\psi_m^{\mathrm{P}}(\tau, z)$ contains all the terms with poles

$$\psi_m^{\mathrm{P}}(\tau, z) = \frac{p_{24}(m+1)}{\eta(\tau)^{24}} \sum_{s \in \mathbb{Z}} \frac{q^{ms^2+s} y^{2ms+1}}{(1 - yq^s)^2}, \tag{2.8}$$

and $\psi_m^{\mathrm{F}}(\tau, z)$ corresponds to the remaining finite piece. Furthermore, the function $p_i(j)$ counts the number of partitions of an integer j in i colours. The double poles in the above equation (2.8) at $y = q^{-s}$ is indicative of the *Wall-crossing phenomenon*. The index is not constant everywhere in moduli space; it jumps discretely across co-dimension one surfaces called 'walls of marginal stability'[11]. The polar piece $\psi_m^{\rm P}(\tau,z)$ captures precisely these jumps. The finite pieces $\psi_m^{\rm F}(\tau,z)$, on the other hand, are a mock-Jacobi forms. The Rademacher expansion that was useful to derive (2.2) is no longer so, for extracting degeneracies of these mock-Jacobi forms. Nevertheless, it has been shown [77] that the Fourier coefficients, $c_m^{\rm F}(n,l)$, associated to $\psi_m^{\rm F}(\tau,z)$ via

$$\psi_m^{\rm F}(\tau,z) \;=\; \sum_{n,l} c_m^{\rm F}(n,l)\, q^n\, y^l\,, \tag{2.9}$$

are related to the degeneracies $d(n,l,m)$ in (2.5) with a specific choice of contour (corresponding to single-center black holes) by

$$d(n,l,m) \;=\; (-1)^{l+1} c_m^{\rm F}(n,l)\,. \tag{2.10}$$

Despite this realization, an actual analytic expression for these degeneracies—such as the one for $\mathcal{N}=8$ theories in (2.2)—has been difficult to achieve. It was shown in [77] that the mock-Jacobi forms $\psi_m^{\rm F}(\tau,z)$ when multiplied by appropriate powers of the Dedekind Eta function, can be written as a sum of an honest weak-Jacobi form (say $\varphi_m^{\rm weak}$) and a mock-Jacobi form (say $\varphi_m^{\rm sub}$), but this time one with sub-leading growth of states when compared to the weak-Jacobi form $\varphi_m^{\rm weak}$. This allows for a very good estimate for the growth of states associated to $\psi_m^{\rm F}(\tau,z)$, especially when the mock-Jacobi forms $\varphi_m^{\rm sub}$ are sufficiently[12] sub-leading.

Macroscopics The macroscopic side of the $K3\times T^2$ compactification has been demonstrated to show all the interesting and complicated structure in comparison to the six-torus. Multi-center configurations are not stable everywhere in the moduli space; they degenerate across walls of marginal stability. And the contributions arising from these multi-center black holes match precisely those coming from the polar piece of the Igusa cusp form, $\psi_m^{\rm P}(\tau,z)$. Consequently, the wall-crossing phenomenon can be re-interpreted as multi-center configurations being stable on one side of the wall while degenerating into constituent single-centers on the other. Moreover, in recent work, the corresponding localization calculations in $\mathcal{N}=4$ supergravity have been carried out [76, 101] to show an explicit matching with the degeneracies obtained from the microscopic estimates. However, there are still minor discrepancies to be ironed out, arising from the mock nature of the forms $\psi_m^{\rm F}(\tau,z)$, which render an exact microscopic counting difficult as we noted earlier. Despite the exceeding success, three significant pieces of the puzzle remain missing—

- The mock-Jacobi forms $\psi_m^{\rm F}(\tau,z)$ can be completed via a 'shadow' function to regain modularity that is ubiquitous in string theory and in the single-center indices in the

[11] There is a concrete mathematical notion of these notions of stability in Algebraic Geometry which unfortunately lies outside the scope of the presentation in this thesis.

[12] This happens for prime m when their growth can be demonstrated to be polynomial at best.

$\mathcal{N} = 8$ theories. However, this completion renders the forms non-holomorphic. A good physical understanding of this non-holomorphicity remains a puzzle.

- A mathematically rigorous way to estimate the growth of states of mock-Jacobi forms in general, and $\psi_m^{\mathrm{F}}(\tau, z)$ in particular, is not known. Therefore, an absolutely exact understanding of states as in the $\mathcal{N} = 8$ theories is very much work in progress.
- While the matching of the leading saddle point from single-center black holes reproduces the expected microscopic answer, the $\mathbb{Z}_c$ orbifolded AdS saddles that reproduce the sub-leading Kloosterman terms such as the ones with $c > 1$ in (2.2) have not been localized on, in supergravity. This appears to be an interesting and rather straight-forward extension of existing literature.
- Finally, while single-center black hole saddles have been explored as far as localization in supergravity is concerned, an honest multi-center localization calculation has not been carried out.

Despite these missing pieces in the literature, we have come to learn some extremely important lessons about the microscopic origin of supersymmetric black hole states. An obvious observation is that an exact identification of states is significantly more challenging, with decreasing supersymmetry. Realizing that going one step further to $\mathcal{N} = 2$ compactifications implies a large number of possible vacua (since there are many threefolds that lead to $\mathcal{N} = 2$ supergravity in four dimensions), one anticipates that a universal formula for single-center states is that much more magnanimous a task. Notwithstanding the apparent insurmountability of the task, we learn an extremely important pragmatic lesson from the more supersymmetric compactifications. It is that picking a black hole and looking for its microstates is difficult. Much like in the $\mathcal{N} = 4$ compactification, identifying the multi-center and subtracting them away from the total partition function appears to be a more pragmatic approach to the problem. For one, it is such a subtraction that allowed for the identification of the single-center mock-Jacobi forms $\psi_m^{\mathrm{F}}(\tau, z)$, which were then approximated by an honest weak-Jacobi form $\varphi_m^{\mathrm{weak}}$ to estimate the growth of states. Moreover, it is imperative for this procedure to work that one understands exactly what it is, that is being computed on the microscopic front. For instance, in the $K3 \times T^2$ compactification, it was crucial to know the significance of the contour of integration in (2.3) which led to hindsight on how the microscopic partition function captures both single- and multi-center states.

As it turns out, in $\mathcal{N} = 2$ compactifications that we will study in the following two chapters, we will need all of this invaluable insight and more to pave way for an understanding of the appropriate micro-states. An answer as comprehensive as in the $\mathcal{N} = 8$ theories or even as satisfactory as the one in $\mathcal{N} = 4$ theories is still distant. However, we will arrive at a stage that lays a concrete road to the identification of appropriate single-center states in $\mathcal{N} = 2$ vacua.

There has been considerable evidence in recent months [102, 103] that subtracting the two-center piece from the total partition function results in a mock-Jacobi form even in $\mathcal{N} = 2$ theories. However, the polar pieces now contain three, four and all higher center configurations. This means that the mock-Jacobi piece obtained from stripping off an $n-$center piece still contains more than mere single-center degeneracies. Presumably, although this is merely educated speculation, stripping off three–center degeneracies yields a function that is mock-Jacobi but still containing the poles of two–center configurations. While starting by stripping off the four–center contributions results in a mock-Jacobi form that contains the poles of three and two–center configurations. And so on. Such intricate structure renders an entirely comprehensive analysis difficult with current techniques.

Therefore, to address this rather intimidating task, we will resort to a modest analysis by picking specific classes of threefolds and explicitly studying the lowest lying states. We will carefully analyse the low-lying spectrum to isolate all possible contributions to the partition function in the regime where no single-center black holes contribute; this is the strictly polar regime of moduli-space where the single-center cosmic censorship bound is violated. This analysis paves a clear path for pushing into the non-polar sector to subtract away all multi-center contributions to be left with the single-center degeneracies. An important ingredient in this analysis will be a technique to compute the degeneracy associated to a given multi-center configuration. After introducing the problem in $\mathcal{N} = 2$ theories from a more self-contained perspective—in the next chapter—we will introduce this technique and use it to identify the low-lying states of the partition function in gravity.

A Cardy's formula - Leading order growth of states

Consider a function that can be written as a q-expansion with one polar term (the Dedekind Eta function is an example):

$$f(\tau) := \sum_{m=-n_0}^{\infty} a_m q^m, \tag{A.1}$$

where $-n_0$ is the power of q in the polar term. Let us assume that this function, $f(\tau)$, transforms under a modular transformation $\tau \to -\frac{1}{\tau}$ as

$$f\left(-\frac{1}{\tau}\right) \sim g\left(\sqrt{-i\tau}\right) f(\tau). \tag{A.2}$$

For the Dedekind Eta function, $g\left(\sqrt{-i\tau}\right)$ is given by $\sqrt{-i\tau}$. The coefficient a_n of the n-th power of q in this expansion gives the number of operators of conformal dimension n, in the CFT. This coefficient can be extracted via a contour integral in the q-plane, of the form

$$a_n \sim \oint f(\tau)\, q^{-n} \frac{dq}{q}. \tag{A.3}$$

Turning this into a τ integral, we obtain

$$\begin{aligned} a_n &\sim \int f(\tau) e^{-2\pi i n\tau} d\tau \\ &\sim \int f\left(-\frac{1}{\tau}\right) g\left(\sqrt{-i\tau}\right)^{-1} e^{-2\pi i n\tau}\, d\tau \\ &\sim \int \left(\sum_{m=-n_0}^{\infty} a_m e^{-\frac{2\pi i m}{\tau}}\right) g\left(\sqrt{-i\tau}\right)^{-1} e^{-2\pi i n\tau}\, d\tau. \end{aligned} \tag{A.4}$$

If $n \gg n_0$, the integral is dominated by the first term in the sum in the integrand:

$$a_n \sim \int a_{-n_0} e^{2\pi i\left(\frac{n_0}{\tau} - n\tau\right)}\, g\left(\sqrt{-i\tau}\right)^{-1} d\tau. \tag{A.5}$$

In a saddle point approximation,

$$\partial_\tau \left(e^{2\pi i\left(\frac{n_0}{\tau} - n\tau\right)}\right)\Big|_{\tau=\tau_*} = 0. \tag{A.6}$$

This implies that

$$\tau_* = \pm i\sqrt{\frac{n_0}{n}}. \tag{A.7}$$

This reduces the equation for a_n to

$$\begin{aligned} a_n &\sim a_{-n_0} e^{2\pi i\left(-2i\sqrt{n_0 n}\right)} \int g\left(\sqrt{-i\tau}\right)^{-1} d\tau \\ &\sim e^{4\pi\sqrt{n_0 n}}. \end{aligned} \tag{A.8}$$

Since the entropy is a logarithm of a_n, contributions in front of the exponential are sub-leading. These factors only contribute to log corrections to the entropy, which along with other sub-leading corrections are captured by the more careful Rademacher expansion. For the purposes of the leading order growth of states, these may be ignored. In this calculation, no knowledge of the specific form of the function $f(\tau)$ was necessary; very general properties were sufficient. A more careful analysis using the famous Hardy-Ramanujan circle method results in Rademacher's exact computation of the degeneracies.

Chapter II

Multi-centers and $\mathcal{N} = 2$ theories

THE microscopic counting of [19, 20] accounts for the number of states, to leading order in charges[1], that yield black hole entropy. Having seen the $\mathcal{N} = 8$ and $\mathcal{N} = 4$ cases, we now return to the $\mathcal{N} = 2$ theories considered in [20]; the microscopics of which are governed by the MSW-CFT we saw in Chapter I.1.1. Much like in the $K3 \times T^2$ compactification, what is counted on the microscopic front is an index—a sum over all 'angular momentum states'—one that includes contributions from single and multi-center states. The macroscopic black hole is a singlet in that it is a static, stationary, spherically symmetric solution to the bulk supergravity equations of motion. While the leading order counting of states matches with the macroscopic entropy, an often under-appreciated problem is the lack of understanding of what each of these states is. One reason for the difficulty in identifying them exactly is that a sum over states of a given representation under the angular momentum group is not a protected quantity. On the macroscopic front, however, a sum over various black hole configurations may seem unnatural. In examples with sufficient amount of supersymmetry, significant progress has been made [77, 78, 89, 101, 104–107] as we saw in the previous chapter. Nevertheless, in cases with lesser supersymmetry, the picture is a lot less clear. In this chapter, we will study a set-up that is least understood in this context—the one of [20].

The MSW CFT is a (0,4) supersymmetric non-linear sigma model that is believed to flow to a conformal fixed point in the IR. The supersymmetric states of this theory can be counted via an index—the modified elliptic genus [108, 109]

$$\mathcal{Z}(q, \bar{q}, \tilde{y}) = \mathrm{Tr}\left(\frac{1}{2}F^2(-1)^F q^{L_0 - \frac{c_L}{24}} \bar{q}^{\bar{L}_0 - \frac{c_R}{24}} \tilde{y}^{2J}\right), \tag{0.1}$$

where $q = e^{2\pi i \tau}$ with τ being the modulus of the torus on which the theory is to live and $\bar{q}$ is its complex conjugate. F refers to the fermion number as in the case of the standard Witten index. c_L and c_R label the left and right moving central charges of the field theory. Finally, $\tilde{y} = e^{2\pi i z}$ is a fugacity associated to the elliptic variable z and is raised

[1] Sub-leading corrections to single-center black hole entropy arising from higher derivative terms have also been studied in the same papers and many more since.

with a 'chemical potential' J associated to the eigenvalue of the generator[2] of the right-moving $U(1)$ algebra arising from the self-dual part of the $(1,1)$ forms of the threefold. BPS excitations of this theory, counted by the above partition function, have been shown to grow—to leading order in charges—exactly as does the entropy of a macroscopic single center supersymmetric black hole in the four dimensional $\mathcal{N} = 2$ supergravity theory [20]. Exciting as that result may already be, the above modified elliptic genus (0.1) in fact enjoys an even richer structure. It is a *weak Jacobi form* of weight $(-\frac{3}{2}, \frac{1}{2})$ and is endowed with a Θ-decomposition[3] in terms of vector valued modular forms $\mathcal{Z}_\gamma$ as [108, 109]

$$\mathcal{Z}(q, \bar{q}, \bar{y}) = \sum_{\gamma=0}^{n} \mathcal{Z}_\gamma(q)\Theta_\gamma(q, \bar{q}, \bar{y}), \tag{0.2}$$

where γ labels the independent elements of the corresponding discriminant group. Loosely speaking, the vector valued modular form $\mathcal{Z}_\gamma$ captures the growth of states of the partition function (0.1) while the Θ_γ functions—forming modular representations of weight $(\frac{1}{2}h^{(1,1)}(CY_3) - 1, \frac{1}{2})$—add to the rich pole structure of the modified elliptic genus. While much more can be said of this decomposition than is within the scope of this thesis, we will restrict our attention to the vector $\mathcal{Z}_\gamma$ which captures the growth of states that endow the macroscopic black holes with their entropy. For simplicity, we will also only consider those compactifications[4] with $h^{(1,1)} = 1$; this allows for a study of uni-modulus supergravity theory on the macroscopic front. Furthermore, given that the Θ_γ functions are then of weight $(0, \frac{1}{2})$, $\mathcal{Z}_\gamma$ would carry modular weight $-\frac{3}{2}$. Finally, $\mathcal{Z}_\gamma$ is also endowed with a q-expansion—the coefficients of which capture a sum over all the black hole microstate degeneracies falling in various representations of the space-time angular momentum—that begins with a negative power of q. The polar sector of the modular form is defined to be the set of all terms in this expansion with negative powers of q; knowledge of all the polar terms uniquely determines the entire modified elliptic genus [108–110].

The leading order growth of coefficients in this q-expansion of $\mathcal{Z}_\gamma$ is what a Cardy-estimate of the growth of states counts. However, as the trace in the definition of the index indicates, all bound states with a total charge equalling that of a single center black hole also contribute to the corresponding term in the q-expansion. While contributions from any one of these bound states may be small, the number of possible configurations clearly grows as the number of partitions of the charge/energy level in question. As Ramanujan

[2]Note that this $U(1)$ generator is to be distinguished from J_3, appearing in the next sections. The latter refers to the angular momentum of the macroscopic black hole dual to a given state in the field theory.

[3]Θ_γ arises in a decomposition of the modular invariant theta function associated to the flux lattice of the Calabi-Yau being compactified on. For details, see [108, 109].

[4]Strictly speaking, it is only in this case of $h^{(1,1)} = 1$ that the above Θ-decomposition is possible.

famously showed, this number grows exponentially; much like the Cardy estimate, one might observe. This raises the following question–

What states is the Cardy formula really counting?

The aim of this chapter is to provide a disambiguation of this issue and work towards an answer to the above question. Ideally, a clinching answer would be a listing of all bound states contributing to a large charge coefficient in the q-expansion of $\mathcal{Z}_\gamma$ leaving an appropriate single-center entropy and the origin of the corresponding states behind. However—unlike in the $\mathcal{N} = 4$ case where only two-center states exist—the exponentially large number of such bound states for a given total charge renders this practically impossible to achieve. One hopes to uncover a structure in the contributions arising from these bound states that may be extrapolated to arbitrary charges. Since the polar terms are the low-lying states and are those that actually entirely determine the modular form uniquely, one may imagine that they provide for a good starting point.

One may in fact opt for a more direct approach to understand single-center black hole entropy: it has been shown [111] that sub-leading corrections to the growth of states of the modified elliptic genus depend on their representation of angular momentum. It is certainly an interesting way forward and deserves more attention than it has received. Notwithstanding this aside, we take the former approach.

In this chapter, we will arrive at a systematic way to identify all the multi-center configurations that enumerate the polar states of the vector valued modular form $\mathcal{Z}_\gamma$ using the equivariant refined index introduced in [112, 113]; as has been noted before [114] no single-center configurations contribute to the polar sector. Along the way we find some interesting results regarding the existence—or lack thereof—of certain three-center configurations involving $D2(\bar{D}2)$ charges. We will work by example to identify all the multi-centers needed to uniquely determine the elliptic genera of the following Calabi-Yau threefolds: the quintic in $\mathbb{P}^4$, the sextic in $\mathbb{WP}_{(2,1,1,1,1)}$, the octic in $\mathbb{WP}_{(4,1,1,1,1)}$ and the dectic in $\mathbb{WP}_{(5,2,1,1,1)}$. All results in this chapter that have been derived before in [109, 110], agree with those references; furthermore, as in the said references, we will use the known Gopakumar-Vafa invariants. These were computed in [115–117] while the relation of Gromov-Witten invariants to Gopakumar-Vafa invariants is excellently reviewed in [118][5]. Finally, the equivalence of these to Donaldson-Thomas invariants was conjectured and proved in [119–122].

The rest of this chapter is organized as follows. In Section 1, we will first review the relevant multi-center configurations of interest and provide an intuitive argument for what

[5]See chapters 33 and 34, in particular.

the appropriate index that counts their interaction degrees of freedom must be; furthermore, we will also spell out the prescription to be used to identify those configurations that contribute to the polar terms of the elliptic genera under consideration. In Section 2, we will explicitly compute the said indices for several examples. We will then conclude with a discussion in Section 3.

1 The refined equivariant index

In this section, we will first review the phase space of multi-center configurations, merely stating results and known facts. Details may be found in [70, 114, 123]. We then move on to a present an intuitive explanation for the appropriate index that counts multi-center degeneracies.

Multi-center configurations in $\mathcal{N} = 2$ supergravity are characterized by a metric ansatz for stationary solutions

$$ds^2 = -e^{2U(\vec{r})}\big(dt + a(\vec{r})\big)^2 + e^{-2U(\vec{r})}d\vec{r}^{\,2}\,, \tag{1.1}$$

with $a(\vec{r})$ denoting a Kaluza-Klein one-form and $U(\vec{r})$ the scale factor. The scalars in the vector multiplet are typically called t^a, with the index a running over the set of all vector multiplets. Since we restrict to Type IIA compactifications with $h^{(1,1)} = 1$, there is only one modulus in the theory allowing for a dropping of the index a. The real and imaginary decomposition of the modulus is labelled as $t = B + iJ$. Denoting the charge lattice by Γ, a given center carries charges that form a vector $\alpha \in \Gamma$; for the case at hand in uni-modulus supergravity, this vector is four-dimensional: (p^0, p, q, q_0). The charges p^0 and p are magnetic in our conventions and correspond to $D6$ and $D4$ brane charges in Type IIA language. Whilst q and q_0 are electric charges corresponding to $D2$ and $D0$ excitations. There is a natural symplectic inner product between two such charge vectors α and $\tilde{\alpha}$

$$\langle \alpha, \alpha' \rangle = q_0 p'^0 + q p' - q' p - q_0' p^0 \tag{1.2}$$

and it is clearly antisymmetric. For a multi-center configuration with total charge $\gamma = \sum_i \alpha_i$, with each center at a location $\vec{r}_i$, the scale factor and the value of the modulus t are uniquely fixed by the 'attractor equations' [123]

$$\begin{aligned} -2e^{-U(\vec{r})}\mathrm{Im}\Big[e^{-i\phi}\Omega(t(\vec{r}))\Big] &= \beta + \sum_{i=1}^{n} \frac{\alpha_i}{|\vec{r} - \vec{r}_i|} \qquad \text{with} \\ \phi &= \arg(Z_\gamma)\,. \end{aligned} \tag{1.3}$$

The moduli space of the scalars in the vector multiplet is a special Kähler manifold that has a principal bundle over its base space with a structure group $Sp(2n_v + 2)$, where n_v is the number of vector multiplets in the theory. Calling the coordinates on the fibers of the appropriate vector bundle X^A and F_A, the manifold affords a nowhere vanishing holomorphic symplectic section. The index A runs over n_v+1 indices; therefore $A \in \{0,1\}$. Now, in the above attractor equations, $\Omega(t(\vec{r})) = -e^{\mathcal{K}/2}\left(X^A, F_A\right)$ is the said symplectic section. $\mathcal{K} = -\ln\left[i\left(F_A\bar{X}^A - \bar{F}_A X^A\right)\right]$ is the Kähler potential associated to $\mathcal{M}_v$. Furthermore, β is a constant vector given in terms of the asymptotic value t_∞ of the modulus by

$$\beta = -2\mathrm{Im}\left[e^{-i\phi}\Omega(t_\infty)\right]. \tag{1.4}$$

In the one-modulus supergravity theory at hand, projective symmetry allows for a fixing of the X^0 coordinate to unity leaving the only modulus $t = X^1/X^0$. The coordinates F_A on the fibers are in fact derived, as $F_A = \partial_A F$, from the prepotential F of the theory:

$$F\left(X^0, X^1\right) = -\frac{k}{6}\frac{\left(X^1\right)^3}{X^0} + \frac{\mathcal{A}}{2}\left(X^1\right)^2 + \frac{c_2\cdot P}{24}X^0X^1 + \text{instantons}. \tag{1.5}$$

Here, we will work in the following normalizations

$$\begin{aligned} \int_{CY_3}\omega\wedge\omega\wedge\omega &= k\,, \\ \int_{CY_3}\omega\wedge c_2(CY_3) &= c_2\cdot P\,, \end{aligned} \tag{1.6}$$

where the ω form a basis of integer two-cycles in the threefold. Finally, the half-integer constant in the quadratic piece of the prepotential is given by $\mathcal{A} = k/2 \bmod 1$. For the purposes of this chapter, the instanton corrections may be ignored[6]. Finally, the one-form $a(\vec{r})$ is determined in terms of the Hodge-star operator of the three flat dimensions by

$$\star_3\,\mathrm{d}a(\vec{r}) = \left\langle d\sum_{i=1}^{n}\frac{\alpha_i}{|\vec{r}-\vec{r}_i|}, \beta + \sum_{i=1}^{n}\frac{\alpha_i}{|\vec{r}-\vec{r}_i|}\right\rangle. \tag{1.7}$$

In what follows, we shall label $|\vec{r}-\vec{r}_i|$ by r_{ij} and $\langle\alpha_i,\alpha_j\rangle$ by α_{ij}. The 'integrability equations'

$$\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{\alpha_{ij}}{r_{ij}} = c_i \qquad \text{with} \qquad c_i = 2\mathrm{Im}\left[e^{-i\phi}Z_{\alpha_i}\right] \tag{1.8}$$

[6] While one may be worried about the validity of the supergravity regime—without instanton corrections—in the case of small charge configurations, they turn out to have a rather specific and easily controlled effect insofar as regularity of solutions is concerned. We will be explicit about this effect in further sections.

ensure the existence of an $a(\vec{r})$ such that the configuration is supersymmetric. Finally, the central charge Z_γ is given by

$$\begin{aligned} Z_\gamma &= \langle \gamma, \Omega(t) \rangle \\ &= e^{\mathcal{K}/2}\Big[p^A F_A - q_A X^A\Big] \\ &= e^{\mathcal{K}/2} X^0 \left[\frac{k}{6} p^0 t^3 - \frac{k}{2} p t^2 - \tilde{q} t - \tilde{q}_0\right], \end{aligned} \tag{1.9}$$

where the charges have been written with a tilde suggestively, to indicate that they are not integer quantized. The exact quantization can be spelled out and we shall do so in Section 2. Furthermore, the Kähler potential can be computed from its definition as

$$\begin{aligned} e^{-\mathcal{K}} &= i\left(\bar{X}^A F_A - X^A \bar{F}_A\right) \\ &= \frac{4}{3}\, k\, \mathrm{J}^3 . \end{aligned} \tag{1.10}$$

Having specified all the quantities appearing in the attractor equations (1.3), there is one additional and extremely important constraint that these multi-center configurations must satisfy; that of regularity. One might impose this by demanding the positivity of the scale factor in front of the $d\vec{r}^2$ term in the metric. The attractor equations can be shown to imply that—for a configuration with i centers located at $\vec{r}_i$—this is equivalent to evaluating the entropy on the regularity vector appearing on the right hand side of the attractor equations (1.3) [70]

$$S\left(\beta + \sum_{i=1}^{n} \frac{\alpha_i}{\left|\vec{r} - \vec{r}_i\right|}\right) > 0, \quad \forall \quad \vec{r} \in \mathbb{R}^3 . \tag{1.11}$$

One may in fact solve for the attractor equations in full generality in uni-modulus supergravity to spell out this entropy function explicitly [124]. We will see the explicit formula in the next section.

At this stage, however, the goal is to understand how one may calculate the total number of degrees of freedom associated to such a gravitational solution. To this end, an 'equivariant refined index' for such bound states as described was proposed in [112, 113]. While we leave the technical derivation of this refined index to those papers, in what follows we will argue for the correctness of their proposed index. This discussion is to give an intuitive picture leaving the more rigorous, technical treatment to those original papers.

Consider the solutions of the integrability equations (1.8). Although the equations are seemingly simple, they are deceptively so. There is no general analytic solution set to these equations. However, for a given configuration, one might numerically solve for the positions r_i of the black hole centers. In general, there is a non-trivial angular momentum associated to every point in the space generated by the solutions of the Denef equations; for a single-center on the other hand, spherical symmetry ensures that this angular momentum is zero. There is also an action of the rotation group $SO(3)$ that leaves the space of solutions invariant; this is just a rotation of the whole configuration of the bound state in space-time. The corresponding study of such spaces, with an action of a group, in the Mathematics literature is that of Hamiltonian spaces and equivariant cohomology. Leaving the intricate details to the excellent review [125], we will resort to a more sketchy and qualitative consideration to tell the number of degrees of freedom to be associated to such bound states. While a two center solution can immediately be imagined, increasing the number of centers in the problem prevents easy visualization. For instance, the integrability equations for a two center problem essentially fix the distance between the two centers[7]. Rotating this configuration in space-time generates a round sphere as the space of solutions; the sphere is clearly smooth and symplectic. To generalize this to phase spaces of solutions of a configuration with higher number of centers is an open problem in Mathematics. Nevertheless, one can write down a symplectic two-form on the phase space of solutions of the integrability equations [126]. It is again a non-trivial task to prove that a given two-form is indeed non-degenerate on the phase space. Therefore, that the phase space is symplectic is best left to be conjectural at this juncture. This phase space is classical. An 'equivariant volume element' of this phase space (read as a volume element that accounts for the non-trivial angular momentum at each point in the space) is one that accounts for the interaction between the black hole centers. The phase space is built out of these equivariant volume elements. This is an extremely important insight. It tells us, among other things, that quantizing this phase space yields a quantum index of the interaction between the black hole centers [113]. Such a quantum index is to keep track of the interaction degrees of freedom of the black holes. While this is a very naive picture, a more rigorous discussion can be found in [113]. In the following, we will take a slightly different perspective from [113] to understand this index.

In mathematical terms, quantization of a phase space that is symplectic, is best understood with the theory of *Geometric Quantization*. An excellent review for aspects relevant to us can be found in [127]. The basic idea is the following - given a line bundle (called the pre-quantum line bundle) and a space of sections of this line bundle (called the pre-quantum space) on the phase space, one can construct a quantum space as a set of subspace of sections of this pre-quantum line bundle that vanishes under the action of a covariant derivative that is defined on the line bundle (via the corresponding connection). Physically speaking, a pre-quantum space can be identified with the space of square integrable sections on an appropriate pre-quantum line bundle. These sections

[7] Up to translations that can be gauged by fixing one of the centers to be at the origin.

would, upon quantization, build up the quantum space - the Hilbert space of states. In the setting at hand, apart from square integrable sections on the line bundle, we also have a spinor bundle consisting of sections corresponding to the fermionic supersymmetry generators in the theory. A clever ploy would be to choose the covariant derivative to be the Dirac operator on the phase space. This is a clever choice for the formally defined equivariant index of the Dirac operator now counts the quantum states in the theory. This is a direct consequence of the definition of the index of the Dirac operator. It is worth understanding this index better for this is what is to be computed, eventually.

Given a vector bundle $E \to M$ on a manifold M with an action of a group G acting on it; consider the action of the group on M such that it lifts to an action on E. The Dirac operator (whose action is assumed to commute with G henceforth) is now defined on the space of sections of this vector bundle as

$$D\colon \Gamma(E) \longrightarrow \Gamma(E). \tag{1.12}$$

By definition, the equivariant index of this Dirac operator, for an element $g \in G$, is

$$\mathrm{Ind}_G(g, D) = \mathrm{Tr}_{\mathrm{Ker}D^+}(g) - \mathrm{Tr}_{\mathrm{Ker}D^-}(g). \tag{1.13}$$

Equivalently, considering the Lie Algebra $\mathfrak{g}$ of G and an element $x = \ln(g) \in \mathfrak{g}$, the equivariant index can be defined as [128]

$$\mathrm{Ind}_G(\exp(x), D) = \frac{1}{(2\pi i)^{\frac{n}{2}}} \int_M Ch_{\mathfrak{g}}(x, E)\hat{A}_{\mathfrak{g}}(x, M), \tag{1.14}$$

where Ch denotes the Chern character and $\hat{A}$ denotes the usual A-roof genus; this is also called Kirilov's formula. For our purposes, in the spirit of the Witten index, picking an element $y^{2J_3} \in G$, where y is a formal generating parameter and J_3 is the third generator of the angular momentum algebra of the rotations in space-time, the index can now be written as[8]

$$g_{\mathrm{ref}}(\{\alpha_i\}, y) = \mathrm{Tr}_{\mathrm{Ker}D^+}\left[(-y)^{2J_3}\right] - \mathrm{Tr}_{\mathrm{Ker}D^-}\left[(-y)^{2J_3}\right]. \tag{1.15}$$

This is the index for a configuration of black hole centers carrying charges α_i that form a bound state satisfying the integrability equations (1.8). g_{ref} stands for the *refined* index; to avoid confusion, we merely stick to conventional notation used in [113]. This can further be shown to reduce to [113]

$$g_{\mathrm{ref}}(\{\alpha_i\}, y) = \int_{\mathcal{M}_n} Ch(v, \mathcal{L})\hat{A}(v, \mathcal{M}_n), \tag{1.16}$$

[8] A dependence on the complexified Kähler parameter t is implicit if one is to work globally in the moduli space; locally, however, the index is constant.

where $\mathcal{L}$ is the line bundle, $\mathcal{M}_n$ is the phase space of an n-centered problem solving the integrability equations and $\nu = \ln y$.

The idea now, is to compute this index via Localization. Knowing the group action on the phase space, a localization technique under an Abelian subgroup of this group ($U(1)$ of $SO(3)$) results in a localization of the black hole centers along a line, say the z axis, with manifest $U(1)$ symmetry; the symmetry being rotations about the axis of localization. This renders a non-vanishing contribution to the index only from the fixed points that are the black hole centers. What was originally a problem in $\mathbb{R}^3$ has now localized to a problem on a line with the centers lying at positions, say z_i. With this knowledge, one may write down a 'superpotential' whose fixed points are given by exactly the fixed points of localization [113]

$$\hat{W}(\lambda, \{z_i\}) = -\sum_{i<j} \alpha_{ij} \mathrm{sign}\left[z_j - z_i\right] \ln\left|z_j - z_i\right| - \sum_i \left(c_i - \frac{\lambda}{n}\right) z_i \,. \tag{1.17}$$

This superpotential is a function of $n+1$ variables: the n centers and a parameter λ. With these considerations, the index can now be written in its computationally easiest form as

$$g_{\mathrm{ref}}(\{\alpha_i\}, y) = \frac{(-1)^{\sum_{i<j}\alpha_{ij}+n-1}}{\left(y - y^{-1}\right)^{n-1}} \sum_p s(p)\, y^{\sum_{i<j}\alpha_{ij}\mathrm{sign}[z_j - z_i]}, \tag{1.18}$$

where p corresponds to a given regular configuration of black hole centers that satisfy the integrability equations and $s(p) = -\mathrm{sign}\det \hat{M}$, with $\hat{M}$ being the Hessian of $\hat{W}(\lambda, \{z_i\})$ with respect to $z_1, \ldots, z_n$. Upon specifying $y = -1$, this g_{ref} is exactly that quantum index which computes the interaction degrees of freedom arising from a given multi center black hole solution to supergravity. Another interpretation of this quantity is that of the Poincaré polynomial associated to the moduli space of the quiver representations: each center in the configuration arises from a D-brane that may be treated as a node with an Abelian gauge group associated to it. With bi-fundamentals extending between the bound centers playing the arrows, these configurations do indeed take the guise of a quiver diagram[129]. Topological invariants associated to the moduli space of representations of these quivers have been shown to be enumerated by this index g_{ref} [130, 131]. With the knowledge of the interaction degrees of freedom between the black hole centers, the total degeneracy associated to a multi-center black hole configuration can now be naturally written as

$$\bar{\Omega}(\{\alpha_i\}; t) = \frac{g_{ref}(\{\alpha_i\}; t)}{|\mathrm{Aut}(\{\alpha_i\})|} \prod_{i=1}^{n} \bar{\Omega}^S(\alpha_i), \tag{1.19}$$

where $\bar{\Omega}(\{\alpha_i\}; t)$ is the total degeneracy associated to the multi-center configuration in question, $\bar{\Omega}^S(\alpha_i)$ corresponds to the rational index associated to a single black hole center carrying charge α_i and the $|\mathrm{Aut}(\{\alpha_i\})|$ factor[9] takes repeated centers into account. The single center indices are input parameters. These rational indices are given, in terms of the integer invariants, by

$$\bar{\Omega}^S(\alpha_i) = \sum_{m|\alpha_i} m^{-1} \frac{y - y^{-1}}{y^m - y^{-m}} \Omega^S(\alpha_i), \tag{1.20}$$

where $\Omega^S(\alpha_i)$ are the integer invariants of the single centers. It may be noted that the product of these rational indices is the mathematical counterpart of the symmetric product of the moduli spaces in the string regime, that contains several singularities [109, 110]. From a supergravity perspective, however, this product can physically be understood as arising from the Bose-Fermi statistics of the interacting single center black holes [112]. This essentially negates all troubles encountered with singularities in the geometric counting.

Finally, a word on the regime of validity of this approach is in order. Owing to the attractor mechanism in four dimensional $\mathcal{N} = 2$ supergravity theories, as the size of the modulus approaches the attractor value, it is fixed by the charges of the single center black hole towards which the modulus is being attracted. In a multi-center configuration however, bound states exist only at large values of the modulus. This is because at smaller values, one is attracted to the basin of attractor of one of the bound state constituents, owing to the attractor mechanism. Therefore, the analysis of multi-center configurations in this thesis is done in the large volume limit: $\mathrm{J} \gg B$.

2 M5-brane elliptic genera from multi-centers

Having—at least heuristically—justified the index that computes the interaction degrees of freedom, in this section we will show how one may identify those mutli-centers that contribute to the polar terms of the MSW elliptic genus. Working by example, we explicitly show that all polar terms of the quintic in $\mathbb{P}^4$, the sextic in $\mathbb{WP}_{(2,1,1,1,1)}$, the octic in $\mathbb{WP}_{(4,1,1,1,1)}$ and the dectic in $\mathbb{WP}_{(5,2,1,1,1)}$ can be reconstructed with this approach. In the next section, we will finish with an argument why this approach is well suited to identifying single-center black hole entropy in the non-polar sector.

To this end, one first needs to identify what the charges of individual terms of the q-expansion of $\mathcal{Z}_\gamma$ must be. Knowing that these charges arise from a $D4 - D2 - D0$ brane construction, the central charge $Z_\gamma(t)$ provides an easy tool for this purpose. D_p branes often support lower dimensional brane charges. A pure $D4$ brane, for instance, supports non-zero $D2$ and $D0$ fluxes [108, 109] to cancel the Freed-Witten anomaly [132].

[9] $|\mathrm{Aut}(\{\alpha_i\})| = \prod_k z_k!$.

Since these branes must form localized objects as black holes in the four dimensional non-compact space in the low energy theory, their extension is entirely confined to the compact Calabi-Yau space. Every Calabi-Yau threefold has a non-vanishing structure-sheaf. Since the $D6$ brane must extend entirely in the threefold, one may view it as the structure-sheaf of the manifold and consequently, there is always one at our disposal. The central charge of a BPS brane is given by the same formula (1.9) as in the supergravity theory. However, the lower dimensional fluxes on the D6 brane induce additional curvature. In addition, if the brane has a non-trivial gauge bundle turned on, the charge vector of the brane would arise from turning on the relevant Chern classes. Taking all of these into consideration, the central charge takes the form [114]

$$Z_\gamma(t) \;=\; -\int_{CY_3} e^{U_1+U_2+U_3} \wedge e^{-t\omega} \wedge \left(1 + \frac{c_2(CY_3)}{24}\right) \tag{2.1}$$

where the U_i represent integer classes in which the Chern classes of the gauge bundle have been expanded as $c_1 = U_1\omega$, $c_2 = U_2\omega$ and $c_3 = U_3\omega$. Expanding the exponentials and using the normalization of (1.6), the central charge reduces to

$$Z_\gamma \;=\; \frac{k}{6}t^3 - \frac{k}{2}U_1 t^2 + \left(\frac{k}{2}U_1^2 + \frac{c_2\cdot P}{24} + U_2\right)t - \left(U_1^3\frac{k}{6} + U_1U_2 + \frac{\mathcal{B}U_1}{24} + U_3\right). \tag{2.2}$$

Using (1.9), this allows for an identification of the corresponding charge vector of a single D6 brane as

$$\begin{aligned}\gamma \;&=\; \left(p^0, p, \tilde{q}, \tilde{q}_0\right)\\ &=\; \left(1, U_1, -\frac{k}{2}U_1^2 - \frac{c_2\cdot P}{24} - U_2, \frac{k}{6}U_1^3 + \frac{c_2\cdot P}{24}U_1 + U_1U_2 + U_3\right)\\ &=\; \left(1, U, -\frac{k}{2}U^2 - \frac{c_2\cdot P}{24}, \frac{k}{6}U^3 + \frac{c_2\cdot P}{24}U\right).\end{aligned} \tag{2.3}$$

where in the last line, we restrict to an Abelian gauge bundle and label the only available integer class U_1 by U. This turns out to be sufficient for the polar sector of interest. Now, solving the attractor equations for a large black hole with the above charges results in a Bekenstein-Hawking entropy $S = \pi\left|Z_\gamma(t_{\text{attractor}})\right|^2$—where $t_{\text{attractor}}$ is the attractor value of the modulus determined in terms of the charges—as follows [124]

$$S \;=\; \pi\sqrt{\mathcal{D}(1, p, \tilde{q}, \tilde{q}_0)}, \tag{2.4}$$

where $\mathcal{D}(1, p, \tilde{q}, \tilde{q}_0)$ is the discriminant function given in terms of the charges as

$$\mathcal{D}(1, p, \tilde{q}, \tilde{q}_0) = \frac{k^2}{9}\left[3\frac{(\tilde{q}p)^2}{k^2} - 18\frac{\tilde{q}_0\tilde{q}p}{k^2} - 9\frac{\tilde{q}_0^2}{k^2} - 6\frac{p^3\tilde{q}_0}{k} + 8\frac{\tilde{q}^3}{k^3}\right] \tag{2.5}$$

for a single center solution. For a multi-center configuration, however, the discriminant is given by (1.11), where the argument of the discriminant is chosen to be the 'regularity vector' appearing in the attractor equations

$$\mathcal{D} = \mathcal{D}\left(\beta + \sum_{i=1}^{n} \frac{\alpha_i}{|\vec{r} - \vec{r}_i|}\right). \tag{2.6}$$

Positivity of the discriminant on the 'regularity vector' ensures regularity of the multi-center configuration.

2.1 Some generalities

It has long been argued that a $D4$ brane splits into a bound state of a $D6$ brane and an anti-$D6$ brane [114]. In an M-Theory setting of the case at hand, it has proved to be very difficult to write down elliptic genera for the MSW CFTs with multiple M5 branes. Therefore, in what follows, we will consider only those with a single M5 brane. This means a unit $D4$ brane charge in the charge vector. Indeed, from the charge vector (2.3), considering a $D6$ brane with one unit flux and a $\bar{D}6$ with no flux yields a $D4$ brane charge vector with induced lower dimensional fluxes:

$$\begin{aligned}
\alpha_{D6} &= \left(1, 1, -\frac{k}{2} - \frac{c_2 \cdot P}{24}, \frac{k}{6} + \frac{c_2 \cdot P}{24}\right) \quad \text{and} \quad \alpha_{\bar{D}6} = \left(-1, 0, \frac{c_2 \cdot P}{24}, 0\right) \quad \text{give} \\
\alpha_{D4} &= \alpha_{D6} + \alpha_{\bar{D}6} = \left(0, 1, -\frac{k}{2}, \frac{k}{6} + \frac{c_2 \cdot P}{24}\right).
\end{aligned} \tag{2.7}$$

Of course, as a consistency check, this must match with the appropriate induced fluxes on the $D4$ brane that cancel the Freed-Witten anomaly; this is indeed satisfied. For example, specifying to the quintic threefold, which has $k = 5$ and $c_2 \cdot P = 50$, this charge vector produces the correct fluxes known from [109]. One can now compute the interaction degrees of freedom between these two centers and check if it matches with what one expects from modularity. Before that however, consider the Θ_γ decomposition of the partition function again:

$$\mathcal{Z}(q, \bar{q}, y) = \sum_{\gamma=0}^{n} \mathcal{Z}_\gamma(q) \Theta_\gamma(q, \bar{q}, y). \tag{2.8}$$

One property of this decomposition is that $\mathcal{Z}_\gamma = \mathcal{Z}_\delta$ for all $\gamma = -\delta$ modulo a pull-back of the second integer cohomology onto the $D4$ brane. For the quintic for instance, $n = 4$ and $\mathcal{Z}_1 = \mathcal{Z}_4$, $\mathcal{Z}_2 = \mathcal{Z}_3$. Now, the pure $D4$ brane degeneracy appears as the first (or most polar) term in the q-expansion of $\mathcal{Z}_0$.

Symmetric product orbifolds and adding $D0$ charges To go to the next term in the expansion, one simply adds $D0$ charge. Thinking geometrically, the $D0$ brane has a moduli space of the entire threefold in consideration and demanding a bound-state with the $D4$ reduces the moduli space of a the latter; the combined moduli space yields the correct degeneracy [109]. Adding more and more $D0$ charges results in symmetric product orbifolds of the moduli space of the $D0$ particles, namely the threefold. Owing to configurations with coinciding branes, one runs into singularities on the moduli space that need to be resolved. As was pointed out in [112], the rational refined indices overcome these subtleties of moduli space singularites. From a multi-center configuration perspective, adding $D0$ charges implies an increase in the number of centers in a configuration. And there are exponentially many of them; the number growing with the number of partitions of the $D0$ charge to be added. Nevertheless, the low-lying spectrum can still be handled. And considering all configurations satisfying regularity, an addition of $D0$ charges takes us towards the non-polar sector of $\mathcal{Z}_0$. We will work out explicit examples in the next subsection to show that counting degrees of freedom associated to all regular configurations produces the correct polar terms.

Rational curves and adding $D2$ charges In order to move 'vertically', so to speak, into the degeneracies in $\mathcal{Z}_1$, one adds $D2$ charges. Thinking geometrically again, adding $D2$ charges is equivalent to demanding that the D4 brane passing through rational curves. So, one computes the moduli space associated to degree 'β' rational curves in conjunction with a demand that the $D4$ brane intersect them. $D2$ fluxes, however, induce $D0$ charges and the amount of induced charge had to be computed using techniques of algebraic geometry. Even in attempts to obtain the elliptic genera from supergravity split-attractor flows [133], the amount of induced charge was needed as an input from geometry to identify the appropriate flows that contribute to the index. Notwithstanding this input, consider the most polar[10] term in the q-expansion of $\mathcal{Z}_1$, say q^{-y}. Writing this term as $q^{-x}q^{z}$, such that $-x+z=-y$ with q^{-x} being the most polar term in $\mathcal{Z}_0$, it turns out to be sufficient to consider added $D0$ charge that corresponds to the positive integer part[11] of z. In the several examples under consideration, it is sufficient to consider rational curves of degree 1 and all polar terms of such kind have $z > 1$; we leave the cases with higher degree rational curves for future work. Positivity of $(z-1)$ has a geometric interpretation: it is that rational curves come with non-trivial moduli spaces only upon an induction of $D0$ charges. In fact, in the theory of Donaldson-Thomas invariants—where a Witten index enumerates invariants $N_{DT}(\beta, n)$ associated to a D2 brane wrapping a curve in the homology class β that intersects a collection of points ascribed to $D0$ branes—there are no topological invariants associated to $N_{DT}(1,0)$ when $z > 1$. That the index is

[10] It may be worth pointing out that not all partition functions necessarily have a polar term in the q-expansion of $\mathcal{Z}_1$.

[11] z is positive in all the examples under consideration.

correctly reproduced by looking at multi-center configurations with added $D0$ charges as we prescribe may be interpreted as supergravity's way of telling us that $N_{DT}(1,0) = 0$ whenever $z > 1$.

In view of the previous discussion on adding $D0$ charges, it is tempting to guess that adding $D2$ charges must involve adding additional centers to charge configurations. Interestingly, a simple argument shows that a generic $D2 - D0$ charge vector never binds to a $D6$ center. Consider generic $D6$ and $D2 - D0$ charge vectors as follows

$$\gamma_1 = \left(1, p, -\frac{p^2}{2}k - \frac{25}{12}, \frac{p^3}{6}k + \frac{25}{12}p\right) \quad \text{and} \quad \gamma_2 = \left(0, 0, q, q_0\right). \tag{2.9}$$

Their symplectic product is given by

$$\gamma_{12} = -\left(q_0 + pq\right) . \tag{2.10}$$

Using the fact that the phase factor associated to them $e^{-i\phi}$ is given by

$$e^{-i\phi} \sim \frac{Z_{(\gamma_1+\gamma_2=\gamma)}}{\left|Z_\gamma\right|} , \tag{2.11}$$

we have that

$$\begin{aligned} \mathrm{Im}\left(e^{-i\phi} Z_\gamma\right) &\sim \mathrm{Im}\left(Z_{\gamma_1} \bar{Z}_{\gamma_2}\right) \\ &\sim \left(q_0 + pq\right) J^3. \end{aligned} \tag{2.12}$$

where the second line is true up to some numerical factors and only holds in the large volume limit $J \gg 0$ for a threefold with positive triple-intersection $k > 0$. Wherever it needs specification, we make an arbitrary choice for the vacuum value of the modulus t at infinity to be $t = 0 + 3i$; this satisfies the large volume condition $J \gg B$. Since the FI constants now have the opposite sign of the symplectic product of the corresponding charges, the integrability equations for the bound state implies that $r_{12} < 0$, which violates regularity. This implies that a bound state of $D6$ with a generic $D2 - D0$ charge never occurs! One might imagine that a three center bound-state of a generic $D2 - D0$ center with $D6 - \bar{D}6$ might still be possible. Although it is hard to prove in full generality, one might take the previous argument as an indication that such three-center bound states generically violate regularity. In the next subsection, we explicitly show that this is true in several examples.

2.2 Explicit elliptic genera for some Calabi-Yau threefolds

The quintic in $\mathbb{P}^4$

The quintic threefold is defined by a degree 5 polynomial in $\mathbb{P}^4$. The topological invariants associated to the quintic are: $\chi(X_5) = -200$, $k = 5$ and $c_2 \cdot P = 50$. Its modified elliptic genus is given by

$$\mathcal{Z}_{X_5}(q,\bar{q},y) = \sum_{\gamma=0}^{4} \mathcal{Z}_\gamma(q)\Theta_\gamma^{(5)}(q,\bar{q},y) \qquad \text{and}$$
$$\Theta_k^{(m)}(q,\bar{q},y) = \sum_{n\in\mathbb{Z}+\frac{1}{2}+\frac{k}{m}} (-1)^{mn} q^{\frac{m}{2}n^2} y^{mn} \tag{2.13}$$

where

$$\mathcal{Z}_0(q) = q^{-\frac{55}{24}}\left(5 - 800q + 58500q^2 + \text{non-polar terms}\right)$$
$$\mathcal{Z}_1(q) = \mathcal{Z}_4(q) = q^{-\frac{83}{120}}(8625 + \text{non-polar terms})$$
$$\mathcal{Z}_2(q) = \mathcal{Z}_3(q) = \text{non-polar terms}\,. \tag{2.14}$$

Pure $D4$ brane The charge vector associated to a Pure $D4$ brane for this compactification can be written from (2.7) with the topological data of the quintic

$$\gamma_1 := \alpha_{D6} = \left(1, 1, -\frac{55}{12}, \frac{35}{12}\right) \quad \text{and} \quad \gamma_2 := \alpha_{\bar{D}6} = \left(-1, 0, \frac{25}{12}, 0\right) \quad \text{give}$$
$$\gamma := \alpha_{D4} = \alpha_{D6} + \alpha_{\bar{D}6} = \left(0, 1, -\frac{5}{2}, \frac{35}{12}\right). \tag{2.15}$$

Computing the discriminant associated to this vector via (2.5), one finds

$$\mathcal{D}_{D4} = -\frac{275}{36}\,, \tag{2.16}$$

which yields an imaginary single-center entropy. This renders this solution un-physical[12]. In order to compute the interaction degrees of freedom, the two-center integrability equations

$$\frac{\gamma_{12}}{z_{12}} = c_1, \tag{2.17}$$

where $z_{12} \in \mathbb{R}$, need to be solved. The required constants, to solve this equation, are tabulated below in Table 1. This results in the following solution

$$z_{12} = \frac{611\sqrt{12665}}{12}\,. \tag{2.18}$$

[12] In fact, computing the discriminant associated to the $D6$ center also yields a negative value: $-3125/1944$. We expect that the ignored instanton corrections to the prepotential lift this sickness; working with this hypothesis, we merely shift the definition of a 'zero discriminant' from $\mathcal{D}_\gamma = 0$ to that of the $D6$ brane. Aside from this subtlety, the instanton corrections play no other role in the analysis.

γ_{12}	Z_1	Z_2	$\alpha = \arg[Z_\gamma]$	$\mathcal{D}_\gamma$	c_1
-5	$\left(\frac{47}{72} + i\frac{7}{24}\right)\sqrt{5}$	$-i\frac{13\sqrt{5}}{24}$	$\tan^{-1}\left(\frac{18}{47}\right)$	$-\frac{275}{36}$	$-\frac{611\sqrt{\frac{5}{2533}}}{12}$

Table 1. Relevant constants for the Pure $D4$ brane.

Since it is only the relative distance between the centers that is important, we fix z_1 to be at the origin. The above solution then implies that z_2 is at a distance of $\pm z_{12}$ from the origin on the axis on which the centers are localized. This leaves us with two possible configurations, namely: 12 and 21, where $z_1 < z_2$ and $z_2 < z_1$ respectively.[13] For consistency, the discriminant associated to the two-center configuration must be positive. This requires the knowledge of β—with an arbitrary choice of the value for the modulus at infinity to be $t = 0 + 3i$ as mentioned before—

$$\beta = \left(\frac{6}{\sqrt{12665}}, \frac{53}{\sqrt{12665}}, -\frac{611}{12}\sqrt{\frac{5}{2533}}, -\frac{76}{\sqrt{12665}}\right), \tag{2.19}$$

Plugging this into the regularity vector, we find that both configurations $\mathcal{D}_{12}$ and $\mathcal{D}_{21}$ are regular everywhere[14] outside the centers; infinities at the location of the centers is expected. Given all the configurations that contribute, using the formula in (1.18)

$$g_{\text{ref}}(\{\gamma_i\}, y) = \frac{(-1)^{\sum_{i<j}\gamma_{ij}+n-1}}{\left(y - y^{-1}\right)^{n-1}} \sum_p s(p)\, y^{\sum_{i<j}\gamma_{ij} sign[z_j - z_i]}, \tag{2.20}$$

the Poincaré polynomial associated to the interaction degrees of freedom of the Pure $D4$ brane realized as a bound state of the $D6$ and $\bar{D6}$ is

$$\begin{aligned} g_{\text{ref}}(\gamma_1, \gamma_2, y) &= \frac{(-1)^{-5+2-1}}{\left(y - y^{-1}\right)}\left(y^5 - y^{-5}\right) \\ &= \left(y^{-4} + y^{-2} + 1 + y^2 + y^4\right), \end{aligned} \tag{2.21}$$

where the sign $s(12)$ was computed to be $+\,1$ from the Hessian of the superpotential in (1.17). Specializing to $y \to (-1)$ results in $g_{\text{ref}} = 5$. Substituting this into (1.19) with the implicit understanding that a $D6$ and a $\bar{D6}$ have refined indices of 1 each[15] yields a final

[13] In the configuration 12, $z_2 = +z_{12}$ and in the configuration 21, $z_2 = -z_{12}$

[14] For simplicity, we check for positivity of the corresponding regularity vector only along the axis of localization.

[15] The structure sheaves have a unit degeneracy.

index of

$$\bar{\Omega}(\gamma_1, \gamma_2; t) = g_{\text{ref}}\, \bar{\Omega}^S_{D6}\, \bar{\Omega}^S_{\bar{D}6} = 5 \times 1 \times 1 = 5. \tag{2.22}$$

This matches the prediction from the string regime and modularity; the most polar term in $\mathcal{Z}_0$ is the pure $D4$ brane. The final index being exactly the same as the norm of the symplectic inner product of the two charge vectors is not a mere coincidence. This is a generic feature of two center solutions to the integrability equations.

$D4$-$D0$ **bound state** As advertised in the previous subsection, the next polar term in $\mathcal{Z}_0$ may be achieved by adding a D0 brane center. Three-center solutions of $D0$ branes bound to $D6$ and $\bar{D}6$ centers have been extensively studied in [113]. While the $D6$ centers considered there were both with $D4$ fluxes turned on, the analysis is largely similar. It has also been previously noted that D_{p-6} branes bound to D_p branes energetically prefer to stay ejected from them as opposed to dissolving as fluxes as preferred by D_{p-2} and D_{p-4} branes. This is consistent with the picture in [113] that adding a $D0$ charge necessarily implies an addition of a new $D0$ center with charge vector $\alpha_{D0} = (0, 0, 0, \pm 1)$. The correct sign may be fixed by noting that adding a positive $D0$ flux on the Pure $D4$ reduces the entropy via a reduction in $\mathcal{D}$. Therefore, in these conventions, a $D4$ brane binds to an anti-$D0$ brane. Therefore, the three-problem of interest now has a third center $\gamma_3 :=$ $(0, 0, 0, -1)$ in addition to the two centers that generated a pure $D4$ brane. These result in a total charge vector given by

$$\gamma := \alpha_{D4-D0} = \left(0, 1, -\frac{5}{2}, \frac{23}{12}\right). \tag{2.23}$$

The corresponding integrability equations take the form

$$\frac{\gamma_{12}}{z_{12}} + \frac{\gamma_{13}}{z_{13}} = c_1, \tag{2.24}$$

$$\frac{\gamma_{23}}{z_{23}} + \frac{\gamma_{21}}{z_{21}} = c_2, \tag{2.25}$$

where $\gamma_{12} = -\gamma_{21}$[16]. We tabulate the relevant data required to solve these equations, in Table 2 and Table 3. Starting far out in the moduli space at $t = 0 + 3i$ again, the corresponding vector for β is

$$\beta = \left(-6\sqrt{\frac{5}{69109}}, \frac{247}{\sqrt{345545}}, -\frac{3211}{12}\sqrt{\frac{5}{69109}}, -\frac{295}{2}\sqrt{\frac{5}{69109}} + \frac{247}{2\sqrt{345545}}\right). \tag{2.26}$$

Solving the Denef equations and writing down those solutions that satisfy the discriminant positivity condition (i.e, $\mathcal{D} > \mathcal{D}_{D6}$) we find Table 4. Gathering all the computations,

[16] The symplectic product of any two charge vectors is antisymmetric.

γ_{12}	γ_{13}	γ_{23}	Z_1	Z_2	Z_3	$\alpha = \arg[Z_\gamma]$
-5	1	-1	$\left(\frac{47}{72} + i\frac{7}{24}\right)\sqrt{5}$	$-i\frac{13\sqrt{5}}{24}$	$\frac{1}{6\sqrt{5}}$	$\tan^{-1}\left(\frac{90}{247}\right)$

Table 2. Relevant data for the $D4 - D0$ state (Part a).

$\mathcal{D}_\gamma$	c_1	c_2
$-\frac{155}{36}$	$-\frac{3139}{12}\sqrt{\frac{5}{69109}}$	$\frac{3211}{12}\sqrt{\frac{5}{69109}}$

Table 3. Relevant data for the $D4$-$D0$ state (Part b).

Configuration	z_1	z_2	z_3	$s(p)$
231	0	−1.33278	−0.65506	-1
312	0	2.07521	−5.42298	1
132	0	1.33278	0.65506	-1
213	0	−2.07521	5.42298	1

Table 4. Configurations contributing to the $D4$-$D0$ state.

we now compute the interaction degrees of freedom for this three center bound state

$$\begin{aligned} g_{\text{ref}}(\gamma_1, \gamma_2, \gamma_3, y) &= \frac{(-1)^{-5+1-1}}{(y - y^{-1})^2}\left(y^5 - y^3 - y^{-3} + y^{-5}\right) \\ &= -\left(y^{-3} + y^{-1} + y^1 + y^3\right). \end{aligned} \tag{2.27}$$

Specializing to $y \to (-1)$ results in $g_{\text{ref}} = 4$. Substituting this into (1.19) and using the fact that the single center refined index for a $D0$ is $\chi(CY_3) = -200$ yields a final index of

$$\begin{aligned} \bar{\Omega}(\gamma_1, \gamma_2, \gamma_3; t) &= g_{\text{ref}}\, \bar{\Omega}^S_{D6}\, \bar{\Omega}^S_{\bar{D}6}\, \bar{\Omega}^S_{D0} \\ &= 4 \times 1 \times 1 \times (-200) \\ &= -800. \end{aligned} \tag{2.28}$$

This too is in perfect agreement with the partition function.

D4-D0-D0 **bound states** There are two possibilities for the next polar state.

- A three center scenario, similar to the $D4$-$D0$ case[17], but with the third center carrying twice the unit $D0$ charge $\alpha_{2D0} = (0,0,0,-2)$. The total charge vector is now

$$\gamma = \left(0, 1, -\frac{5}{2}, \frac{11}{12}\right). \tag{2.29}$$

The corresponding integrability equations take the form

$$\frac{\gamma_{12}}{z_{12}} + \frac{\gamma_{13}}{z_{13}} = c_1, \tag{2.30}$$

$$\frac{\gamma_{23}}{z_{23}} + \frac{\gamma_{21}}{z_{21}} = c_2. \tag{2.31}$$

The relevant data required to solve these equations is collected in the following tables.

γ_{12}	γ_{13}	γ_{23}	Z_1	Z_2	Z_3	$\alpha = \arg[Z_\gamma]$
-5	2	-2	$\left(\frac{47}{72} + i\frac{7}{24}\right)\sqrt{5}$	$-i\frac{13\sqrt{5}}{24}$	$\frac{1}{3\sqrt{5}}$	$\tan^{-1}\left(\frac{90}{259}\right)$

Table 5. Relevant data for the $D4$-$2D0$ state (Part a).

$\mathcal{D}_\gamma$	c_1	c_2
$-\frac{35}{36}$	$-\frac{3223}{12}\sqrt{\frac{5}{75181}}$	$\frac{3367}{12}\sqrt{\frac{5}{75181}}$

Table 6. Relevant data for the $D4$-$2D0$ state (Part b).

The corresponding vector for β is

$$\beta = \left(-6\sqrt{\frac{5}{75181}}, \frac{259}{\sqrt{375905}}, -\frac{3367}{12}\sqrt{\frac{5}{75181}}, -\frac{295}{2}\sqrt{\frac{5}{75181}} + \frac{259}{2\sqrt{375905}}\right). \tag{2.32}$$

Solving the integrability equations and writing down those solutions that satisfy the discriminant positivity condition, we find the values in Table 7. The associated Poincaré polynomial is now

$$g_{\text{ref}}(\gamma_1, \gamma_2, \gamma_3, y) = \frac{(-1)^{-5+2-2}}{(y - y^{-1})^2}\left(y^5 - y^1 - y^{-1} + y^{-5}\right)$$

[17] This is an example of a scenario where $\mathcal{D} > \mathcal{D}_{D6}$ and yet it corresponds to a purely multi-center solution.

Configuration	z_1	z_2	z_3	$s(p)$
231	0	-0.446522	-0.222042	-1
312	0	1.95346	-5.41678	1
132	0	0.446522	0.222042	-1
213	0	-1.95346	5.41678	1

Table 7. Configurations contributing to the $D4$-$2D0$ state.

$$= -\left(y^{-3} + 2y^{-1} + 2y^{1} + y^{3}\right). \tag{2.33}$$

Therefore $g_{\text{ref}} = 6$ while the refined index for the $2D0$ center is given by

$$\bar{\Omega}^{S}_{2D0} = \chi(CY_3) + \frac{\chi(CY_3)}{4} = -250. \tag{2.34}$$

This yields a final index of

$$\begin{aligned}\bar{\Omega}(\gamma_1, \gamma_2, \gamma_3; t) &= g_{\text{ref}}\, \bar{\Omega}^{S}_{D6}\, \bar{\Omega}^{S}_{\bar{D}6}\, \bar{\Omega}^{S}_{2D0} \\ &= 6 \times 1 \times 1 \times (-250) \\ &= -1500.\end{aligned} \tag{2.35}$$

- A four center scenario with two explicit $D0$ centers:

The contributing centers are the previous $D6$ and $\bar{D}6$ centers with two explicit unit charge $D0$ charge vectors. The total charge vector is clearly the same as before. A detailed computation is no more illuminating to present here; the resulting g_{ref} in this scenario is half that of the previous case, owing to the halving of the symplectic products. The degeneracy for this four center $D6$-$\bar{D}6$-$D0$-$D0$ solution is $g_{\text{ref}} = 3$. This yields a final index of

$$\begin{aligned}\bar{\Omega}(\gamma_1, \gamma_2, \gamma_3, \gamma_4; t) &= \frac{g_{\text{ref}}}{2}\, \bar{\Omega}^{S}_{D6}\, \bar{\Omega}^{S}_{\bar{D}6}\, \bar{\Omega}^{S}_{D0}\, \bar{\Omega}^{S}_{D0} \\ &= \frac{3}{2} \times 1 \times (-200) \times (-200) \\ &= 60000,\end{aligned} \tag{2.36}$$

where the factor of half comes from the automorphism arising from the two identical $D0$ centers. This results in a total contribution of -1500 + 60000 = 58500, towards this state. All these numbers are clearly consistent with the modular prediction for $\mathcal{Z}_0$.

Once one has identified the appropriate centers that are of interest, the authors of [134] have developed a Mathematica code for the computation of the Poincare polynomials. The code is attached to their paper.

$D6$-$\bar{D}6$-$D2_{D0}$ **bound states** To move into the polar sector of $\mathcal{Z}_1$, now, one must add $D2$ charges. Naively, this added charge may merely be an increase in the $D2$ component of either of the $D6$ brane charges or act as an additional third center, with possible additional induced $D0$ centers. The split attractor flow allows for such flows into many channels [114]. In fact, in the large volume limit—the one we stick to in this thesis—one can even compute the index across the wall of marginal stability along the flow [135]; for an end point with three centers—which we will think of being the two $D6$ centers along with a generic $D2_{D0}$ center $(0, 0, q, q_0)$—is given by [135]

$$\begin{aligned}\Omega((12)3;t) \;=\; & \frac{1}{4}(-1)^{\gamma_{12}+\gamma_{31}+\gamma_{23}}\,\gamma_{(1+2)3}\cdot\gamma_{12}\cdot\Omega(\gamma_1)\cdot\Omega(\gamma_2)\cdot\Omega(\gamma_3)\\ & \left(\underbrace{\mathrm{sgn}\big[\mathrm{Im}[Z(\gamma_1+\gamma_2,t)\bar{Z}(\gamma_3,t)]\big]}_{a}+\underbrace{\mathrm{sgn}[\gamma_{(1+2)3}]}_{b}\right)\\ & \left(\underbrace{\mathrm{sgn}\big[\mathrm{Im}[Z(\gamma_1,t_1)\bar{Z}(\gamma_2,t_1)]\big]}_{c}+\underbrace{\mathrm{sgn}[\gamma_{12}]}_{d}\right). \end{aligned} \tag{2.37}$$

Specifying the quintic data, we find

$$\begin{aligned}\mathrm{sgn}[a] &= \mathrm{sgn}\left[q_0-\frac{235}{12}\right],\\ \mathrm{sgn}\big[b\big] &= -\mathrm{sgn}\big[5+q_0\big]\\ \mathrm{sgn}\big[d\big] &= -\mathrm{sgn}\big[q+q_0\big],\\ \mathrm{sgn}[c] &= \mathrm{sgn}\left[\frac{-72q_0^2-360q_0+408qq_0+5785q}{1728(5+q_0)}\right],\end{aligned} \tag{2.38}$$

upon computing the corresponding quantities in the underbraces. For a non-vanishing index, $\mathrm{sgn}[a]$ and $\mathrm{sgn}\big[b\big]$ must have the same sign (and similarly with $\mathrm{sgn}[c]$ and $\mathrm{sgn}\big[d\big]$). $\mathrm{sgn}[a]$ and $\mathrm{sgn}\big[b\big]$ have the same sign iff $-5 < q_0 < 20$, where we use the fact that q & $q_0 \in \mathbb{Z}$. Therefore, if this condition is satisfied, $\mathrm{sgn}[a] + \mathrm{sgn}\big[b\big] = -2$. For the total contribution to the index to be positive[18], $\mathrm{sgn}[c]$ and $\mathrm{sgn}\big[d\big] < 0$. Since $\gamma_{12} = -q - q_0$,

$$\mathrm{sgn}\big[d\big] = \mathrm{sgn}[\gamma_{12}] = -\mathrm{sgn}\big[q+q_0\big]. \tag{2.39}$$

[18]Considerations similar to those that will follow, rule out the case when the contribution is negative too.

Now $\text{sgn}[d] < 0$ implies $q > -q_0$. Putting all the pieces together, the allowed values for the center γ_2 such that there is a non-vanishing contribution to the index are collected in Table 8 of Appendix A. It is evident that in the direction of the physical $D0$ charges that bind with the $D4$, there are no non-vanishing $D2$ charges to form a three-center black hole bound state. Nevertheless, one might still investigate if $D0$ charges of the opposite sign can form the third center with non-vanishing $D2$ charges. As it turns out, none of the allowed values in Table 8 result in a positive discriminant everywhere outside the location of the centers. An example of this is shown in Figure 1 where the discriminant function (2.6) associated to a three center configuration with charges $D6$ and $\bar{D6}$ as in (2.15) and a third $D2_{D0}$ center with $(0, 0, -1, 2)$ is plotted against the axis of localization of the centers. Owing to the negative discriminant of the Pure D4 brane (arising from the ignoring of instanton corrections to the prepotential), one expects that the discriminant is negative at the locations of the $D6$ and $\bar{D6}$ centers. However, as is evident from the plot, the discriminant dips below zero even near the third center corresponding to the $D2_{D0}$ charge vector. This charge vector has zero discriminant and therefore must not go down to negative infinity as it does in the plot. One may easily check that in fact all allowed values of the $D2_{D0}$ center listed in Table 8 violate regularity.

This rules out the possibility of having a three center bound state with non-vanishing $D2$ charges. This may be seen as a more precise vindication of the naive argument we saw in the previous subsection.

In so far as the modified elliptic genus is concerned, this means that a polar term with $D2$ charges can occur only as a two-center configuration where one of the centers has additional $D2$ and $D0$ fluxes. To identify which of the two centers picks up the additional lower dimensional charges, we again look for the charge vectors whose discriminant increases upon the addition of the said charges to find the configuration to be

$$D6: \quad \gamma_1 = \left(1, 1, -\frac{55}{12}, \frac{23}{12}\right) \qquad \text{and} \qquad \bar{D6}: \quad \gamma_2 = \left(-1, 0, \frac{13}{12}, 0\right). \tag{2.40}$$

Since this is now a two center problem, g_{ref} is given by the symplectic product of the charge vectors $g_{\text{ref}} = |\gamma_{12}| = 3$. Therefore, the final index is given by

$$\begin{aligned} \bar{\Omega}(\gamma_1, \gamma_2; t) &= g_{\text{ref}}\, \bar{\Omega}_1^S\, \bar{\Omega}_2^S \\ &= 3 \times 2875 \times 1 \\ &= 8625, \end{aligned} \tag{2.41}$$

where the factor of 2875 comes from the Donaldson-Thomas invariants associated to the $D6$ with a $D2$ flux and a point p.

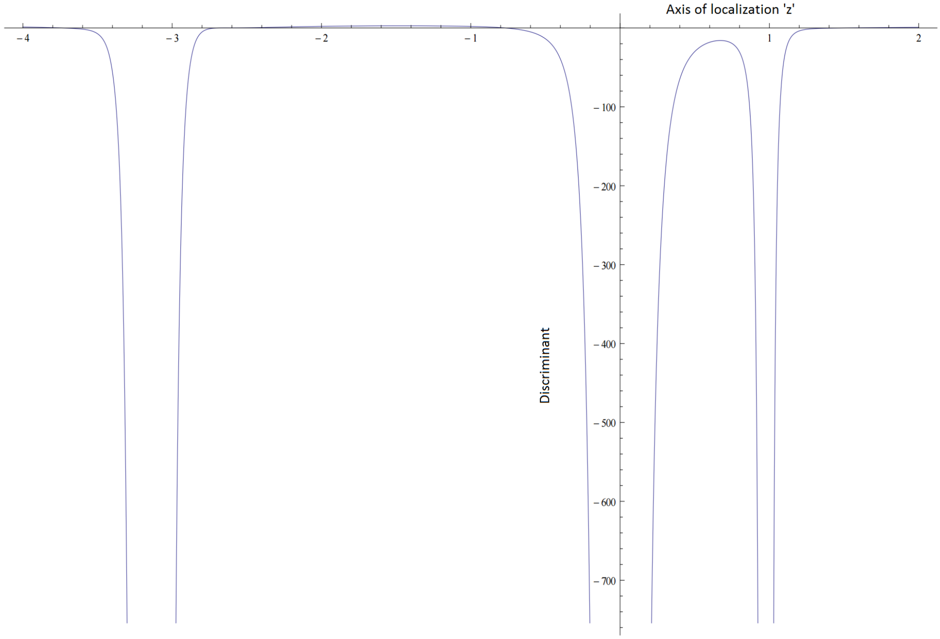

Figure 1. Discriminant function of a $D6-\bar{D}6-D2_{D0}$ three-center configuration with charges $D6$ and $\bar{D}6$ as in (2.15) and $D2_{D0}$ charge vector $(0,0,-1,2)$ plotted against the axis of localization of the centers.

X_6 in $\mathbb{WP}_{(2,1,1,1,1)}$

The sextic is a degree 6 hypersurface in $\mathbb{WP}_{(2,1,1,1,1)}$. For the purposes of this thesis, the topological invariants associated to the sextic we need are: $\chi(X_6) = -204$, $k = 3$, $c_2 \cdot P = 42$ and $N_{DT}(1,1) = 7884$. Its modified elliptic genus is given by

$$\mathcal{Z}_{X_6}(q,\bar{q},y) = \sum_{\gamma=0}^{2} \mathcal{Z}_\gamma(q)\Theta^{(3)}_\gamma(q,\bar{q},y) \tag{2.42}$$

where

$$\begin{aligned} \mathcal{Z}_0(q) &= q^{-\frac{45}{24}}(4 - 612q + \text{non-polar terms}) \\ \mathcal{Z}_1(q) &= Z_2(q) = q^{-\frac{5}{24}}(15768 + \text{non-polar terms}) . \end{aligned} \tag{2.43}$$

The charge vector of the Pure $D4$ brane is given by

$$\alpha_{D6} = \left(1,1,-\frac{13}{4},\frac{9}{4}\right) \quad \text{and} \quad \alpha_{\bar{D}6} = \left(-1,0,\frac{7}{4},0\right) \quad \text{give}$$

$$\alpha_{D4} = \alpha_{D6} + \alpha_{\bar{D}6} = \left(0, 1, -\frac{3}{2}, \frac{9}{4}\right), \tag{2.44}$$

Omitting explicit detail, the associated Poincaré polynomial is

$$g_{\text{ref}}(D4, y) = -\left(y^{-3} + y^{-1} + y^{1} + y^{3}\right). \tag{2.45}$$

which yields the correct final index of 4. Adding a $D0$ brane, yields

$$g_{\text{ref}}(D4 - D0, y) = y^{-2} + 1 + y^{2} \tag{2.46}$$

which gives a final index of $3 \times -204 = -612$. Finally, adding a $D2$ charge, the two centers are

$$D6\colon \quad \gamma_1 = \left(1, 1, -\frac{13}{4}, \frac{5}{4}\right) \qquad \text{and} \qquad \bar{D6}\colon \quad \gamma_2 = \left(-1, 0, \frac{3}{4}, 0\right). \tag{2.47}$$

with

$$g_{\text{ref}}(D4 - D2_{D0}, y) = 2 \tag{2.48}$$

yielding a final index of $2 \times 7884 = 15768$.

X_8 in $\mathbb{WP}_{(4,1,1,1,1)}$

The octic threefold is a degree 8 hyperplane in $\mathbb{WP}_{(4,1,1,1,1)}$ its relevant topological invariants are: $\chi(X_8) = -296$, $k = 2$, $c_2 \cdot P = 44$ and $N_{DT}(1,1) = 29504$. Its modified elliptic genus is given by

$$\mathcal{Z}_{X_8}(q, \bar{q}, y) = \sum_{\gamma=0}^{4} \mathcal{Z}_\gamma(q) \Theta^{(2)}_\gamma(q, \bar{q}, y) \tag{2.49}$$

where

$$\begin{aligned} \mathcal{Z}_0(q) &= q^{-\frac{23}{12}}(4 - 888q + \text{non-polar terms}) \\ \mathcal{Z}_1(q) &= q^{-\frac{1}{6}}(59008 + \text{non-polar terms}). \end{aligned} \tag{2.50}$$

The Pure $D4$ brane is now

$$\begin{aligned} \alpha_{D6} &= \left(1, 1, -\frac{17}{6}, \frac{13}{6}\right) \quad \text{and} \quad \alpha_{\bar{D}6} = \left(-1, 0, \frac{11}{6}, 0\right) \quad \text{give} \\ \alpha_{D4} &= \alpha_{D6} + \alpha_{\bar{D}6} = \left(0, 1, -1, \frac{11}{6}\right), \end{aligned} \tag{2.51}$$

with an associated Poincaré polynomial

$$g_{\text{ref}}\big(D4, y\big) \;=\; -\Big(y^{-3} + y^{-1} + y^{1} + y^{3}\Big)\,. \tag{2.52}$$

and final index of 4. Adding a $D0$ brane, yields

$$g_{\text{ref}}\big(D4 - D0, y\big) \;=\; y^{-2} + 1 + y^{2} \tag{2.53}$$

which gives a final index of $3 \times -296 = -888$. Finally, adding a $D2$ charge, the two centers are

$$D6: \quad \gamma_1 = \left(1, 1, -\frac{17}{6}, \frac{7}{6}\right) \qquad \text{and} \qquad \bar{D6}: \quad \gamma_2 = \left(-1, 0, \frac{5}{6}, 0\right). \tag{2.54}$$

with

$$g_{\text{ref}}\big(D4 - D2_{D0}, y\big) \;=\; 2 \tag{2.55}$$

yielding a final index of $2 \times 29504 = 59008$.

X_{10} **in** $\mathbb{WP}_{(5,2,1,1,1)}$

The dectic is a degree 10 hypersurface in $\mathbb{WP}_{(5,2,1,1,1)}$. The relevant topological invariants associated to the dectic are: $\chi(X_{10}) = -288$, $k = 1$ and $c_2 \cdot P = 34$. Its modified elliptic genus is given by

$$\begin{aligned} \mathcal{Z}_{X_5}\big(q, \bar{q}, y\big) \;&=\; \frac{\eta(q)^{-35}}{576}\big[541 E_4(q)^4 + 1187 E_4(q) E_6(q)^2\big]\Theta_1(\bar{q}, y) \\ &=\; q^{-\frac{35}{24}}\left(3 - 576q + \text{non-polar terms}\right). \end{aligned} \tag{2.56}$$

A Pure $D4$ brane in this example is

$$\begin{aligned} \alpha_{D6} \;&=\; \left(1, 1, -\frac{23}{12}, \frac{19}{12}\right) \quad \text{and} \quad \alpha_{\bar{D}6} \;=\; \left(-1, 0, \frac{17}{12}, 0\right) \quad \text{give} \\ \alpha_{D4} \;&=\; \alpha_{D6} + \alpha_{\bar{D}6} \;=\; \left(0, 1, -\frac{1}{2}, \frac{19}{12}\right). \end{aligned} \tag{2.57}$$

The Poincaré polynomial is

$$g_{\text{ref}}\big(D4, y\big) \;=\; y^{-2} + 1 + y^{2}\,. \tag{2.58}$$

And the corresponding index is 3. Adding a $D0$ brane, yields

$$g_{\text{ref}}\big(D4 - D0, y\big) \;=\; -y^{-1} - y^{1} \tag{2.59}$$

which gives a final index of $2 \times -288 = -576$. Clearly, all results exactly build the polar terms under consideration in the examples.

3 Discussion

In this chapter, we have identified all multi-center configurations (whose total charge vectors violate the naive single-center cosmic censorship bound) that build the polar sector of several elliptic genera of Calabi-Yau threefolds with maximal holonomy. It is natural to expect that once one moves into the non-polar sector of the theory, when total charge vectors no longer violate the cosmic-censorship bound, single center black holes begin to contribute. Exactly what states these constitute is not fully known. Several interesting suggestions have been made [136–139] in the literature. Nevertheless, large charge single-center black hole entropy has not been easy to understand concretely, with these suggestions.

With the prescription we have proposed in this chapter, one may now seek to push into the non-polar sector of the elliptic genus to understand single-center black hole entropy. Naively, the approach from split-flows proposed in [133, 139] might have been a good starting point to push deep into the non-polar sector. As one increases charge, there is an expectation that an increasing number of multi-center configurations must contribute to the index. For instance, moving on from a $D4$-$D0$ charge vector to a $D4$-$2D0$ charge vector, one expects two different contributions: one from a three center $D6$-$\bar{D}6$-$2D0$ solution and another from a four center $D6$-$\bar{D}6$-$D0$-$D0$ configuration. The authors of [133, 139], however, argue for only a single flow. On the contrary, we saw explicitly in (2.35) and (2.36) that both the expected configurations do indeed contribute to produce the correct polar term. In extension, enumeration of all multi-center configurations can systematically be done with the approach we present in this thesis. It may be noted that a more general prescription might be needed to incorporate higher degree rational curves to include more $D2$ charges with appropriately induced $D0$ charges. Nevertheless, it is our hope that this enables for a better understanding of the non-polar sector.

Although we have refrained from stressing on them, there are several aspects of purely mathematical interest that are very closely related to the study of BPS states mentioned in this chapter. The elliptic genera we studied so far encode topological invariants of the moduli space of the derived category of coherent sheaves on a Calabi-Yau threefold. Some questions in this field related to this thesis are: What is the generating function for the Euler numbers of the moduli space of stable sheaves (seen as objects in the derived category of coherent sheaves where stability is usually thought to be Bridgeland stability as in the Kontsevich-Soibelman setup) on a smooth three-dimensional quasi-projective variety? What is the generating function for the Betti numbers for the same? Göttsche has answered both these questions for sky-scraper sheaves on smooth two-dimensional quasi-projective varieties [140]. In the mid-nineties, Cheah [141] managed to write a generating function (the McMahon function) for the Euler numbers of the moduli space of stable sky-scraper sheaves on smooth three dimensional quasi-projective varieties.

A refinement of Cheah's result in the spirit of Göttsche is expected to be related to single-center black hole entropies. While hoping for a general result might be far fetched from the explicit multi-center prescription presented in this chapter, we speculate that it may well prove to be very helpful in conjuring up and testing conjectures [136, 137] in this regard.

Another branch of mathematical interest that is closely related is the study of Poincaré polynomials of quiver representation spaces. A Kähler manifold is endowed with a natural Sl_2 Lefschetz action on the cohomology. And the generating function of the Euler numbers mentioned above captures this action because Euler numbers are, after all, characterized by the cohomology. However, the refined generating function of the Betti numbers exactly organizes BPS states into different representations of the Lefschetz action. States invariant under this action have been conjectured to be special, in that they are expected to capture single-center black hole entropy. With the results presented in this chapter, one may identify the Lefschetz singlets in the low-lying non-polar terms to test the conjecture of [136, 138] that Pure-Higgs states make up single-center indices in all the above examples. The Poincaré polynomials studied in the mathematics literature that encode these invariants are based on Reinike's solution to the Harder-Narasimhan recursion for quivers without oriented closed loops. However, the indices used in this thesis, originally proposed in [134, 135], applies to those with or without closed loops. It would be interesting to understand a mathematical counterpart of the latter, as extensions of Reinike's results. On the other hand, one may seek to understand a pattern of growth of multi-center entropies to compare against asymptotic behaviour of states under various representations that has been predicted in [111].

A Three-center $D6$-$\bar{D}6$-$D2_{D0}$ configurations in the quintic

The allowed $D2$-$D0$ charges for a generic $D6$-$\bar{D}6$-$D2_{D0}$ three-center configuration to have a non-vanishing index are collected in Table 8 below. None of these allowed values

q_0	q
-4	None
-3	None
-2	None
-1	None
0	None
1	0
2	-1, 0
3	-2, -1, 0
4	-3, -2, -1, 0
5	-4, -3, -2, -1, 0
6	-5, -4, -3, -2, -1, 0
7	-6, -5, -4, -3, -2, -1, 0
8	-7, -6, -5, -4, -3, -2, -1, 0
9	-8, -7, -6, -5, -4, -3, -2, -1, 0
10	-9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1
11	-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1
12	-11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1
13	-12, -11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1
14	-13, -12, -11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1
15	-14, -13, -12, -11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1
16	-15, -14, -13, -12, -11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1
17	-16, -15, -14, -13, -12, -11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2
18	-17, -16, -15, -14, -13, -12, -11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2
19	-18, -17, -16, -15, -14, -13, -12, -11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2

Table 8. Allowed values for q and q_0.

satisfy regularity, proving the non-existence of the corresponding three-center solutions.

Part Two

The non-supersymmetric and the dynamical

Chapter III

Near-extremal black hole entropy

ENTROPY is a quantity that appears in the study of finite-temperature systems in statistical physics. And yet, we found it easier to study it in supersymmetric configurations which have vanishing temperature. No physical process can lead us to absolute zero temperature, as thermodynamics has come to teach us. Therefore, it is imperative that we are able to use the intuition gained from the study of supersymmetric black holes to study those that have non-zero temperature. In fact, it was shown in several papers starting with [104, 142–146] that the near-extremal generalization of the leading order entropy we saw in (I.1.27) and (I.1.35) is given by

$$S_{\text{non-extremal}} = 2\pi\left(\sqrt{\frac{c_L n_L}{6}} + \sqrt{\frac{c_R n_R}{6}}\right), \tag{0.1}$$

where the additional term arises from the right-moving excitations in the CFT. This formula is valid (microscopically) only when $n_L \gg 1$ and $n_R \gg 1$. And on the black hole front, the additional term may be seen as coming from the parameter of non-extremality (which is either the temperature, T, or the distance, r_0, between the inner and outer horizons of the charged black hole). Since supersymmetry allowed us to leave the right-moving sector in the ground state, excitations of it naturally imply a breaking of supersymmetry. Finally, the momentum in the CFT (or equivalently the electric charge) is given by $q_0 = n_L - n_R$. The macroscopic analysis is valid only in the near-extremal limit $n_L \gg n_R$.

Despite this simple generalization, true dynamics of the black hole arise from non-trivial interaction between the horizon and fundamental matter in the theory, which are not captured by a mere identification of states. Therefore, we would like to study such non-extremal black hole entropy in theories with light, charged matter in the spectrum of the gravitational theory. And yet, we would still like to have the microscopic field theory tractable. Such instances are hard to come by and it is the aim of this chapter to take strides in this direction. We will first illustrate how light, charged matter in the spectrum leads to interesting dynamics and how powerful such considerations can be in teaching us about the microscopic dynamics of black holes.

1 An invitation to dynamics of a black hole

It is important to note that the black hole solutions whose near-extremal entropy is captured by (0.1) are physically different from those captured by (I.1.27). Despite the small temperature or near-extremality, they are not to be seen as a perturbation from either the supersymmetric or extremal black holes. Nevertheless, from (0.1), we see that there exists a smooth limit to the supersymmetric entropy formula by taking $n_L \gg n_R \gg 1$. Ensuring a fixed $n_R \gg 1$, considering larger and larger n_L takes us closer and closer to the supersymmetric case. The strict BPS entropy is achieved by a ground state on the right moving sector. But it is no small perturbation to jump from a ground state to a large ensemble of states in the right moving sector. However, the only way to approach an extremal black hole (to arbitrary proximity), starting from a near extremal one, is by considering a large ensemble of states in the right moving sector and taking an even larger ensemble on the left moving sector. Therefore, one is not a 'small perturbation' of the other, by any measure. Another, perhaps more straight forward, way to see this is by recalling that a supersymmetric black hole has an infinitely long throat in the geometry, while a near extremal hole has a finite throat. No matter how close one is to the supersymmetric limit, a near extremal throat is infinitely smaller than the supersymmetric one. Therefore, it could not possibly be a perturbation. Therefore, to understand dynamics of a black hole with charged matter, it is imperative that we pick a black hole that is arbitrarily close to extremality and consider how its entropy changes upon, say, throwing a particle into it.

Constraints in a Reissner-Nordström background

To get a flavour for the nature of microscopic features we might hope to learn via such dynamics, for simplicity, let us start with a Reissner-Nordström black hole as was done in [147]. The temperature of such a black hole is given by[1]

$$T = \frac{\sqrt{f}}{4\pi M\left(1 - \frac{Q^2}{2M^2} + \sqrt{f}\right)}, \tag{1.1}$$

where $f = 1 - \frac{Q^2}{M^2}$. Q is the charge of the black hole while M is its mass. The change in temperature, if the mass and charge of the black hole change by δM and δQ, respectively, is then

$$\delta T \approxeq \frac{1}{2\pi M^2} f^{-\frac{1}{2}} \left(\delta M - \frac{Q}{M}\delta Q\right). \tag{1.2}$$

Here, we have assumed that the changes are small compared to the mass and charge of the black hole, such that the original background is still a good approximation. Barring any exotic features, we expect the temperature of the near extremal black hole to monotically

[1] See, for instance, equation (1) of [147].

decrease in the approach to extremality as particles of opposite charge are absorbed. Therefore, in the process of moving out of extremality, we expect that $\frac{\delta T}{T}$ is positive. Therefore, we see that $\delta Q - \delta M > 0$ if we pick a black hole with $M > 0$ and $Q > 0$. Now, consider

$$\frac{\delta T}{T} = \frac{2\left(1 - \frac{Q^2}{2M^2} + \sqrt{f}\right)\left(q\frac{Q}{M} - m\right)}{Mf}, \tag{1.3}$$

where $\delta Q = -q$ with $q > 0$ and $\delta M = -m$ with $m > 0$. The reason for such a consideration is the following. Due to vacuum polarization in the presence of a strong electric field at the horizon, we expect the oppositely charged particles (of those that are pair created, say,) to fall into the black hole while the like-charged ones to fly off to infinity. Since the mass of the black hole is measured at infinity, we also expect the mass of the black hole to reduce by the mass of the particle that is detected at infinity. Therefore, upon pair production of a particle of mass m and charge q, the mass and charge of the black hole are expected to change to $(M - m)$ and $(Q - q)$, respectively. Since the pair created particle does not have a horizon of its own, we have $q > m$ which implies that $(M - m) > (Q - q)$. For a statistical interpretation of the black hole to be reasonable, we would like to demand that $\frac{\delta T}{T} \ll 1$. This means,

$$q\frac{Q}{M} - m \ll \frac{Mf}{2\left(1 - \frac{Q^2}{2M^2} + \sqrt{f}\right)}. \tag{1.4}$$

We can rewrite this as

$$q\frac{Q}{M} - m \ll \frac{Mf}{\left(1 + 2\sqrt{f} + f\right)} = Mf\left(1 + \sqrt{f}\right)^{-2} \sim Mf\left(1 - 2\sqrt{f}\right).$$

For a large black hole that is very close to extremality, this relation approximately reduces to

$$q\frac{Q}{M} - m \ll Mf = M\left(1 - \frac{Q^2}{M^2}\right). \tag{1.5}$$

Let us now define a quantity that naively measures the 'extent of non-extremality' as $E := M - Q$. This measures non-extremality since extremal black holes satisfy $M = Q$. This definition of E implies

$$\frac{E}{M} = 1 - \frac{Q}{M}.$$

Therefore, in terms of E, the inequality (1.5) can be rewritten as

$$q\left(1 - \frac{E}{M}\right) - m \ll M\left(1 + \frac{Q}{M}\right)\left(1 - \frac{Q}{M}\right) = E\left(2 - \frac{E}{M}\right),$$

which can now be simplified to

$$q - m - q\frac{E}{M} \ll 2E\,, \tag{1.6}$$

where we have again used the fact that the black hole under consideration is very close to extremality, enabling us to drop terms such as $\frac{E^2}{M}$, for small E. We can now rewrite the above inequality as

$$q - m \ll E\left(2 + \frac{q}{M}\right). \tag{1.7}$$

Noticing that $M \gg q$, we finally obtain the following constraint:

$$E \gg \frac{1}{2}(q - m) > 0. \tag{1.8}$$

This imposes a rather severe constraint. Given that we have picked a black hole that is arbitrarily close to extremality, we know that $E \ll 1$. Therefore, a fulfilling of the above inequality and the validity of a statistical interpretation of near-extremal black holes are at obvious odds with each other. The above derivation can be seen as a rephrasing of the essence of [147]. However, for the black hole of our interest, we will see that string theory allows for a large ensemble of states that allow for a convenient relaxation of the above inequality while still allowing for the black hole to be arbitrarily near extremality. From a stringy point of view, we have already seen this possibility in the form of the inequality $n_L \gg n_R \gg 1$, where we see that the degree of extremality can still be very small provided $n_L \gg n_R$ while still exciting a large ensemble of states via $n_R \gg 1$.

In the rest of this chapter, we will set up a near-extremal black hole arising from threefold compactifications while still aiming to keep the microscopic picture of the MSW-CFT under control.

2 Rholography and gauged supergravity

As we saw in Part One of this thesis, among the successes of string (M-) theory, the interpretation of black hole entropy as resulting from microscopic degrees of freedom of D-branes (M-branes) stands out. The black hole solutions described in [19, 20] can be uplifted to black strings in one higher dimension, where the near-horizon limit contains an AdS_3 factor in the supersymmetric case, or a BTZ-factor for near-extremal black holes. One can then use the Cardy formula for the dual conformal field theories, which are based on $(0, 4)$ [20] and $(4, 4)$ [19] superconformal field theories in two dimensions. In the near-extremal case, one makes use of the correspondence between BTZ geometries and thermal conformal field theories, which does not rely on any supersymmetry [146].

Among the plethora of asymptotically flat black holes whose microscopics have been studied, there have been none in gauged supergravity, in which there is light charged matter in the supergravity spectrum. Such a situation would be needed to study absorption and reflection coefficients of charged matter by the black hole, or to compute black hole discharge through Schwinger processes. In this chapter, we will not go as far as this, but we present a construction of a class of black hole solutions in gauged supergravity with flat Minkowski vacua. In other words, we present a framework in which these processes could be studied. Microscopically, we find that they are described by $N = 4$ superconformal systems ($(0,4)$ or $(4,4)$) with twisted boundary conditions, characterized by a parameter, called ρ in the original work of Schwimmer and Seiberg [148]. The twist involves the outer automorphism group of the superconformal algebra, and the corresponding ρ-algebra is inequivalent to the usual NS and R-sectors and its spectral flow [148]. To the best of our knowledge, this ρ-algebra has never found an interesting application, but in this chapter we show that it governs the microscopics of asymptotically flat black holes in gauged supergravities.

We will first concern ourselves with M-theory on a compact Calabi-Yau threefold (CY_3, henceforth). This results in five-dimensional $\mathcal{N} = 2$ supergravity coupled to vector- and hypermultiplets. The low-energy limit of the M5-brane put on a compact divisor in the CY_3 is a black string solution to the said supergravity theory. A further compactification of this black string along an S^1 is a black hole solution of a four-dimensional $\mathcal{N} = 2$ supergravity theory. As was shown in [20], the macroscopic entropy of such a BPS black hole is microscopically realized as a set of microstates within the conformal field theory living on the worldsheet of the M5-string. This theory has been called the MSW-CFT in the literature; it is a (0,4) superconformal field theory (SCFT) in 2 dimensions. The M5-string worldsheet effective action has been studied in detail, in [149] for instance.

While we will still persist with five-dimensional ungauged supergravity theory resulting from compactifications of M-theory, we wish now to consider a more general Scherk-Schwarz reduction along the additional S^1 to arrive at a four-dimensional theory. Such a consideration is not new. In [150, 151], it was shown that imposing Scherk-Schwarz twisted boundary conditions results in gauged supergravities theory in four dimensions with positive definite scalar potentials with Minkowski vacua. Our strategy will be to find a Scherk-Schwarz twist and the corresponding gauging that preserves the black hole solutions from the untwisted case. As we will show, such a twist can be done by using the R-symmetry group. R-symmetry in supergravity is in general not a symmetry of the action, but classically - ignoring quantum corrections - it often is. We can say it is an approximate symmetry, valid in the classical supergravity regime, and use it in the Scherk-Schwarz twist. Since the theory is now gauged, the spectrum is non-trivial and there are light, R-charged particles in it. Owing to the fact that the circle on which we Scherk-Schwarz reduce is exactly the spatial circle of the M5-string, boundary conditions can consistently be imposed in the microscopics to match the macroscopic supergravity setup. In essence, a twisted generalization of the MSW-CFT results. The interplay between R-symmetry, the ρ-twist and the use of holography motivates us to name our construction "R-Holography", "ρ-lography", or simply "Rholography".

In Section 3, we review the ρ-algebras of [148] and show how the boundary conditions are implemented in the bulk on the gravitini living in AdS_3. Moreover, we will study the implications of the ρ-twist for the ground state energy of the right moving sector of the MSW-CFT. This will allow us to determine the entropy of the field theory in a thermal state, as a function of ρ, using Cardy's celebrated formula.

Further on, in Section 4, we will identify the appropriate Scherk-Schwarz twist in the five-dimensional supergravity theory that corresponds to the microscopic ρ-twist and carry out the reduction to four dimensions. As we will show, the relevant twist uses the R-symmetry that acts on the supersymmetry generators of the five-dimensional $\mathcal{N} = 2$ supergravity theory. After identifying a flat Minkowski vacuum we compute the supergravity spectrum and we construct a non-extremal black hole in the gauged four-dimensional theory. Uplifting this black hole to five dimensions allows us to define a consistent near-horizon limit in which we realize a thermal BTZ geometry; but now with a ρ-twisted angular coordinate. To leading order, its entropy matches the Cardy formula of the dual ρ-twisted MSW-CFT. Phrased differently, we conjecture that the ρ-twisted MSW-CFT is dual to M-theory on[2] $AdS_3^\rho \times S^2 \times CY_3$. The superscript on the AdS_3 merely refers to the twisted boundary conditions along the angular coordinate in AdS_3.

Finally, we present a different example in Section 5 based on the D1-D5 system in type IIB on $K3 \times S^1$, which is of equal interest as the MSW setup. The line of thought is exactly the same as in the MSW system, except that the R-symmetry is now larger, i.e. $SO(5)_R$ instead of $SU(2)_R$. Furthermore, the CFT is $(4,4)$ instead of $(0,4)$ allowing for ρ-twists on either of the chiral sectors.

As is evident through this introduction, we will consistently lay emphasis on the two punchlines of this work. One is the microscopic description of a large class of asymptotically flat black holes in R-gauged supergravities. The other is the microscopic realization of the ρ-algebras and their conjectured bulk duals.

[2]Scherk-Schwarz reductions often break supersymmetry spontaneously [152], and the vacuum might be unstable. To deal with this properly, one has to actually start with a T^6 instead of a CY_3, so that partial supersymmetry can be preserved. This effect is however not relevant for the present leading order calculations.

3 Rholography

The M5-brane breaks half of the supersymmetry available in M-theory. It carries a chiral $(0,2)$ SCFT in six dimensions. The Lorentz group breaks to $Spin(1,5)$ with an additional $USp(4)$ R-symmetry owing to the transverse directions. Its world volume theory consists of 5 scalars X $\{= X^a,\ a = 1,\dots,5\}$ (corresponding to the transverse directions of the brane), four six-dimensional Weyl spinors ψ $\{= \psi_i,\ i = 1,\dots,4\}$ that obey a symplectic reality condition and an anti-symmetric two form B_2 whose field strength is self-dual. Considering M-theory on a CY_3 background and placing the M5-brane on a holomorphic compact divisor inside the threefold reduces the symmetry of the world-volume theory. The $USp(4) \simeq SO(5)$ breaks to a $Spin(3) \times Spin(2)$ symmetry—the $Spin(3)$ comes from the position of the brane in non-compact space while the $Spin(2)$ is owed to the position of the brane in the CY_3. Furthermore, the six dimensional local Lorentz group breaks to $Spin(1,1) \times Spin(4)$, and reduces to a $Spin(1,1) \times SU(2) \times U(1)$ symmetry on the M5-string worldsheet [109]. We may now gauge fix the world-sheet coordinates to align with the target space coordinates to realize the $Spin(1,1)$ Lorentz symmetry on the world-sheet. This is the MSW-CFT and its world-sheet field content can be obtained from the reduction of the M5-brane world-volume fields [149]. In this chapter, we will entirely focus our attention on two symmetries of this field theory—the $Spin(3)$ that manifests itself as a local $SU(2)$ Kac-Moody algebra in the field theory, and the global $SU(2)$ flavor symmetry. The latter is actually only a symmetry of the algebra, and not necessarily of the CFT. It is the outer automorphism group of the superconformal algebra. It may happen that for large value of the central charge, the outer automorphism group may actually become a symmetry. In the dual bulk, this is the classical supergravity regime. We get back to this point later. Since all the supersymmetry generators are in the right moving sector of the CFT, we can study this $N = 4$ superconformal algebra in its own right. For notational ease, we will call the Kac-Moody gauge group $SU(2)_\eta$ and the outer automorphism group $SU(2)_\rho$. Together, they form the total automorphism group $SO(4)$ of the small $N = 4$ superconformal algebra [148].

It is worth understanding the presence of these symmetries in the different theories of interest. From the black string perspective, the $SU(2)_\eta$ local gauge symmetry is realized as the spherical symmetry of the horizon. It sits inside the Lorentz group of the five-dimensional supergravity theory and similarly, it is also the rotational symmetry goup of a spherical black hole in the four-dimensional supergravity theory. The outer automorphism group, also called the global $SU(2)_\rho$ flavor symmetry when it is a symmetry, however, has roots in the CY_3. As we discuss in Section 4, it is the $SU(2)$ R-symmetry of five-dimensional $\mathcal{N} = 2$ supergravity acting on the supersymmetry generators. Upon compactifying on a circle, we will perform the Scherk-Schwarz twist with respect to a $U(1)$ subgroup of this $SU(2)$ R-symmetry. As we will show, in the supergravity regime, this subgroup is actually a symmetry of the action. In the $N = 4$ CFT, as has been studied in [148], a twisting of the Abelian subgroup of the local $SU(2)_\eta$ is just a gauge symmetry; it can be undone by spectral flow. However, a twisting of the Abelian subgroup of the global $SU(2)_\rho$ symmetry results in an infinite family of $(0,4)$ algebras parametrized by the twisting parameter ρ. It cannot be undone because the $U(1)_\rho \subset SU(2)_\rho$ is only an approximate symmetry, in much the same way as the bonus symmetry discussed in [153]. Hence there is no current algebra associated to it, and hence no spectral flow. The twist is 'felt' by all the fields in the CFT that transform non-trivially under the $SU(2)_\rho$. This includes, in particular, the supercharges that transform under a doublet representation. Since the twist is under an Abelian subgroup of the R-symmetry, the corresponding five-dimensional supergravity theory realizes it as a specific Scherk-Schwarz reduction on the circle; one that corresponds to an R-gauging in four dimensions. This is a $U(1)$ gauged supergravity theory in four dimensions, and it already indicates that the twist parameter ρ must be related to the $U(1)$ gauge coupling constant. Much like in the un-twisted case, the twisted CFT counts the microstates associated to a black hole in this gauged supergravity theory.

We will now make the discussion more concrete, by first presenting the small $N = 4$ superconformal algebra with ρ-twist in Section 3.1, and the structure of the holographic dual in Section 3.2.

3.1 Small $N = 4$ superconformal algebra with ρ-twist

We will call the Virasoro generators L_m (and their corresponding stress tensor $L(z)$), the Kac-Moody generators T^i (with $i = 1,2,3$) and the four supercharges G^{aA} (with $a = 1,2$ and $A = \pm$). Here, i is an $SU(2)_\eta$ triplet index, a an $SU(2)_\eta$ doublet index and A an $SU(2)_\rho$ doublet index. The Operator Product Expansions (OPEs) can be determined from [148, 154, 155]. Dropping the regular terms when $z \to w$, they are

$$\begin{aligned}
L(z)L(w) &= \frac{\partial_w L(w)}{z-w} + \frac{2L(w)}{(z-w)^2} + \frac{\frac{1}{2}c_R}{(z-w)^4}\,,\\
G^{a\pm}(z)G^{b\mp}(w) &= \delta^{ab}\left(\frac{2L(w)}{z-w} + \frac{\frac{2}{3}c_R}{(z-w)^3}\right)\\
&\qquad + (\sigma^i)^{ab}\left(\frac{2\,\partial_w T^i(w)}{z-w} + \frac{4i\,T^i(w)}{(z-w)^2}\right),\\
T^i(z)T^j(w) &= i\frac{\varepsilon^{ijk}\,T^k(w)}{z-w} + \frac{\frac{1}{12}c_R\,\delta^{ij}}{(z-w)^2}\,,\\
L(z)G^{a\pm}(w) &= \frac{\partial_w G^{a\pm}(w)}{z-w} + \frac{\frac{3}{2}G^{a\pm}(w)}{(z-w)^2}\,,
\end{aligned} \tag{3.1}$$

$$
\begin{aligned}
L(z)T^i(w) &= \frac{\partial_w T^i(w)}{z-w} + \frac{T^i(w)}{(z-w)^2}\,, \\
T^i(z)G^{a+}(w) &= \frac{\frac{1}{2}G^{b+}(w)\left(\sigma^i\right)_b{}^a}{z-w}\,, \\
T^i(z)G^{a-}(w) &= -\frac{\frac{1}{2}\left(\sigma^i\right)^a{}_b G^{b-}(w)}{z-w}\,,
\end{aligned}
$$

where $(\sigma^i)^a{}_b$ are the Pauli matrices.

As shown in [148, 154], the total automorphism group of these OPEs (and the algebra generated by them) is $SO(4) = SU(2)_\eta \times SU(2)_\rho$. The inner outomorphism group is $SU(2)_\eta$ and corresponds to the current algebra while the outer automorphism group is the global $SU(2)_\rho$. Twists under the Abelian subgroups of the two $SU(2)$ groups are generated by [148, 154] –

$$
\begin{aligned}
G^{1\pm}(ze^{2\pi i}) &= e^{\mp i\pi(\rho+\eta)}\, G^{1\pm}(z)\,, \\
G^{2\pm}(ze^{2\pi i}) &= e^{\mp i\pi(\rho-\eta)}\, G^{2\pm}(z)\,, \\
T^{\pm}(ze^{2\pi i}) &= e^{\pm 2\pi i\eta}\, T^{\pm}(z)\,,
\end{aligned}
\tag{3.2}
$$

where $T^\pm = T^1 \pm iT^2$, while $T^3(z)$ and $L(z)$ are left to be periodic. The resulting mode expansion for the supercharges is, therefore,

$$
\begin{aligned}
G^{1\pm}(z) &= \sum_{m\in\mathbb{Z}} G^{1\pm}_{m\pm\frac{\rho+\eta}{2}+\frac{1}{2}}\, z^{-m\mp\frac{\rho+\eta}{2}-2}\,, \\
G^{2\pm}(z) &= \sum_{m\in\mathbb{Z}} G^{2\pm}_{m\pm\frac{\rho-\eta}{2}+\frac{1}{2}}\, z^{-m\mp\frac{\rho-\eta}{2}-2}\,.
\end{aligned}
\tag{3.3}
$$

The usual NS and R sectors have $\rho = 0$, with $\eta = 0$ and $\eta = 1$ respectively. These result in half-integer ($\eta = 0$) and integer ($\eta = 1$) modes for the supercharges, respectively. For $\rho \neq 0$, one gets inequivalent algebras. In this chapter, we will exclusively work with non-zero ρ.

Any particular mode can be extracted out of this Laurent series by an appropriate Cauchy integral as

$$
\begin{aligned}
G^{1\pm}_{m\pm\frac{\rho+\eta}{2}+\frac{1}{2}} &= \frac{1}{2\pi i}\int \mathrm{d}z\, z^{m\pm\frac{\rho+\eta}{2}+1}\, G^{1\pm}(z)\,, \\
G^{2\pm}_{m\pm\frac{\rho-\eta}{2}+\frac{1}{2}} &= \frac{1}{2\pi i}\int \mathrm{d}z\, z^{m\pm\frac{\rho-\eta}{2}+1}\, G^{2\pm}(z)\,.
\end{aligned}
\tag{3.4}
$$

The anti-commutation relations for the supercharges can now be calculated from this mode expansion and the OPE in (3.1), using Cauchy's theorem. The result is:[3]

$$\left\{ G^{1+}_{m+\frac{\rho+\eta}{2}+\frac{1}{2}}\,,\; G^{1-}_{n-\frac{\rho+\eta}{2}-\frac{1}{2}} \right\} = 2\,L_{m+n} \;+\; 2(m-n+1+(\rho+\eta))\,T^3_{m+n} \\ +\; \frac{c_R}{12}\left[(2m+1+(\rho+\eta))^2-1\right]\delta_{m+n,0}\,. \qquad (3.5)$$

$$\left\{ G^{2+}_{m+\frac{\rho-\eta}{2}+\frac{1}{2}}\,,\; G^{2-}_{n-\frac{\rho-\eta}{2}-\frac{1}{2}} \right\} = 2\,L_{m+n} \;-\; 2(m-n+1+(\rho-\eta))\,T^3_{m+n} \\ +\; \frac{c_R}{12}\left[(2m+1+(\rho-\eta))^2-1\right]\delta_{m+n,0}\,. \qquad (3.6)$$

We know that the η twist is a gauge redundancy and therefore causes spectral flow. Any physical quantity must be independent of η. The gauge independent, spectral flow invariant quantities do not depend on η and are defined by the relations [148]

$$\begin{aligned} L_n(\rho,\eta) &= L_n(\rho) - \eta\,T^3_n(\rho) + \eta^2\,\frac{c_R}{12}\,\delta_{n,0} \\ T^3_n(\rho,\eta) &= T^3_n(\rho) - \eta\,\frac{c_R}{6}\,\delta_{n,0} \\ T^{\pm}_{n\pm\eta}(\rho,\eta) &= T^{\pm}_n(\rho)\,, \end{aligned} \qquad (3.7)$$

for the bosonic operators, and

$$G^{1\pm}_{n\pm\frac{\rho+\eta}{2}+\frac{1}{2}}(\rho,\eta) = G^{1\pm}_{n\pm\frac{\rho}{2}+\frac{1}{2}}(\rho) \quad \text{and} \quad G^{2\pm}_{n\pm\frac{\rho-\eta}{2}+\frac{1}{2}}(\rho,\eta) = G^{2\pm}_{n\pm\frac{\rho}{2}+\frac{1}{2}}(\rho)\,, \qquad (3.8)$$

for the modes of the supercharges. Therefore, we see that one way to arrive at the gauge independent quantities from the gauge dependent one, is by setting $\eta = 0$—this is what we do in the following. The parameter ρ takes values $0 \le \rho \le 2$, but without loss of generality we can restrict $0 \le \rho \le 1$ as follows from the periodicity conditions.

From the algebra, we can now derive the unitarity constraints on a highest weight state labelled by the eigenvalues (h, l) of L_0 and T^3_0 respectively. This analysis was done in [154], and we state the result here:

$$l < \frac{c_R}{12}\,, \qquad h \ge (1-\rho)\,l + \frac{c_R}{12}\,\rho\,(1-\frac{\rho}{2})\,,$$

[3] In this relation, instead of $G^{1-}_{n-\frac{\rho+\eta}{2}+\frac{1}{2}}$, note that we have used a shifted mode $G^{1-}_{n-\frac{\rho+\eta}{2}-\frac{1}{2}}$ such that the latter is the complex conjugate of the generator $G^{1+}_{m+\frac{\rho+\eta}{2}+\frac{1}{2}}$, with $m = -n$. This merely shifts the Laurent expansion appropriately.

$$l = \frac{c_R}{12}\,, \qquad h = \frac{c_R}{12}\,(1 - \frac{\rho^2}{2})\,. \tag{3.9}$$

Since we are interested in black holes with zero angular momentum, we must take $l = 0$. The ground state energy then is

$$h_0 \;=\; \frac{c_R}{6}\left(\frac{\rho}{2} - \frac{\rho^2}{4}\right), \qquad \big(L_0 - h_0\big)|0\rangle = 0\,. \tag{3.10}$$

Acting with raising operators in the algebra on this vacuum state, one obtains representations with integer shifts from this ground state. Therefore, a generic state in this sector has a conformal dimension[4] $n_R = N_R + h_0$. Therefore, the entropy of the field theory in an excited state with conformal dimensions n_L and n_R in the Cardy regime is given by

$$\begin{aligned} S_{CFT} &= 2\pi\left(\sqrt{\frac{c_L}{6}n_L} \;+\; \sqrt{\frac{c_R}{6}n_R}\right) \\ &= 2\pi\left(\sqrt{\frac{c_L}{6}n_L} \;+\; \sqrt{\frac{c_R}{6}\left(N_R + \frac{c_R}{6}\left(\frac{\rho}{2} - \frac{\rho^2}{4}\right)\right)}\right). \end{aligned} \tag{3.11}$$

We shall see that this matches with the expectation from the bulk theory, in Section 4, where the momentum along the string is identified with the electric charge of the black hole. Since the field theory is that of an M5-string, the momentum along the string can be calculated to be

$$\begin{aligned} L_0 \;-\; \bar{L}_0 \;&=\; n_L \;-\; n_R \\ &=\; n_L \;-\; \left(N_R + \frac{c_R}{6}\left(\frac{\rho}{2} - \frac{\rho^2}{4}\right)\right). \end{aligned} \tag{3.12}$$

This momentum is no longer integer-quantized—it is shifted by the ground state energy h_0. It is worth noticing that the ground state energy vanishes for $\rho = 2$, but this value is equivalent to $\rho = 0$ as one can see from (3.10). The maximum value arises for $\rho = 1$, namely $h_0 = c/24$. This is precisely the same shift for the ground state energy between the Ramond and Neveu-Schwarz sector. This is not surprising, since $\eta = 0$ and $\rho = 1$ is equivalent to the Ramond sector which has $\eta = 1$ and $\rho = 0$.

3.2 $AdS_3^\rho \times S^2$ bulk duals

The twisting of the supercharges in the small $N = 4$ superconformal algebra raises the question of what the corresponding operation is in the dual bulk theory that lives on AdS_3.

[4] Here, n_L and N_R are integers while h_0 is a continuous parameter in the space of algebras defined by ρ.

A systematic study of the asymptotic dynamics and symmetries of three-dimensional extended supergravity on AdS_3 was made in [156] in the Chern-Simons formulation. The AdS_3 superalgebra that corresponds to the small $N = 4$ superconformal algebra is $SU(1,1|2)/U(1)$ and contains an "inner" $SU(2)$ symmetry that is dual to the $SU(2)_\eta$ current algebra. Furthermore, it was shown that the twisting of the $SU(2)_\rho$ outer automorphism group corresponds to twisting the periodicity conditions on the gravitini. Indeed, AdS_3 has the topology of a disc times a real line, with coordinates (r, θ) and t, and the supergravity fields in three dimensions must be given periodicity conditions in θ in such a way that the supergravity Lagrangian remains invariant (see Section 6 in [156]). In our notation, following (3.2) for $\eta = 0$, and suppressing the coordinates r and t, this means,

$$\psi_\mu^{a\pm}(\theta + 2\pi) = e^{\mp i\pi\rho}\psi_\mu^{a\pm}(\theta)\,. \tag{3.13}$$

The three-dimensional gravitini are in general denoted by ψ_μ^{aA}, where the superscripts denote the representation of the R-symmetry in three dimensions. In general, the R-symmetry in $N = 4, D = 3$ is $SO(4)_R$, but R-symmetry in supergravity is not always a symmetry of the Lagrangian, only of the superalgebra. However, as mentioned in the Introduction and the beginning of this section, our three-dimensional supergravity comes from $\mathcal{N} = 2$ in five dimensions, where the R-symmetry is only $SU(2)_R$. The five-dimensional theory is defined on $AdS_3 \times S^2$, and after reducing to three dimensions, the R-symmetry enlarges to

$$D = 5: \quad SU(2)_R \qquad \Rightarrow \qquad D = 3: \quad SO(4)_R = SU(2)_\rho \times SU(2)_\eta\,. \tag{3.14}$$

Here, the $SU(2)_\eta$ is now a symmetry and it is gauged, with $SU(2)_\eta$ Chern-Simons gauge fields that are dual to the current algebra in the small $N = 4$ superconformal algebra. The $SU(2)_\rho$ is the $SU(2)_R$ from five dimensions. It corresponds to the outer automorphism group in the dual CFT. Both these groups ($SU(2)_\rho$ and $SU(2)_R$) are outer automorphisms and are in general not symmetries of the Lagrangian. It is now clear that the index $a = 1, 2$ denotes the two-dimensional representation of $SU(2)_\eta$ and $A = 1, 2$ (or in complexified notation $A = +, -$) the one of $SU(2)_\rho$. Hence, on the one hand, twisting the periodicity conditions with a $U(1) \subset SU(2)_\rho$ implies twisting a $U(1) \subset SU(2)_R$ in five dimensions. On the other hand, twisting the periodicity conditions on the gravitini with a $U(1) \subset SU(2)_\eta$ can be undone by a gauge transformation or field redefintition in the bulk. In the boundary CFT, this corresponds to spectral flow in the current algebra.

While the analysis in [156] was done for pure Chern-Simons supergravity, we assume here that it can be extended to include also matter multiplets and that our reduction from five dimensions can be recasted in this language. This would mean that all fields in five dimensions that have R-charge, will be subject to boundary conditions similar to (3.13). For hypermultiplets, we discuss this in the next section.

Piecing all the above together, we may now conjecture that M-theory on $AdS_3^\rho \times S^2 \times CY_3$ is dual to the ρ twisted MSW-CFT, which we denote by $(0,4)_\rho$ CFT. By AdS_3^ρ, we mean AdS_3 with ρ-twisted boundary conditions along the angular coordinate in AdS_3. This is what we call "Rholography". Consequently, the $(0,4)_\rho$ theory in an excited state at finite temperature—as considered above—accounts for the entropy of a macroscopic excited state above the AdS_3^ρ vacuum. In Section 4, we will show that this excited macroscopic state is precisely a massive, non-extremal BTZ$^\rho$ black hole, as one might expect; of course, this BTZ$^\rho$ geometry will also be one with a twisted angular direction. As we will show, in turn, this BTZ$^\rho$ geometry appears in the uplift of the four-dimensional black hole using the Scherk-Schwarz mechanism.

In closing, let us note that the discussion in this section is rooted in a chiral $N = 4$ SCA in two dimensions; therefore, its scope is certainly not limited to just the $(0,4)$ MSW CFT. Let us consider, for instance, the D1-D5 CFT of Strominger and Vafa. It is a $(4,4)$ theory. $\frac{1}{2}$-BPS states in this theory correspond to space-time $\frac{1}{4}$-BPS states. Such $\frac{1}{2}$-BPS states are counted by keeping one of the chiral sectors in the vacuum (using supersymmetry), while exciting the other chiral sector. As was shown in [19], such a count precisely matches the macroscopic entropy of $\frac{1}{4}$-BPS black holes in five-dimensional $\mathcal{N} = 4$ supergravity obtained from a Type IIB compactification on a Calabi-Yau twofold times a circle. As was later pointed out in [143], exciting both chiral sectors of this two dimensional $(4,4)$ theory counts microstates of near-extremal black holes.

The reasoning behind rholography works very similarly as for the case discussed before. Compactifications of type IIB on $K3$ yield six-dimensional chiral $(0,2)$ supergravity. The R-symmetry is $SO(5)_R \simeq USp(4)_R \simeq Sp(2)_R$. We then reduce on six-dimensional backgrounds of the type $AdS_3 \times S^3$, and in three dimensions with sixteen supercharges, the R-symmetry is in general $SO(8)_R$. The R-symmetry is in general not a symmetry, but since we reduced on S^3, an $SO(4)$ subgroup is a symmetry and is gauged. The analogous (to (3.14)) decomposition of the total R-symmetry group is now

$$D = 6: \quad SO(5)_R \qquad \Rightarrow \qquad D = 3: \quad SO(8)_R \to SO(4)_\rho \times SO(4)_\eta \,. \tag{3.15}$$

The $SO(4)_\eta$ is a gauge symmetry and produces two sets of $SU(2)$ current algebras that are present in the left and right-moving sectors of the dual CFT. The $SO(4)_\rho$ further decomposes in two outer automorphism groups of the left and right-moving sectors, and each can be used to give twisted boundary conditions with parameters, say ρ_L and ρ_R. As we will argue in Section 5, twisting both sectors would spontaneously break all the supersymmetry of the vacuum in the macroscopic five-dimensional supergravity theory. But the qualitatively new feature arising from considering the $(\rho_L = 0, \rho_R \neq 0)$ D1-D5 system is that the vacuum in the corresponding supergravity theory still breaks supersymmetry spontaneously; but this time, only partially so. Clearly, this results in exactly the same formula (3.11) for the microscopic entropy.

While we will move on to the macroscopic discussion corresponding to the MSW CFT in the next section, we will comment on the microscopic counterpart of the D1-D5 CFT in Section 5.

4 Black holes from M-theory and Scherk-Schwarz reductions

The four-dimensional black holes we wish to describe in this chapter arise from M-theory compactifications on $CY_3 \times S^1$. Their microscopic entropy is governed by the MSW (0,4) CFT, and the M5-string is compactified on the S^1. As explained in the introduction, we extend the discussion here by imposing a non-trivial Scherk-Schwarz twist along the S^1. The twist group element is chosen to be in the $U(1)_R$ subgroup of the $SU(2)_R$ R-symmetry in the five-dimensional supergravity theory. Hence, it acts on the five-dimensional supercharges that transform as a doublet. This way, as we review in the subsection to follow, we generate gauged supergravity in four dimensions with a positive definite scalar potential with a Minkowski vacuum. In the example of this section, the vacuum spontaneously breaks supersymmetry from $\mathcal{N} = 2$ to $\mathcal{N} = 0$. In our analysis, in this section, we will ignore radiative quantum corrections to the potential and possible worries about instabilities of the vacuum[5]. The supersymmetry breaking scale will be proportional to the twist parameter that plays the role of the gauge coupling constant in gauged supergravity. We assume it to be very small, such that quantum corrections are suppressed. Furthermore, we assume the S^1 radius R to be much larger than the length scale of the CY_3, i.e. $R^6 \gg Vol_{CY_3} \gg l_{11}^6$, where l_{11} is the eleven-dimensional Planck length. In this regime, the supergravity approximation is valid. All particles that carry R-charge in five dimensions (gravitinos, gaugini, and the hypermultiplets) will become massive in four dimensions, with masses set by the supersymmetry breaking scale—so they will be light. The black holes that we wish to construct are therefore solutions of four-dimensional gauged supergravity, and our set-up allows us to study them in the presence of light charged matter. Since supersymmetry is broken, the only sensible thing to do is to construct non-extremal solutions, though our microscopic matching only works in the near-extremal limit. The uplift of this solution to five dimension is a black

[5] There exist Scherk-Schwarz reductions with R-symmetry twists with supersymmetry preserving vacua, as we discuss in the next section. The reader who is very worried about radiative corrections and instabilities of the supersymmetry breaking vacuum mentioned above, might find the example of the next section more appealing. There, (half of the) supersymmetry in the vacuum is preserved and calculations are under better control. Alternatively, one might start with M-theory on a T^6 instead of a CY_3, such that partial supersymmetry can remain after the Scherk-Schwarz twist.

string with twisted boundary conditions, and a near horizon geometry that contains a BTZ factor in the near-extremal limit. This near horizon geometry has a holographic dual which is governed by a CFT with a ρ-algebra of symmetries as in Section 2, at finite (and small) temperature.

For practical purposes, we choose a CY_3 with small Hodge numbers, $h_{1,1} = h_{1,2} = 1$. Such Calabi-Yau manifolds were constructed in [157]. As a consequence, the low energy effective action is five-dimensional supergravity coupled to two hypermultiplets and without any vector multiplets. The Scherk-Schwarz reduction to four dimensions can in this example be carried out in great detail. Nevertheless, we expect our conclusions to hold more generally, for any CY_3, and as a result for more general hypermultiplet couplings. We therefore start Section 4.1 with some general statements about Scherk-Schwarz reductions in supergravity, and then specify our model in more detail. In Section 4.2, we discuss black hole solutions while in Section 4.3, we uplift them to five dimensions and argue for a match of their macroscopic entropy with the Cardy-formula (3.11).

4.1 R-Symmetry and Scherk-Schwarz reduction

A generic compactification of M-theory on a CY_3 yields an effective five-dimensional theory of $\mathcal{N} = 2$ supergravity coupled to $h_{1,1} - 1$ vector multiplets and $h_{1,2} + 1$ hypermultiplets [158]. Further compactification on a circle S^1 gives an additional Kaluza-Klein vector multiplet (so $h_{1,1}$ in total) and the same number of hypermultiplets as in five dimensions. The effect of doing a Scherk-Schwarz twist on S^1 is to yield four-dimensional *gauged* $\mathcal{N} = 2$ supergravity with a gauge group $U(1)$. Our setup follows the treatment and the analysis of [150, 151], and we use the conventions of [151]. The five-dimensional metric is decomposed as

$$\mathrm{d}s^2_{(5)} = R^{-1}\mathrm{d}s^2_{(4)} + R^2(\mathrm{d}z + A^0)^2 \,, \tag{4.1}$$

where $z \sim z + 2\pi$ is the coordinate along the circle, and R denotes the radius of the circle above a base point x. All length scales are measured in terms of the eleven-dimensional Planck units. Finally, A^0 is the Kaluza-Klein vector that we also call the four-dimensional graviphoton. Five-dimensional gauge fields decompose as[6]

$$A^I_{(5)} = A^I_{(4)} + a^I(\mathrm{d}z - A^0) \,, \tag{4.2}$$

[6]For a supergravity theory obtained as a compactification of M-theory on a CY_3 with Hodge numbers $h_{1,1}$ and $h_{1,2}$, the indices in (4.11) are $\Lambda, \Sigma \in \{0, 1, \ldots n_v\}$ and $I, J \in \{1, \ldots n_v\}$, with n_v the number of vector multiplets, and $u, v \in \{1, \ldots, 4n_h\}$, with $n_h = h_{1,2} + 1$ the number of hypermultiplets. In five dimensions, we have $n_v = h_{1,1} - 1$ and in four dimensions, we have $n_v = h_{1,1}$. The number of hypermultiplets stays the same in five and four dimensions.

with a^I four-dimensional scalars. They combine into complex scalars with the real scalars h^I of the five-dimensional vector multiplet

$$t^I = a^I - iR\, h^I \,. \tag{4.3}$$

All these fields have zero $SU(2)_R$ R-charge in five dimensions, so they reduce to four dimensions just like in a Kaluza-Klein reduction. Their zero modes are massless. The non-trivial Scherk-Schwarz twist here is performed only on those quantities that transform under the R-symmetry. These include the supercharges, hence the fermions, and the hypermultiplet scalars. These fields get a non-trivial z-dependence, different from a Kaluza-Klein expansion of a periodic field. As a consequence, what used to be the massless zero modes in a Kaluza-Klein scheme, now become massive modes, with masses proportional to the twist parameter. These modes are taken to be very light compared to the higher Kaluza-Klein modes. This can be achieved by taking the twist parameter to be small. To be more concrete, we can define the Scherk-Schwarz twist on the supercharges Q^A; $A = 1, 2$, which form a doublet under $SU(2)_R$, as

$$Q^A(x^\mu, z + 2\pi) \;=\; \left(e^{2i\pi\alpha\sigma_3}\right)^A{}_B\, Q^B(x^\mu, z)\,, \tag{4.4}$$

for a Scherk-Schwarz phase α belonging to the $U(1)_R \subset SU(2)_R$, and with σ_3 being the third Pauli matrix. A similar transformation holds for the gravitini ψ^A_μ and for the gaugini λ^{AI}. Comparing with the twist on the worldsheet supercharges in (3.2) with $\eta = 0$, we identify

$$\alpha = \frac{\rho}{2}\,. \tag{4.5}$$

The justification for this was given before, namely that we identify the bulk $SU(2)_R$ symmetry with the worldsheet $SU(2)_\rho$ outer automorphism group. This is because the S^1 we twist on, is the same as the S^1 we wrap the M5-string around. Similarly, the S^1 we twist on is the same S^1 that becomes part of the AdS_3 in the near horizon geometry of the black string. In essence, the coordinate z is equal to θ used in (3.13). So the periodicity conditions we used on the supercharges (4.4) and gravitini are also the same as in (3.13).

Any (complex) field $\Phi(x, z)$ with twisted periodicity conditions has a mode expansion

$$\Phi(x, z) = e^{i\alpha z} \sum_{n=-\infty}^{+\infty} \Phi_n(x) e^{inz}\,. \tag{4.6}$$

In a Scherk-Schwarz reduction, we restrict to the $n = 0$ mode in the expansion. In other words, we give the five-dimensional field a particular z-dependence that satisfies

$$\Phi(x, z) = e^{i\alpha z}\Phi_0(x) \qquad \Longrightarrow \qquad \partial_z\Phi = i\alpha\Phi\,. \tag{4.7}$$

The effect of this is that the four-dimensional field becomes both charged and massive, with $m^2 = q^2$ in the appropriate units. The masses will be proportional to α and inversely proportional to the radius R, and we give an explicit example at the end of this subsection. Applied to the case at hand, we get

$$\partial_z Q^A = i\alpha {\sigma_3}^A{}_B Q^B\,, \qquad \partial_z \psi^A_\mu = i\alpha {\sigma_3}^A{}_B \psi^B_\mu\,, \qquad \partial_z \lambda^{AI} = i\alpha {\sigma_3}^A{}_B \lambda^{BI}\,. \tag{4.8}$$

These fermionic fields transform with the same Scherk-Schwarz phase, because they are in the same (doublet) representation of the $SU(2)_R$ symmetry.

In the hypermultiplet sector, both scalars and fermions transform under this twist. The scalars parametrize a quaternion-Kähler manifold of dimension $4n_h$, with metric h_{uv}, and the holonomy group is contained in $SU(2)_R \times USp(2n_h)$. For a given hypermultiplet scalar manifold which is a coset of the form G/H, the maximal compact subgroup always contains an $SU(2)_R$ factor. So, homogeneous quaternion-Kähler manifolds always contain $SU(2)_R$ isometries, and hence the Scherk-Schwarz twist can be implemented using the $U(1)_R \subset SU(2)_R$ Killing vector (we add a subscript “0" to the Killing vector for later notational purposes),

$$\partial_z q^u = \alpha k^u_0(q)\,, \tag{4.9}$$

so the Scherk-Schwarz twist is in general non-linearly realized on the real hypermultiplet scalars. One can write down a similar formula for the hyperini, using the results of [159, 160]. Since this is not very insightful, we refrain from giving explicit expressions here.

In general, Scherk-Schwarz twists lead to gauged supergravities in one dimension lower, with supersymmetry preserved at the level of the action. Gauged supergravities have scalar potentials V_g which are positive-definite for Scherk-Schwarz reductions. Furthermore, they typically allow Minkowski vacua with spontaneously broken supersymmetry. The original references on the topic are [152, 161]. Some other useful literature can be found in e.g. [162, 163].

In four-dimensional $N = 2$ supergravity, the bosonic sector of the theory is generically (for electric gaugings) described by the action [164]

$$\begin{aligned} S_{4d} = \int \frac{R}{2} \star 1 - \gamma_{i\bar{k}} dt^i \wedge \star d\bar{t}^{\bar{k}} + \frac{1}{4}\mathcal{I}_{\Lambda\Sigma} F^\Lambda \wedge * F^\Sigma + \frac{1}{4}\mathcal{R}_{\Lambda\Sigma} F^\Lambda \wedge F^\Sigma + \\ + h_{uv} Dq^u \wedge \star Dq^v - V_g \star 1\,, \end{aligned} \tag{4.10}$$

and the potential has a universal form for generic gaugings described by [164][7]

$$V_g = 2g^2\Big(\gamma_{i\bar{k}}k^i_\Lambda k^{\bar{k}}_\Sigma + 4h_{uv}k^u_\Lambda k^v_\Sigma\Big)\bar{L}^\Lambda L^\Sigma + 2g^2\Big(U^{\Lambda\Sigma} - 3\bar{L}^\Lambda L^\Sigma\Big)P^x_\Lambda P^x_\Sigma\,. \tag{4.11}$$

where $i, j = 1, .., n_v$, $u, v = 1, .., 4n_h$, $\Lambda, \Sigma = 1, .., n_v + 1$. In this formula, g is the gauge coupling, k^i_Λ and k^u_Λ are Killing vectors of the special Kähler and quaternionic isometries respectively, $\gamma_{i\bar{k}}$ is the metric of the special Kähler manifold with holomorphic coordinates t^i, and h_{uv} the metric of the quaternionic manifold with coordinates q^u. Notice that the hypermultiplet scalars now appear with a covariant derivative $D_\mu q^u = \partial_\mu q^u + k^u_\Lambda A^\Lambda_\mu$, since they are charged under the Kaluza-Klein field, as discussed above. The symplectic sections L^Λ are defined from the holomorphic ones by $L^\Lambda = e^{K/2}X^\Lambda$, where K is the Kähler potential, and we use special coordinates such that $X^0 = 1$. For more on conventions and properties on special geometry, see [164]. $P^x_\Lambda; x = 1, 2, 3$ are the moment maps that can be computed from the quaternionic Killing vectors. Finally, $U^{\Lambda\Sigma}$ is the symmetric tensor defined on any special Kähler manifold. The precise definition is not important here, since the last term in (4.11) will vanish in our case.

In our setup, only the Kaluza-Klein vector A^0 from (4.1) is involved in the gauging, and this gauge field is labeled by indices $\Lambda, \Sigma = 0$. Even if other gauge fields are present, they do not take part in the gauging in the sense that no fields are charged under them. The only relevant moment map is therefore P^x_0, and thus the only relevant Killing vector of quaternionic isometries is k^u_0, which we specify below. Moreover, by properties of special geometry it holds that $(U^{00} - 3\bar{L}^0L^0) \equiv 0$ in the large radius limit, so the last terms in the potential (4.11) vanish.

Since we will perform a Scherk-Schwarz twist with respect to the R-symmetry, and the scalars in the vector multiplet have no R-charge, the corresponding four-dimensional spectrum should have scalars in the vector multiplets that remain massless and uncharged. This is simply achieved by choosing the gauging of a compact $U(1)$ isometry in the hypermultiplet scalar manifold only, thus implying $k^i_\Lambda = 0$ for every $\Lambda = 0, 1, .., n_v$. The potential we consider in this work is then of the no-scale form

$$V_g = 2g^2(4h_{uv}k^u_0k^v_0)\bar{L}^0L^0 = 8g^2h_{uv}k^u_0k^v_0e^K\,, \tag{4.12}$$

and is positive definite. Using the relations, in the conventions of [151],

$$e^{-K} = 8R^3\,, \qquad \sqrt{-g_{(5)}} = \frac{\sqrt{-g_{(4)}}}{R}\,, \tag{4.13}$$

[7] We use the conventions of [151], which differ from [164] by factors of two in the potential and gauge kinetic terms. One can switch between the conventions by rescaling our four-dimensional metric $g \to \frac{1}{2}g$ and then multiplying the action by 2. This has the effect of rescaling our potential with an overall factor of $\frac{1}{2}$ and our gauge kinetic terms with an overall factor of 2, while the scalar kinetic terms and Einstein-Hilbert term, normalized as $\mathcal{L} = \frac{1}{2}\sqrt{-g}R(g)$, remain the same.

one can interpret this as a potential coming from the dimensional reduction of the hypermultiplet scalars' kinetic terms [150]. Indeed, using (4.9), we find

$$\sqrt{-g_{(5)}}h_{uv}\partial_z q^u \partial_z q^v g^{zz} = \sqrt{-g_{(4)}}\, V_g = \sqrt{-g_{(4)}}\,\frac{g^2}{R^3}h_{uv}k_0^u k_0^v \,. \tag{4.14}$$

From this, one can see two possible types of vacua, both of which are Minkowski. The first one is to have the Killing vectors finite and non-zero in the vacuum; the potential is then of the runaway type and the theory decompactifies. We are not considering this option since in our case, the Killing vectors of the R-symmetry will have fixed points and vanish in the vacuum. R is then a flat direction, and the potential is called no-scale. Therefore, we can freely take the radius to be large, such that $R^6 \gg Vol_{CY_3}$.

The masses of the particles in the spectrum follow from expanding fluctuations around the vacuum to quadratic order, and involve the derivatives of the Killing vectors which need not vanish in the vacuum. We refer to [164] for general expressions of the mass matrices. Furthermore, the Scherk-Schwarz reduction also generates terms proportional to the Kaluza-Klein vector A^0, from which one can determine that the charge[8] is equal to the mass, $m^2 = q^2$. Finally, for (4.14) to hold, we identify the Scherk-Schwarz twist parameter with the gauge coupling constant

$$\alpha = g \qquad \Longrightarrow \qquad g = \frac{\rho}{2}\,. \tag{4.15}$$

It is important to notice that in the vacuum, the bosonic part of the Lagrangian becomes that of ungauged supergravity. Indeed, in the vacuum, the potential vanishes and all covariant derivatives on the hypermultiplet scalars become ordinary ones since the covariant derivatives involve Killing vectors that vanish in the vacuum. The hypers can therefore be frozen to their vevs. The scalars in the vector multiplets remain neutral. As a consequence, any bosonic solution of the equations of motion in ungauged supergravity without hypermultiplets can be imported into the R-gauged supergravity theory. This observation will be important when we discuss black hole solutions in Section 4.2.

Example

The derivation of the scalar potential in the four-dimensional theory holds for any choice of $U(1)$ Scherk-Schwarz gauging from five to four dimensions, gauged by the graviphoton

[8] In computing the charge, one must take care of the correct normalization of the Kaluza-Klein vector. In the conventions of [151], the kinetic term for A^0 is $\mathcal{L} = -\frac{R^3}{8}F_{\mu\nu}F^{\mu\nu}$, so one needs to rescale the gauge fields $A^0 \to \frac{\sqrt{2}}{R^{3/2}}A^0$ to have a canonically normalized Maxwell field.

(Kaluza-Klein vector A^0), for a generic CY_3-compactification. To exemplify our strategy further, we now choose a particular model, namely the case in which the CY_3 has $h_{1,1} = h_{1,2} = 1$, as discussed at the beginning of this section. Such a compactification gives a five-dimensional $\mathcal{N} = 2$ supergravity theory with no vector-multiplets and $n_h = 2$ hypermultiplets whose scalar manifold is the c-map of $SU(1,1)/U(1)$. This has been extensively studied in [165] and [166], for example. A result of these studies is that the quaternionic manifold is $G_{2(2)}/SO(4)$ where $SO(4) = SU(2)_R \times SU(2)$. We parametrize it by introducing coordinates

$$q^u = (\phi, \varphi, \chi, a, \xi^0, \xi^1, \tilde{\xi}_0, \tilde{\xi}_1) \,. \tag{4.16}$$

Here, φ and χ form a complex structure modulus, the ξ and $\tilde{\xi}$ come from the periods of three-form in eleven dimensions restricted to the CY_3, and a is the dual of the three-form, restricted to five dimensions. Finally, the (dimensionless) volume-modulus of the CY_3—measured in terms of eleven-dimensional Planck units—is given by

$$\mathrm{Vol}_{CY_3} = e^{-2\phi} \,. \tag{4.17}$$

In these coordinates, the metric is

$$\begin{aligned} h_{uv} dq^u dq^v \;=\;& d\phi^2 + 3(d\varphi)^2 + \tfrac{3}{4} e^{4\varphi} (d\chi)^2 \\ &+ \tfrac{1}{4} e^{4\phi} \left[da + \xi^0 d\tilde{\xi}_0 + \xi^1 d\tilde{\xi}_1 - \tilde{\xi}_0 d\xi^0 - \tilde{\xi}_1 d\xi^1 \right]^2 \\ &+ \tfrac{1}{2} e^{2\phi - 6\varphi} (d\xi^0)^2 + \tfrac{1}{2} e^{2\phi - 2\varphi} \left[d\xi^1 - \sqrt{3}\chi d\xi^0 \right]^2 \\ &+ \tfrac{1}{2} e^{2\phi + 2\varphi} \left[d\tilde{\xi}_1 - \sqrt{3}\chi^2 d\xi^0 + 2\chi d\xi^1 \right]^2 \\ &+ \tfrac{1}{2} e^{2\phi + 6\varphi} \left[d\tilde{\xi}_0 + \sqrt{3}\chi d\tilde{\xi}_1 - \chi^3 d\xi^0 + \sqrt{3}\chi^2 d\xi^1 \right]^2 ; \end{aligned} \tag{4.18}$$

see equation (5.4) of [167] for a similar parametrisation.

The scalar potential can be found from the Killing vector belonging to the $U(1)_R \subset SU(2)_R \subset SO(4)$ isometry. The explicit form for this Killing vector is given in the appendix, using the parametrization (4.18) for the metric on the coset $G_{2(2)}/SO(4)$. A Minkowski vacuum is then obtained for the values of the fields which are a vanishing locus for the Killing vector $k_0^u = 0$, and thus, for the choice (A.3) in appendix A,

$$\chi = a = \xi^0 = \xi^1 = \tilde{\xi}_0 = \tilde{\xi}_1 = 0 \,, \quad e^{4\phi} = \gamma^2 \delta^4 \,, \quad e^{4\varphi} = 3\gamma^2 \,. \tag{4.19}$$

The parameters γ and δ specify the choice of the Killing vector as seen inside $G_{2(2)}$—this may be seen from (A.1) and (A.1). The volume, therefore, may be chosen to be large by specifying an appropriate Killing vector with large δ, for example.

Expanding around the vacuum, one can determine the masses of the hypermultiplet scalars. In four-dimensional Planck units[9] , they are found to be

$$m^2_{(0)} = \frac{g^2}{R^3} , \tag{4.20}$$

and are fully degenerate, i.e. all eight hyperscalars have the same mass. In the fermionic sector, all the fields are charged under $U(1)_R \subset SU(2)_R$. The gravitini undergo a super-Higgs mechanism and become massive by eating up the gaugini. Their mass eigenvalues can be computed from the moment maps (see e.g. [164] for more details on the gravitino mass matrix). In four-dimensional Planck units, we again find

$$m^2_{(3/2)} \quad = \quad \frac{g^2}{R^3} . \tag{4.21}$$

The gravitini mass sets the supersymmetry breaking scale. It is very small in the regime we are working in, namely large radius R and small coupling g. This provides an argument why radiative corrections might be suppressed.

The fermionic sector in the hypermultiplets contains two Dirac spinors, one for each hypermultiplet. Equivalently, there are four chiral components $\zeta_\alpha; \alpha = 1, ..., 4$. Their Dirac masses are found to be

$$m_{(1/2)} = 0 , \qquad \text{and} \qquad m_{(1/2)} = m_{(3/2)} . \tag{4.22}$$

The chiral components in each hypermultiplet then have the same masses, but with double degeneracy.

4.2 Black holes in R-gauged supergravity

The example and general considerations in the previous subsection illustrate the following: After freezing the hypermultiplets to their expectation values, the Killing vectors vanish and so does the scalar potential. Turning to the bosonic sector described by the action (4.10), the resulting supergravity Lagrangian after freezing the hypers (4.19) is precisely that of ungauged supergravity. The covariant derivatives on the hypermultiplet scalars become ordinary derivatives, and so the hypermultiplets decouple classically. We have already mentioned that the scalar of the vector multiplet is a flat direction in the Minkowski vacuum obtained by Scherk-Schwarz twist on the $U(1)_R$ isometry. In particular, the vector multiplet equations of motion decouple from the hypermultiplet

[9]The scalar potential in (4.12) contains a κ_4^{-2}, so all masses scale with the four-dimensional Planck mass in our model. The gauge coupling constant g is dimensionless.

ones, and are totally insensitive to the Scherk-Schwarz twist: they are effectively the same as the equations of ungauged supergravity. Therefore, every solution of the ungauged supergravity bosonic Lagrangian for the metric and the scalars of the vector multiplets is also automatically a solution of the Scherk-Schwarz reduced theory around the Minkowski vacuum where the hypermultiplets are stabilized. This has been discussed in the context of near-horizon supersymmetry already in [168] and more recently in the context of near-horizon dimensional reduction in [169]. Constructions of black hole solutions in gauged supergravities with maximal supersymmetry preserving vacua can be found in [170]. The black holes we consider here live in supersymmetry breaking vacua. The energy scale set by the temperature of the black hole is supposed to be larger than the supersymmetry breaking scale, yet still low enough such that the specific heat remains positive. For black hole temperatures lower than the supersymmetry breaking scale, the massive modes first need to be integrated out.

We then consider a non-extremal black hole—a solution of the theory (4.10) around the Minkowski vacuum coupled to $n_v = 1$ vector multiplet. This corresponds to the dimensional reduction of five-dimensional minimally coupled supergravity. We further truncate to zero axions and consider the case of one electric charge q_0, and one magnetic charge, p^1, with the scalar field t being the coordinate of $SU(1,1)/U(1)$. The black hole is a solution of the Einstein, scalar and Maxwell equations[10]

$$\begin{aligned}
R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} &= g_{\mu\nu}\left(-\gamma_{t\bar{t}}\partial_\mu t\partial^\mu \bar{t} + \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma\,\mu\nu}\right) \\
&\quad - \mathcal{I}_{\Lambda\Sigma}F^{\Lambda}_{\alpha\mu}F^{\Sigma\,\alpha}{}_{\nu} + 2\gamma_{t\bar{t}}\partial_\mu t\partial_\nu \bar{t}\,, \\
\gamma^{t\bar{t}}\partial_t(\mathcal{I}_{\Lambda\Sigma})F^{\Lambda}_{\mu\nu}F^{\Sigma\,\mu\nu} &= -\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu \bar{t}\right) - \gamma^{t\bar{t}}\partial_{\bar{t}}\gamma_{t\bar{t}}\,\partial_\mu \bar{t}\partial^\mu t\,, \qquad (4.23) \\
\gamma_{t\bar{t}} = \frac{3}{4\mathrm{Im}(t)^2} &= (\gamma^{t\bar{t}})^{-1}\,, \\
\partial_\mu\left(\sqrt{-g}\mathcal{I}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}\right) &= 0\,,
\end{aligned}$$

where there is no summation on the t and $\bar{t}$ indices since the scalar manifold is of complex dimension 1. The setup of this solution corresponds to a particular case of [171].

The metric of the black hole solution in the region outside the horizon is given by

$$\mathrm{d}s^2_{(4)} = -e^{2U(r)}\mathrm{d}t^2 + e^{-2U(r)}\mathrm{d}r^2 + e^{-2U(r)}f(r)\mathrm{d}\Omega^2_{(2)}\,, \qquad (4.24)$$

with $f(r) = (r-r_+)(r-r_-)$ and $\mathrm{d}\Omega^2_{(2)} = \mathrm{d}\theta^2 + \sin\theta^2\mathrm{d}\phi^2$. We have denoted the inner and

[10] Here we switch back to the conventions used in [164] that are most commonly used in the black hole literature.

outer black hole horizons by $r_\pm = r_* \pm r_0$, while $r_*^2 = 2\sqrt{|q_0(p^1)^3|}$ refers to the radius of the extremal solution obtained by taking the limit $r_0 \to 0$. The warp factor $U(r)$ and the purely imaginary scalar field—parametrized as $t(r) = -i\lambda(r)$—are determined in terms of two harmonic functions as

$$e^{-2U(r)} = \frac{r-r_-}{r-r_+} 4\sqrt{\mathcal{I}_0(\mathcal{I}_1)^2}\,, \qquad \lambda(r) = \sqrt{\frac{\mathcal{I}_0}{\mathcal{I}_1}}\,, \tag{4.25}$$

where

$$\begin{aligned} \mathcal{I}_0 &= \frac{R^{3/2}}{2} \frac{r - r_* + r_0\sqrt{1+\frac{2q_0^2}{R^3 r_0^2}}}{r - r_* + r_0}\,, \\ \mathcal{I}_1 &= \frac{1}{2\sqrt{R}} \frac{r - r_* + r_0\sqrt{1+\frac{2(p^1)^2 R}{r_0^2}}}{r - r_* + r_0}\,. \end{aligned} \tag{4.26}$$

We note that the scalar field at infinity becomes the dilaton of the Minkowski vacuum discussed in the previous section, R, which is a free parameter. The gauge fields of the theory are—with $F^\Lambda = \frac{1}{2} F^\Lambda_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$—

$$\begin{aligned} F^0 &= \frac{q_0}{R^3} \frac{1}{\left(r - r_* + r_0\sqrt{1+\frac{2q_0^2}{R^3 r_0^2}}\right)^2} dt \wedge dr\,, \\ F^1 &= p^1 \sin\theta d\theta \wedge d\phi\,. \end{aligned} \tag{4.27}$$

One then finds the entropies associated to the inner and outer horizons to be

$$\frac{S_\pm}{\pi} = \left(r_0 \pm \sqrt{r_0^2 + \frac{2q_0^2}{R^3}}\right)^{1/2} \left(r_0 \pm \sqrt{r_0^2 + 2(p^1)^2 R}\right)^{3/2}. \tag{4.28}$$

The non-extremal parameter is related to the thermodynamic quantities of the black hole by $r_0 = 2S_+T$, with the temperature being $T = \frac{\kappa}{2\pi} = \frac{r_+ - r_-}{4\pi r_+^2}$. In the extremal case, $r_0 = 0$, the radius R drops out of the entropy formula and we obtain the well-known result

$$S = 2\pi\sqrt{q_0(p^1)^3}\,, \tag{4.29}$$

that has been reproduced microscopically for BPS black holes in ungauged supergravity. The mass of the non-extremal black hole is

$$M = \tfrac{1}{4}\left[3\sqrt{r_0^2 + 2(p^1)^2 R} + \sqrt{r_0^2 + \frac{2q_0^2}{R^3}}\right]. \tag{4.30}$$

This solution has a smooth $T \to 0$ limit but, in the absence of supersymmetry, its stability is no longer guaranteed. Therefore, on physical grounds, we choose to work with a non-extremal black hole.

4.3 Uplift to 5 dimensions

Turning the circle reduction to the four-dimensional theory (4.10) around, the 4D black hole (4.24)—(4.27) can be uplifted to a five-dimensional black string. We will now demonstrate that, close to extremality, the near horizon region of this black string displays a BTZ factor. In this 5D near-horizon region, the scalar $\lambda(r)$ supporting the back string becomes independent of the radial variable r. For simplicity of presentation, we set the scalar to constant already in four-dimensions before uplifting. We have verified that this gives the same result as uplifting the full black hole solution (4.24)—(4.27) and then taking the scalars to be constant, since every correction to the near-horizon physics from the running scalars starts at higher orders.

We thus set the scalar λ in (4.25) to its attractor value, $\lambda = R = \sqrt{\frac{q_0}{p^1}}$, everywhere. It seems that this choice fixes the dilaton of the Minkowski vacuum; however, one must remember that the constant scalars case is just a shortcut to identify the 5d near horizon region. So, in this case, fixing the value of R has no physical meaning and one should simply treat this as a calculational trick. Trading the non-extremality parameter r_0 for the mass M through (4.30) and changing to a new radial variable

$$\tilde{r} = r - r_* + M \,, \tag{4.31}$$

the solution (4.24)—(4.27) becomes Reissner-Nordström,

$$\begin{aligned} \mathrm{d}s^2_{(4)} &= -\Big(1 - \tfrac{2M}{\tilde{r}} + \tfrac{r_*^2}{\tilde{r}^2}\Big)\mathrm{d}t^2 + \frac{\mathrm{d}\tilde{r}^2}{1 - \frac{2M}{\tilde{r}} + \frac{r_*^2}{\tilde{r}^2}} + \tilde{r}^2 \mathrm{d}\Omega^2_{(2)} \,, \\ F &= \frac{2r_*}{\tilde{r}^2}\, \mathrm{d}t \wedge \mathrm{d}\tilde{r} \,, \end{aligned} \tag{4.32}$$

where $F \equiv \lambda^{3/2} F^0 = \frac{1}{\sqrt{3}}\lambda^{1/2} * F^1$. Using the formulae (4.1) and (4.2) for a single vector multiplet, this solution uplifts on the circle parametrised by the angle z to the five-dimensional black string

$$\begin{aligned} \mathrm{d}s^2_{(5)} &= \sqrt{\tfrac{p^1}{q_0}}\Big(- \big(1 - \tfrac{2M}{\tilde{r}} + \tfrac{r_*^2}{\tilde{r}^2}\big)\mathrm{d}t^2 + \frac{\mathrm{d}\tilde{r}^2}{1 - \frac{2M}{\tilde{r}} + \frac{r_*^2}{\tilde{r}^2}} + \tilde{r}^2 \mathrm{d}\Omega^2_{(2)}\Big) \\ &\quad + \tfrac{q_0}{p^1}\big(\mathrm{d}z + \sqrt{\tfrac{2(p^1)^3}{q_0}}\, \tfrac{1}{\tilde{r}}\, \mathrm{d}t\big)^2 \,, \end{aligned} \tag{4.33}$$

$$F_{(5)} = \sqrt{6}p^1 \sin\theta \, d\theta \wedge d\phi ,$$

where $F_{(5)}$ is the field strength of the gauge field in (4.2).

Let us now exhibit how the announced BTZ factor arises from the solution (4.33) in the near horizon region, close to extremality. To this end, we rescale

$$\begin{aligned}
\tilde{r} &\longrightarrow r_* + \epsilon\rho ,\\
M &\longrightarrow r_* + \epsilon^2 \frac{\rho_0^2}{2r_*} ,\\
t &\longrightarrow \frac{1}{\epsilon} r_*^2 \tau ,\\
z &\longrightarrow \left(\frac{p^1}{q_0}\right)^{3/4} \left(r_* \varphi - t\right)
\end{aligned} \tag{4.34}$$

following e.g. [172] and then let $\epsilon \longrightarrow 0$. In this near horizon, near extremal limit, the five-dimensional metric (4.33) becomes

$$\begin{aligned}
ds_{(5)}^2 &= 2(p^1)^2 \left(-(\rho^2 - \rho_0^2) d\tau^2 + \frac{d\rho^2}{(\rho^2 - \rho_0^2)} + (d\varphi - \rho \, d\tau)^2 \right)\\
&\quad + 2(p^1)^2 \, d\Omega_{(2)}^2 ,
\end{aligned} \tag{4.35}$$

which is the direct product of a BTZ metric and a two-sphere S^2, of radius $2(p^1)^2$. To see this more explicitly, we identify $\rho_0 = \frac{r_+ + r_-}{2\ell}$ and make a further change of coordinates

$$\begin{aligned}
\rho &= \frac{1}{\ell(r_+ - r_-)} \left(r^2 - \frac{1}{2}(r_+^2 + r_-^2) \right),\\
\tau &= 2\left(-\frac{t}{\ell} + \phi \right),\\
\varphi &= \frac{r_- - r_+}{\ell} \left(\frac{t}{\ell} + \phi \right),
\end{aligned} \tag{4.36}$$

with r_+ and r_- being the outer and inner horizons respectively and $\ell^2 = 8\left(p^1\right)^2$ being the square of the radius of AdS, to rewrite the metric (4.35) as

$$\begin{aligned}
ds_{(5)}^2 &= \left(-\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\ell^2 r^2} dt^2 + \frac{\ell^2 r^2 dr^2}{(r^2 - r_+^2)(r^2 - r_-^2)} + r^2 \left(d\phi - \frac{r_+ r_-}{\ell r^2} dt \right)^2 \right)\\
&\quad + 2(p^1)^2 \, d\Omega_{(2)}^2 .
\end{aligned} \tag{4.37}$$

The contribution in brackets can now be recognised as the standard non-extremal, rotating BTZ metric with radius fixed by ℓ, mass and angular momentum given by

$$M_{BTZ} = \frac{r_+^2 + r_-^2}{\ell^2} , \qquad J_{BTZ} = \frac{2r_+ r_-}{\ell} . \tag{4.38}$$

Our results are consistent with the general black string solutions discussed in [173]. The total entropy of the uplifted, five-dimensional solution is now

$$\begin{aligned} S &= \frac{1}{4G_5}\,\text{Area}(S^2)\,2\pi r_+ \\ &= \frac{1}{4G_3}\,2\pi r_+ . \end{aligned} \tag{4.39}$$

The metric (4.37) can be written entirely in terms of M_{BTZ} and J_{BTZ} as

$$\mathrm{d}s^2_{(5)} = \left(-f(r)\,\mathrm{d}t^2 + f^{-1}(r)\mathrm{d}r^2 + r^2\left(\mathrm{d}\phi + N_\phi \mathrm{d}t\right)^2\right) + 2(p^1)^2\,\mathrm{d}\Omega^2_{(2)}\,, \tag{4.40}$$

where

$$f(r) = -M_{BTZ} + \frac{r^2}{\ell^2} + \frac{J^2_{BTZ}}{4r^2} \qquad N_\phi = -\frac{J_{BTZ}}{2r^2}. \tag{4.41}$$

This is written in conventions where the AdS_3 mass is −1, as opposed to $-\frac{1}{8G_3}$. One may restore the factors of G_3 by

$$f(r) = -8G_3 M_{BTZ} + \frac{r^2}{\ell^2} + \frac{16G_3^2 J^2_{BTZ}}{r^2}, \qquad N_\phi = -\frac{4G_3 J_{BTZ}}{r^2}\,, \tag{4.42}$$

and is now identical (up to a shift in the radial variable) to the metric written in [146].

Given that the BTZ geometry arises in the bulk supergravity, following the results of [146] and [174], it is clear that the entropies of the macroscopic solution and the microscopic field theory match with each other. In fact, the central charges c_L and c_R of the CFT do not feel the boundary conditions, so they can be used again in the Cardy formula. However, the conventional argument—in [19, 20], for instance—is that given a macroscopic black hole with certain (electric) charge, one may choose a conformal field theory with states carrying the same momentum that reproduces the macroscopic entropy. It is crucial, therefore, that the quantization conditions on the black hole charge and the field theory momentum are the same. In the case of supersymmetric black holes, both were integers and consequently consistent with each other. We saw in (3.12) that the momentum along the string is quantized; this becomes the four-dimensional electric charge,

$$q_0 = n_L - \left(N_R + \frac{c_R}{6}\left(\frac{\rho}{2} - \frac{\rho^2}{4}\right)\right). \tag{4.43}$$

To leading order in $g = \rho/2$, this is $a + b\,\frac{\rho}{2} = a + b\,g$, where a and b are integers[11]. Therefore, it is important that our macroscopic black hole satisfies this condition. We will

[11] As shown in [175], $\frac{c_R}{6}$ is an integer.

now present a quick argument why the black hole (4.24) does satisfy this quantization condition.

For the black hole under consideration to be a physically reasonable one, it needs to have been formed by a collapse of particles within the theory. Elementary zero-mode particles in our theory have charges proportional to the gauge coupling constant[12] g. The most general black hole in this theory could conceivably be formed by a collapse of Kaluza-Klein particles with integer charges and Scherk-Schwarz particles with charges proportional to g. Picking a black hole formed by $n_L - N_R$ Kaluza-Klein particles and $\frac{c_R}{6}$ Scherk-Schwarz particles, it has an electric charge that is exactly consistent with the quantization condition on the microscopic momentum (3.12), to leading order in ρ. It would be interesting to understand the macroscopic origin of the term in (3.12) that is quadratic in g. For black hole temperatures larger than the supersymmetry breaking scale, this term is irrelevant. For lower temperatures, this correction can perhaps be understood after integrating out the hypermultiplets in a one-loop approximation. We leave this interesting point for future work.

5 Extensions to supersymmetric vacua

The construction we have presented so far needs attention to one further detail. We have considered a non-extremal black hole in a vacuum that spontaneously breaks supersymmetry. It is, therefore, important that the vacuum is at least sufficiently stable to allow for the formation of such a large black hole.To avoid possible problems with instabilities, we now present an alternative example in which supersymmetry is only partially broken in the vacuum. Since the discussion is very similar to the previous section, we will be rather brief and sketchy, only concentrating on the main steps.

Let us consider Type IIB Superstring theory on a $K3$ surface, preserving sixteen supercharges. This yields a six dimensional chiral $(0,4)$ supergravity theory supplemented with a moduli space, parametrized by the scalar fields,

$$\mathcal{M} = \frac{SO(5,21)}{SO(5)_R \times SO(21)}, \tag{5.1}$$

where the $SO(5)_R \simeq USp(4)_R$ is the R-symmetry. This R-symmetry group contains two compact $U(1)$ subgroups, labelled by say, $U(1)_{\rho_L}$ and $U(1)_{\rho_R}$,

$$U(1)_{\rho_L} \times U(1)_{\rho_R} \subset SO(5)_R . \tag{5.2}$$

[12]The factors of R^3 in the charges arise from issues of canonical normalization of the four-dimensional vectors. These are clearly not present in the five-dimensional 'normalizations'.

One may now repeat the construction we have presented in this chapter, and compactify further on a circle with a Scherk-Schwarz twist, this time down to five dimensions. This procedure leads to a Scherk Schwarz reduced gauged $\mathcal{N} = 4$ supergravity in five dimensions. For toroidal compactifications that result in maximal supersymmetry in six dimensions, such partial supersymmetry breaking flat vacua have been shown to exist[176]. For theories arising from K3 compactifications, a similar feature has been shown in [177]. Applied to the case at hand, one can twist the six-dimensional supercharges with respect to $U(1)_{\rho_L} \times U(1)_{\rho_R} \subset SO(5)_R$, with twist parameters ρ_L and ρ_R. If both parameters are switched on, supersymmetry is completely broken in the vacuum. However, if we set, say $\rho_L = 0$, supersymmetry is only partly broken—and two of four gravitinos remain massless:

$$SO(5)_R \simeq USp(4)_R \longrightarrow USp(2)_R \simeq SU(2)_R \, . \tag{5.3}$$

Further details on the spectrum can be found in [177].

Therefore, setting $\rho_L = 0$ leaves us with an $\mathcal{N} = 2$ preserving Minkowski vacuum in five dimensions. Given that a stable vacuum is now guaranteed, it is no longer problematic to consider a non-extremal black hole excitation above this vacuum. In fact, one may even stick to the extremal case. Following up on the spectrum computed in [177], for example, it is straightforward to check that the appropriate quantization condition on the electric charge of these black holes is consistent with the expectation from the ρ-algebras.

In such a set up, an extension of the Rholographic picture is simple too. A black string solution of the six dimensional supergravity theory has an $AdS_3 \times S^3$ horizon. In fact, this was the set up considered in the classic example of [19]. Its Rholographic counterpart would be the $\rho_{L/R}$-twisted non-extremal excitation on the $AdS_3^{\rho_{L/R}}$ vacuum. The field theory living on its boundary is a $(4,4)$ D1-D5 CFT. It contains two chiral $\mathcal{N} = 4$ superconformal algebras in two dimensions. For the Rholographic extension of which, as discussed at the end of Section 3, one may consider a $\rho_{L/R}$-algebra extension on either of the chiral components of this CFT. Therefore, a $\rho_{L/R}$-twisted D1-D5 CFT is conjecturally dual to Type IIB Superstring theory on an $AdS_3^{\rho_{L/R}} \times S^3 \times K3$.

It is worth noting that the D1-D5 CFT has local gauge symmetry that leads to spectral-flow, much like in the case of the MSW CFT. While there was one set of Kac-Moody currents corresponding to the $SU(2)$ gauge symmetry in the MSW CFT (corresponding to rotational symmetry on the S^2 of the AdS_3 horizon), the D1-D5 CFT has two such current algebras corresponding to rotational symmetry on the S^3, with an isometry group $SO(4) \simeq SU(2) \times SU(2)$. It must be stressed that the Scherk-Schwarz twist on the world-sheet does not involve the current algebras. Rather, it uses the outer automorphism groups of the left and right moving sectors, which we call $SU(2)_{\rho_L} \times SU(2)_{\rho_R}$. It is clear then that the twists on the worldsheet supercharges is with respect to the subgroups

$$U(1)_{\rho_L} \times U(1)_{\rho_R} \subset SU(2)_{\rho_L} \times SU(2)_{\rho_R} \,, \tag{5.4}$$

and if we want to preserve some supersymmetry in the bulk, we set one of the twist parameters to zero, e.g. $\rho_L = 0$. The concerned reader may consider this example to be on more firm ground, as far as stability of the vacuum is concerned. In fact, it would be interesting to compute black hole discharge rates and R-charged particle scattering processes using conformal field theory techniques for the ρ-algebras. It would also be interesting to explore the consequences of, and find more evidence for, the Rholographic picture. We leave these interesting questions for future research.

A Compact gauging of $G_{2(2)}/SO(4)$

Here we identify the relevant $U(1)_R$ of the model discussed in the main text and then compute its associated Killing vector and moment map. Let H_1, H_2 be the two Cartans and E_i, F_i, $i = 1, \dots, 6$, the positive and negative root generators of the split real form $G_{2(2)}$. The maximally compact subgroup SO(4) is generated by

$$K_1 = E_1 - \delta^2\gamma^{-2} F_1\,, \qquad K_2 = E_2 - \gamma^2 F_2\,, \qquad K_3 = E_3 - \delta^2 F_3\,,$$
$$K_4 = E_4 - \gamma^2\delta^2 F_4\,, \qquad K_5 = E_5 - \gamma^4\delta^2 F_5\,, \qquad K_6 = E_6 - \gamma^2\delta^4 F_6\,,$$

for any non-zero real constants γ and δ. Indeed, the further combinations

$$\begin{aligned} J_1 &= \tfrac{1}{2}\delta^{-1}\left(\gamma^{-2}K_5 - \sqrt{3}K_3\right)\,, \\ J_2 &= \tfrac{1}{2}\gamma^{-1}\left(\delta^{-2}K_6 + \sqrt{3}K_2\right)\,, \\ J_3 &= \tfrac{1}{2}\gamma\delta^{-1}\left(K_1 - \gamma^{-2}\sqrt{3}K_4\right) \end{aligned} \tag{A.1}$$

and

$$\begin{aligned} L_1 &= \tfrac{1}{2}\delta^{-1}\left(3\gamma^{-2}K_5 + \sqrt{3}K_3\right)\,, \\ L_2 &= \tfrac{1}{2}\gamma^{-1}\left(3\delta^{-2}K_6 - \sqrt{3}K_2\right)\,, \\ L_3 &= \tfrac{1}{2}\gamma\delta^{-1}\left(3K_1 + \gamma^{-2}\sqrt{3}K_4\right) \end{aligned} \tag{A.2}$$

can be checked to generate two copies of SU(2), for any γ and δ. This is most straightforwardly seen using an explicit matrix realisation of the $G_{2(2)}$ generators, like *e.g.* the one given in appendix C of [178]. A calculation similar to that of that appendix allows us to establish that the $SU(2)_R \approx Sp(1)$ corresponding to the R-symmetry is generated by J_x, $x = 1, 2, 3$. Any of the J_x can thus be picked up as the relevant $U(1)_R$ to gauge our model with. For definiteness, we choose[13] J_3.

We now turn to the calculation of the Killing vector associated to J_3. The Killing vectors of hypermultiplet spaces in the image of the c-map have been given in terms of special geometry data in [179] (see [180] for a recent update). Here, rather than using those general formulae, we play the following trick, based on the homogeneity of $G_{2(2)}/SO(4)$, to read off the Killing vector associated to a specific generator. If $\mathcal{V}(q^u)$ is the right, say, coset representative and $\sharp$ denotes the $G_{2(2)}$-generalised transpose (see e.g. [178] for the details), then $P = \frac{1}{2}\left(d\mathcal{V}\,\mathcal{V}^{-1} + (d\mathcal{V}\,\mathcal{V}^{-1})^\sharp\right)$ is a one-form valued on the Lie algebra $\mathfrak{g}_{2(2)}$ of $G_{2(2)}$. For any real one-form A, the one-form $\hat{P} = \frac{1}{2}\left(D\mathcal{V}\,\mathcal{V}^{-1} + (D\mathcal{V}\,\mathcal{V}^{-1})^\sharp\right)$, with

[13] We have explicitly verified that graviphoton gaugings along J_1 only, along J_2 only or along J_3 only are physically indistinguishable, as they should.

$D\mathcal{V}\mathcal{V}^{-1} \equiv (d\mathcal{V} + gA\,\mathcal{V}\,J_3)\mathcal{V}^{-1}$, is also $\mathfrak{g}_{2(2)}$-valued. Here we have found it useful to stick in a (coupling) constant g. We can then expand $\hat{P}$ in the basis H_1, H_2, E_i, F_i of $\mathfrak{g}_{2(2)}$ to read off the covariant derivative $Dq^u = dq^u + gA\,k^u$ and the components of the k^u of the Killing vector associated to the generator J_3. In fact, we have repeated this exercise for all 14 generators of $G_{2(2)}$ to compute all Killing vectors of $G_{2(2)}/SO(4)$, and have explicitly verified that these vectors do indeed leave the metric $h_{uv}dq^u dq^v = \frac{1}{4}\mathrm{Tr}(PP)$ invariant. Obviously, the same process can be followed to compute the Killing vectors of any (non-compact) homogeneous space.

Performing the suitable coordinate transformation that brings the metric $h_{uv}dq^u dq^v = \frac{1}{4}\mathrm{Tr}(PP)$ obtained from the coset approach into the c-map form (4.18), we thus find that the Killing vector $k_0 = k_0^u\,\partial_u$ associated to the $U(1)_R$ generator J_3 has the following components k_0^u along the coordinates (4.16):

$$
\begin{aligned}
k_0^\phi &= -2^{-\frac{3}{2}}\,3^{-\frac{3}{4}}\,\gamma^{-1}\delta\left(\xi^0 + 3\sqrt{3}\,\delta^2\,\tilde{\xi}_1\right),\\
k_0^\varphi &= 2^{-\frac{3}{2}}\,3^{-\frac{3}{4}}\,\gamma^{-1}\delta\left(\xi^0 - 4\sqrt{3}\,\gamma^2\,\chi\,\xi^1 - \sqrt{3}\,\gamma^2\,\tilde{\xi}_1\right),\\
k_0^\chi &= -2^{-\frac{1}{2}}\,3^{-\frac{5}{4}}\,\gamma^{-1}\delta\Big(\sqrt{3}\,\chi\,\xi^0 - \left(1 - 6\gamma^2 e^{-4\varphi} + 6\gamma^2\chi^2\right)\xi^1 + 3\sqrt{3}\,\gamma^2\,\tilde{\xi}_0\\
&\qquad\qquad - 3\chi\,\gamma^2\,\tilde{\xi}_1\Big),\\
k_0^a &= 2^{-\frac{3}{2}}\,3^{-\frac{11}{4}}\,\gamma^{-1}\delta^{-1}\Big(9\delta^2 a(\xi^0 + 3\sqrt{3}\,\gamma^2\,\tilde{\xi}_1)\\
&\quad - 9\delta^2 e^{-2\phi-2\varphi}\chi\,\xi^0\left[9\gamma^2 + 18\gamma^2 e^{4\varphi}\chi^2 - e^{8\varphi}\chi^2(1 - 9\gamma^2\chi^2)\right]\\
&\quad + 9\sqrt{3}e^{-2\phi-2\varphi}\,\xi^1\left[3\gamma^2\delta^2 + \sqrt{3}e^{2\phi+2\varphi} + 12\gamma^2\delta^2 e^{4\varphi}\chi^2 - \delta^2 e^{8\varphi}\chi^2(1 - 9\chi^2)\right]\\
&\quad - 9\,\tilde{\xi}_0\left[3\sqrt{3}\gamma^2 + \delta^2 e^{-2\phi+6\varphi}(1 - 9\chi^2)\right]\\
&\quad + 9\sqrt{3}\,\delta^2 e^{-2\phi+2\varphi}\chi\,\tilde{\xi}_1\left[6\gamma^2 - e^{4\varphi}(1 - 9\gamma^2\chi^2)\right]\\
&\quad + 9\delta^2\xi^0\left[\xi^0\tilde{\xi}_0 + \xi^1\tilde{\xi}_1 - 3\sqrt{3}\,\gamma^2\,\tilde{\xi}_0\,\tilde{\xi}_1\right]\\
&\quad - 9\delta^2\xi^1\left[54\gamma^2\xi^0\tilde{\xi}_0 - 2\sqrt{3}(\xi^1)^2 - 9\sqrt{3}\,\gamma^2(\tilde{\xi}_1)^2\right]\Big),\\
k_0^{\xi^0} &= 2^{-\frac{3}{2}}\,3^{-\frac{3}{4}}\,\gamma^{-1}\delta^{-1}\Big(2\delta^2(\xi^0)^2 - 6\gamma^2\delta^2(\xi^1)^2 + 3\sqrt{3}\,\gamma^2 - \delta^2 e^{-2\phi+6\varphi}(1 - 9\gamma^2\chi^2)\Big),\\
k_0^{\xi^1} &= -2^{-\frac{3}{2}}\,3^{-\frac{3}{4}}\,\gamma^{-1}\delta\Big(3\sqrt{3}\,\gamma^2\,a - 2\xi^0\xi^1 + 3\sqrt{3}\,\gamma^2\,\xi^0\tilde{\xi}_0 - 5\sqrt{3}\,\gamma^2\,\xi^1\tilde{\xi}_1\\
&\qquad\qquad - 6\sqrt{3}\,\gamma^2\,\chi e^{-2\phi+2\varphi} + \sqrt{3}e^{-2\phi+6\varphi}\chi(1 - 9\gamma^2\chi^2)\Big),\\
k_0^{\tilde{\xi}_0} &= 2^{-\frac{3}{2}}\,3^{-\frac{3}{4}}\,\gamma^{-1}\delta\Big(a - \xi^0\tilde{\xi}_0 - \xi^1\tilde{\xi}_1 + 6\sqrt{3}\,\gamma^2\,\tilde{\xi}_0\tilde{\xi}_1 + 9\gamma^2\chi e^{-2\phi-2\varphi}\\
&\qquad\qquad + 18\gamma^2 e^{-2\phi+2\varphi}\chi^3 - e^{-2\phi+6\varphi}\chi^3(1 - 9\gamma^2\chi^2)\Big),
\end{aligned}
$$

$$\begin{aligned} k_0^{\xi_1} = -2^{-\frac{3}{2}}\,3^{-\frac{5}{4}}\,\gamma^{-1}\delta^{-1}\Big(&2\delta^2(\xi^1)^2 - 12\sqrt{3}\,\gamma^2\delta^2\,\xi^1\xi_0 - 6\,\gamma^2\delta^2\,(\xi_1)^2 \\ &- 3\sqrt{3} + 9\,\gamma^2\delta^2\,e^{-2\phi-2\varphi} + 36\,\gamma^2\delta^2\,e^{-2\phi+2\varphi}\chi^2 \\ &- 3\,\delta^2\,e^{-2\phi+6\varphi}\chi^2(1-9\gamma^2\chi^2)\Big)\,. \end{aligned} \tag{A.3}$$

It is now straightforward to doublecheck by standard methods that this vector leaves the metric (4.18) invariant, and thus is indeed Killing, and that it vanishes at (4.19).

We have also computed the moment map P_0^x, $x = 1, 2, 3$, corresponding to this isometry. Since the full expression is not very illuminating, we only give its value at the vacuum (4.19), which is the only quantity needed for all our analyses. With the normalisation of [164], we obtain

$$P_0^x = (0, 0, 2)\,, \tag{A.4}$$

independent of γ and δ. Since the moment map is independent of these factors, so is the mass (charge) spectrum.

Chapter IV

Dynamics of the Schwarzschild horizon

The emergence of string theory, holography [21, 31, 32] and gauge-gravity duality [33–35] has shed significant light on the information-loss problem. In fact, it is often claimed that if one were to believe gauge-gravity duality, the paradox is solved 'in-principle' as the boundary theory is unitary by construction and the duality states an equivalence (at the level of partition functions) between the gravitational and boundary field theories. Nevertheless, the strength of this claim is questionable [181, 182] and even within the best understood examples of gauge-gravity duality, there is no general consensus on the exact process of information retrieval. Furthermore, the best understood examples of the said duality, while providing for a very useful toolbox, typically involve bulk space-times with a negative cosmological constant and are far from the real world. Technology at this stage is far from established to reliably understand more realistic space-times. Additionally, why intricate details of string theory or the duality may be absolutely necessary for our understanding of the evolution of general gravitational dynamics is not apparent. While the fuzzball program [183–187] provides some arguments for why stringy details may be important, it is fair to say that there is no general consensus on the matter.

Years before gauge-gravity duality was proposed and was seen as a possible resolution to the information paradox, there was an alternative suggestion by 't Hooft [31, 188, 189]. The proposal was to consider particles of definite momenta 'scattering' off a black-hole horizon. These particles were to impact the out-going Hawking quanta owing to their back-reaction on the geometry. With the knowledge that the black hole is made out of a large, yet *finite*, number of in-states, one may scatter particles of varying momenta repeatedly, until all in-states that may have made up the black hole have been exhausted. This led to a construction of an S-Matrix that maps in to out states. This matrix was shown to be unitary. A further advancement for spherically symmetric horizons was made recently [190–192], where a partial wave expansion allowed for an explicit writing of the S-Matrix for each spherical harmonic. However, this construction has its own short-comings. It presumes that the S-Matrix can be split as

$$S_{\text{total}} = S_{-\infty}\, S_{\text{horizon}}\, S_{+\infty}\,, \tag{0.1}$$

where $S_{\pm\infty}$ correspond to matrices that map asymptotic in-states to in-going states near the horizon and outgoing states near the horizon to asymptotic out-states respectively.

And S_{horizon} is the S-Matrix that captures all the dynamics of the horizon. Whether such an arbitrarily near-horizon region captures all the dynamics of the black hole is not entirely clear. The construction is also done in a 'probe-limit' in that the back-reaction is not taken to impact the mass of the black hole. Only its effect on outgoing particles is captured. Furthermore, throwing a particle into a black hole is not an exactly spherically symmetric process. While a non-equilibrium process initiated by the in-going particle does break spherical symmetry, it may be expected that the black hole settles down into a slightly larger, spherically symmetric solution after some characteristic time-scale that depends on the interactions between the various degrees of freedom that make up the black hole. This scattering can be decomposed into partial waves. And the different waves are assumed to evolve independently. However, one expects that the partial waves are not independent and that they indeed 'interact' in a generic evolutionary process; it is not clear how one may incorporate this interaction in this construction. Furthermore, the scattering algebra possibly would need modification in a more general setup. Another important limitation is that the back-reaction calculations ignore transverse effects [193, 194] which grow in increasing importance as we approach Planckian scales. Finally, while it may not be a fundamental difficulty, the splitting of the wave function via (0.1) needs further investigation.

As is evident from the above, it is surprisingly easy to criticize even the most promising approaches to quantum black hole physics. In this chapter, we seek to address some of the criticisms of the S-Matrix approach to quantum black holes. Inspired by old ideas from non-critical 2d string theory [195–202], we construct a theory—of a collection of quantum mechanics models with inverted harmonic oscillator potentials—that exactly reproduces the S-Matrix of 't Hooft for every partial wave; the inverted potentials arise naturally to allow for scattering states, as opposed to bound states in a conventional harmonic oscillator. The intrinsically quantum nature of the model dispenses with the critique that the S-Matrix of 't Hooft is a 'classical' one. With any toy model, it may be hard to establish the validity of its applicability to black hole physics. However, in our construction, all the observables (S-Matrix elements) are exactly identical to those of 't Hooft's S-Matrix; thereby avoiding any ambiguity of its validity. Furthermore, we observe that in-states must contain an approximately constant number density over a wide range of frequencies in order for the scattered out-states to appear (approximately) thermal; this condition was also noted in the 2d string theory literature. Finally, and perhaps most significantly, we show that our model captures an exponentially growing degeneracy of states.

It may be added that aside from the approaches mentioned earlier, there have been many attempts to construct toy-models to study black hole physics [203–208]. The hope being that 'good' toy models teach us certain universal features of the dynamics of black hole horizons.

This chapter is organized as follows. In Section 1, we briefly review gravitational back-reaction and ’t Hooft’s S-Matrix construction along with its partial wave expansion. Our derivation is slightly different to the one of ’t Hooft [190] in that our derivation relies only on the algebra associated to the scattering problem. Therefore, the ‘boundary conditions’ of the effective bounce, as was imposed in ’t Hooft’s construction is built in from the start via the back-reaction algebra (1.13). In Section 2, we present our model and compute the corresponding scattering matrix to show that it explicitly matches the one of ’t Hooft. In Section 3, we make an estimate of the high energy behaviour of the total density of states to argue that the model indeed describes the existence of an intermediate black-hole state. We conclude with a discussion and some future perspectives in Section 4.

A brief summary of results There are two main results of this chapter: one is a re-writing the degrees of freedom associated to ’t Hooft’s black hole S-Matrix in terms of inverted harmonic oscillators; this allows us to write down the corresponding Hamiltonian of evolution explicitly. The second, related result is an identification of a connection to 2d string theory which in turn allows us to show that there is an exponential degeneracy of how a given total initial energy may be distributed among many partial waves of the 4d black hole; much as is expected from the growth of states associated to black hole entropy. At various points in Sections 2 and 3, we review some aspects of matrix models and 2d string theory in detail. While we expect some consequences for these theories based on our current work, we do not have any new results within the framework of 2d black holes or matrix models in this chapter.

1 Back-reaction and the Black Hole S-Matrix

Consider a vacuum solution to Einstein’s equations of the form:

$$ds^2 = 2A(u^+, u^-)du^+ du^- + g(u^+, u^-)\, h(\Omega) d\Omega^2 , \tag{1.1}$$

where u^+, u^- are light-cone coordinates, $A(u^+, u^-)$ and $g(u^+, u^-)$ are generic smooth functions of those coordinates and $h(\Omega)$ is the metric tensor depending on only the $(d-2)$ transverse coordinates Ω. It was shown in [194] that an in-going massless particle with momentum p^- induces a shock-wave at its position specified by Ω and $u^- = 0$. The shock-wave was shown to change geodesics such that out-going massless particles feel a kick—of the form $u^- \to u^- + 8\pi G\, p^-_{\text{in}} \hat{f}(\Omega, \Omega')$—in their trajectories at $u^- = 0$, where $\hat{f}$ depends on the spacetime in question. If we were to associate a putative S-Matrix to the dynamics of the black hole, the said back-reaction may be attributed to this S-Matrix in the following manner. Consider a generic in-state $|\text{in}_0\rangle$ that collapsed into a black hole and call the corresponding out-state after the complete evaporation of the black hole $|\text{out}_0\rangle$. The S-Matrix maps one into the other via: $S\,|\text{in}_1\rangle = |\text{out}_1\rangle$. Now the back-reaction effect may be treated as a tiny modification of the in-state as $|\text{in}_0\rangle \to \left|\text{in}_0 + \delta p^-_{\text{in}}(\Omega)\right\rangle$, where $\delta p^-_{\text{in}}(\Omega)$ is the momentum of an in-going particle at position Ω on the horizon. Consequently, the action of the S-Matrix on the modified in-state results in a different out-state which is acted upon by an operator that yields the back-reacted displacement:

$$S\left|\text{in}_0 + \delta p^-_{\text{in}}(\Omega)\right\rangle \;=\; e^{-i\delta p^+_{\text{out}}(\Omega')\delta u^-_{\text{out}}}\left|\text{out}_0\right\rangle, \tag{1.2}$$

where the operator acting on the out-state above is the 'displacement' operator written in Fourier modes. Now, we may repeat this modification arbitrarily many times. This results in a cumulative effect arising from all the radially in-going particles with a distribution of momenta on the horizon. Therefore, writing the new in- and out-states—with all the modifications included—as $|\text{in}\rangle$ and $|\text{out}\rangle$ respectively, we have

$$\langle\text{out}|S|\text{in}\rangle \;=\; \langle\text{out}_0|S|\text{in}_0\rangle\,\exp\left[-i8\pi G\int d^{d-2}\Omega'\, p^+_{\text{out}}(\Omega')\,\hat{f}(\Omega,\Omega')\,p^-_{\text{in}}(\Omega)\right].$$

Should we now *assume* that the Hilbert space of states associated to the black-hole is completely spanned by the in-going momenta and that the Hawking radiation is entirely spanned by the out-state momenta, we are naturally led to a unitary S-Matrix given by

$$\left\langle p^+_{\text{out}}\middle|S\middle|p^-_{\text{in}}\right\rangle \;=\; \exp\left[-i8\pi G\int d^{d-2}\Omega'\, p^+_{\text{out}}(\Omega')\,\hat{f}(\Omega,\Omega')\,p^-_{\text{in}}(\Omega)\right]. \tag{1.3}$$

There is an overall normalization factor (vacuum to vacuum amplitude) that is undetermined in this construction. The assumption that the black hole Hilbert space of states is spanned entirely by the in-state momenta p^-_{in} is equivalent to postulating that the said collection of radially in-going, gravitationally back-reacting particles collapse into a black hole. While this may seem a reasonable assumption, it is worth emphasizing that there is no evidence for this at the level of the discussion so far. We have not modeled a collapsing problem. We will see in Section 3 that our proposed model in Section 2 provides for a natural way to study this further. And significantly, we give non-trivial evidence that the derived S-Matrix possibly models a collapsing black-hole.

1.1 Derivation of the S-Matrix

We now return to the back-reaction effect at a semi-classical level in order to derive an explicit S-Matrix using a partial wave expansion in a spherically symmetric problem. For the back-reacted metric—after incorporating the shift $u^- \to u^- + f(\Omega,\Omega')$ into (1.1)—to still satisfy Einstein's equations of motion, the following conditions need to hold at $u^- = 0$ [194]:

$$8\,\pi\, p^-_{\text{in}}\, A\left(u^{+,-}\right)^2 \delta^{(d-2)}(\Omega,\Omega') \;=\; \frac{A(u^{+,-})}{g(u^{+,-})}\,\triangle_\Omega\, f(\Omega,\Omega')$$

$$
\begin{aligned}
&- \left(\frac{d-2}{2}\right) \frac{\partial_{u^+}\partial_{u^-} g(u^{+,-})}{g(u^{+,-})} f(\Omega,\Omega') \qquad (1.4)\\
\partial_{u^-} A(u^{+,-}) \;&=\; 0 \;=\; \partial_{u^-} g(u^{+,-}),
\end{aligned}
$$

where $\triangle_\Omega$ is the Laplacian on the $(d-2)$-dimensional metric $h(\Omega)$. We concern ourselves with the Schwarzschild black-hole, written in Kruskal-Szekeres coordinates as

$$
ds^2 \;=\; -\frac{32\, G^3\, m^3}{r} e^{-r/2Gm} du^+\, du^- + r^2 d\Omega^2 \,. \tag{1.5}
$$

For the above metric (1.5), at the horizon $r = R = 2Gm$, the conditions (1.4) were shown [194] to reduce to

$$
\triangle_S\, (\Omega) f(\Omega,\Omega') \;\coloneqq\; (\triangle_\Omega - 1) f(\Omega,\Omega') \;=\; -\kappa\, \delta^{(d-2)}(\Omega,\Omega'), \tag{1.6}
$$

with the implicit dependence of r on u^+ and u^- given by

$$
u^+\, u^- \;=\; \left(1 - \frac{r}{2Gm}\right) e^{-r/2Gm}, \tag{1.7}
$$

and $\kappa = 2^4\, \pi\, e^{-1}\, G\, R^2\, p_{\text{in}}^-$. These seemingly ugly coefficients may easily be absorbed into the stress-tensor on the right hand side of the Einstein's equations. Now, the cumulative shift experienced by an out-going particle, say u_{out}^-, is given by a distribution of in-going momenta on the horizon

$$
u_{\text{out}}^-(\Omega) \;=\; 8\pi G R^2 \int d^{d-2}\Omega'\, \tilde{f}(\Omega,\Omega')\, p_{\text{in}}^-(\Omega'), \tag{1.8}
$$

where $\kappa\, \tilde{f}(\Omega,\Omega') = f(\Omega,\Omega')$. Similarly, we have the complementary relation for the momentum of the out-going particle, say p_{out}^+ given in terms of the position u_{in}^+ of the in-going particle:

$$
u_{\text{in}}^+(\Omega) \;=\; -8\pi G R^2 \int d^{d-2}\Omega'\, \tilde{f}(\Omega,\Omega')\, p_{\text{out}}^+(\Omega') \tag{1.9}
$$

The expressions (1.8) and (1.9) may be seen as 'boundary conditions' of an effective bounce off the horizon. However, this intuition is rather misleading and we will refrain from this line of thought. Nevertheless, what is striking to note is that the momentum of the in-state is encoded in the out-going position of the Hawking radiation while the position of the in-state is encoded in the momentum of the out-going Hawking state! However, so far, the quantities $u_{\text{in/out}}^\pm$ are dimensionless while $p_{\text{in/out}}^\mp$ are densities of momenta with mass dimensions four. Therefore, to appropriately interpret these as positions and momenta, we rescale them as $u_{\text{in/out}}^\pm \to R u_{\text{in/out}}^\pm$ and $p_{\text{in/out}}^\mp \to R^{-3} p_{\text{in/out}}^\mp$ [191]. Notwithstanding this rescaling, we continue to use the same labels for the said quantities in order to avoid clutter of notation. Now, using the canonical commutation relations, respectively, for the out and in particles[1]

$$\left[\hat{u}^-(\Omega), \hat{p}^+(\Omega')\right] = \left[\hat{u}^+(\Omega), \hat{p}^-(\Omega')\right] = i\,\delta^{(d-2)}(\Omega - \Omega'), \tag{1.10}$$

we may derive the algebra associated to the black hole scattering. We do this in a partial wave expansion—in four dimensions—as

$$\hat{u}^\pm(\Omega) = \sum_{lm} \hat{u}^\pm_{lm}\, Y_{lm}(\Omega) \quad \text{and} \quad \hat{p}^\pm(\Omega) = \sum_{lm} \hat{p}^\pm_{lm}\, Y_{lm}(\Omega)\,. \tag{1.11}$$

Working with these eigenfunctions of the two-sphere Laplacian and using 1.6 we can write the back-reaction equations (1.8) and (1.9) as

$$\hat{u}^\pm_{lm} = \mp \frac{8\pi G}{R^2\left(l^2 + l + 1\right)} \hat{p}^\pm_{lm} =: \mp\lambda\, \hat{p}^\pm_{lm}\,. \tag{1.12}$$

In terms of these partial waves, we may now write the scattering algebra as

$$\left[\hat{u}^\pm_{lm}, \hat{p}^\mp_{l'm'}\right] = i\delta_{ll'}\delta_{mm'} \tag{1.13}$$

$$\left[\hat{u}^+_{lm}, \hat{u}^-_{l'm'}\right] = i\,\lambda\,\delta_{ll'}\delta_{mm'} \tag{1.14}$$

$$\left[\hat{p}^+_{lm}, \hat{p}^-_{l'm'}\right] = -\frac{i}{\lambda}\,\delta_{ll'}\delta_{mm'} \tag{1.15}$$

A few comments are now in order. Since the different spherical harmonics do not couple in the algebra, we will drop the subscripts of l and m from here on. Furthermore, we see that the shift-parameter λ 'morally' plays the role of Planck's constant $\hbar$, but one that is now l dependent. Moreover, we see that wave-functions described in terms of four phase-space variables are now pair-wise related owing to the back-reaction (1.12). Finally, it is important to note that each partial wave does not describe a single particle but a specific profile of a density of particles. For instance, the s-wave with $l = 0$ describes a spherically symmetric density of particles.

Since the operators $\hat{u}^\pm$ and $\hat{p}^\pm$ obey commutation relations associated to position and momentum operators, we see that the algebra may be realized with $\hat{u}^- = -i\lambda\partial_{u^+}$ in the u^+ basis and $\hat{u}^+ = i\lambda\partial_{u^-}$ in the u^- basis. A similar realization is evident for the momentum operators. Moreover, we may now define the following inner-products on the associated Hilbert space of states that respect the above algebra:

$$\left\langle u^\pm \middle| p^\mp \right\rangle = \frac{1}{\sqrt{2\pi}} \exp\left(iu^\pm p^\mp\right) \tag{1.16}$$

[1] To avoid clutter in notation, we drop the in/out labels on positions and momenta of particles. u^+ and u^- always refer to ingoing/outgoing positions, respectively. Consequently, p^- and p^+ are always associated with ingoing/outgoing momenta, respectively.

$$\langle u^+ | u^- \rangle = \frac{1}{\sqrt{2\pi\lambda}} \exp\left(i\frac{u^+u^-}{\lambda}\right) \tag{1.17}$$

$$\langle p^+ | p^- \rangle = \sqrt{\frac{\lambda}{2\pi}} \exp\left(i\lambda p^+ p^-\right) \tag{1.18}$$

Using (1.17), for instance, we may write the out-going wave-function—travelling along the coordinate u^- after scattering—in terms of the in-going one travelling along u^+ as

$$\langle u^- | \psi \rangle =: \psi^{\text{out}}(u^-) = \int_{-\infty}^{\infty} \frac{du^+}{\sqrt{2\pi\lambda}} \exp\left(-i\frac{u^+u^-}{\lambda}\right) \psi^{\text{in}}(u^+)\,. \tag{1.19}$$

One can immediately see that this mapping is Unitary just being a fourier transform. To derive another useful form of the S-Matrix associated to the scattering, we first move to Eddington-Finkelstein coordinates:

$$u^+ = \alpha^+ e^{\rho^+}, \quad u^- = \alpha^- e^{\rho^-} \quad p^+ = \beta^+ e^{\omega^+} \quad \text{and} \quad p^- = \beta^- e^{\omega^-} \tag{1.20}$$

where $\alpha^\pm = \pm 1$ and $\beta^\pm = \pm 1$ to account for both positive and negative values of the phase space coordinates u^+, u^-, p^+ and p^-. The normalization of the wave-function as

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} \left|\psi(u^+)\right|^2 du^+ \\
&= \int_{-\infty}^{0} \left|\psi(u^+)\right|^2 du^+ + \int_{0}^{\infty} \left|\psi(u^+)\right|^2 du^+ \\
&= -\int_{\infty}^{-\infty} \left|\psi^+\left(-e^{\rho^+}\right)\right|^2 e^{\rho^+} d\rho^+ + \int_{-\infty}^{\infty} \left|\psi^+\left(+e^{\rho^+}\right)\right|^2 e^{\rho^+} d\rho^+ \\
&= \sum_{\alpha=\pm} \int_{-\infty}^{\infty} \left|\psi^+\left(\alpha e^{\rho^+}\right)\right|^2 e^{\rho^+} d\rho^+
\end{aligned}$$

suggests the following redefinitions for the wave-function in position and momentum spaces

$$\psi^\pm\left(\alpha^\pm e^{\rho^\pm}\right) = e^{-\rho^\pm/2} \phi^\pm(\alpha^\pm, \rho^\pm) \quad \& \quad \tilde{\psi}^\pm\left(\beta^\pm e^{\omega^\pm}\right) = e^{-\omega^\pm/2} \tilde{\phi}^\pm(\beta^\pm, \omega^\pm)\,.$$

Therefore, using (1.19), we may write $\phi^{\text{out}}(\alpha^-, \rho^-)$ as:

$$\begin{aligned}
\phi^{\text{out}}(\alpha^-, \rho^-) &= \frac{1}{\sqrt{2\pi\lambda}} \int_{-\infty}^{\infty} du^+ \, e^{\frac{\rho^+ + \rho^-}{2}} \exp\left(-i\frac{u^+u^-}{\lambda}\right) \phi^{\text{in}}(\alpha^+, \rho^+) \\
&= \sum_{\alpha^+=\pm} \int_{-\infty}^{\infty} \frac{du^+}{\sqrt{2\pi}} e^{\frac{\rho^+ + \rho^- - \log\lambda}{2}} \exp\left(-i\alpha^+\alpha^- e^{\rho^+ + \rho^- - \log\lambda}\right)
\end{aligned}$$

$$\begin{aligned}
&\phi^{\text{in}}(\alpha^+,\rho^+) \\
&= \sum_{\alpha^+=\pm}\int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi}}\,\exp\Big(\frac{x}{2}-i\alpha^+\alpha^- e^x\Big) \\
&\qquad \phi^{\text{in}}(\alpha^+, x+\log\lambda-\rho^-)\,,
\end{aligned} \tag{1.21}$$

where in the last line, we introduced $x := \rho^+ + \rho^- - \log\lambda$. This equation may be written in matrix form as

$$\begin{pmatrix} \phi^{\text{out}}(+,\rho^-) \\ \phi^{\text{out}}(-,\rho^-) \end{pmatrix} = \int_{-\infty}^{\infty} dx \begin{pmatrix} A(+,+,x) & A(+,-,x) \\ A(-,+,x) & A(-,-,x) \end{pmatrix} \begin{pmatrix} \phi^{\text{in}}(+,x+\log\lambda-\rho^-) \\ \phi^{\text{in}}(-,x+\log\lambda-\rho^-) \end{pmatrix} \tag{1.22}$$

where we have defined the quantity

$$A(\gamma,\delta,x) := \frac{1}{\sqrt{2\pi}}\,\exp\Big(\frac{x}{2}-i\,\gamma\,\delta\,e^x\Big)\,, \tag{1.23}$$

with $\gamma = \pm$ and $\delta = \pm$. This integral equation may further be simplified by moving to Rindler plane waves:

$$\phi^{\text{out}}(\pm,\rho^-) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk_-\,\phi^{\text{out}}(\pm,k_-)\,e^{ik_-\rho^-} \tag{1.24}$$

$$\phi^{\text{in}}(\pm,x+\log\lambda-\rho^-) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk_{\tilde{x}}\,\phi^{\text{in}}(\pm,k_{\tilde{x}})e^{-ik_{\tilde{x}}(x+\log\lambda-\rho^-)} \tag{1.25}$$

$$A(\gamma,\delta,x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk_x\,A(\gamma,\delta,k_x)e^{ik_x x} \tag{1.26}$$

This allows us to write the above matrix equation (1.22) as

$$\begin{pmatrix} \phi^{\text{out}}(+,k) \\ \phi^{\text{out}}(-,k) \end{pmatrix} = e^{-ik\log\lambda}\begin{pmatrix} A(+,+,k) & A(+,-,k) \\ A(-,+,k) & A(-,-,k) \end{pmatrix}\begin{pmatrix} \phi^{\text{in}}(+,k) \\ \phi^{\text{in}}(-,k) \end{pmatrix} \tag{1.27}$$

where $A(\gamma,\delta,k)$ can be computed from the inverse Fourier transform of (1.23) using a coordinate change $y = e^x$ and the identity

$$\int_0^{\infty} dy\,e^{i\sigma y}y^{-ik-\frac{1}{2}} = \Gamma\Big(\frac{1}{2}-ik\Big)\,e^{i\sigma\frac{\pi}{4}}\,e^{k\sigma\frac{\pi}{2}}\,,\quad \text{where}\quad \sigma = \pm\,. \tag{1.28}$$

Carrying out this computation, we find the following S-Matrix:

$$S(k_l,\lambda_l) = e^{-ik_l\log\lambda_l}\begin{pmatrix} A(+,+,k_l) & A(+,-,k_l) \\ A(-,+,k_l) & A(-,-,k_l) \end{pmatrix}$$

$$= \frac{1}{\sqrt{2\pi}}\Gamma\left(\frac{1}{2} - ik_l\right)e^{-ik_l \log \lambda_l}\begin{pmatrix} e^{-i\frac{\pi}{4}}\, e^{-k_l\frac{\pi}{2}} & e^{i\frac{\pi}{4}}\, e^{k_l\frac{\pi}{2}} \\ e^{i\frac{\pi}{4}}\, e^{k_l\frac{\pi}{2}} & e^{-i\frac{\pi}{4}}\, e^{-k_l\frac{\pi}{2}} \end{pmatrix} \tag{1.29}$$

In this expression, we have reinstated a subscript on k and λ to signify that they depend on the specific partial wave in question. One may additionally diagonalize this matrix by noting that

$$A(+,+,k) = A(-,-,k) \quad \text{and} \quad A(+,-,k) = A(-,+,k)\,. \tag{1.30}$$

With this observation, we see that the diagonalization of the S-Matrix is achieved via the redefinitions

$$\begin{aligned} \phi_1^+(k) &= \phi^+(+,k) + \phi^+(-,\rho^+)\,, & \phi_2^+(k) &= \phi^+(+,k) - \phi^+(-,k) \\ A_1(k) &= A(+,+,k) + A(+,-,k)\,, & A_2(k) &= A(+,+,k) - A(+,-,k)\,. \end{aligned}$$

It may be additionally checked that this matrix is unitary. As already mentioned, while it may not be clear whether this matrix is applicable to the formation and evaporation of a physical black hole, a conservative statement that can be made with certainty is the following: all information that is thrown into a large black hole is certainly recovered in its entirety, at least when the degrees of freedom in question are positions and momenta. It would be interesting to generalize this to degrees of freedom carrying additional conserved quantities like electric charge, etc. On the other hand, there is a certain property of the S-Matrix that may be puzzling at first sight. Positive Rindler energies k imply that the off-diagonal elements in the S-Matrix are dominant with exponentially suppressed diagonal elements. While negative Rindler energies reverse roles. One way to interpret this feature is to think of an eternal black hole where dominant off-diagonal elements suggest that information about in-going matter from the right exterior is carried mostly by out-going matter from the left exterior. However, in a physical collapse, there is only one exterior. It has been suggested by 't Hooft that one must make an antipodal mapping between the two exteriors to make contact with the one-sided physical black hole; we discuss this issue in Section 4.

2 The model

Asking two simple questions allows us to almost entirely determine a quantum mechanical model that corresponds to the black hole scattering matrix of the previous section. The first question is 'what kind of a quantum mechanical potential allows for scattering states?' The answer is quite simply that it must be an unstable potential. The second question is 'what quantum mechanical model allows for energy eigenstates that resemble those of Rindler space?' The answer, as we will show in this section, is a model of waves scattering off an inverted harmonic oscillator potential. Using this intuition, we will now construct the model and show that it explicitly reproduces the desired S-Matrix. Having constructed the model, we will then proceed to compare it to 2d string theory models. The construction of our model and intuition gained from a comparison to 2d string theory/matrix quantum mechanics models [196, 199, 202] allows us to study time delays and degeneracy of states in the next section.

Inverted quadratic potentials, at a classical level, fill up phase space with hyperbolas as opposed to ellipses as in the case of standard harmonic oscillator potentials. Since we have a tower of 4d partial waves in the black hole picture, each of them results in a phase space of position and momentum and consequently a collection of inverted harmonic oscillators, one for each partial wave. Since the black hole scattering of 't Hooft mixes positions and momenta, we are naturally led to consider the description of scattering in phase space.

2.1 Construction of the model

We first start with a phase space parametrized by variables x_{lm} and p_{lm}. To implement the appropriate scattering off the horizon, we start with the same black hole scattering algebra: $[\hat{x}_{lm}, \hat{p}_{l'm'}] = i\lambda\delta_{mm'}\delta_{ll'}$, with $\lambda = c/(l^2 + l + 1)$ with $c = 8\pi G/R^2$. We will return to how this parameter might naturally arise in a microscopic setting in Section 4. Standard bases of orthonormal states are $|x; l, m\rangle$ and $|p; l, m\rangle$; these are coordinate and momentum eigenstates respectively, with

$$\langle l, m; x|p; l, m\rangle = \frac{1}{\sqrt{2\pi\lambda}} e^{ipx/\lambda}\, \delta_{mm'}\delta_{ll'} \, . \tag{2.1}$$

Since our interest is in the scattering of massless particles, it will turn out to be convenient to use light-cone bases $|u^{\pm}; l, m\rangle$ which are orthonormal eigenstates of the light-cone operators:

$$\hat{u}^{\pm}_{lm} = \frac{\hat{p}_{lm} \pm \hat{x}_{lm}}{\sqrt{2}} \qquad \text{and} \qquad \left[\hat{u}^{+}_{lm}, \hat{u}^{-}_{l'm'}\right] = i\lambda\delta_{ll'}\delta_{mm'} \, . \tag{2.2}$$

While they look similar to creation and annihilation operators of the ordinary harmonic oscillator, $\hat{u}^{\pm}$ are in truth hermitian operators themselves; and are not hermitian conjugate to each other. Therefore, the states $|u^{\pm}; l, m\rangle$ are reminiscent of coherent states. These plus and minus bases will be useful in describing the in and outgoing states of the upside down harmonic oscillator. For definiteness, we will choose for the ingoing states to be described in terms of the $u^{+}_{l,m}$ basis while for the outgoing ones to be in terms of the $u^{-}_{l,m}$ basis. As in the previous section, we will work in the simplification where different oscillators (partial waves) do not interact and will therefore omit the partial wave labels in all places where they do not teach us anything new. Furthermore, as before, from the commutation relations we may define the following inner product on the Hilbert space of states

$$\langle u^+|u^-\rangle = \frac{1}{\sqrt{2\pi\lambda}} \exp\left(\frac{iu^+u^-}{\lambda}\right), \tag{2.3}$$

that expresses the Fourier transform kernel between the two bases. We may again realize the algebra if $\hat{u}^-$ acts on $\langle u^+|u^-\rangle$ and $\langle u^+|x\rangle$ as $-i\lambda\partial_{u^+}$ while $\hat{u}^+$ acts on $\langle u^-|u+\rangle$ and $\langle u^-|x\rangle$ as $i\lambda\partial_{u^-}$. To endow the model with dynamics, we now turn to the Hamiltonian for each oscillator/partial wave

$$\begin{aligned} H_{lm} &= \frac{1}{2}\left(p_{lm}^2 - x_{lm}^2\right) \\ &= \frac{1}{2}(u_{lm}^+u_{lm}^- + u_{lm}^-u_{lm}^+) . \end{aligned} \tag{2.4}$$

which may also be written as

$$H = \mp i\lambda\left(u^{\pm}\partial_{u^{\pm}} + \frac{1}{2}\right) \tag{2.5}$$

in the $u^{\pm}$ bases where we drop the l, m indices. Physically the wave-function can be taken to correspond to a wave coming from the right which after scattering splits into a transmitted piece that moves on to the left and a reflected piece that returns to the right. The other wave function can be obtained from this one by a reflection $x \to -x$. The light-cone coordinates describe these left/right movers and simplify the description of scattering since the Schrödinger equation becomes a first order partial differential equation. Moreover, the energy eigenfunctions are simply monomials of $u^{\pm}$ while in the x representation the energy eigenfunctions are more complicated parabolic cylinder functions. In particular, for each partial wave the Schrödinger equation in light-cone coordinates is:

$$i\lambda\,\partial_t\psi_{\pm}(u^{\pm}, t) = \mp i\lambda\,(u^{\pm}\partial_{u^{\pm}} + 1/2)\psi_{\pm}(u^{\pm}, t) \tag{2.6}$$

with solutions

$$\psi_{\pm}(u^{\pm}, t) = e^{\mp t/2}\,\psi_{\pm}^0(e^{\mp t}u^{\pm}) . \tag{2.7}$$

This can also be written in bra/ket notation as:

$$\langle u^{\pm}|\psi^{\pm}(t)\rangle = \langle u^{\pm}|e^{\frac{i}{\lambda}\hat{H}t}|\Psi_0^{\pm}\rangle = e^{\mp\frac{t}{2}}\langle e^{\mp t}u^{\pm}|\Psi_0^{\pm}\rangle . \tag{2.8}$$

The time evolution for the basis states is given by

$$e^{\frac{i}{\lambda}Ht}|u^{\pm}\rangle = e^{\pm\frac{t}{2}}|e^{\pm t}u^{\pm}\rangle$$

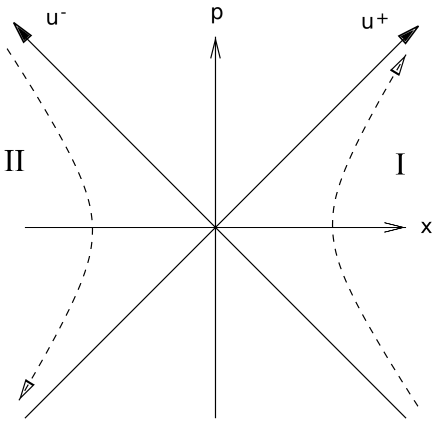

Figure 1. The scattering diagram.

$$\begin{aligned} \langle u^{\pm}|e^{\frac{i}{\lambda}Ht} &= e^{\mp\frac{t}{2}}\langle e^{\mp t}u^{\pm}| \\ \langle u^{+}|e^{\frac{i}{\lambda}Ht}|u^{-}\rangle &= \frac{1}{\sqrt{2\pi\lambda}}e^{-\frac{t}{2}}\exp\left(\frac{i}{\lambda}u^{+}u^{-}e^{-t}\right) \end{aligned} \tag{2.9}$$

In the conventions of Figure 1, it is easy to see that ingoing states can be labelled by the u^+ axis while the outgoing ones by the u^- axis. Since the potential is unbounded, the Hamiltonian has a continuous spectrum. In the u^+ representation the energy eigenstates with eigenvalue ϵ are

$$\frac{1}{\sqrt{2\pi\lambda}}(u^+)^{i\frac{\epsilon}{\lambda}-\frac{1}{2}}\ .$$

The singularity at $u^+ = 0$ leads to a two fold doubling of the number of states. This is understood to be arising from the existence of the two regions (I - II) in the scattering diagram. From now on we use $|\epsilon,\alpha^+\rangle_{\rm in}$ and $|\epsilon,\alpha^-\rangle_{\rm out}$ for the in and outgoing energy eigenstates with the labels $\alpha^+ = \pm$, $\alpha^- = \pm$ to denote the regions I and II. While we have four labels, we are still only describing waves in the two quadrants (I-II) with two of them for ingoing waves and two for outgoing ones. The in-states may be written as

$$\langle u^+|\epsilon,+\rangle_{\rm in} = \begin{cases} \frac{1}{\sqrt{2\pi\lambda}}(u^+)^{i\frac{\epsilon}{\lambda}-\frac{1}{2}} & u^+ > 0 \\ 0 & u^+ < 0 \end{cases}$$

$$\langle u^+|\epsilon,-\rangle_{\rm in} = \begin{cases} 0 & u^+ > 0 \\ \frac{1}{\sqrt{2\pi\lambda}}(-u^+)^{i\frac{\epsilon}{\lambda}-\frac{1}{2}} & u^+ < 0 \end{cases}$$

describing left and right moving ingoing waves for the regions I and II respectively. Similarly, the natural out basis is written as

$$\langle u^-|\epsilon,+\rangle_{\text{out}} = \begin{cases} \frac{1}{\sqrt{2\pi\lambda}}(u^-)^{-i\frac{\epsilon}{\lambda}-\frac{1}{2}} & u^- > 0 \\ 0 & u^- < 0 \end{cases}$$

$$\langle u^-|\epsilon,-\rangle_{\text{out}} = \begin{cases} 0 & u- > 0 \\ \frac{1}{\sqrt{2\pi\lambda}}(-u^-)^{-i\frac{\epsilon}{\lambda}-\frac{1}{2}} & u^- < 0 \end{cases}$$

to describe the right and left moving outgoing waves for the regions I and II respectively. Therefore, time evolution of the energy eigenstates

$$\langle u^+|\epsilon,+\rangle_{\text{in}}(t) = \frac{1}{\sqrt{2\pi\lambda}} e^{-i\frac{\epsilon}{\lambda}t}(u^+)^{i\frac{\epsilon}{\lambda}-\frac{1}{2}} = \frac{1}{\sqrt{2\pi\lambda}} e^{-\frac{\rho^+}{2}} e^{-i\frac{\epsilon}{\lambda}t} e^{i\frac{\epsilon}{\lambda}\rho^+} \tag{2.10}$$

implies that they correspond to the Rindler relativistic plane-waves[2] moving with the speed of light in the tortoise-coordinates if we identify the quantum mechanical time with Rindler time $t = \tau$ and the inverted harmonic oscillator energy with the Rindler momentum via $\kappa\lambda = \epsilon$. This means that the energy of the eigenstates of the non-relativistic inverted oscillator, when multiplied by λ, can also be interpreted as the energy/momentum of the Rindler relativistic plane waves of the previous section. This allows us to write down any ingoing state in terms of these Rindler plane waves. As we have seen, the unitary operator relating the $u^\pm$ representations is given by the fourier kernel (2.3) on the whole line that acts on a state as

$$\psi_{\text{out}}(u^-) = \left[\hat{S}\psi_{\text{in}}\right](u^-) = \int_{-\infty}^{\infty} \frac{du^+}{\sqrt{2\pi\lambda}} e^{\frac{-iu^+u^-}{\lambda}} \psi_{\text{in}}(u^+). \tag{2.11}$$

It is now clear that repeating the calculations of the previous section results in the same S-Matrix, rather trivially. However, to make the connection to the eigenstates of the inverted harmonic oscillator transparent, we will derive it in a more conventional manner. To represent the action of the kernel on energy eigenstates, we split it into a 2×2 matrix that relates them as follows:

$$\begin{pmatrix} |\epsilon,+\rangle_{out} \\ |\epsilon,-\rangle_{out} \end{pmatrix} = \hat{S} \begin{pmatrix} |\epsilon,+\rangle_{in} \\ |\epsilon,-\rangle_{in} \end{pmatrix} \tag{2.12}$$

The fastest method to find each entry is to compute the in-going energy eigenstates in the out-going position basis and vice versa using the insertion of a complete set of states of the form

$$\langle u^-|\epsilon\rangle_{\text{in}} = \int_{-\infty}^{\infty} du^+ \langle u^-|u^+\rangle \langle u^+|\epsilon\rangle_{\text{in}} . \tag{2.13}$$

[2]Normalised in the $u^\pm$ basis.

The results are

$$\langle u^-|\epsilon,\pm\rangle_{\text{in}} = \lambda^{\frac{i\epsilon}{\lambda}} e^{\frac{\mp i\pi}{4}} e^{\pm\frac{\pi\epsilon}{2\lambda}} \Gamma\left(\frac{1}{2}+i\frac{\epsilon}{\lambda}\right) \frac{(\alpha^-|u^-|)^{-i\frac{\epsilon}{\lambda}-\frac{1}{2}}}{\sqrt{2\pi\lambda}} \tag{2.14}$$

$$\langle u^+|\epsilon,\pm\rangle_{\text{out}} = \lambda^{\frac{-i\epsilon}{\lambda}} e^{\frac{\pm i\pi}{4}} e^{\pm\frac{\pi\epsilon}{2\lambda}} \Gamma\left(\frac{1}{2}-i\frac{\epsilon}{\lambda}\right) \frac{(\alpha^+|u^+|)^{i\frac{\epsilon}{\lambda}-\frac{1}{2}}}{\sqrt{2\pi\lambda}} . \tag{2.15}$$

Each of these equations gives two results for each sign[3] to yield:

$$\begin{aligned} \mathcal{S} &= \frac{1}{\sqrt{2\pi}} \exp\left(-i\frac{\epsilon}{\lambda}\log\lambda\right)\Gamma\left(\tfrac{1}{2}-i\frac{\epsilon}{\lambda}\right)\begin{pmatrix} e^{-i\frac{\pi}{4}} e^{-\frac{\pi\epsilon}{2\lambda}} & e^{i\frac{\pi}{4}} e^{\frac{\pi\epsilon}{2\lambda}} \\ e^{i\frac{\pi}{4}} e^{\frac{\pi\epsilon}{2\lambda}} & e^{-i\frac{\pi}{4}} e^{-\frac{\pi\epsilon}{2\lambda}} \end{pmatrix} \\ &= e^{i\Phi(\epsilon)} \exp\left(-i\frac{\epsilon}{\lambda}\log\lambda\right)\begin{pmatrix} \frac{e^{-i\pi/4}}{\sqrt{1+e^{2\pi\epsilon/\lambda}}} & \frac{e^{i\pi/4}}{\sqrt{1+e^{-2\pi\epsilon/\lambda}}} \\ \frac{e^{i\pi/4}}{\sqrt{1+e^{-2\pi\epsilon/\lambda}}} & \frac{e^{-i\pi/4}}{\sqrt{1+e^{2\pi\epsilon/\lambda}}} \end{pmatrix}, \end{aligned} \tag{2.16}$$

with the scattering phase $\Phi(\epsilon)$ being defined as

$$\Phi(\epsilon) = \sqrt{\frac{\Gamma\left(\frac{1}{2}-i\frac{\epsilon}{\lambda}\right)}{\Gamma\left(\frac{1}{2}+i\frac{\epsilon}{\lambda}\right)}} . \tag{2.17}$$

Identifying parameters as $k_l\, \lambda_l = \epsilon_l$, we see that this precisely reproduces the S-Matrix derived in the previous section for every partial wave. In this model, it is clear that the competition between reflection and transmission coefficients is owed to the energy of the waves being scattered being larger than the tip of the inverted potential.

2.2 A projective light-cone construction

Although we had good reason to expect such an inverse harmonic oscillator realization of the black hole S-Matrix, there is, in fact, another way to derive it—using what is called a projective light-cone construction. This construction was first studied by Dirac and [209, 210] provide a good modern introduction to the topic. The essential idea is to embed a null hyper-surface inside Minkowski space to study how linear Lorentz symmetries induce non-linearly realized conformal symmetries on a (Euclidean) section of the embedded surface. This allows us to relate the Rindler Hamiltonian -which can then be related directly to the Hamiltonian of the quantum mechanics model that describes the scattering- with the Dilatation operator on the horizon. In a black hole background this construction

[3] For negative signs, one makes use of $(-1)^{i\epsilon/\lambda-1/2} = e^{-i\pi/2}e^{-\pi\epsilon/\lambda}$.

is of course expected to hold only locally in the near horizon region. We first introduce $X = (x^\mu, x^{d-1}, x^d)$ with $\mu = 1, .., d-2$ (note that μ is a Euclidean index), where the light-cone coordinates are defined as $x^\pm = x^d \pm x^{d-1}$. Here, x^d serves as the time coordinate [4]. The Minkowski metric η_{MN} in these coordinates is given as

$$ds^2 \;=\; -dx^+dx^- + dx_\mu dx^\mu\,, \tag{2.18}$$

which has an $SO(d-1,1)$ Lorentz symmetry. There is an isomorphism between the corresponding Lorentz algebra and the Euclidean conformal algebra in $d-2$ dimensions. To state this isomorphism, we first label the $d-2$-dimensional Euclidean conformal group generators as:

$$\begin{aligned}
P_\mu &= i\partial_\mu && \text{corresponding to translations,}\\
M_{\mu\nu} &= i\big(x_\mu\partial_\nu - x_\nu\partial_\mu\big) && \text{to rotations,}\\
D &= ix^\mu\partial_\mu && \text{to dilatations, and}\\
K_\mu &= i\Big(2x_\mu(x^\nu\partial_\nu) - x^2\partial_\mu\Big) && \text{to special conformal transformations}\,.
\end{aligned}$$

The identification is now given as follows:

$$J_{\mu\nu} \;=\; M_{\mu\nu}\,,\quad J_{\mu+} \;=\; P_\mu\,,\quad J_{\mu-} \;=\; K_\mu\,,\quad J_{+-} \;=\; D\,, \tag{2.19}$$

where the $SO(3,1)$ Lorentz generators J_{MN} are given by

$$J_{MN} \;=\; x_M p_N - x_N p_M\,. \tag{2.20}$$

These satisfy the $SO(3,1) \simeq SL(2,\mathbb{C})$ algebra. In particular the Dilatation operator on the two dimensional horizon is

$$D \;=\; J_{+-} \;=\; x_+p_- - x_-p_+ \;=\; \frac{1}{\lambda}\big(u^+u^- + u^-u^+\big) \;=\; \frac{1}{\lambda}H\,, \tag{2.21}$$

where in the second equality we used $u^\pm \;=\; x_\pm$ to connect to the light-cone coordinates of the previous sub-section and in the third equality, we made use of the back-reaction relations (1.12). Interestingly enough, we see that an appropriately scaled Dilatation operator together with the back-reaction relations gives us exactly the Hamiltonian of the inverted oscillator. The scaling is also neatly realized in the relation between the quantum mechanical energy ϵ and the Rindler energy κ to relate the two S-Matrices.

This construction via the light-cone projection could possibly shed more light on the relation between the black hole S-Matrix and string theoretic amplitudes. In the early

[4] The null cone is described by the equation $X^2 = 0$ and a Euclidean section can be given as $x^+ = f(x^\mu)$.

papers on black hole scattering [31, 188, 189], a striking similarity between the S-Matrix and stringy amplitudes was observed. The role of the string worldsheet was attributed to the horizon itself. It was noted that the string tension was imaginary. In the construction above, we found that the induced conformal symmetry on the horizon is Euclidean and that the Dilatation operator is mapped to the time-evolution operator (Rindler Hamiltonian) of the 4d Lorentzian theory. This led us to the unstable potential of the inverse harmonic oscillator. It may well be that the apparently misplaced factors of i in the string tension is owed to the Euclidean nature of conformal algebra on the horizon. It would also be interesting to understand the role of possible infinite-dimensional local symmetries on the horizon/worldsheet [211, 212] from the point of view of the quantum mechanics model, elaborating on the null cone construction. We leave this study to future work.

While the model is seemingly very simple, this is not the first time that such a model has been considered to be relevant for black hole physics [201, 213]. However, previous considerations have found that these models do not correspond to 2d black hole formation owing to an insufficient density of states in the spectrum. Refining these considerations with the intuition that each oscillator as considered in this section corresponds to a partial wave of a 4d black hole, we find that our model may indeed be directly related to 4d black holes formed by physically collapsing matter. We provide evidence for this in Section 3. In order to move on to which, however, it will be very useful for us to review the 2d string theory considerations of the past; this is what we now turn to.

2.3 Relation to matrix models and 2-d string theory

Hermitian Matrix Quantum Mechanics (MQM, henceforth) in the inverted harmonic oscillator was studied in connection with $c = 1$ Matrix models and string theory in two dimensions. For more details, we refer the reader to [195, 214]. Here, we briefly review these results in order to point out various similarities and differences with our work. The Lagrangian of MQM is of the form

$$L = \frac{1}{2}\mathrm{Tr}\left[(D_t M)^2 + M^2\right] \qquad \text{with} \qquad D_t = \partial_t - iA_t\,, \tag{2.22}$$

where A_t is a non-dynamical gauge field. The $N\times N$ Hermitian Matrices transform under $U(N)$ as $M \to U^\dagger M U$. The role of the non-dynamical gauge field is to project out the non-singlet states in the path integral. Diagonalization of the matrices results in a Vandermonde factor in the path integral measure:

$$\mathcal{D}M = \mathcal{D}U \prod_i dx_i \prod_{i<j}(x_i - x_j)^2\,. \tag{2.23}$$

This indicates a natural fermionic redefinition of the wave-functions into Slater determinants (in a first quantised description). The Hamiltonian of the system is, therefore, in terms of N free fermions:

$$\hat{H}\,\tilde{\Psi} \;=\; -\left(\frac{\hbar^2}{2}\sum_{i=1}^{N}\partial_{x_i}^2 + \frac{1}{2}x_i^2\right)\tilde{\Psi} \quad \text{with} \quad \tilde{\Psi}(x_i) = \prod_{i<j}(x_i - x_j)\Psi(x_i)\,. \tag{2.24}$$

with $\tilde{\Psi}(x_i)$ being the redefined fermionic wave-functions. Filling up the 'Fermi-sea' up to a level μ, allows for a definition of the vacuum. Clearly, all fermions are subject to the same chemical potential μ that is typically considered to be below the tip of the inverted oscillator. A smooth string world-sheet was argued to be produced out of these matrices in a double-scaling limit $\mu \to 0, \hbar \to 0$ with a fixed inverse string-coupling defined by the ratio $\mu/\hbar \sim 1/g_s$. In this double-scaling limit, this theory describes string theory on a 2d linear dilaton background with coordinates described by time t and the Liouville field ϕ. The matrix model/harmonic oscillator coordinate x is conjugate to the target space Liouville field via a non-local integral transformation [215]. In contrast to this picture, owing to a one-one correspondence between the 2d harmonic oscillators and 4d partial waves in our model, this integral transform is unnecessary. However, it has been argued in string theory that only the quadratic tip is relevant in this double-scaling limit, even in the presence of a generic inverted potential, emphasizing the universality of the quadratic tip. Whilst we do not have a similar stringy argument, we expect the ubiquitous presence of the quadratic potential to persist in our construction owing to the ubiquitous presence of the Rindler horizon in physical black holes formed from collapsing matter. A modern discourse with emphasis on the target space interpretation of the matrix model as the effective action of N $D0$ branes may be found in [216]. A natural second quantized string field theory description of the system where the fermionic wave-functions are promoted to fermionic fields may be found in [195, 217, 218] and references therein. A satisfactory picture of free fermionic scattering in the matrix model was given in [196] via the following S-Matrix relation:

$$\hat{S} \;=\; i_{b\to f} \circ \hat{S}_{ff} \circ i_{f\to b} \tag{2.25}$$

where even though the asymptotic tachyonic states are bosonic, one is instructed to first fermionize, then scatter the fermions in the inverted quadratic potential and then to bosonize again. The total S-matrix is unitary if the fermionic scattering is unitary and the bosonization spans all possible states. The logic of this expression resembles that of 't Hooft's S-matrix, where one first expands a generic asymptotic state into partial waves, expresses them in terms of near horizon Rindler parameters, scatters them with the given S-matrix that is similar to the one of 2d string theory before transforming back to the original asymptotic coordinates. At the level of the discussion now, it may already be noted that one important difference between the 2d string-theoretic interpretation of the matrix model and our 4d partial wave one is the nature of the transformations that relate asymptotic states to the eigenstates of the inverted harmonic oscillator. Additionally, and perhaps more importantly, in our construction, we have an entire collection of such harmonic oscillators/matrix models parametrised by l, m that conspire to make up a 4d black hole. We present concrete evidence for this by studying time-delays and degeneracy of states in Section 3. There are further differences between the 2d string theories and our construction, in order to present which, we need to proceed to a study of the spectrum of states in our model; this enables us to study growth of states in the two models. Finally, we also comment on a possible second quantization and appropriate MQM interpretation of our model in Section 4.

3 Combining the oscillators (partial waves)

On the side of the macroscopic black hole in Section 1, the calculation was done in an approximation where there is a pre-existing black hole into which degrees of freedom are thrown (as positions and momenta). It was then evident that the information that was sent into the black hole is completely recovered since the S-Matrix was unitary. Furthermore, the back-reaction computation told us exactly how this information is retrieved: in-going positions as out-going momenta and in-going momenta as out-going positions. However, a critical standpoint one may take with good reason would be to say that this is not good enough to tell us if a physical collapse of a black hole and complete evaporation of it is a unitary process. The calculation has not modeled a collapsing problem.

The picture to have in a realistic collapse is that of an initial state that evolves in time to collapse into an intermediate black hole state which then subsequently evaporates to result in a final state that is related to the initial one by a unitary transformation. Naturally, the corresponding macroscopic picture is that of a strongly time-dependent metric. Heuristically, one may think of the total S-Matrix of this process as being split as

$$\hat{S} \;=\; \hat{S}_{\mathscr{I}^- \to \mathrm{hor}^-}\, \hat{S}_{\mathrm{hor}^- \to \mathrm{hor}^+}\, \hat{S}_{\mathrm{hor}^+ \to \mathscr{I}^+} \tag{3.1}$$

where $\hat{S}_{\mathscr{I}^- \to \mathrm{hor}^-}$ corresponds to evolution from asymptotic past to a (loosely defined) point in time when gravitational interactions are strong enough for the collapse to begin, $\hat{S}_{\mathrm{hor}^- \to \mathrm{hor}^+}$ to the piece that captures all the 'action'—insofar as collapse and evaporation are concerned—take place and finally $\hat{S}_{\mathrm{hor}^+ \to \mathscr{I}^+}$ represents the evolution of the evaporated states to future infinity. The horizon—being a teleological construction that can be defined only if one knows the global structure of spacetime—has a time dependent size and location in a collapse/evaporation scenario but for us will nevertheless comprise the locus of spacetime points where the backreaction effects are important. Therefore, we use subscripts $\mathrm{hor}^{\pm}$ to refer to it, at different points in time, in the above heuristic split.

Thought of the total evolution this way, it is clear that the most important contribution arises from the part of the matrix that refers to the region in space-time where gravitational back-reaction cannot be ignored. The other pieces are fairly well-approximated by quantum field theory on an approximately fixed background. Nevertheless, in the intermediate stage, the metric is strongly time-dependent.

At the outset, let it be stated that we will not get as far as being able to derive this metric from the quantum mechanics model. We may ask if there are generic features of the black hole that we have come to learn from semi-classical analyses that can also be seen in this model. We will focus on two important qualitative aspects of (semi-classical) black holes:

Time-delay A physical black hole is not expected to instantaneously radiate information that has been thrown into it. There is a time-delay between the time at which radiation begins to be received by a distant observer and the time at which one may actually recover in-going information. In particular, given an in-state that collapses into a black hole, we expect that the time-scale associated to the scattering process is 'long'. In previous studies of 2d non-critical string theory, it was found that with a single inverted harmonic oscillator, the associated time-delay is not long enough to have formed a black hole [197, 200, 201]. However, with the recognition that each oscillator corresponds to a partial wave and that a collection of oscillators represents a 4d black hole, we see that the black hole degeneracy of states arises from the entire collection while the time-delay associated to each oscillator is the time spent by an in-going mode in the scattering region; the latter being more reminiscent of what one might call 'scrambling time'.

Approximate thermality As Hawking famously showed [18], the spectrum of radiation looks largely thermal for a wide range of energies. One way to probe this feature is via the number operator—which, for a finite temperature system, can be written as $\langle \hat{N}(\omega) \rangle = \rho(\omega) f(\omega)$ with $\rho(\omega)$ being the density of states and $f(\omega)$ the appropriate thermal distribution for Fermi/Bose statistics. Given that the S-Matrix is unitary, we know that this notion of temperature and thermality of the spectrum is only approximate. Notwithstanding this, a detector at future infinity should register this approximately thermal distribution for a large frequency range.

In what follows, we will study whether the S-Matrix corresponding to our collection of oscillators in the model presented in Section 2 displays both these properties.

3.1 Time delays and degeneracy of states

We have seen that the total scattering matrix associated to four-dimensional gravity can be seen as arising via a collection of inverted harmonic oscillators, each with a different algebra differentiated by λ_l in, say, (1.13). One canonical way to study life-times in scattering problems in quantum mechanics is via the time-delay matrix, which is defined as:

$$\Delta t_{ab} = \mathrm{Re}\left(i \sum_c S^\dagger(k_l, \lambda_l)_{ac} \left(\frac{dS(k_l, \lambda_l)}{dk} \right)_{cb} \right). \tag{3.2}$$

Each matrix element above encodes the time spent by a wave of energy k_l in the scattering region in the corresponding channel. The trace of this matrix, called Wigner's time delay τ_l, captures the total characteristic time-scale associated to the entire scattering process. Said another way, should we start with a generic in-state that undergoes scattering and is then retrieved in the asymptotic future as some out-state, the trace of the above matrix associates a life-time to the intermediate state [219, 220]. For large energies k_l, using $S(k_l, \lambda_l)$ in (1.29), the Wigner time-delay associated to the scattering of a single oscillator can be calculated to scale as $\tau \sim \log(\lambda_l\, k_l)$. This is the same result as was found in the 2d string theory literature [197, 200, 201] and was argued to not be long-enough for black hole formation. Based on these black hole non-formation results in the matrix quantum mechanics, it was suggested that studying the non-singlet sectors would shed light on 2d black hole formation[205, 221]. Despite some efforts in relating the adjoint representations with long-string states [213], a satisfactory Lorentzian description is still missing. Anticipating our result prematurely, our model does not suffer from these difficulties as it is to describe a 4d black hole with a collection of oscillators. Merely the s-wave oscillator in our model would mimic the singlet sector in matrix quantum mechanics[5].

The above time delay τ may also be interpreted as a density of states associated to the system. The inverted potential under consideration implies a continuous spectrum. In order to discretize which, to derive the density of states, the system must be stabilized—by putting it in a box of size Λ, for instance. Demanding that the wavefunctions vanish at the wall and regulating the result by subtracting any cut-off dependent quantities, the density of states may be computed from the scattering phase Φ defined via $\mathcal{S}(k_l, \lambda_l) = \exp\left[i\, \Phi(k_l, \lambda_l)\right]$ as $\rho(\epsilon_l) = d\Phi/d\epsilon_l$ [222]. The result is exactly the same as what we get from computing the time delay using the scattering matrix (1.29) and the time-delay equation (3.2) to find a Di-Gamma function $\psi^{(0)}$

$$\begin{aligned} \rho(\epsilon_l) = \tau_l &= \frac{2}{\lambda_l} \mathrm{Re}\left[\psi^{(0)}\left(\frac{1}{2} - i \frac{\epsilon_l}{\lambda_l} \right) + \log(\lambda_l) \right] \\ &= \mathrm{Re}\left[\sum_{n=0}^{\infty} \frac{2}{i\epsilon_l - \lambda_l\left(n + \frac{1}{2}\right)} + \frac{2}{\lambda_l} \log(\lambda_l) \right]. \end{aligned} \tag{3.3}$$

[5]It would be very interesting if higher l modes can be described as non-singlets of a matrix model.

This density of states may be used to define a partition function for each partial wave (with Hamiltonian $\hat{H}_{lm}$), where the energy eigenstates contributing to the partition function will have been picked out by the poles of the density $\rho(\epsilon_l)$. However, in our model, we see that there are many oscillators in question. Should we start with an in-state made of a collection of all oscillators instead of a single partial wave, we may first write down the total S-Matrix as a product of the individual oscillators as

$$\mathcal{S}_{\text{tot}} = \prod_{l=0}^{\infty} \mathcal{S}(k_l, \lambda_l), \tag{3.4}$$

assuming that different partial waves do not interact. One may correct for this by adding interaction terms between different oscillators. To compute the time-delay associated to a scattering of some in-state specified by a given total energy involves an appropriately defined Wigner time-delay matrix as

$$\tau_{\text{tot}} = \text{Tr}\left[\text{Re}\left(-i\left(\mathcal{S}^{\dagger}_{\text{tot}}\right)_{ac}\left(\frac{d\mathcal{S}_{\text{tot}}}{dE_{\text{tot}}}\right)_{cb}\right)\right]. \tag{3.5}$$

where this equation makes sense only if we have defined a common time evolution and unit of energy for the total system/collection of partial waves. We will elaborate on this in a while. Now, even in the spherically symmetric approximation, to write the total S-Matrix as a function of merely one coarse-grained energy E_{tot} is not a uniquely defined procedure. However, our intuition that each partial wave may be thought of as a single-particle oscillator allows us to compute the density of states in a combinatorial fashion. We will see that the degeneracy of states associated to an intermediate long-lived thermal state arises from the various ways in which one might distribute a given total energy among the many available oscillators. Given a total energy E_{tot}, we now have the freedom to describe many states, each with a different distribution of energies into the various available oscillators. From the poles in the density defined in (3.3), we see that each oscillator has energies quantized as[6]

$$\epsilon_l = i\lambda_l\left(n_l + \frac{1}{2}\right). \tag{3.6}$$

This allows us to measure energies in units of c, where c is defined implicitly via $\lambda_l(l^2 + l + 1) = c$. Therefore, in these units, the energies are 'quantized' as

$$\frac{\epsilon_l}{i\,c} = \frac{1}{l^2 + l + 1}\left(n_l + \frac{1}{2}\right). \tag{3.7}$$

[6]The seemingly disconcerting factor of i is just owed to the fact that we have scattering states as opposed to bound ones.

Now, given some total energy E_{tot}, we see that any oscillator may be populated with a single particle state carrying energy such that $n_l = E_{\text{tot}}(l^2 + l + 1)$, where we leave out the half integer piece for simplicity. Importantly, we see that there exist 'special' states coming from very large l-modes even for very small energies. For example, an energy of 1 could arise from a very large l-mode with the excitation given by $n_l = (l^2 + l + 1)$. This is rather unsatisfactory for one expects that it costs a lot of energy to create such states. Moreover, there is an interplay between the log term in the growth of states and the behaviour of the DiGamma function that we are unable to satisfactorily take into account. There is an additional problem which is that the energy of each partial wave is measured in different units that are l dependent; this means that they also evolve with different times. We thus conclude that this is not the correct way to combine the different oscillators.

There is a rather beautiful way to resolve all these three problems via a simple change of variables that we turn to next. It will allow us to interpret the above cost of energy as relative shifts of energies with respect to a common ground state. Additionally these relative shifts also cure the above interplay; there will simply be no log term in the density of states. Finally, this will also introduce a canonical time evolution for the entire system, resulting in one common unit of energy.

3.2 Exponential degeneracy for the collection of oscillators

In order to combine the different oscillators and define a Hamiltonian for the total system we need to get rid of the l dependence in the units of energy used for different oscillators. It turns out that this is possible by rewriting the black hole algebra. Moreover using these new variables, the relation between 't Hooft's black hole S-Matrix for an individual partial wave and the one of 2d string theory of type II [196] can be made manifest. To make this connection transparent, we again start with a collection of inverse harmonic oscillators and the following Hamiltonian for the total system

$$\begin{aligned} H_{tot} &= \sum_{l,m} \frac{1}{2}\left(\tilde{p}_{lm}^2 - \tilde{x}_{lm}^2\right) \\ &= \sum_{l,m} \frac{1}{2}\left(\tilde{u}_{lm}^+ \tilde{u}_{lm}^- + \tilde{u}_{lm}^- \tilde{u}_{lm}^+\right), \end{aligned} \tag{3.8}$$

but this time imposing the usual λ-independent commutation relations $[\tilde{u}_{lm}^+ \tilde{u}_{l'm'}^-] = i\delta_{ll'}\delta_{mm'}$. The λ dependence will come through via an assignment of a chemical potential $\mu(\lambda)$ for each oscillator; this assignment is to be thought of as a different vacuum energy for each partial wave. Following [196], one may then derive an S-Matrix for this theory. To match this to the one of 't Hooft for any given partial wave, one must identify the chemical potential and energy parameters as $\mu = 1/\lambda$ and the Rindler energy $k = \omega + \mu = \omega + 1/\lambda$.

It is worth noting that in the reference cited above, only energies below the tip of the inverted potential were considered, resulting in a dominant reflection coefficient. In contrast 't Hooft's partial waves carry energies higher than the one set by the tip of the potential. Consequently, to make an appropriate identification of 2d string theory with the partial wave S-Matrix, an interchanging of the reflection and transmission coefficients is necessary. From a matrix quantum mechanics point of view, it may additionally be noted that the partial wave parameter λ may be absorbed in either the Planck's constant or the chemical potential to leave the string coupling of each partial wave fixed as $g_s \sim \hbar/\mu \sim c/(l^2+l+1)$. This indicates that as we increase the size of the black hole or we consider higher l partial waves the corresponding string coupling becomes perturbatively small.
Writing out the energies of the various partial waves with the above identification, we have

$$k_l = \omega_l + \frac{l^2+l+1}{c}, \qquad \text{and} \qquad E_{\text{tot}}^{\text{Rindler}} = \sum_l k_l \,. \tag{3.9}$$

At this stage, the labels ω_l are continuous energies. However, discretizing the spectrum as before, by putting the system in a box, we arrive at discrete energies[7]

$$c\, E_{\text{tot}} = \sum_l \left[i\, c \left(n_l + \tfrac{1}{2} \right) + l^2 + l + 1 \right], \tag{3.10}$$

for every individual oscillator. Without a detour into this 2d string theory literature, we may have alternatively arrived at this spectrum from the quantum mechanics model in 2 via the following identifications:

$$\epsilon_l \longrightarrow 1 + \lambda_l\, \omega_l \qquad \text{and} \qquad \lambda_l \longrightarrow \frac{1}{\mu_l} \,. \tag{3.11}$$

While the model presented in Section 2 makes the algebra manifest, the above identification of parameters to relate to the model with a λ-independent algebra makes the physical interpretation of the relative shifts in energies between the partial waves manifest and allows for a consistent definition of time and energy for the total system. This allows us to rewrite our S-Matrix $S(\epsilon_l/\lambda_l)$ as a function of two variables ω_l and μ_l as $S(\omega_l, \mu_l)$. With this change of variables, we recover exactly the S-Matrix of the 2d matrix models discussed in the literature and therefore, now allows us to interpret μ as a chemical potential of the theory. However, since μ_l is now l dependent in our collection of models, it gives us a natural way to interpret how the combined system of oscillators behave. To excite a very large l oscillator, one first has to provide sufficient energy that is equal to $\mu_l \sim (l^2+l+1)$. Therefore, we naturally see that exciting a large l-oscillator

[7] Note again the relative factor of i that indicates that the harmonic oscillator levels have to do with decaying/scattering states while l's are bound states.

costs energy! The physical spectrum may be depicted as in Figure 2, where we depict an arbitrarily chosen ground-state energy with $E = 0$, each oscillator labelled by l and excitations above them by n_l. The various oscillators are shifted by a chemical potential. And the vacuum is defined to be the one with all Rindler energies k_l set to zero. Now, given an initial state carrying a total energy of E_{tot}, we are left with a degeneracy of states that may be formed by distributing this energy among the many available oscillators. The larger this energy, the more oscillators we may distribute it into and hence the larger the degeneracy. The degeneracy associated to equation (3.10), without the chemical potential shift, is merely asking for the number of sets of all integers $\{n_l\}$ that add up to E_{tot}. These are the celebrated partitions into integers that—as Ramanujan showed—grow exponentially. Clearly, for large total energy, our degeneracy grows similarly at leading order. However, the chemical potential shift slows down the growth polynomially compared to the partitioning into integers owing to the fact that for a given E_{tot}, only approximately $\sqrt{E_{\text{tot}}}$ number of oscillators are available. It is worthwhile to note that, in this simplistic analysis, we have ignored the degeneracy arising from the m quantum number; accounting for which clearly increases the growth of states. Therefore, we already see that the model allows for collapse in that it supports an exponential growth of density of states! This shares striking resemblance to the Hagedorn growth of density of states in black holes.

As a conservative estimate, we may start with some total energy E_{tot} and a fixed set of oscillators that are allowed to contribute to it. This allows us to sum over the contribution arising from the $(l^2 + l + 1)c^{-1}$ piece in (3.10) to be left with some subtracted total energy $\tilde{E}_{\text{tot}}$ that is to be distributed among the n_l excitations over each of the available oscillators. Clearly, this grows exponentially much as the partitions into integers does, with the subtracted energy $\tilde{E}_{\text{tot}}$. This is given by the famous Hardy-Ramanujan formula for the growth of partitions of integers:

$$p(n) \sim \exp\left(\pi\sqrt{\frac{2n}{3}}\right). \tag{3.12}$$

Identifying n with the integer part of $\tilde{E}_{\text{tot}}$, we see the desired exponential growth. And considering that the same total energy may be gained from choosing different sets

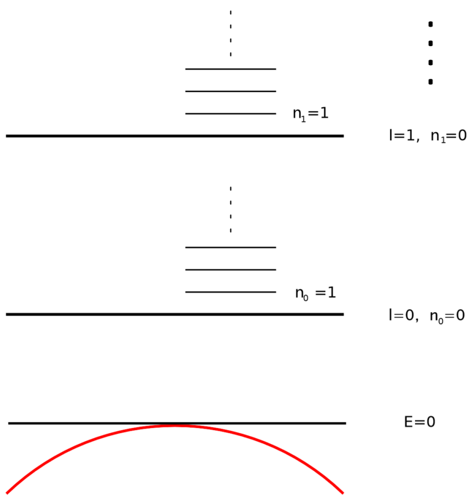

Figure 2. Spectrum of the collection of oscillators. The red curve is indicative of the potential with the horizontal solid lines indicating the various energy levels available.

of oscillators to start with, increases this degeneracy further, in equal measure. While imposing the antipodal identification of 't Hooft—which we discuss in Section 4—reduces this degeneracy, the exponential growth of states remains. How one may derive the Schwarzschild entropy from this degeneracy requires a truly microscopic understanding of the parameter λ. We suggest a way forward towards the end of this chapter but leave a careful study to future work.

4 Discussion

In this chapter, we have constructed a quantum mechanics model that reproduces 't Hooft's black hole S-Matrix for every partial wave using which, we provided non-trivial evidence that it corresponds to a black hole S-Matrix; one that can be formed in a time-dependent collapsing process owing to the appropriate density of states. Several questions, though, remain unanswered. The only degrees of freedom in question were momenta and positions of ingoing modes. One may add various standard model charges, spin, etc. to see how information may be retrieved by the asymptotic observer.

Dynamically speaking, gravitational evolution is expected to be very complicated in real-world scenarios. We have merely approximated it to one where different spherical harmonics do not interact. While incorporating these interactions may be very difficult to conceive in gravity, they are rather straightforward to implement in the quantum mechanical model; one merely introduces interaction terms coupling different oscillators. Exactly what the nature of these interactions is, is still left open.

The complete dynamics of the black hole includes a change in mass of the black hole during the scattering process. In this chapter, we chose to work in an approximation where this is ignored. The corresponding approximation in the inverted oscillators is that the potential is not affected by the scattering waves. In reality, of course, the quadratic potential changes due to the waves that scatter off it. The change in the form of the inverted potential due to a scattering mode can be calculated [223]. We hope to work on this in the future and we think that this gives us a natural way to incorporate the changes to the mass of the black hole. Another possible avenue for future work is to realise a truly microscopic description of the S-matrix, either in the form of a matrix model or a non-local spin model having a finite-dimensional Hilbert space from the outset, where the inverse harmonic potential or emergent $SL(2,\mathbb{R})$ symmetries are expected to arise after an averaging over the interactions between the microscopic degrees of freedom. Some models with these properties can be found in [224–227].

Antipodal entanglement Unitarity of the S-Matrix demands that both the left and right exteriors in the two-sided Penrose diagram need to be accounted for; they capture

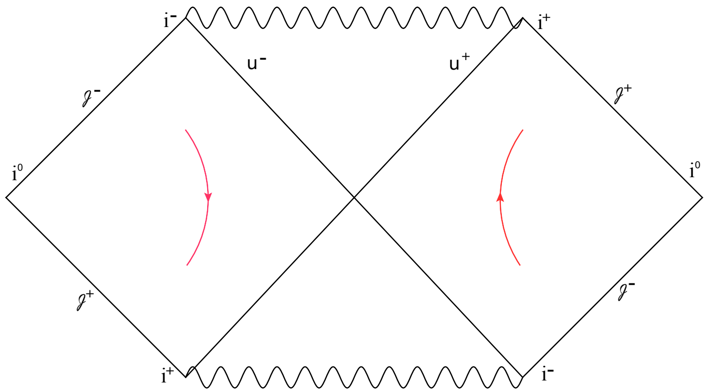

Figure 3. The Penrose diagram. Each point in the conformal diagram originally corresponds to a different sphere. After antipodal identification, the points (u^+, u^-) and $(-u^+, -u^-)$ correspond to antipodal points on a common sphere as given in (4.1). The red lines indicate the arrows of time.

the transmitted and reflected pieces of the wave-function, respectively. In the quantum mechanics model, there appears to be an ambiguity of how to associate the two regions I and II of the scattering diagram in Fig. 1 to the two exteriors of the Penrose diagram. We saw, in the previous section, that the quantum mechanical model appears to support the creation of physical black holes by exciting appropriate oscillators. Therefore, in this picture there is necessarily only one physical exterior. To resolve the issue of two exteriors, it was proposed that one must make an antipodal identification on the Penrose diagram [191]. Unitarity is arguably a better physical consistency condition than a demand of the maximal analytic extension. The precise identification is given by $x \longrightarrow Jx$ with[8]

$$J : \quad (u^+, u^-, \theta, \phi) \longleftrightarrow \left(-u^+, -u^-, \pi - \theta, \pi + \phi\right) . \tag{4.1}$$

Note that J has no fixed points and is also an involution, in that $J^2 = 1$. Such an identification implies that spheres on antipodal points in the Penrose diagram are identified with each other. In particular, this means

$$u^{\pm}(\theta, \phi) = -u^{\pm}(\pi - \theta, \pi + \phi) \qquad \text{and} \qquad p^{\pm}(\theta, \phi) = -p^{\pm}(\pi - \theta, \pi + \phi) .$$

Therefore, noting that the spherical harmonics then obey $Y_{l,m}(\pi - \theta, \pi + \phi) = (-1)^l Y_{l,m}(\theta, \phi)$, we see that only those modes with an l that is odd contribute. However, owing to the validity of the S-Matrix only in the region of space-time that is near the horizon, this identification is presumably valid only in this region. Global identifications of

[8]Note that the simpler mapping of identifying points in I, II via $(u^+, u^-, \theta, \phi) \leftrightarrow (-u^+, -u^-, \theta, \phi)$ is singular on the axis $u^+, u^- = 0$.

the two exteriors have been considered in the past [228–230]. The physics of the scattering, with this identification is now clear. In-going wave-packets move towards the horizon where gravitational back-reaction is strongest according to an asymptotic observer. Most of the information then passes through the antipodal region and a small fraction is reflected back. Turning on quantum mechanics implies that ingoing position is imprinted on outgoing momenta and consequently, an highly localised ingoing wave-packet transforms into two outgoing pieces—transmitted and reflected ones—but both having highly localised momenta. Their positions, however, are highly de-localised. This is how large wavelength Hawking particles are produced out of short wavelength wave-packets and an IR-UV connection seems to be at play. Interestingly, the maximal entanglement between the antipodal out-going modes suggests a wormhole connecting each pair [231]; the geometric wormhole connects the reflected and transmitted Hilbert spaces. Furthermore, as the study of the Wigner time-delay showed, the reflected and transmitted pieces arrive with a time-delay that scales logarithmically in the energy of the in-going wave. This behaviour appears to be very closely related to scrambling time (not the lifetime of the black hole) and we leave a more detailed investigation of this feature to the future. One may also wonder why transmitted pieces dominate the reflected ones. It may be that the attractive nature of gravity is the actor behind the scene.

Approximate thermality We now turn to the issue of thermality of the radiated spectrum. Given a number density, say $N^{\text{in}}(k)$ as a function of the energy k, we know that there is a unitary matrix that relates it to radiated spectrum. This unitary matrix is precisely the S-Matrix of the theory. The relation between the in and out spectra is given by $N^{\text{out}}(k) = S^\dagger N^{\text{in}}(k) S$. Using the explicit expression for the S-Matrix (1.29), we find

$$N^{\text{out}}_{++}(k) = \frac{N^{\text{in}}_{++}(k)}{1+e^{2\pi k}} + \frac{N^{\text{in}}_{--}(k)}{1+e^{-2\pi k}} \tag{4.2}$$

$$N^{\text{out}}_{--}(k) = \frac{N^{\text{in}}_{--}(k)}{1+e^{2\pi k}} + \frac{N^{\text{in}}_{++}(k)}{1+e^{-2\pi k}}\,, \tag{4.3}$$

where N^{in}_{++} and N^{in}_{--} are the in-going number densities from either side of the potential. We see that indeed the scattered pulse emerges with thermal factors $1+e^{\pm 2\pi k}$. For most of the radiated spectrum to actually be thermal, we see that N^{in}_{++} and N^{in}_{--} must be constant over a large range of energies. This was observed to be the case in the context of 2d string theory, starting from a coherent pulse, seen as an excitation over an appropriate Fermi-sea vacuum [197, 200, 201]. In our context, since we do not yet have a first principles construction of the appropriate second quantised theory, this in-state may be chosen. For instance, a simple pulse with a wide-rectangular shape would suffice. One may hope to create such a pulse microscopically, by going to the second quantised description and creating a coherent state. Alternatively, one may hope to realize a matrix quantum mechanics model that realizes a field theory in the limit of large number of particles. After all, we know that each oscillator in our model really corresponds to a partial wave and not a single particle in the four dimensional black hole picture.

Second Quantization v/s Matrix Quantum Mechanics Given the quantum mechanical model we have studied in this chapter, we may naively promote the wave-functions ψ_{lm} into fields to obtain a second quantized Lagrangian:

$$\mathcal{L} = \sum_{l,m} \int_{-\infty}^{\infty} du^\pm \, \psi^\dagger_{lm}(u^\pm, t) \left[i\partial_t + \frac{i}{2}\left(u^\pm \partial_{u^\pm} + \partial_{u^\pm} u^\pm\right) + \mu_l \right] \psi_{lm}(u^\pm, t) \, .$$

With a change of variables to go to Rindler coordinates,

$$\psi^{(\text{in/out})}_{lm}(\alpha^\pm, \rho^\pm, t) = e^{\rho^\pm/2} \psi_{lm}(u^\pm = \alpha^\pm e^{\rho^\pm}, t) \, ,$$

the Lagrangian becomes relativistic

$$\mathcal{L} = \sum_{l,m} \int_{-\infty}^{\infty} d\rho^\pm \sum_{\alpha^\pm = 1,2} \Psi^{\dagger (\text{in/out})}_{lm}(\alpha^\pm, \rho^\pm, t) \left(i\partial_t - i\partial_{\rho^\pm} + \mu_l\right) \Psi^{(\text{in/out})}_{lm}(\alpha^\pm, \rho, t) \, ,$$

where the label 'in' (out) corresponds to the + (−) sign. The form of the Lagrangian being first order in derivatives indicates that the Rindler fields are naturally fermionic. In this description we have a collection of different species of fermionic fields labelled by the $\{l, m\}$ indices. And the interaction between different harmonics would correspond to interacting fermions of the kind above. The conceptual trouble with this approach is that each "particle" to be promoted to a field is in reality a partial wave as can be seen from the four-dimensional picture. Therefore, second quantizing this model may not be straightforward [192]. It appears to be more appealing to think of each partial wave as actually arising from an N-particle matrix quantum mechanics model which in the large-N limit yields a second quantized description. Since N counts the number of degrees of freedom, it is naturally related to c via

$$\frac{1}{N^2} \sim c = \frac{8\pi G}{R^2} \sim \frac{l_P^2}{R^2} \, .$$

Therefore, N appears to count the truly microscopic Planckian degrees of freedom that the black hole is composed of. The collection of partial waves describing the Schwarzschild black hole would then be a collection of such N-particle matrix quantum mechanics models. Another possibility is to describe the total system in terms of a single matrix model but including higher representations/non-singlet states to describe the higher l modes. This seems promising because if one fixes the ground state energy of the lowest $l = 0$ (or $l = 1$ after antipodal) oscillator, the higher l oscillators have missing poles in their density of states compared to the $l = 0$, much similar to what was found for the adjoint and higher representations in [232]. Finally we note that we can combine the chemical potential with the oscillator Hamiltonian to get

$$\hat{H}_{\text{tot}} = \sum_{l,m} \left[\frac{1}{2} \left(\hat{p}_{lm}^2 - \hat{x}_{lm}^2 \right) + \frac{R^2}{8\pi G} \left(\hat{L}^2 + 1 \right) \right],$$

with $\hat{L}^2 = \sum_i \hat{L}_i^2$ giving the magnitude of angular momentum of each harmonic. One can then perform a matrix regularisation of the spherical harmonics following [233, 234] which replaces the spherical harmonics $Y_{lm}(\theta, \phi)$ with $N \times N$ matrices $\mathbb{Y}_{lm}$ where $l \leq N - 1$. This naturally sets a cut-off on the spherical harmonics from the onset. To sharpen any microscopic statements about the S-matrix, one might first need to derive an MQM model that regulates Planckian effects.

Chapter V

Exploring the principles of semi-holography

Field theory duals appear to count states corresponding to various black holes as we saw in Chapters I, II and III. On the other hand—as we saw in Chapter IV—dynamics of black holes appear to be better captured by their event horizons which were easier to study in the Schwarzschild background; where even the existence of an appropriate field theory dual is a challenging open question. Bottom-up holography provides for an interesting middle ground to bridge this gap. Several examples of renormalization group (RG) flows between fixed points (CFTs) have been studied in the literature. Studying these UV-IR flows at intermediate scales facilitates an ideal ground for two purposes. Firstly, it allows us to understand gravitational dynamics in various states (including the thermal ones corresponding to black holes) as radial evolutions in the bulk. After all, the physics of the horizon is expected to kick in, not arbitrarily close to the horizon, but at a scale that is set by the size of the horizon; there is emergent physics at this scale, if one were to approach it from the vacuum at infinity. Secondly, the converse problem of understanding QCD-like theories at intermediate scales using the bulk becomes more tractable via classical gravity equations, in the large N limit. As advertised in the Preface, we will largely focus on the field theoretic aspects in this chapter, using semi-holography.

The organisation of this chapter is as follows. In Section 1, we will review the present formulation of semi-holography and then argue for the need for generalising it in order for it to be an effective theory in a wide range of energy scales. In particular, we will advocate that we need a more democratic formulation where we do not give precedence to either the perturbative or to the non-perturbative degrees of freedom. Although we will call the perturbative sector as the ultraviolet sector and non-perturbative sector as the infrared sector, it is to be noted that non-perturbative effects are present even at high energy scales although these are suppressed. In principle, both sectors contribute at any energy scale although one of the sectors may give dominant contributions at a specific energy scale. Since semi-holography is a framework that is constructed at intermediate energy scales, it better treats both the ultraviolet and infrared sectors, or rather the perturbative and non-perturbative sectors in a democratic manner. Eventually the parameters of the non-perturbative sector should be determined (perhaps not always uniquely) by the perturbative sector or vice versa. We will argue this democratic formulation is actually necessary since otherwise we cannot perform non-perturbative renormalisation of the effective parameters. We will also sketch how the democratic formulation should work.

In Section 2, we will show how the requirement that there exists a local and conserved energy-momentum tensor constrains the effective parameters and semi-holographic coupling between the perturbative and the non-perturbative sectors. Thus we will realise a concrete democratic formulation of semi-holography at arbitrary energy scales.

In Section 3, we will illustrate the construction of semi-holography with a bi-holographic toy model in which the perturbative UV dynamics of semi-holography will be replaced by a strongly coupled holographic theory that admits a classical gravity description on its own. The infrared sector will be even more strongly coupled and also holographic. We will explicitly demonstrate the following features.

- Some simple consistency conditions can determine the hard-soft couplings between the two sectors and the parameters of the IR theory as functions of the parameters of the UV theory.

- The behaviour of the hard-soft couplings in the limit $\Lambda \to \infty$ is state-independent and can be obtained from the construction of the vacuum state. However, the running of the hard-soft couplings with the scale is state-dependent.

- The parameters defining the holographic IR classical gravity theory is fixed once and for all through the construction of the vacuum state of the full theory. However, the gravitational fields of this IR classical gravity theory undergo state-dependent field redefinitions in excited states.

- The UV and IR classical gravity theories are both sick in the sense that the respective geometries have edge singularities (not naked curvature singularities though) arising from geodesic incompleteness. The possibility of smooth gluing of their respective edges that removes the singularities in both plays a major role in determining the full theory.

We will also examine how we can define RG flow in the bi-holographic theory.

In Section 4, we will indicate how the steps of the construction of the bi-holographic toy theory can be applied also to the construction of the semi-holographic framework for QCD and also discuss the complications involved. Finally, we will conclude with discussions on the potential phenomenological applications of the bi-holographic framework.

1 Democratising semiholography

1.1 A brief review

Let us begin by sketching a first construction of a semi-holographic model for pure large N QCD based on a similar model [48, 55, 235] for the quark-gluon plasma (QGP) formed in heavy-ion collisions. The effective action for pure large N QCD at a scale Λ can be proposed to be:

$$S^{\mathrm{QCD}}[A^a_\mu, \Lambda] = S^{\mathrm{pQCD}}[A^a_\mu, \Lambda] + W^{\mathrm{hQCD}}\left[\tilde{g}_{\mu\nu}[A^a_\mu, \Lambda], \delta\tilde{g}_{\mathrm{YM}}[A^a_\mu, \Lambda], \tilde{\theta}[A^a_\mu, \Lambda]\right], \tag{1.1}$$

where the exact Wilsonian effective action of QCD denoted as S^{QCD} at a scale Λ is composed of two parts: (i) the perturbative QCD effective action S^{pQCD} at the scale Λ obtained from Feynman diagrams, and (ii) non-perturbative terms (leading to confinement) which cannot be obtained from Feynman diagrams but can be described by an emergent holographic strongly coupled QCD-like theory. The latter part of the full action is then given by W^{hQCD}, the generating functional of the connected correlation functions of the emergent strongly coupled holographic QCD-like theory whose marginal couplings—namely $\tilde{g}_{\mathrm{YM}}$ and $\tilde{\theta}$ (or rather, their expansions around infinity and zero respectively) and the effective background metric $\tilde{g}_{\mu\nu}$ in which it lives are functionals of the perturbative gauge fields A^a_μ and the scale Λ. In order that a holographic theory can capture non-perturbative effects at even high energy scales, it must have a large number of fields as we will discuss in Section 4. Nevertheless, at high energy scales the non-perturbative contributions are insignificant. We will argue that the semi-holographic construction can be useful at intermediate energy scales where the non-perturbative effects can also be captured by a few gravitational field via holography to a good degree of approximation.

It is to be noted that W^{hQCD} should be defined with an appropriate vacuum subtraction so that it vanishes when the modifications in the couplings $\delta\tilde{g}_{\mathrm{YM}}$ and $\tilde{\theta}$ vanish, and when[1] $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$. Asymptotically when $\Lambda \to \infty$, the sources for the emergent holographic QCD are also expected to vanish, so that the full action receives perturbative contributions almost exclusively. In the infra-red, however, the holographic contributions are expected to dominate.

In the large N limit, the emergent holographic QCD is expected to be described by a

[1] More generally, the subtraction should ensure that W^{hQCD} vanishes when $\tilde{g}_{\mu\nu}$ is identical to the fixed background metric where all the degrees of freedom live.

classical gravitational theory. Therefore,

$$W^{\mathrm{hQCD}}\Big[\tilde{g}_{\mu\nu}[A^a_\mu], \delta\tilde{g}_{\mathrm{YM}}[A^a_\mu], \tilde{\theta}_{\mathrm{YM}}[A^a_\mu]\Big] = S^{\mathrm{on-shell}}_{grav}\Big[\tilde{g}_{\mu\nu} = g^{(\mathrm{b})}_{\mu\nu}, \delta\tilde{g}_{\mathrm{YM}} = \phi^{(b)}, \tilde{\theta} = \chi^{(\mathrm{b})}\Big], \quad (1.2)$$

i.e. W^{hQCD} is to be identified with the on-shell action $S^{\mathrm{on-shell}}_{grav}$ of an appropriate five-dimensional classical gravity theory consisting of *at least* three fields, namely the metric G_{MN}, the dilaton Φ and the axion $\mathcal{X}$. Furthermore, the leading behaviour of the bulk metric is given by its identification with the boundary metric $g^{(\mathrm{b})}_{\mu\nu}$, whilst $\delta\tilde{g}_{\mathrm{YM}}$ is identified with the boundary value $\phi^{(\mathrm{b})}$ of the bulk dilaton Φ and $\tilde{\theta}$ is identified with the boundary value $\chi^{(\mathrm{b})}$ of the bulk axion $\mathcal{X}$. For reasons that will soon be elucidated, one may now postulate that:

$$g^{(\mathrm{b})}_{\mu\nu} = \eta_{\mu\nu} + \gamma t^{\mathrm{pQCD}}_{\mu\nu}, \text{ with } t^{\mathrm{pQCD}}_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S^{\mathrm{pQCD}}[A^a_\mu, \Lambda]}{\delta g_{\mu\nu}}\Big|_{g_{\mu\nu}=\eta_{\mu\nu}} \quad (1.3a)$$

$$\phi^{(\mathrm{b})} = \beta h^{\mathrm{pQCD}}, \quad \text{with} \quad h^{\mathrm{pQCD}} = \frac{\delta S^{\mathrm{pQCD}}[A^a_\mu, \Lambda]}{\delta g^{\mathrm{YM}}[\Lambda]}, \quad (1.3b)$$

$$\chi^{(\mathrm{b})} = \alpha a^{\mathrm{pQCD}}, \quad \text{with} \quad a^{\mathrm{pQCD}} = \frac{\delta S^{\mathrm{pQCD}}[A^a_\mu, \Lambda]}{\delta\theta}. \quad (1.3c)$$

The couplings α, β and γ have been called hard-soft couplings. These of course cannot be new independent parameters, but rather of the functional forms $(1/\Lambda^4)f(\Lambda_{\mathrm{QCD}}/\Lambda)$ which should be derived from first principles. Furthermore, $f(0)$ must be finite so that the non-perturbative contributions to the full action vanish in the limit $\Lambda \to \infty$ reproducing asymptotic freedom. If S^{pQCD} were just the classical Yang-Mills action, then [55]:

$$\begin{aligned} t^{\mathrm{pQCD}}_{\mu\nu} &= \frac{1}{N_c}\mathrm{tr}\Big(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}\eta_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\Big), \\ h^{\mathrm{pQCD}} &= \frac{1}{4N_c}\mathrm{tr}\Big(F_{\alpha\beta}F^{\alpha\beta}\Big), \\ a^{\mathrm{pQCD}} &= \frac{1}{4N_c}\mathrm{tr}\Big(F_{\alpha\beta}\tilde{F}^{\alpha\beta}\Big). \end{aligned} \quad (1.4)$$

It can be readily shown that in consistency with the variational principle, the full semi-holographic action (1.1) can be written in the following form:

$$S[A^a_\mu, \Lambda] = S^{\mathrm{pQCD}}[A^a_\mu, \Lambda] + \frac{1}{2}\int \mathrm{d}^4x\, \overline{\mathcal{T}}^{\mu\nu} g^{(b)}_{\mu\nu} + \int \mathrm{d}^4x\, \overline{\mathcal{H}}\phi^{(\mathrm{b})} + \int \mathrm{d}^4x\, \overline{\mathcal{A}}\chi^{(b)}, \quad (1.5)$$

where

$$\overline{\mathcal{T}}^{\mu\nu} = 2\frac{\delta S_{grav}^{\text{on-shell}}}{\delta g_{\mu\nu}^{(\text{b})}}, \quad \overline{\mathcal{H}} = \frac{\delta S_{grav}^{\text{on-shell}}}{\delta \phi^{(\text{b})}}, \quad \overline{\mathcal{A}} = \frac{\delta S_{grav}^{\text{on-shell}}}{\delta \chi^{(\text{b})}}, \tag{1.6}$$

are the self-consistent expectation values of the holographic operators which are non-linear and non-local functionals of the sources[2] $g_{\mu\nu}^{(\text{b})}$, $\phi^{(\text{b})}$ and $\chi^{(\text{b})}$.

The reason for postulating the hard-soft interactions to be of the forms given by equations (1.3a) to (1.3c) can now be readily explained. We need to solve for the full dynamics in an iterative fashion (assuming that the iteration indeed converges). This means that the dynamics of the perturbative sector is modified by the holographic operators which appear as self-consistent mean fields as in (1.5). The holographic operators are in turn obtained by solving classical gravity equations with sources given by (1.3a), (1.3b) and (1.3c), which are determined by the perturbative gauge fields. It should therefore be guaranteed that both sectors must be solvable at each step in the iteration including perturbative quantum effects. Therefore, both the perturbative and non-perturbative sectors should be renormalizable at each step of the iteration so that one can solve for the dynamics of both without introducing any new coupling. The modified perturbative action (1.5) in the limit $\Lambda \to \infty$ is indeed a *marginal* deformation of the standard perturbative QCD action since the added terms involve $t_{\mu\nu}^{\text{cl}}$, h^{cl} and a^{cl} which are all possible (scalar and tensor) gauge-invariant operators of mass dimension four. Furthermore, this is also why the gravitational theory contains sources for only the dimension four operators as in (1.2), despite the possible existence of many other (massive) gravitational fields generating (non-perturbative) condensates of higher dimensional operators without additional sources.

Finally, it is important to reiterate that the importance of the hard-soft couplings given in (1.3a), (1.3b) and (1.3c) relies on the emergence of an intermediate scale $\Lambda_{\text{I}} > \Lambda_{\text{QCD}}$ between the energy scales where we can rely exclusively on either perturbative QCD or chiral Lagrangian effective field theories which can be reproduced from holographic models such as [37–39]. This intermediate scale Λ_{I} is most likely where the perturbative gauge coupling is of order unity but not too large. At this intermediate scale itself, the hard-soft couplings should give significant modifications to perturbative dynamics.

In the context of an application to QGP, S^{pQCD} can be replaced by the glasma effective action, i.e. a classical Yang-Mills action for the small-x saturated gluons (which form a weakly coupled over-occupied system) with colour sources provided by the large-x[3] ($x >$

[2] It is assumed here that the full theory lives in flat Minkowski space with metric $\eta_{\mu\nu}$. It is easy to generalise the construction to any metric on which the full degrees of freedom live. For details, see [55].

[3] Here x denotes the fraction of hadronic longitudinal momentum carried by the partonic gluon.

x_0) gluons (frozen on the time-scale of the collisions) with a distribution function whose evolution with the cut-off x_0 can be followed perturbatively [236–238]. In this case, one can show that (i) the full system has a well-defined local energy momentum tensor that is conserved in flat space, and (ii) the iterative method of solving the full dynamics indeed converges at least in some simple test cases [55].

1.2 Why and how should we democratise semi-holography?

The present formulation of semi-holography as discussed above makes two central assumptions which are:

1. The perturbative sector and the holographic theory dual to the non-perturbative sector are both deformed marginally with scalar and tensorial couplings which are functionals of the operators of the other sector.
2. The full action (as for instance (1.1) in the case of pure large-N QCD) can be written as a functional of the perturbative fields only.[4]

In what follows, we will argue that if semi-holography needs to work in the sense of a non-perturbative effective framework, we must replace the second assumption with the simple assertion that:

- The full theory must have a conserved and local energy-momentum tensor which can be constructed without the need to know the explicit Lagrangian descriptions of the effective ultraviolet or infrared dynamics.

It is to be noted that the formulation discussed above already leads to a local and conserved energy-momentum tensor of the combined system which can be constructed explicitly. As shown in [55], this can be obtained from the complete action (1.1) by differentiating it with respect to the fixed background $g_{\mu\nu}$ on which the full system lives. One can prove that this energy-momentum tensor is conserved in the fixed background $g_{\mu\nu}$ when the full system is solved, i.e when the Ward identity of the CFT in the dynamical background $\tilde{g}_{\mu\nu}$ and the modified dynamical equations of the perturbative sector are satisfied simultaneously. *The crucial point is that we must not insist that we can derive the full energy-momentum tensor from an action such as* (1.1) *which is a functional of the ultraviolet fields only. We will argue that such a demand automatically arises from a democratic formulation of semi-holography where the full energy-momentum tensor can be constructed directly from the Ward identities of the perturbative and non-perturbative*

[4] Note that W^{hQCD} in (1.1) is after all a functional of the sources alone which in turn are functionals of the perturbative gauge fields. Furthermore, the term *perturbative fields* may also seem problematic because after their coupling with the non-perturbative sector, these do not remain strictly perturbative. Nevertheless, if we solve the two sectors via iteration as discussed before, we can treat the dynamics of these fields perturbatively (with self-consistently modified couplings) at each step of the iteration as discussed before.

sectors in respective background metrics and sources that are determined by the operators of the other sector. This full energy-momentum tensor should be conserved locally in the fixed background metric where the full system lives.

We lay out our reasons for advocating democratic formulation of semi-holography. The first one is somewhat philosophical. In the case we are assuming that the strongly coupled non-perturbative sector is described by a holographically dual classical gravity theory, we are not making any explicit assumption of the Lagrangian description of this dual holographic theory. It could be so in certain situations we do not know the explicit Lagrangian description of the perturbative sector as well.[5] We should be able to formulate an explicit semi-holographic construction in such a situation also. Furthermore, the basic idea of semi-holography is to take advantage of dualities. It is quite possible that the ultraviolet is strongly coupled instead of the infrared and we should take advantage of a weakly coupled (perhaps holographic) dual description of the ultraviolet. In that case, the original Lagrangian description of the UV even if known will not be useful. Therefore, we better have a broader construction which can work without the need of knowing an explicit Lagrangian description of the perturbative sector or that of the holographic theory dual to the non-perturbative sector. This implies we need to treat both on equal footing.

The second reason for advocating democratic formulation is more fundamental. Let us take the example where in the infrared the theory flows to a strongly coupled holographic conformal field theory (IR-CFT) from a weakly coupled fixed point (UV-CFT) in the ultraviolet. There will be a specific UV-IR operator map in such a theory relating operators in the UV fixed point to those defined at the IR fixed point via scale-evolution as explicitly known in the case of RG flows between minimal-model two-dimensional conformal field theories [240–242]. Typically a relevant operator in the UV will flow to an irrelevant operator in the IR. In fact the entire flow will be generated by a relevant deformation of the UV fixed point – the operator(s) generating such a deformation will become irrelevant at the IR fixed point. Let the UV deformation be due to a coupling constant g multiplying a relevant operator O^{UV} which will flow to an irrelevant operator O^{IR} of the strongly coupled holographic IR-CFT that is represented by a bulk field Φ. A naive way to formulate semi-holography in such a case will be:

$$S = S^{\mathrm{UV-CFT}} + g \int \mathrm{d}^4x\, O^{\mathrm{UV}} + S^{\mathrm{grav}}[\phi^{(\mathrm{b})} = \lambda O^{\mathrm{UV}}], \tag{1.7}$$

[5]Note that by assuming that we have a weakly coupled perturbative description we do not necessarily commit ourselves to a Lagrangian description also. Such a perturbation series can be obtained by a chain of dualities without a known Lagrangian description as in the case of some quiver gauge theories [239].

where $\phi^{(b)}$ is the coefficient of the leading asymptotic term of the bulk field Φ, λ is a dimensionful hard-soft coupling constant and we have suppressed the scale dependence. The immediate problem is that we have turned on an irrelevant deformation of the dual holographic theory as

$$S^{\text{grav}}[\phi^{(b)} = \lambda O^{\text{UV}}] = \lambda \int \mathrm{d}^4x\, O^{\text{UV}} O^{\text{IR}} \tag{1.8}$$

represents an irrelevant deformation of the IR-CFT since O^{IR} is an irrelevant operator. This contradicts our assumption that both sectors should be renormalizable after being mutually deformed by the operators of the other. Furthermore, turning on non-trivial sources of irrelevant operators leads to naked gravitational singularities in holography and the removal of asymptotic anti-de Sitter behaviour of the spacetime. We can therefore argue that the formulation of the full semi-holographic theory in terms of an action which is a functional of the UV variables as in (1.7) should be abandoned as it leads to such a contradiction. As we will show later in the democratic formulation, we will be able to generate a non-trivial expectation value $\langle O^{\text{IR}} \rangle$ without sourcing it.

Similarly, in the case of QCD where the infrared dynamics has a mass gap, non-perturbative vacuum condensates of operators of high mass dimensions are crucial for the cancellation of renormalon Borel poles of perturbation series [243–245]. This implies that we need to couple irrelevant operators of the infrared holographic theory with the gauge-invariant marginal operators of perturbative QCD as we will discuss in Section 4. This is not quite possible within the present formulation for the same reasons mentioned above.

Let us then sketch how the democratic formulation should be set-up. Let us denote $S^{(1)}$ as the quantum effective perturbative action and $S^{(2)}$ as the quantum effective action of the holographic theory dual to the non-perturbative sector both defined at the same energy scale Λ. Furthermore, for simplicity let us assume that the two sectors couple via their energy-momentum tensors and a scalar operator in each sector. The democratic formulation postulates that the individual actions are deformed as follows:[6]

$$S^{(1)} = S^{(1)}[g^{(1)}_{\mu\nu}, J^{(1)}], \quad S^{(2)} = S^{(2)}[g^{(2)}_{\mu\nu}, J^{(2)}] \tag{1.9}$$

[6]Note that we have not turned on sources for the elementary fields. Therefore, $S^{(i)}$ can denote either W, the generating functional of the connected correlation functions, or Γ, the 1-PI (one-particle irreducible) effective action, which is the Legendre transform of W. In absence of sources for elementary fields, $W = \Gamma$. Below, the effective actions of both theories have been defined in specific background metrics and with specific couplings (denoted as $J^{(i)}$ and corresponding to specific composite operator vertices) which are functionals of the operators of the two sectors. These effective actions can be defined even when the Lagrangian descriptions (i.e. representations of the two sectors via some elementary fields) are unknown.

with

$$\begin{aligned} g^{(i)}_{\mu\nu} &= \eta_{\mu\nu} + h^{(i)}_{\mu\nu}[t^{(i)}_{\mu\nu}, O^{(i)}], \\ J^{(i)} &= F^{(i)}[t^{(i)}_{\mu\nu}, O^{(i)}], \end{aligned} \tag{1.10}$$

where i denotes 1, 2. This means that the two sectors couple only via their effective sources and background metrics. Furthermore *we should not allow redundant dependencies* meaning that $h^{(1)}_{\mu\nu}$ and $F^{(1)}$ can depend on $t^{(1)}_{\mu\nu}$ and $O^{(1)}$ only such that when $t^{(2)}_{\mu\nu} = O^{(2)} = 0$, then it should also follow that $h^{(1)}_{\mu\nu} = F^{(1)} = 0$. The aims will be:

1. To determine the functional forms of $h^{(i)}_{\mu\nu}$ and $F^{(i)}$ by requiring the existence of a local energy-momentum tensor of the full theory conserved in the background metric $\eta_{\mu\nu}$ where the full theory lives and disallowing redundant dependencies, and
2. To determine the theory $S^{(2)}$ and the hard-soft coupling constants appearing in $h^{(i)}_{\mu\nu}$ and $F^{(i)}$ as functions of the parameters of the perturbative sector, i.e. the parameters in $S^{(1)}$.

Remarkably, we will see in the following section that the requirement of the existence of a local and conserved energy-momentum tensor of the full system along with some other simple assumptions constrains the functional forms of $h^{(i)}_{\mu\nu}$ and $F^{(i)}$ such that we can only have a few possible hard-soft coupling constants relevant for physics (including non-perturbative effects) at given energy scales. The scale ($\Lambda-$)dependence of $h^{(i)}_{\mu\nu}$ and $F^{(i)}$ should be only through the hard-soft coupling constants. *If the operator $O^{(2)}$ in the holographic theory dual to the infrared is an irrelevant operator then we should demand $J^{(2)} = 0$. The functional form of $F^{(2)}$ will then play a major role in determining how the hard-soft coupling constants and parameters of the holographic theory (i.e. parameters in $S^{(2)}$) are determined by the parameters in $S^{(1)}$.* In order to demonstrate how we can achieve the second task of determining the hard-soft coupling constants and the parameters in $S^{(2)}$ in principle, we will construct a toy model in Section 3. Later in Section 4 we will outline how we can achieve this in the case of QCD. This of course will be a difficult problem in practice, and therefore we will postpone this to the future.

As will be clear in the next section, even if we can choose an arbitrary background metric $g_{\mu\nu}$ instead of $\eta_{\mu\nu}$ for the full system, we can construct the combined local and conserved energy-momentum tensor. Furthermore, it will be trivial to generalise the construction to the case where there are multiple relevant/marginal scalar operators in the perturbative sector. We will not consider the case when the perturbative sector has relevant/marginal vector operators and tensor operators other than the energy-momentum tensor. We will postpone such a study to the future. The phenomenological semi-holographic constructions discussed in the previous subsection will turn out to be special cases of the more general scenario to be described below.

2 Coupling the hard and soft sectors

In what follows, we will study how we can determine the most general form of couplings between the hard and soft sectors following (1.9) and (1.10) such that there exists a local and conserved energy-momentum tensor of the full system in the fixed background metric. We will not assume any Lagrangian description of either sector in terms of elementary quantum fields and therefore we will only use the local Ward identities of each sector. Furthermore, we will disallow redundant dependencies in the coupling functions which simply redefine the respective effective UV and IR theories as discussed above.

For the moment, we will assume that the UV theory (perturbative sector) has one relevant operator $O^{(1)}$ which couples to a relevant/marginal/irrelevant operator $O^{(2)}$ in the IR theory (the holographic non-perturbative sector). Furthermore, we should also take into account the energy-momentum tensor operators $T^{(1)\mu\nu}$ and $T^{(2)\mu\nu}$ in the coupling of the UV and IR theories. As mentioned earlier, it will be clear later how we can generalise our results to the case of multiple relevant and marginal scalar operators in the perturbative sector each coupling to multiple operators in the non-perturbative sector.

2.1 Simple scalar couplings

The simplest possible consistent coupling of the UV and IR theories leading to a conserved local energy-momentum tensor of the full system is given by:

$$g^{(1)}_{\mu\nu} = g^{(2)}_{\mu\nu} = g_{\mu\nu}, \quad J^{(1)} = \alpha_0 O^{(2)}, \quad J^{(2)} = \alpha_0 O^{(1)}. \tag{2.1}$$

Above α_0 is once again a scale-dependent dimensionful hard-soft coupling constant. In the case we have a fixed point both in the UV and in the IR, we can postulate

$$\alpha = A_0 \frac{1}{\Lambda_{\mathrm{I}}^{\Delta^{\mathrm{UV}}+\Delta^{\mathrm{IR}}-d}} + \frac{1}{\Lambda^{\Delta^{\mathrm{UV}}+\Delta^{\mathrm{IR}}-d}}\, f\left(\frac{\Lambda}{\Lambda_{\mathrm{I}}}\right), \tag{2.2}$$

where A_0 is a dimensionless constant, Δ^{UV} is the scaling dimension of the UV operator $O^{(1)}$ at the UV fixed point, Δ^{IR} is the scaling dimension of the IR operator $O^{(2)}$ at the IR fixed point and d is number of spacetime dimensions. Furthermore, Λ_{I} is an emergent intermediate energy-scale (but different from Λ_{QCD} in case of QCD). If the UV and/or IR limits are not conformal, then the $\Lambda-$dependence of α_0 should be more complicated. At present we will not bother about $\Lambda-$dependence of the hard-soft couplings although we should keep in mind that the effective $\Lambda-$dependence of the couplings of the two sectors arises via them.

Both the UV and IR theories will have their respective Ward identities:

$$\nabla_\mu T^{(1)\mu}{}_{\nu} = O^{(1)}\nabla_\nu J^{(1)} \qquad \text{and} \qquad \nabla_\mu T^{(2)\mu}{}_{\nu} = O^{(2)}\nabla_\nu J^{(2)}, \tag{2.3}$$

where ∇ is the covariant derivative constructed in the fixed background metric $g_{\mu\nu}$, $T^{(1)\mu}{}_{\nu} = T^{(1)\mu\rho} g_{\rho\nu}$, etc. These identities will be satisfied once we have solved the full dynamics self-consistently.

It is clear that these Ward identities together with $J^{(1)}$ and $J^{(2)}$ specified via (2.1) imply the existence of $T^{\mu}{}_{\nu}$ defined as

$$T^{\mu}{}_{\nu} := T^{(1)\mu}{}_{\nu} + T^{(2)\mu}{}_{\nu} - \alpha O^{(1)} O^{(2)} \delta^{\mu}{}_{\nu} \tag{2.4}$$

which satisfies the combined Ward identity

$$\nabla_{\mu} T^{\mu}{}_{\nu} = 0. \tag{2.5}$$

Therefore, $T^{\mu\nu} = T^{\mu}{}_{\rho} g^{\rho\nu}$ can be identified with the local conserved energy-momentum tensor of the full system. *The crucial point is that the forms of the sources specified via* (2.1) *imply that the respective Ward identities* (2.3) *add to form a total derivative and thus results in a conserved energy-momentum tensor* (2.4) *for the full system.*

Of course, we can make other choices for (2.1). One example is

$$g^{(1)}_{\mu\nu} = g^{(2)}_{\mu\nu} = g_{\mu\nu}, \; J^{(1)} = \alpha O^{(2)} + \frac{1}{2}\tilde{\alpha}_1 O^{(1)}, \; J^{(2)} = \alpha O^{(1)} + \frac{1}{2}\tilde{\alpha}_2 O^{(2)}. \tag{2.6}$$

In this case, we would have obtained

$$T^{\mu}{}_{\nu} = T^{(1)\mu}{}_{\nu} + T^{(2)\mu}{}_{\nu} - \left(\alpha O^{(1)} O^{(2)} - \tilde{\alpha}_1 {O^{(1)}}^2 + \tilde{\alpha}_2 {O^{(2)}}^2\right) \delta^{\mu}{}_{\nu}. \tag{2.7}$$

This would have led to redundancies as $\tilde{\alpha}_i$ simply lead to redefinitions of the UV and IR theories. Therefore, without loss of generality we will not allow such parameters. Another possibility is:

$$g^{(1)}_{\mu\nu} = g^{(2)}_{\mu\nu} = g_{\mu\nu}, \quad J^{(1)} = \alpha_1 {O^{(2)}}^2 O^{(1)}, \quad J^{(2)} = \alpha_1 {O^{(1)}}^2 O^{(2)}, \tag{2.8}$$

with α_1 being an appropriate scale-dependent constant. In this case,

$$T^{\mu}{}_{\nu} = T^{(1)\mu}{}_{\nu} + T^{(2)\mu}{}_{\nu} - \frac{3}{2}\alpha_1 \left(O^{(1)} O^{(2)}\right)^2 \delta^{\mu}{}_{\nu}. \tag{2.9}$$

In fact, one can more generally choose[7]

$$g^{(1)}_{\mu\nu} = g^{(2)}_{\mu\nu} = g_{\mu\nu},$$

[7] It is easy to see that the general class of such simple scalar couplings should be such that $O^{(1)}[J^{(1)}, J^{(2)}]\, \mathrm{d}J^{(1)} + O^{(2)}[J^{(1)}, J^{(2)}]\, \mathrm{d}J^{(2)}$ should be a total differential in which we have inverted the functions $J^{(1)}[O^{(1)}, O^{(2)}]$ and $J^{(2)}[O^{(1)}, O^{(2)}]$ to obtain $O^{(1)}[J^{(1)}, J^{(2)}]$ and $O^{(2)}[J^{(1)}, J^{(2)}]$. If $J^{(1)}[O^{(1)}, O^{(2)}]$ and $J^{(2)}[O^{(1)}, O^{(2)}]$ are analytic in $O^{(1)}$ and $O^{(2)}$ at $O^{(1)} = O^{(2)} = 0$, we obtain the general expressions below.

$$J^{(1)} = \sum_{k=0}^{\infty} \alpha_k {O^{(2)}}^{k+1} {O^{(1)}}^{k}, \tag{2.10}$$
$$J^{(2)} = \sum_{k=0}^{\infty} \alpha_k {O^{(1)}}^{k+1} {O^{(2)}}^{k}.$$

and this will lead to a conserved energy-momentum tensor for the full system given by

$$T^{\mu}{}_{\nu} = T^{(1)\mu}{}_{\nu} + T^{(2)\mu}{}_{\nu} - \sum_{k=0}^{\infty} \frac{2k+1}{k+1} \alpha_k \left(O^{(1)}O^{(2)}\right)^{k+1} \delta^{\mu}{}_{\nu}. \tag{2.11}$$

satisfying:

$$\nabla_{\mu} T^{\mu}{}_{\nu} = 0. \tag{2.12}$$

Our general expectation is that in QCD, $\langle O^{(i)} \rangle \approx \Lambda_{\rm QCD}^{\kappa}$ whereas $\alpha_k \approx \Lambda_{\rm I}^{\kappa'}$ (for appropriate κ and κ'), where $\Lambda_{\rm I}$ is a state-dependent scale such that $\Lambda_{\rm QCD} \ll \Lambda_{\rm I}$. In case of the vacuum, $\Lambda_{\rm I}$ could be the scale where the strong coupling is order unity (i.e. neither too small nor too large) and in case of QGP formed in heavy ion collisions $\Lambda_{\rm I}$ could be the saturation scale. If this assumption is true, one can make a useful truncation in k in (2.10), as the terms neglected will be suppressed by higher powers of $\Lambda_{\rm QCD}/\Lambda_{\rm I}$. In this case, semi-holography will turn out to be an useful effective non-perturbative framework. Of course the hard-soft couplings α_ks and the condensates $\langle O^{(i)} \rangle$s are both scale-dependent. However, as long as their scale dependence do not spoil the above justification for the truncation of terms that appear in the couplings of the two sectors, semi-holography can be used as an effective framework at least for a class of processes.

It is to be noted that only in the simplest case, i.e. when $\alpha_k = 0$ for $k \neq 0$, we may be able to reproduce the full energy-momentum tensor (2.11) and the sources (2.10) from an action. In this case, the action is given by:

$$S = S^{(1)} + S^{(2)} + \alpha_0 \int d^d x \, O^{(1)} O^{(2)}. \tag{2.13}$$

For other cases, one can reproduce the energy-momentum tensor but cannot reproduce the right sources. However, even in the case $\alpha_k = 0$ for $k \neq 0$, the action (2.13) can reproduce the energy-momentum tensor only when $O^{(1)}$ and $O^{(2)}$ are composites of elementary scalar fields with no derivatives involved. This observation has been made earlier in [246] in a different context. The lesson is that an action of the form (2.13) does not exist in the general semi-holographic formulation of non-perturbative dynamics. In fact, if such an action existed, it would have been problematic as it would have implied doing a naive path integral over both UV and IR fields. This would not have been desirable because the IR degrees of freedom are *shadows* of the UV degrees of freedom in the sense that they do not have independent existence. After all, the IR theory and hard-soft couplings should be determined by the coupling constants of perturbation theory governing the UV dynamics. Since the hard-soft couplings are state-dependent, these and the parameters of the IR theory, generally speaking, should be determined by how perturbative dynamics describe the state or rather participate in the process being measured. We will examine this feature in our toy model illustration in the following section.

2.2 More general scalar couplings

So far, we have considered the cases in which the effective background metrics for the UV and IR theories are identical to the fixed background metric $g_{\mu\nu}$ in which all degrees of freedom live. Here, we will examine the cases when the effective background metrics $g^{(1)}_{\mu\nu}$ and $g^{(2)}_{\mu\nu}$ of the UV and IR theories are different and have operator-dependent scale factors. In this case, we will consider

$$\begin{aligned} \tilde{g}^{(1)}_{\mu\nu} &= g_{\mu\nu} e^{2\sigma^{(1)}[O^{(i)}, T^{(i)}]}\,, \\ \tilde{g}^{(2)}_{\mu\nu} &= g_{\mu\nu} e^{2\sigma^{(2)}[O^{(i)}, T^{(i)}]}\,, \\ J^{(i)} &= J^{(i)}[O^{(j)}, T^{(j)}]\,. \end{aligned} \tag{2.14}$$

Above, $T^{(i)} \equiv T^{(i)\mu\nu} g^{(i)}_{\mu\nu}$ is the trace of the energy-momentum tensors in the respective effective background metrics. Once again, we will disallow redundant dependencies of the sources on the operators.

Let us first establish a useful identity. The Ward identity for the local conservation of energy and momentum in the background metric $\tilde{g}_{\mu\nu} = g_{\mu\nu} e^{2\sigma}$, i.e.

$$\tilde{\nabla}_\mu T^\mu{}_\nu = O \tilde{\nabla}_\nu J, \tag{2.15}$$

where $\tilde{\nabla}$ is the covariant derivative constructed from $\tilde{g}$ can be rewritten as

$$\nabla_\mu \Big(T^\mu{}_\nu e^{d\sigma} \Big) - \frac{1}{d} (\mathrm{Tr}\, T)\, \nabla_\nu e^{d\sigma} - e^{d\sigma} O \nabla_\nu J = 0\,, \tag{2.16}$$

where ∇ is built out of $g_{\mu\nu}$ and d is the number of spacetime dimensions. The Ward identity in this form will be useful for the construction of the energy-momentum tensor of the full system which should be locally conserved in the background $g_{\mu\nu}$.

The general consistent scalar-type couplings which give rise to a conserved energy-momentum tensor of the full system then are of the form:

$$e^{d\sigma^{(1)}} = 1 + d\beta \Big(T^{(2)} + O^{(2)} \Big), \tag{2.17a}$$

$$
\begin{aligned}
e^{d\sigma^{(2)}} &= 1 + d\beta\Big(T^{(1)} + O^{(1)}\Big), & \text{(2.17b)}\\
J^{(1)} &= \frac{1}{d}\ln\Big(1 + d\beta\Big(T^{(2)} + O^{(2)}\Big)\Big)\\
&\quad + \sum_{k=0}^{\infty}\alpha_k O^{(2)k+1}\Big(1 + d\beta\Big(T^{(1)} + O^{(1)}\Big)\Big)^{k+1} O^{(1)k}\Big(1 + d\beta\Big(T^{(2)} + O^{(2)}\Big)\Big)^{k}, & \text{(2.17c)}\\
J^{(2)} &= \frac{1}{d}\ln\Big(1 + d\beta\Big(T^{(1)} + O^{(1)}\Big)\Big)\\
&\quad + \sum_{k=0}^{\infty}\alpha_k O^{(1)k+1}\Big(1 + d\beta\Big(T^{(2)} + O^{(2)}\Big)\Big)^{k+1} O^{(2)k}\Big(1 + d\beta\Big(T^{(1)} + O^{(1)}\Big)\Big)^{k}. & \text{(2.17d)}
\end{aligned}
$$

Clearly when $\beta = 0$ we revert back to the case (2.10) discussed in the previous subsection.

Using (2.16), we can rewrite the Ward identities in the respective UV and IR theories in the form:

$$
\begin{aligned}
\nabla_\mu\Big(T^{(1)\mu}{}_{\nu} e^{d\sigma^{(1)}}\Big) - \frac{1}{d}T^{(1)}\,\nabla_\nu e^{d\sigma^{(1)}} - e^{d\sigma^{(1)}}O^{(1)}\nabla_\nu J^{(1)} &= 0,\\
\nabla_\mu\Big(T^{(2)\mu}{}_{\nu} e^{d\sigma^{(2)}}\Big) - \frac{1}{d}T^{(2)}\,\nabla_\nu e^{d\sigma^{(1)}} - e^{d\sigma^{(2)}}O^{(2)}\nabla_\nu J^{(2)} &= 0, \qquad \text{(2.18)}
\end{aligned}
$$

where $T^{(i)\mu}{}_{\nu} \equiv T^{(i)\mu\nu} g^{(i)}_{\mu\nu}$. Substituting (2.14), and then (2.17a), (2.17b), (2.17c) and (2.17d) in the above equations, we find that

$$
\begin{aligned}
T^{\mu}{}_{\nu} &= T^{(1)\mu}{}_{\nu}\Big(1 + d\beta\Big(T^{(2)} + O^{(2)}\Big)\Big) + T^{(2)\mu}{}_{\nu}\Big(1 + d\beta\Big(T^{(1)} + O^{(1)}\Big)\Big)\\
&\quad - \beta\Big(T^{(1)} + O^{(1)}\Big)\Big(T^{(2)} + O^{(2)}\Big)\delta^{\mu}{}_{\nu}\\
&\quad - \sum_{k=0}^{\infty}\frac{2k+1}{k+1}\alpha_k\Big[O^{(1)}O^{(2)}\Big(1 + d\beta\Big(T^{(1)} + O^{(1)}\Big)\Big)\\
&\qquad\qquad \Big(1 + d\beta\Big(T^{(2)} + O^{(2)}\Big)\Big)\Big]^{k+1}\delta^{\mu}{}_{\nu}, \qquad \text{(2.19)}
\end{aligned}
$$

satisfies the combined Ward identity:

$$
\nabla_\mu T^{\mu}{}_{\nu} = 0. \qquad \text{(2.20)}
$$

Therefore, $T^{\mu\nu} \equiv T^{\mu}{}_{\rho} g^{\rho\nu}$ is the energy-momentum tensor of the combined system.

For each pair of scalar operators in the UV and IR theories, we can then have the more general scalar couplings β and α_k. It should already be evident at this stage how the existence of a local energy-momentum tensor of the full system restricts the hard-soft couplings via the functional forms of the effective sources and background metrics. In practice, for a wide range of energy scales, we should only require a finite number of hard-soft couplings for reasons mentioned previously.

2.3 Tensorial couplings

To explore how the effective background metric can be tensorially modified as opposed to being modified by an overall scale factor, we will exploit another identity. Let $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ be two different metric tensors such that one can be smoothly deformed to the other and $z^{\mu}{}_{\nu} \equiv g^{\mu\rho}\tilde{g}_{\rho\nu}$. Then we can show that the Levi-Civita connections Γ constructed from g and $\tilde{\Gamma}$ constructed from $\tilde{g}$ are related by the identity:

$$\tilde{\Gamma}^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} + \frac{1}{2}\left(\nabla_{\mu}\,(\ln\,\mathrm{z})^{\rho}{}_{\nu} + \nabla_{\nu}\,(\ln\,\mathrm{z})^{\rho}{}_{\mu} - \nabla^{\rho}\,(\ln\,\mathrm{z})_{\mu\nu}\right). \tag{2.21}$$

In order to prove this, one can substitute $\tilde{g}_{\mu\nu}$ by $g_{\mu\nu} + \delta g_{\mu\nu}$ in the above equation, expand both sides of the equation in $\delta g_{\mu\nu}$, and finally confirm that the identity indeed holds to all orders in this expansion.

Using the above identity, one can show that the Ward identity,

$$\tilde{\nabla}_{\mu} T^{\mu}{}_{\nu} = O\nabla_{\nu} J, \tag{2.22}$$

in the background $\tilde{g}$ can then be rewritten as

$$\nabla_{\mu}\left(T^{\mu}{}_{\nu}\sqrt{\det z}\right) - \frac{1}{2}T^{\alpha}{}_{\beta}\sqrt{\det z}\,\nabla_{\nu}(\ln\,\mathrm{z})^{\beta}{}_{\alpha} - \sqrt{\det z}\,O\,\nabla_{\nu} J \;=\; 0 \tag{2.23}$$

in the background metric g.

The tensorial couplings then turn out to be given by the following form of the effective sources[8]

$$z^{(1)\mu}{}_{\nu} \;=\; \exp\left[\left(2\gamma_1\left(T^{(2)\mu}{}_{\nu} - \frac{1}{d}T^{(2)}\delta^{\mu}{}_{\nu}\right) + 2\gamma_2 T^{(2)\mu}{}_{\nu}\right)\sqrt{\det z^{(2)}}\right] \tag{2.24a}$$

$$z^{(2)\mu}{}_{\nu} \;=\; \exp\left[\left(2\gamma_1\left(T^{(1)\mu}{}_{\nu} - \frac{1}{d}T^{(1)}\delta^{\mu}{}_{\nu}\right) + 2\gamma_2 T^{(1)\mu}{}_{\nu}\right)\sqrt{\det z^{(1)}}\right] \tag{2.24b}$$

$$J^{(1)} \;=\; J^{(2)} \;=\; 0. \tag{2.24c}$$

In order to define $z^{(i)\mu}{}_{\nu} \equiv g^{\mu\rho}g^{(i)}_{\rho\nu}$ above, we have chosen a fixed background metric g on which the full system lives. Furthermore, $T^{(i)\mu}{}_{\nu} \equiv T^{(i)\mu\nu}g^{(i)}_{\mu\nu}$. The above tensorial couplings arise from essentially two available tensor structures, namely the traceless part of the energy-momentum tensor and the energy-momentum tensor itself of the complementary theory, giving rise to the two hard-soft coupling constants γ_1 and γ_2. We call these couplings tensorial because these do not involve any scalar operator.

[8] As before, $T^{(i)\mu}{}_{\nu} = T^{(i)\mu\rho}g^{(i)}_{\rho\nu}$.

The determinants $\det z^{(1)}$ and $\det z^{(2)}$ can be obtained by first evaluating the left and right hand sides of eqs. (2.24a) and (2.24b) which yields:

$$\begin{aligned} \mathrm{detz}^{(1)} &= \exp\Big[2\gamma_2 T^{(2)}\sqrt{\mathrm{detz}^{(2)}}\Big], \\ \mathrm{detz}^{(2)} &= \exp\Big[2\gamma_2 T^{(1)}\sqrt{\mathrm{detz}^{(1)}}\Big]. \end{aligned} \tag{2.25}$$

Clearly then, $\det z^{(1)}$ and $\det z^{(2)}$ are solutions of[9]

$$\begin{aligned} \mathrm{detz}^{(1)} &= \exp\Big[2\gamma_2 T^{(2)} \exp\Big[\gamma_2 T^{(1)}\sqrt{\mathrm{detz}^{(1)}}\Big]\Big], \\ \mathrm{detz}^{(2)} &= \exp\Big[2\gamma_2 T^{(1)} \exp\Big[\gamma_2 T^{(2)}\sqrt{\mathrm{detz}^{(2)}}\Big]\Big] \end{aligned} \tag{2.26}$$

These solutions must be substituted in (2.24a) and (2.24b) to finally obtain the complete expressions of the effective background metrics as functionals of the energy-momentum tensors of the two sectors.

As a consequence of the above tensorial couplings, we now find that the energy-momentum tensor of the full system takes the form

$$\begin{aligned} T^{\mu}{}_{\nu} &= T^{(1)\mu}{}_{\nu}\sqrt{\det z^{(1)}} + T^{(2)\mu}{}_{\nu}\sqrt{\det z^{(2)}} \\ &\quad - \gamma_1\sqrt{\det z^{(1)}}\sqrt{\det z^{(2)}}\Big(T^{(1)\alpha}{}_{\beta} - \frac{1}{d}T^{(1)}\delta^{\alpha}{}_{\beta}\Big)\Big(T^{(2)\beta}{}_{\alpha} - \frac{1}{d}T^{(2)}\delta^{\beta}{}_{\alpha}\Big)\delta^{\mu}{}_{\nu} \\ &\quad - \gamma_2\sqrt{\det z^{(1)}}\sqrt{\det z^{(2)}}T^{(1)\alpha}{}_{\beta}T^{(2)\beta}{}_{\alpha}\delta^{\mu}{}_{\nu}\,. \end{aligned} \tag{2.27}$$

satisfying the combined Ward identity (2.20) in the fixed background g.

2.4 Combining general scalar and tensorial couplings

Having independently identified the scalar and tensorial hard-soft couplings, we can put them together to obtain a general class of scalar plus tensorial couplings and the resulting combined energy-momentum tensor of the full theory. In order to do so, we begin by define two functions U and V as below (with $z^{(i)\mu}{}_{\nu} \equiv g^{\mu\rho}g^{(i)}_{\rho\nu}$):

$$U := d\beta\Big(T^{(1)} + \mathrm{O}^{(1)}\Big)\sqrt{\mathrm{detz}^{(1)}} \text{ and } V := d\beta\Big(T^{(2)} + \mathrm{O}^{(2)}\Big)\sqrt{\mathrm{detz}^{(2)}}\,. \tag{2.28}$$

Analogous to (2.14), we impose that the scale factors $e^{d\sigma^{(1)}}$ and $e^{d\sigma^{(2)}}$ in the UV and IR theories should assume the forms:

$$e^{d\sigma^{(1)}} = 1 + e^{-d\sigma^{(2)}}V \qquad \text{and} \qquad e^{d\sigma^{(2)}} = 1 + e^{-d\sigma^{(1)}}U\,, \tag{2.29}$$

[9]It is easy to check that real and positive solutions of these equations exist at least perturbatively in γ_2.

Equivalently, we can impose:

$$e^{d\sigma^{(1)}} = \frac{1+V-U}{2} + \sqrt{U + \left(\frac{1+V-U}{2}\right)^2}, \tag{2.30a}$$

$$e^{d\sigma^{(2)}} = \frac{1+U-V}{2} + \sqrt{V + \left(\frac{1+U-V}{2}\right)^2}. \tag{2.30b}$$

to relate the scale factors with $\det z^{(1)}$. As an aside, it may be worth noting that the two terms under the square-roots in (2.30a) and (2.30b) can be checked to be equal. Finally, we demand that the effective background metrics and sources are given by the expressions:[10]

$$\begin{aligned}
z^{(1)\mu}{}_{\nu} &= \exp\left[\left(2\gamma_1\left(T^{(2)\mu}{}_{\nu} - \frac{1}{d}T^{(2)}\delta^{\mu}{}_{\nu}\right) + 2\gamma_2 T^{(2)\mu}{}_{\nu}\right)\sqrt{\det z^{(2)}}\right] e^{2\sigma^{(1)}},\\
z^{(2)\mu}{}_{\nu} &= \exp\left[\left(2\gamma_1\left(T^{(1)\mu}{}_{\nu} - \frac{1}{d}T^{(1)}\delta^{\mu}{}_{\nu}\right) + 2\gamma_2 T^{(1)\mu}{}_{\nu}\right)\sqrt{\det z^{(1)}}\right] e^{2\sigma^{(2)}},\\
J^{(1)} &= \sigma^{(1)} + \sum_{k=0}^{\infty}\alpha_k O^{(2)k+1}\left(\mathrm{det} z^{(2)}\right)^{\frac{k+1}{2}} O^{(1)k}\left(\mathrm{det} z^{(1)}\right)^{\frac{k}{2}},\\
J^{(2)} &= \sigma^{(2)} + \sum_{k=0}^{\infty}\alpha_k O^{(1)k+1}\left(\mathrm{det} z^{(1)}\right)^{\frac{k+1}{2}} O^{(2)k}\left(\mathrm{det} z^{(2)}\right)^{\frac{k}{2}},
\end{aligned} \tag{2.31}$$

where $\sigma^{(1)}$ and $\sigma^{(2)}$ are given by (2.28), (2.30a) and (2.30b). As in the case discussed in the previous subsection, we need to solve $\mathrm{det}z^{(i)}$s self-consistently by evaluating the determinants of the left and right hand sides of the first two equations above.[11] It may readily be checked that when the tensorial couplings are turned off by setting $\gamma_1 = \gamma_2 = 0$, we revert back to the case discussed in Section 2.2 where we have obtained:

$$\begin{aligned}
\sqrt{\mathrm{det} z^{(1)}} &= 1 + d\beta\left(T^{(2)} + O^{(2)}\right) = e^{d\sigma^{(1)}} \qquad \text{and}\\
\sqrt{\mathrm{det} z^{(2)}} &= 1 + d\beta\left(T^{(1)} + O^{(1)}\right) = e^{d\sigma^{(2)}}.
\end{aligned}$$

The total conserved energy-momentum tensor is given by:

$$\begin{aligned}
T^{\mu}{}_{\nu} = {} & T^{(1)\mu}{}_{\nu}\sqrt{\det z^{(1)}} + T^{(2)\mu}{}_{\nu}\sqrt{\det z^{(2)}}\\
& - \gamma_1\sqrt{\det z^{(1)}}\sqrt{\det z^{(2)}}\left(T^{(1)\alpha}{}_{\beta} - \frac{1}{d}T^{(1)}\delta^{\alpha}{}_{\beta}\right)\left(T^{(2)\beta}{}_{\alpha} - \frac{1}{d}T^{(2)}\delta^{\beta}{}_{\alpha}\right)\delta^{\mu}{}_{\nu}
\end{aligned}$$

[10]As before, $T^{(i)\mu}{}_{\nu} = T^{(i)\mu\rho}g^{(i)}_{\rho\nu}$.

[11]Once again we can check that sensible solutions exist at least perturbatively in the hard-soft couplings.

$$- \gamma_2 \sqrt{\det z^{(1)}} \sqrt{\det z^{(2)}} T^{(1)\alpha}{}_{\beta} T^{(2)\beta}{}_{\alpha} \delta^{\mu}{}_{\nu}$$
$$- \beta \left(T^{(1)} + O^{(1)} \right) \left(T^{(2)} + O^{(2)} \right) \sqrt{\det z^{(1)}} \sqrt{\det z^{(2)}} e^{-d(\sigma^{(1)} + \sigma^{(2)})} \delta^{\mu}{}_{\nu}$$
$$- \sum_{k=0}^{\infty} \frac{2k+1}{k+1} \alpha_k \left(O^{(1)} O^{(2)} \sqrt{(\det z^{(1)})(\det z^{(2)})} \right)^{k+1} \delta^{\mu}{}_{\nu}. \tag{2.32}$$

which satisfies the Ward identity (2.20) in the fixed background $g_{\mu\nu}$. We note that for each pair of scalar operators in the UV and IR theories, we can then have the more general tensor and scalar couplings γ_2, γ_1, β and α_k. In the following section, we will present a toy example to demonstrate how we can determine the hard-soft couplings and the parameters of the IR theory as functionals of the perturbative coupling constants. In particular, the sources for irrelevant scalar operators in the IR theory should vanish – this will play a major role in determining the hard-soft coupling constants.

Finally, we would like to emphasise that the special phenomenological semi-holographic constructions [48, 55, 235] discussed in Section 1.1 are special instances of the general hard-soft coupling scheme discussed in this section. These special instances naturally follow if the hard-soft couplings are small and we retain only such leading coupling terms.[12] As discussed before, such phenomenological constructions can be well justified in a certain range of energy scales.

At the end of Section 3.3, we will show how the general coupling rules are modified when the full theory couples to external scalar sources. In fact, this investigation will allow us to define all scalar operators in the full theory as appropriate weighted sums of the effective UV and IR operators.

3 A bi-holographic illustration

In this section, we construct a complete toy theory to illustrate the principles of the semi-holographic framework. In our toy theory, we will see how the UV dynamics determines both the hard-soft couplings and the IR theory. We will also see why the bulk fields of the dual IR holographic theory should undergo state-dependent field-redefinitions in the semi-holographic construction although the classical gravitational theory dual to the IR is itself not state-dependent. Furthermore, we will find that $\Lambda \to \infty$ behaviour of the hard-soft couplings can be obtained from the vacuum but their runnings with the scale can be state-dependent.

[12] In the model for heavy-ion collisions discussed in Section 1.1, we have two pairs of scalar operators. The first pair are the perturbative and *shadow* glueball condensates, and the second pair are the perturbative and *shadow* Pontryagin charge densities.

The basic simplification in our toy theory consists of replacing the perturbative UV dynamics by a strongly coupled holographic theory admitting a dual classical gravity description on its own. The IR dynamics will be given by a *different* even more strongly coupled holographic theory with a *different* dual classical gravity description. The advantage of this *biholographic* set-up will be that the UV-IR operator map can be simplified by construction. As we will see in the following section, this map will be immensely complex in QCD which is asymptotically free (although we can still proceed systematically). Our bi-holographic construction is designed to establish the conceptual foundations of semi-holography.

The spirit of our bi-holographic construction is captured in Fig. 1. The UV dynamics is represented by the blue $(d+1)$−dimensional holographic emergent universe which covers the radial domain $-\infty < u < 0$ and the IR dynamics is represented by the red $(d+1)$−dimensional holographic emergent universe which covers the radial domain $0 < u < \infty$, with u being the holographic radial coordinate denoting the scale. Each of these geometries is asymptotically AdS and their individual conformal boundaries are at $u = \pm\infty$ respectively. Although the bulk fields are governed by different classical gravity theories in the two different universes, these transit smoothly at the gluing surface $u = 0$. In each universe, we can use the standard rules of holographic duality [33–35] to extract the effective UV and IR sources and expectation values of the operators from the behaviour of the bulk fields in the respective asymptotic regions. However, the boundary conditions of the two asymptotic regions $u \to \pm\infty$ are correlated by the general consistent coupling rules of the previous section which leads to the existence of a conserved local energy-momentum tensor of the full dual quantum many-body system.[13]

Crucially, the UV and IR theories cure each other. Individually, the UV and IR universes are singular (in a specific sense to be discussed later) if extended in the regions

[13]This feature distinguishes our construction from those described in [246–248] where two or more holographic CFTs are coupled by gluing the boundaries of their dual asymptotically AdS geometries. In our case, the AdS spaces are glued in the interior reflecting that the dual theories *glue* together to form a complete and consistent theory. The AdS boundaries in our case are then coupled non-locally in the sense that the sources specified at the boundaries should be correlated by the rules found in the previous section. Such type of non-local couplings (related to multi-trace couplings of operators in future and past directed parts of the Schwinger-Keldysh contour in the dual theory) have been also recently discussed in [249] in the context of eternal AdS black holes which have two distinct conformal boundaries. It has been shown that such couplings lead to formation of traversable wormholes leading to a concrete realisation of the ER = EPR conjecture [231] stating that quantum entanglement of degrees of freedom (i.e. Einstein-Podolsky-Rosen pairs) leads to formation of Einstein-Rosen bridges (i.e. wormholes) between distinct space-time regions. In our case, this wormhole is perhaps engineered by our coupling rules as suggested by the construction in [249] reflecting the entanglement between UV and IR degrees of freedom of the dual system. It is worthwhile to note in this context that although the coupling of the boundaries is non-local, it is strongly constrained in our construction by the existence of a local and conserved energy-momentum tensor in the full dual many-body system.

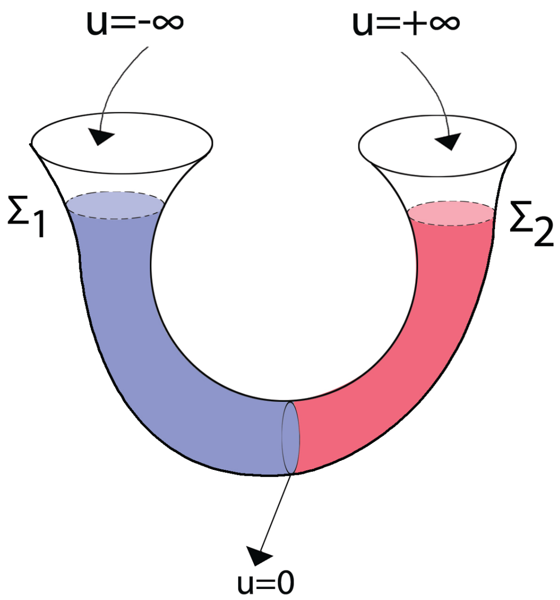

Figure 1. Our biholographic toy theory is described by two different holographic UV (blue) and IR (red) universes with different classical gravity laws. However these are smoothly glued at $u = 0$. A scale Λ in the full theory in a certain RG scheme is represented by data on the two appropriate hypersurfaces Σ_1 and Σ_2 belonging to the two universes as described in the text. In the UV, most contributions come from the UV blue universe. This explains the nomenclature.

$u > 0$ and $u < 0$ respectively. However the smooth gluing at $u = 0$ implies that the full holographic construction giving the dynamics of the quantum many-body system has no singularity. Therefore, the IR theory indeed completes the UV theory almost in the same manner in which non-perturbative dynamics cures the Borel singularities of perturbation theory. We can also view the IR Universe extending along $0 < u < \infty$ as a second cover the UV universe $-\infty < u < 0$ which thus becomes bi-metric. We will discuss this point of view later.

As mentioned above, the leading asymptotic behaviour of bulk fields giving the effective sources and effective background metric of the respective theories are *coupled*, or rather correlated by the general semi-holographic construction rules established in the previous section. Therefore, the full theory admits a local energy momentum tensor conserved in the actual fixed background metric where all the degrees of freedom live.Here, we will take this fixed background metric to be $\eta_{\mu\nu}$.

At this stage, we should clarify in which sense we are using the terms *UV theory* and *IR theory*. After all, the full energy-momentum tensor constructed in the previous section receives contributions from both UV and IR theories at any scale. In our case, it means that the microscopic energy-momentum tensor of the dual many-body quantum system will be a complicated combination of the energy-momentum tensors and other data obtained from the sub-leading asymptotic modes of both UV and IR universes. Nevertheless, we will see that the contribution to the the energy-momentum tensor at $\Lambda = \infty$ coming solely from the IR universe is zero in the vacuum state. Furthermore, the scale factors of the effective UV and IR metrics (identified with the boundary metrics of the UV and IR universes) will turn out to be dynamically determined such that the effective UV metric will be slightly compressed and effective IR metric will be slightly dilated compared to the fixed background Minkowski space. This explains our UV and IR nomenclatures. Note these feature are also present in the semi-holographic constructions as discussed before – the vanishing of the scale-dependent hard-soft couplings in the UV ensures that perturbative contributions dominate in the UV as should be the case in asymptotically free theories like QCD. In our bi-holographic construction, although the contributions of the UV universe will dominate, the hard-soft couplings will be finite in the limit $\Lambda \to \infty$.

Furthermore, in our bi-holographic construction, a scale Λ in the full theory is represented by the data on the union of two appropriate hypersurfaces Σ_1 and Σ_2 in the UV and IR universes respectively in a specific type of RG scheme as shown in Fig. 1. We will discuss this issue in more details later. The IR hypersurface $u = 0$ will represent an endpoint of the RG flow. In fact the geometry near $u = 0$ can be described as an infrared AdS space with zero volume, and so we will argue that it is a fixed point. More generally, however $u = 0$ could be a *wall* representing confinement in the dual theory.

Conceptually, our bi-holographic construction is thus very different from how RG flow is represented in standard holographic constructions such as that described in [250]. In these cases, the full emergent spacetime is described by a *single* gravitational theory and it also has a *single* conformal boundary. Furthermore, although the spacetime has another AdS region in the IR (deep interior), this matches with the rear part of a pure AdS geometry, i.e. the part that contains the Poincare horizon and not the asymptotic conformal boundary. Physically these geometries represent deformation of the UV fixed point in the dual field theory by a relevant operator as a result of which it flows to a different IR fixed point – a geometric c−function can also be constructed [250] reproducing the central charges of the UV and IR fixed points. Our bi-holographic construction however should not be thought of as a flow from an UV to an IR fixed point driven by a relevant deformation.[14] Rather the UV region in our case represents a strongly coupled version of usual perturbative dynamics, and the IR region represents the non-perturbative sector that exists as a *shadow* of the perturbative degrees of freedom in many-body quantum systems. In our case, the shadow IR theory and the hard-soft coupling constants will be determined by the parameters of the UV gravitational theory via:

1. the general coupling rules of the previous section ensuring the existence of a conserved energy-momentum tensor of the full system,
2. the vanishing of the sources for irrelevant IR operator(s), and
3. the continuity of bulk fields and their radial derivatives up to appropriate orders at the matching hypersurface $u = 0$.

3.1 A useful reconstruction theorem

We can readily proceed with some simplifying assumptions. The first assumption is that both the UV and IR holographic classical gravity descriptions are provided by Einstein-dilaton theories consisting of a scalar field with (different) potentials and minimally coupled to gravity. The gravitational theories are then individually described by the respective actions:

$$S_{\rm grav}^{\rm UV,IR} = \frac{1}{16\pi G_N}\int {\rm d}^{d+1}x\,\sqrt{-G}\Big(R - G^{MN}\partial_M\Phi\partial_N\Phi - 2V^{\rm UV,IR}(\Phi)\Big), \tag{3.1}$$

with d denoting the number of spacetime dimensions of the dual quantum many-body system. Note Φ is dimensionless in the above equation.

[14] The general semi-holographic construction can of course apply to such a case particularly if the UV fixed point is weakly coupled.

To describe the dual vacuum state, it will be convenient to use the *domain-wall coordinates* in which the bulk metric and scalar fields assume the forms:[15]

$$ds^2 = du^2 + e^{2\rho(u)}\eta_{\mu\nu}dx^\mu dx^\nu, \qquad \Phi = \Phi(u). \tag{3.2}$$

The conformal boundaries are at $u = \pm\infty$. We choose $d = 4$. In the domain-wall coordinates, the gravitational equations of motion for ρ and Φ in the domain $-\infty < u < 0$ can be written in the form:

$$\rho'' = -\frac{1}{3}\Phi'^2, \tag{3.3}$$

$$\frac{3}{2}\rho'' + 6\rho'^2 = -V^{\mathrm{UV}}(\Phi). \tag{3.4}$$

The equations of motion for ρ and Φ in the domain $0 < u < \infty$ are exactly as above but with V^{UV} replaced by V^{IR}.

Instead of choosing $V^{\mathrm{UV}}(\Phi)$ and $V^{\mathrm{IR}}(\Phi)$ in order to specify the UV and IR gravitational theories, we will take advantage of the so-called *reconstruction theorem* which states that there exists a unique map between a choice of the radial profile of the scale factor, i.e. $\rho(u)$ and the bulk scalar potential $V(\Phi)$ which supports it.[16] The proof of this theorem is straightforward. Suppose we know $\rho(u)$. We can then first use eq. (3.3) to construct $\Phi(u)$. However, this requires an integration constant. If the spacetime has a conformal boundary, at $u = -\infty$ for instance, the behaviour of ρ near $u = -\infty$ should be as follows:

$$\rho(u) = \rho_0 - \frac{u}{L} - \rho_\delta \exp\left(2\delta\frac{u}{L}\right) + \text{subleading terms}, \tag{3.5}$$

with $\delta > 0$ and $\rho_\delta > 0$. The latter restriction results from the requirement that ρ'' should be negative in order for a solution for Φ to exist as should be clear from (3.3). One can readily check from (3.3) that the asymptotic behaviour of Φ should be:

$$\Phi(u) = \Phi_0 \pm 2\sqrt{3\rho_\delta}\exp\left(\delta\frac{u}{L}\right) + \text{subleading terms}. \tag{3.6}$$

If $\delta \neq d$, then $2\sqrt{3\rho_\delta}$ should correspond to the non-normalisable mode or the normalisable mode of Φ. The constant Φ_0 is merely an integration constant.

We can then invert this function $\Phi(u)$ to obtain $u(\Phi)$. Furthermore, from eq. (3.4), we can readily obtain $V(u)$ substituting $u(\Phi)$ in which yields $V(\Phi)$. Thus we construct

[15] It has been observed that the domain-wall radial coordinate u can be directly related to the energy-scale of the dual theory [54, 251, 252].

[16] As far as we are aware of, this theorem was first stated in the context of cosmology in [253].

the $V(\Phi)$ corresponding to a specific $\rho(u)$. This ends the proof of the reconstruction theorem. Note this proof assumes that the inverse function $u(\Phi)$ exists. We have to ensure that this is indeed the case.

The crucial point is that one can readily see from (3.4) that asymptotically (i.e. near $u = \infty$ or equivalently near $\Phi \approx \Phi_0$), $V(\Phi)$ should have the expansion

$$V(\Phi) = -\frac{6}{L^2} + \frac{1}{2}m^2(\Phi - \Phi_0)^2 + O(\Phi - \Phi_0)^3, \tag{3.7}$$

when $d = 4$ with

$$m^2L^2 = \delta(\delta - 4). \tag{3.8}$$

This implies that $V(\Phi)$ should have a critical point at $\Phi = \Phi_0$ in order that the geometry can become asymptotically anti-de Sitter. Furthermore, the field $\Phi - \Phi_0$ and not Φ corresponds to the dual operator O with scaling dimension δ or $4 - \delta$ when $\delta \neq 4$. Therefore, without loss of generality when $\delta \neq 4$, we can always employ the field redefinition $\Phi \to \Phi - \Phi_0$ and set the integration constant to be zero. When $\delta = 4$, Φ is massless and we need to use the holographic correspondence to figure out what the integration constant should be since Φ_0 then corresponds to a marginal coupling of the dual field theory. Usually, it is put to zero even in this case. We will not deal with the massless scalar case here.

3.2 The bi-holographic vacuum

We first focus on constructing the bi-holographic vacuum state. Let us begin by individually choosing an ansatz for the UV and IR gravitational theories. We take the advantage of the reconstruction theorem described above by making an ansatz for $\rho(u)$ in the respective domains instead of doing so for the potentials $V^{\mathrm{UV,IR}}(\Phi)$. For the sake of convenience we also put $8\pi G_N = 1$. We will set $d = 4$.

Since both the UV and IR gravitational theories have the same G_N as clear from (3.1) and their AdS radii will turn out to be of the same order if not equal, we can take the large N limit in both sectors simultaneously. Therefore, we can not only suppress quantum gravity loops in each gravitational theory, but also hybrid ones. This justifies our assumption that quantum gravity effects can be ignored in both gravitational theories.

The UV domain

To take advantage of the reconstruction theorem, we choose the scale factor profile in the (dual UV) domain $-\infty < u < 0$ to be:

$$\rho^{\mathrm{UV}}(u) \;=\; A_0 - \frac{u}{L^{\mathrm{UV}}} - A_1 \tanh\Big(\frac{u}{L^{\mathrm{UV}}}\Big) + A_2 \tanh\Big(2\frac{u}{L^{\mathrm{UV}}}\Big). \tag{3.9}$$

We can motivate the choice of the tanh functions as follows. These lead to right exponentially subleading asymptotic behaviour at $u = -\infty$ and leads to $\rho'' = 0$ at $u = 0$. Crucially, if we choose our parameters A_1 and A_2 such that $\rho'' < 0$ for $u < 0$, then typically $\rho'' > 0$ should follow for $u > 0$. This does not lead to a curvature singularity, however (3.3) implies that Φ has no real solution for $u > 0$, i.e. it signals the end of UV spacetime at $u = 0$. This leads to a singularity in the sense of *geodetic incompleteness*, because any freely falling observer can reach the edge $u = 0$ from a finite value of u in finite proper time. This singularity is eventually cured by the emergence of the IR universe.

The above choice for $\rho^{\mathrm{UV}}(u)$ leads to a unique solution for $\Phi^{\mathrm{UV}}(u)$ whose asymptotic (i.e. $u \to -\infty$) behaviour is:

$$\Phi^{\mathrm{UV}}(u) \;=\; U_1 \exp\Big(\frac{u}{L^{\mathrm{UV}}}\Big) + \cdots + V_1 \exp\Big(3\frac{u}{L^{\mathrm{UV}}}\Big) + \cdots , \tag{3.10}$$

with[17]

$$U_1 = -2\sqrt{6}\sqrt{A_1}, \quad V_1 = 4\sqrt{\frac{2}{3}}\frac{A_1 + A_2}{\sqrt{A_1}}. \tag{3.11}$$

We can readily obtain $V^{\mathrm{UV}}(\Phi)$ too from (3.4) as described above but it's complete explicit form will not be of much importance for us. The only information in $V^{\mathrm{UV}}(\Phi)$ which is significant for us is that the mass of Φ^{UV} which is given by $m^2 L^{\mathrm{UV}2} = -3$ which also implies that the corresponding operator $O^{(1)}$ has scaling dimension $\Delta^{\mathrm{UV}} = 3$ at the UV fixed point, and is therefore a relevant operator. Furthermore, $U_1 \neq 0$ implies that the UV fixed point is subjected to a relevant deformation and $U_1 L^{\mathrm{UV}-1}$ is the relevant coupling[18] as will be clear once we transform to the Fefferman-Graham coordinates. The Fefferman-Graham radial coordinate z in which the metric assumes the form:

$$\mathrm{d}s^2 = \left(\frac{L^{\mathrm{UV}2}}{z^2}\right)\left(\mathrm{d}z^2 + e^{\tilde{\rho}^{\mathrm{UV}}(z)}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu\right) \tag{3.12}$$

is related to u by $z = L^{\mathrm{UV}} \exp(u/L^{\mathrm{UV}})$. In the Fefferman-Graham coordinates we obtain:

$$\begin{aligned} e^{2\tilde{\rho}^{\mathrm{UV}}(z)} &= e^{2(A_0+A_1-A_2)}\left(1 - 4A_1\frac{z^2}{L^{\mathrm{UV}2}} + 4\left(A_1 + 2A_1^2 + A_2\right)\frac{z^4}{L^{\mathrm{UV}4}} + \cdots\right), \\ \Phi^{\mathrm{UV}}(z) &= -2\sqrt{6}\sqrt{A_1}\frac{z}{L^{\mathrm{UV}}} + 4\sqrt{\frac{2}{3}}\frac{A_1 + A_2}{\sqrt{A_1}}\frac{z^3}{L^{\mathrm{UV}3}} + \cdots . \end{aligned} \tag{3.13}$$

[17] U_1 is determined up to a sign. We make a choice of sign here.

[18] Although the full theory is not a relevant deformation of a UV fixed point as discussed before, the UV holographic theory individually can be described in such terms. This is owing to the fact that the holographic geometry will be asymptotically AdS.

We can readily perform holographic renormalisation in the Fefferman-Graham coordinates to extract the sources and the expectation values of the operators in the UV description [254–258]. The scale factor $\sigma^{(1)}$ in the boundary metric $g^{(1)}_{\mu\nu} = e^{2\sigma^{(1)}}\eta_{\mu\nu}$ and the source $J^{(1)}$ for the scalar operator $O^{(1)}$ are given by:

$$\sigma^{(1)} = A_0 + A_1 - A_2, \quad J^{(1)} = -2\sqrt{6}\sqrt{A_1}L^{\mathrm{UV}^{-1}}. \tag{3.14}$$

The expectation values of the trace of the energy-momentum tensor $T^{(1)}$ (defined as $T^{(1)} := T^{(1)\mu\nu}g^{(1)}_{\mu\nu}$ as in the previous section) and $O^{(1)}$ are given by:[19]

$$T^{(1)} = 16(A_1 + A_2)L^{\mathrm{UV}^{-4}}, \quad O^{(1)} = -4\sqrt{\frac{2}{3}}\frac{A_1 + A_2}{\sqrt{A_1}}L^{\mathrm{UV}^{-3}}. \tag{3.15}$$

As a consistency check, we note that the CFT Ward identity $T^{(1)} = J^{(1)}O^{(1)}$ which is scheme-independent is indeed satisfied by the above values. We also note that when $\Delta = 3$, we also have the possibility of alternative quantisation in which case the field Φ corresponds to an operator with $\Delta = 1$ (which saturates the unitarity bound on lowest possible dimensions of scalar primary operators in a CFT) and the roles of $J^{(1)}$ and $O^{(1)}$ can be interchanged. Here, we perform the more usual quantisation.

The IR domain

In the infrared domain $0 < u < \infty$, we choose the scale factor $\rho^{\mathrm{IR}}(u)$ to be:

$$\rho^{\mathrm{IR}}(u) = B_0 - \frac{u}{L^{\mathrm{IR}}} - B_1 \tanh\left(5\frac{u}{L^{\mathrm{IR}}}\right) + B_2 \tanh\left(10\frac{u}{L^{\mathrm{IR}}}\right), \tag{3.16}$$

with $L^{\mathrm{IR}} < 0$ so that the conformal boundary is indeed at $u = \infty$. The choice of $tanh$ functions can be motivated by similar arguments presented in the UV case – we need an edge singularity at $u = 0$ which is cured by the gluing to the UV universe. This choice then implies that $\Phi(u)$ has the asymptotic expansion:[20]

$$\Phi^{\mathrm{IR}}(u) = V_2 \exp\left(5\frac{u}{L^{\mathrm{IR}}}\right) + \cdots, \quad \text{with} \quad V_2 = -2\sqrt{6}\sqrt{B_1}. \tag{3.17}$$

[19] We have used the minimal subtraction scheme in which we do not obtain any new parameter from the regularisation procedure as no finite counterterm is invoked. In this case, the scheme dependence arises only from a finite counterterm proportional to $J^{(1)^4}$. It is dropped in the minimal subtraction scheme. There is a beautiful independent justification of the minimal subtraction scheme [251, 259] (see also [54] for an yet another perspective) as only in this scheme one can define holographic c−function and beta functions which satisfy identities analogous to those in the field-theoretic local Wilsonian RG flows constructed by Osborn [260].

[20] Once again V_2 is determined up to a sign. We make a choice of sign here.

Therefore Φ in the IR region corresponds to an irrelevant operator with $\Delta = 5$. We can now define a new Fefferman-Graham coordinate via $u = L_{\rm IR}\log(\tilde{z})$ suitable for the IR asymptotia which is at $\tilde{z} = 0$. The asymptotic expansions are:

$$\tilde{z}^2 e^{2\rho_{\rm IR}(\tilde{z})} = e^{2(B_0+B_1-B_2)}\left(1 - 4B_1\frac{\tilde{z}^{10}}{L^{{\rm IR}^{10}}} + \cdots\right) \tag{3.18}$$

$$\phi_{\rm IR}(\tilde{z}) = V_2\frac{\tilde{z}^5}{L^{{\rm IR}^5}} + \cdots . \tag{3.19}$$

With our choice of ρ in the IR theory, the scalar source is vanishing while the scalar vev is parametrised by V_2. Indeed if the scalar source would not have vanished, it would have lead to a runaway asymptotic behaviour causing a curvature singularity. As the dual operator is irrelevant, its source should vanish as otherwise we cannot find the corresponding state in the holographic correspondence as well. The effective IR metric, which is the boundary metric of the IR asymptotic region is given by $e^{2(B_0+B_1-B_2)}\eta_{\mu\nu}$, while the IR stress-tensor is vanishing. Thus, we obtain

$$\sigma^{(2)} = B_0 + B_1 - B_2, \quad J^{(2)} = 0, \tag{3.20}$$

and

$$T^{(2)} = 0, \quad O^{(2)} = 2\sqrt{6}\sqrt{B_1}L^{{\rm IR}^{-5}}. \tag{3.21}$$

Gluing and determining the full theory

For the full construction, we need to consider the hard-soft couplings. We make a simplistic assumption that the tensorial hard-soft couplings γ_1 and γ_2 are zero. We also make another assumption that we can set all scalar hard-soft couplings α_k to zero except for α_0. Therefore, α_0 and β are the only non-vanishing hard-soft couplings in our construction. In what follows, we will denote α_0 by α for notational convenience. Our coupling rules thus (resulting from setting $\alpha_k = 0$ for $k \neq 0$ and $d = 4$ in (2.17c) and (2.17d)) are:

$$e^{4\sigma^{(1)}} = 1 + 4\beta\left(T^{(2)} + O^{(2)}\right), \tag{3.22a}$$

$$e^{4\sigma^{(2)}} = 1 + 4\beta\left(T^{(1)} + O^{(1)}\right), \tag{3.22b}$$

$$J^{(1)} = \frac{1}{4}\ln\left(1 + 4\beta\left(T^{(2)} + O^{(2)}\right)\right) + \alpha O^{(2)}, \tag{3.22c}$$

$$0 = \frac{1}{4}\ln\left(1 + 4\beta\left(T^{(1)} + O^{(1)}\right)\right) + \alpha O^{(1)}. \tag{3.22d}$$

In the final equation above we have used $J^{(2)} = 0$, i.e. the source of the irrelevant IR operator must vanish.

Furthermore, we impose that the UV and IR geometries can be smoothly glued along their edges which coincide at $u = 0$. This matching should cure the respective *edge* singularity (resulting from geodetic incompleteness) as discussed above. Therefore, the metric and the bulk scalar field, and also their radial derivatives up to appropriate orders should be continuous at $u = 0$. There is one subtle point we need to take into account during the gluing procedure. The asymptotic region $u = \infty$ in the IR domain naturally corresponds to UV rather than IR, however in the full theory it represents IR contributions. It is then natural to reverse the scale (radial) orientation of the IR geometry while gluing it to the UV geometry at $u = 0$. Equivalently, we should set ρ to $-\rho$ in the IR geometry before we glue it to the IR. One can then also think that the ρ travels back to $-\infty$ from 0, so that the IR geometry gives another cover of the spacetime whose full extension is $-\infty < u < 0$ and an observer simply can pass smoothly from the UV cover to the IR cover. The spacetime is thus bi-metric. The smooth gluing of the two Universes is ensured if at $u = 0$:[21]

$$\rho^{\mathrm{UV}} = \rho^{\mathrm{IR}}, \quad \rho^{\mathrm{UV}\prime} = -\rho^{\mathrm{IR}\prime}, \quad \Phi^{\mathrm{UV}} = \Phi^{\mathrm{IR}}, \quad \Phi^{\mathrm{UV}\prime} = -\Phi^{\mathrm{IR}\prime}. \tag{3.23}$$

By our choices of ρ, $\rho'' = 0$ at $u = 0$ whether we approach from the UV side or the IR side and therefore we automatically obtain from (3.3) that $\Phi' = 0$ from both ends and is hence continuous. Effectively we thus have only two matching conditions, namely

$$\rho^{\mathrm{UV}} = \rho^{\mathrm{IR}} \quad \text{and} \quad \rho^{\mathrm{UV}\prime} = -\rho^{\mathrm{IR}\prime} \quad \text{at } u = 0. \tag{3.24}$$

The matching of Φ is ensured via a field redefinition. As discussed in Section 3.1, we can always redefine Φ as $\Phi - \Phi_0$ with Φ_0 being the asymptotic value of Φ (which after the redefinition becomes zero) so that the potential $V(\Phi)$ has no tadpole term at $\Phi = 0$. However, if there are two asymptotic boundaries, as in our construction, we can do this redefinition in one asymptotic region only. In this case, the integration constant Φ_0 in the other asymptotic region should be set by continuity. We will choose $\Phi_0 = 0$ in the IR end and obtain the value of Φ_0 at the UV end. We need to check that Φ_0 which can be obtained by integrating (3.3) should be finite. This will indeed be the case.

It is fairly obvious that L^{UV} sets the dimensions of all dimensionful parameters in the field theory including α and β, the hard-soft couplings. Without loss of generality, we can set $L^{\mathrm{UV}} = 1$. The (dimensionless) parameters which determine our UV theory are A_0, A_1 and A_2. However, A_0 only contributes to the scale factor ($\sigma^{(1)}$) of the effective metric of the UV theory and does not play any role in determining V^{UV}. So we can regard A_1 and A_2 as the true UV parameters. The other parameters are α and β (the hard-soft

[21] A more diffeomorphism invariant statement is that on the hypersurface $u = 0$, the induced metric obtained from the UV and IR sides should match, and the Brown-York stress tensors should also match but with a flipped sign of the IR term. Such a type of gluing with reversed orientation of one manifold has also been considered in [261] in the context of constructing holographic bulk analogue of Schwinger-Keldysh time contour (which reverses and flows back in time.)

couplings) which are dimensionful and $\delta \equiv L^{\mathrm{IR}}/L^{\mathrm{UV}}$, B_0, B_1 and B_2 (determining the IR theory) which are dimensionless. Including all parameters of the UV and IR theories and the hard-soft couplings we have in total 9 parameters.

The set of parameters should be such that we must satisfy the 4 coupling equations (3.22a), (3.22b),(3.22c) and (3.22d), and the 2 matching equations in (3.24). Since our 9 parameters should satisfy 6 equations, we can determine 6 of our parameters from 3. We choose the 3 parameters which determine the rest to be the UV parameters A_0, A_1 and A_2. In practice, it is easier to choose A_1, B_2 and δ which gives the ratio of the UV and IR scales instead as the set of independent parameters. Nevertheless, we can check that it is equivalent to making the right choices for A_is and then determining B_2 and δ. We will proceed by choosing $A_1 = B_2 = 1$ and $\delta = -4.91$ (recall that our parametrisation (3.16) require L^{IR} to be negative). $|\delta| > 1$ implies that the IR theory is more strongly coupled. It is convenient to first utilise the matching equations (3.24) to obtain:

$$B_0 = A_0\,, \quad B_1 = -\frac{1}{5}(1 + 10B_2 + \delta(1 + A_1 - 2A_2)). \tag{3.25}$$

We then utilise (3.22b) and (3.22c) to note that α and β should be given by:

$$\alpha = \frac{4J^{(1)} - \ln(1 + 4\beta(T^{(2)} + O^{(2)}))}{4O^{(2)}(1 + 4\beta(T^{(2)} + O^{(2)}))}, \quad \text{and} \quad \beta = \frac{e^{4\sigma^{(2)}} - 1}{4(T^{(1)} + O^{(1)})}. \tag{3.26}$$

The right hand sides above are given by the parameters of the UV and IR theory via (3.14), (3.15), (3.20) and (3.21). Therefore we obtain α and β in terms of other parameters, namely A_is, B_is and δ. Since B_0 and B_1 are given by (3.25), and the values of A_1, B_2 and δ have been fixed, α and β are now functions of A_0 and A_2.

Substituting (3.25) and (3.26) in (3.22a) and (3.22d), and using the fixed values of A_1, B_2 and δ, we can determine the values of A_0 and A_2 numerically. These numerical values are then used to obtain B_0 and B_1 from (3.25). Finally, we can use (3.26) to determine α and β.

Doing so, we obtain $A_2 = -0.25$. As discussed above, we can now also claim that we have actually set $A_1 = 1$, $A_2 = -0.25$ and $\delta = -4.91$ and have determined all other parameters in terms of these. In the end, we obtain $A_0 = -1.25$, $B_0 = -1.25$, $B_1 = 0.25$ and $B_2 = -1$. Furthermore, the hard-soft couplings in units $L^{\mathrm{UV}} = 1$ are

$$\alpha = 5.7 \times 10^3, \quad \text{and} \quad \beta = 1.2 \times 10^{-4}. \tag{3.27}$$

Of course we have determined these values of α and β in the limit $\Lambda \to \infty$ of the dual field-theoretic system. It is indeed a bit surprising that α is so enormously large and β is so tiny. This completes determining all parameters of the IR theory and hard-soft couplings in terms of the dimensionless UV parameters A_0, A_1 and A_2. It is also interesting to note that as a result of our solutions we obtain

$$\sigma^{(1)} = -1.03 \times 10^{-7}, \quad \text{and} \quad \sigma^{(2)} = 1.16 \times 10^{-3}. \tag{3.28}$$

This implies the effective UV metric is slightly compressed and the effective IR metric is slightly dilated compared to the background flat Minkowski space as claimed before.

Finally the other non-vanishing effective sources and vevs in units $L^{\mathrm{UV}} = 1$ turn out to be:

$$T^{(1)} = 12.06, \quad J^{(1)} = -2\sqrt{6}, \quad O^{(1)} = -2.46, \quad O^{(2)} = -8.6 \times 10^{-4}. \tag{3.29}$$

It is also reassuring to see that the effective IR vev is small in units $L^{\mathrm{UV}} = 1$ compared to the effective UV vev.

The most interesting feature is the behaviour of ρ'' which has been plotted in Fig. 2. It is clear from the figure that the gluing cures the edge singularities of each component Universe arising from the geodetic incompleteness – if extended to $u > 0$ and $u < 0$, ρ'' becomes positive in the UV and IR universes respectively.

We plot the scale factor $\rho(u)$ in Fig. 3 and $\Phi(u)$ in Fig. 4. $\Phi(u)$ is obtained by integrating (3.3). The integration constant Φ_0 in the UV region which is simply $\Phi(u = \infty)$ can be determined to be about 6.91.

Finally, we plot $V(\Phi)$ in units $L^{\mathrm{UV}} = 1$ as a function of Φ in Fig. 5. We find that V is $V-$shaped. The asymptotic values of $V(\Phi)$ in each component Universe is -6 where $V(\Phi)$ has critical points. The critical point in the UV universe is at $\Phi = 0$ and that in the IR universe is at $\Phi = \Phi_0 \approx 6.91$. Crucially, $V(\Phi)$ has a minima at $u = 0$ where $\Phi \approx 3.43$. Here $V(\Phi)$ is not differentiable, but still it is kind of a critical point as in the asymptotic regions. Furthermore, as clear from Figs. 2 and 4 that at $u = 0$, $\rho'' = 0$ and $\Phi' = 0$ like in the two asymptotic regions also. Therefore, *we can think of the region $u = 0$ as an AdS space of zero volume*. This hints that our full theory flows to an infrared fixed point. We leave a more detailed analysis to the future.

If we take the perspective mentioned before that the UV (blue) and IR (red) universes are two covers of $-\infty < u \leq 0$ joined smoothly at $u = 0$, then clearly the two covers do not only have two different metrics but also two different potentials for the scalar field Φ.

One final remark regarding determining all parameters of the IR theory and the hard-soft couplings as functions of the parameters of the UV theory is that we have assumed that the irrelevant IR operator coupling to the UV operator has dimension 5. Clearly, if we

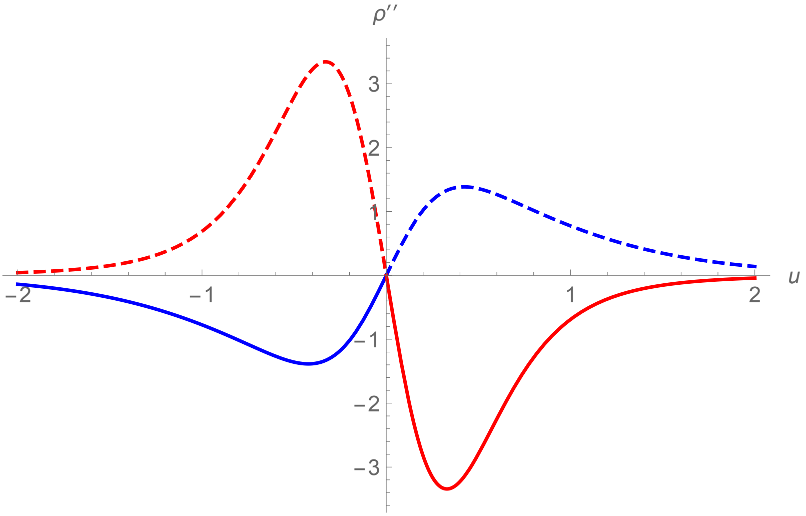

Figure 2. Plot of $\rho''(u)$; the blue curve refers to the UV while the red one refers to the IR. The dashed lines indicate the regions where $\rho'' > 0$ – here the solutions to Φ do not exist and hence these regions should be discarded. It is clear then how the gluing cures the edge singularities arising from the geodetic incompleteness in each individual component Universe.

change the dimension of the IR operator to 6 as for instance and modify our ansatz (3.16) accordingly, we will still be able to repeat the same exercise to obtain the new IR parameters and the hard-soft couplings. Thus the IR theory that completes the UV theory is unique up to certain assumptions of which the most crucial one is the scaling dimension of the irrelevant IR operator. In the case of semi-holographic framework for QCD, it will turn out that the dimensions of the IR operators to which the UV operator couples to will be fixed by perturbation theory itself. We will discuss this in the next section.

3.3 Excited states

As we have defined the bi-holographic theory and have explicitly constructed the vacuum state, we can proceed to compute physical observables of excited states. Let us first see how we can compute small fluctuations about the vacuum state. The parameters of the IR theory and the hard-soft couplings α and β (at $\Lambda = \infty$) have been determined once and for all in terms of the parameters of the UV theory. In fact these parameters together define the biholographic theory. Let us first consider scalar fluctuations, i.e. $\delta\rho^{\mathrm{UV,IR}}(u,x)$ and $\delta\Phi^{\mathrm{UV,IR}}(u,x)$ in the UV and IR universes. As a result of the fluctuations, we generate $\delta\sigma^{(1)}(x)$, $\delta\sigma^{(2)}(x)$, $\delta T^{(1)}(x)$, $\delta O^{(1)}(x)$ and $\delta O^{(2)}(x)$. As discussed before, we should have

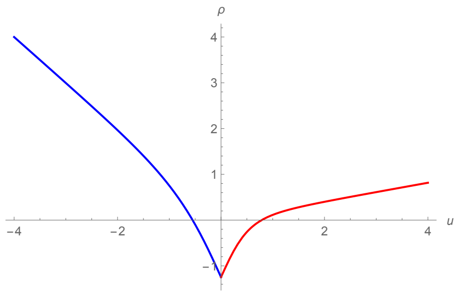

Figure 3. Plot of the $\rho(u)$; the blue curve refers to the UV region while the red one refers to the IR region. ρ' is continuous at $u = 0$ if we flip the sign of ρ' on the IR side.

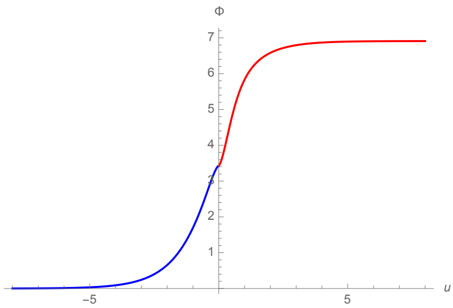

Figure 4. Plot of the scalar field $\phi(u)$; the blue curve refers to the UV region while the red one refers to the IR region. Note that $\Phi' = 0$ at $u = 0$ where $\Phi \approx 3.43$. The integration constant Φ_0 (see text) at $u = \infty$ is simply $\Phi(u = \infty)$ which is approximately 6.91.

$\delta J^{(2)}(x) = 0$, and the CFT Ward identity then implies that $\delta T^{(2)}(x) = 0$. In any case, we should solve the fluctuations so that perturbations of both sides of the semi-holographic coupling equations (3.22a), (3.22b), (3.22c) and (3.22d) match. Crucially, we note that we are neither perturbing the fixed background metric $\eta_{\mu\nu}$ where the conserved energy-momentum tensor of the full system lives, nor adding any external source to the system. Individually in each Universe, we get two conditions each for each of the two sources (boundary metric and scalar source) from the coupling equations. The remaining conditions that we should impose will be that the perturbations must not affect the smooth gluing of the two Universes at $u = 0$. To this end, we will demand that at $u = 0$

$$\begin{aligned} \delta\rho^{\mathrm{UV}}(x) &= \delta\rho^{\mathrm{IR}}(x), \\ \delta\rho'^{\mathrm{UV}}(x) &= -\delta\rho'^{\mathrm{IR}}(x) \quad \text{and} \\ \delta\Phi'^{\mathrm{UV}}(x) &= -\delta\Phi'^{\mathrm{IR}}(x)\,. \end{aligned} \tag{3.30}$$

Note that we have reversed the orientation of the radial direction in the IR universe before gluing as before.[22] In order to ensure the continuity of Φ at $u = 0$, we have to readjust the integration constant for Φ in the IR, i.e introduce an appropriate $\delta\Phi_0(x) \equiv \delta\Phi(u = \infty, x)$. This means that the potential $V^{\mathrm{IR}}(\Phi - \Phi_0 - \delta\Phi_0(x))$ is the

[22] A more diffeomorphism invariant statement is that the induced metrics and the Brown-York tensors on both sides should match at $u = 0$ after we flip the sign of the Brown-York tensor on the IR side.

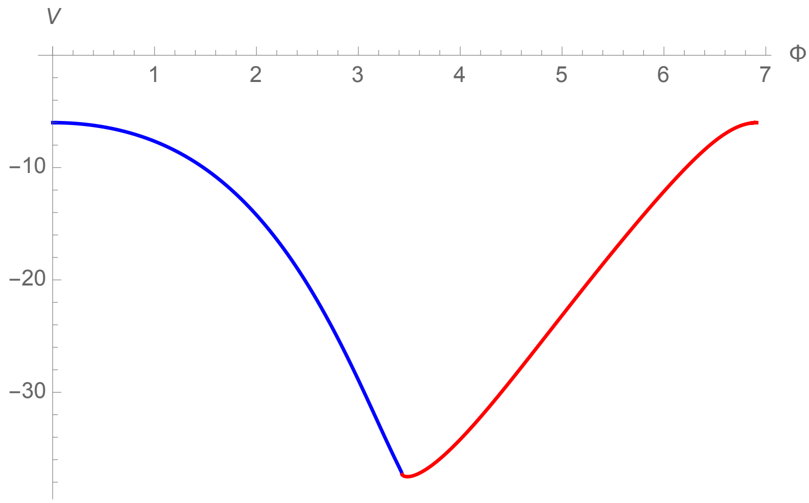

Figure 5. Plot of $V(\Phi)$ (in units $L^{\mathrm{UV}} = 1$) as a function of Φ; the blue curve refers to the UV region while the red one refers to the IR region. The kink in the middle where $V(\Phi)$ has a minima corresponds to $u = 0$ where $\Phi = 3.43$ approximately. Here $V(\Phi)$ is not differentiable. The two asymptotic values of $V(\Phi)$ at the critical points $\Phi = 0$ and $\Phi \approx 6.91$ respectively are -6.

same function as in the vacuum although the IR field Φ has now been redefined.[23] This redefinition means that V^{IR} has no tadpole term. However, this field-redefinition thus affect the definition of the IR theory in a subtle but concrete way. One can check that with these conditions, we can completely determine any fluctuation about the vacuum state and compute the perturbation of the full energy-momentum tensor of the dual system, etc.

It is easy to generalise the above discussion to the case of tensor and vector fluctuations of the vacuum state. Furthermore, we can similarly consider fluctuations of other bulk fields which vanish in the vacuum solution. In order to generalise our construction of excited states which are not small departures from the vacuum, we can still use the general coupling conditions (3.22a), (3.22b), (3.22c) and (3.22d). However, we cannot use $u = 0$ as the matching hypersurface as the domain-wall coordinates in which we have constructed the vacuum solution will be ill-defined beyond some patches of the UV and IR components individually. In the general case, *we postulate that the UV and IR universes should be glued at their edge hypersurfaces where there is no curvature singularity*

[23]Note this canonical V^{IR} which remains state invariant is different from the red curve in Fig. 5. In order to see this form we simply need to compute $V(\Phi - 6.91)$ in the red region.

but beyond which solutions for the matter fields cease to exist. This gluing will then remove the edge singularities in each individual component which arises from geodetic incompleteness as in the vacuum case. In order for the postulate to make sense, *we would require that edge singularities should appear in each component Universe much before any curvature singularity can occur.* Although the matter fields will not exist beyond the edge singularities, the individual UV and IR metrics can also be continued in the unphysical regions as we have seen in the case of the vacuum. Event horizons can lie either in the physical or in the unphysical parts of each component Universe system. At this stage, we are not sure what should be the general thermodynamic description of bi-holographic thermal states, although armed with our well-defined full energy-momentum tensor we can in principle study this question. It will be also fascinating to understand non-equilibrium behaviour of bi-holographic systems. We leave such investigations for the future.

We need to discuss though how we can couple external sources to the full bi-holographic system. Since we have already considered the case of consistent coupling rules when the fixed background metric is an arbitrary curved metric in the previous section, we need to understand only how to introduce other external scalar sources and external gauge potentials. We consider the case of external scalar sources only, as we have not studied the case of vector-type couplings. In presence of an external scalar source $J^{\rm ext}$, we need to modify the general coupling rules (3.22a), (3.22b), (3.22c) and (3.22d) to:

$$e^{4\sigma^{(1)}} = 1 + 4\beta\left(T^{(2)} + O^{(2)}\right), \tag{3.31a}$$

$$e^{4\sigma^{(2)}} = 1 + 4\beta\left(T^{(1)} + O^{(1)}\right), \tag{3.31b}$$

$$J^{(1)} = \frac{1}{4}\ln\left(1 + 4\beta\left(T^{(2)} + O^{(2)}\right)\right) + \alpha O^{(2)} + J^{\rm ext}, \tag{3.31c}$$

$$0 = \frac{1}{4}\ln\left(1 + 4\beta\left(T^{(1)} + O^{(1)}\right)\right) + \alpha O^{(1)} + J^{\rm ext}. \tag{3.31d}$$

One can readily check that as a result of the above, the Ward identity of the full system will be modified to:

$$\partial_\mu T^\mu{}_\nu = O\partial_\nu J^{\rm ext}, \tag{3.32}$$

where $T^\mu{}_\nu$ will be given by the more general expression (2.19) (with $g_{\mu\nu} = \eta_{\mu\nu}$ and $d = 4$) and

$$O = O^{(1)}e^{4\sigma^{(1)}} + O^{(2)}e^{4\sigma^{(2)}}. \tag{3.33}$$

In fact, this gives as a way to define the full operator O of the biholographic (or semiholographic) system as a combination of the individual operators of the two sectors. More generally, O will be[24]

$$O = O^{(1)}\sqrt{\det z^{(1)}} + O^{(2)}\sqrt{\det z^{(2)}}, \tag{3.34}$$

i.e. the sum of the individual operators weighted by the individual volume density factors of the effective metrics (recall $\sqrt{\det z^{(i)}} = \sqrt{\det g^{(i)}}/\sqrt{\det g}$). Thus J^{ext} couples democratically also to both sectors – the relative strengths of the couplings being determined dynamically by the compression/dilation factors of the volume densities of the individual effective metrics as compared to the fixed background metric. With the coupling rules now set by (3.31a), (3.31b), (3.31c) and (3.31d), we can repeat the discussion before about how to compute small perturbations of the biholographic vacuum state and also other states far away from the vacuum.

3.4 The highly efficient RG flow perspective

A natural question to ask is how we can achieve a RG flow description of the biholographic theory. The right framework is indeed *highly efficient RG flow* as introduced in [53, 54] (for a recent short review see [56]) which has been shown to reproduce the traditional holographic correspondence. In particular, this framework will allow us to define a conserved energy-momentum tensor of the full system at each scale without the need for introducing an action formalism. One of the key points of construction of highly efficient RG flow is that we should allow also the background metric $g_{\mu\nu}(\Lambda)$ and sources $J(\Lambda)$ evolve with the scale Λ as a *state-independent* functionals $g_{\mu\nu}[T^{\alpha\beta}(\Lambda), O(\Lambda), \Lambda]$ and $J[T^{\alpha\beta}(\Lambda), O(\Lambda), \Lambda]$ of the scale and effective operators so that at each scale Λ, the Ward identity

$$\nabla_{(\Lambda)\mu} T^{\mu}{}_{\nu}(\Lambda) = O(\Lambda)\nabla_{(\Lambda)\nu} J(\Lambda) \tag{3.35}$$

is satisfied in the effective background $g_{\mu\nu}(\Lambda)$. Thus the effective background metric preserves the Ward identity. Such a RG flow can be non-Wilsonian and an explicit construction can be achieved in the large N limit by defining single-trace operators via collective variables (instead of the elementary quantum fields) which parametrise their expectation values in all states. A highly efficient RG flow leads to a $(d+1)$–dimensional spacetime with $g_{\mu\nu}(\Lambda)$ being essentially identified with $\Lambda^{-2}\gamma_{\mu\nu}$, with $\gamma_{\mu\nu}$ being the induced metric on the hypersurface $r = \Lambda^{-1}$ at a constant value of the radial coordinate that is identified with the inverse of the scale. Furthermore, the dual $(d+1)$– dimensional metric will follow diffeomorphism invariant equations with a specific type of gauge fixing that can be decoded from a deformed form of Weyl invariance associated with the

[24] We can readily verify that in the most general case we need to add J^{ext} both to $J^{(1)}$ and $J^{(2)}$ in (2.31) to obtain the general consistent coupling rules in the presence of an external source J^{ext}. It does not affect the general expression (2.32) of the full energy-momentum tensor, however the Ward identity that it satisfies now should be $\nabla_\mu T^{\mu}{}_{\nu} = O\nabla_\nu J^{\text{ext}}$ in the fixed background metric g.

corresponding highly efficient RG flow.

In the bi-holographic case, the *highly efficient RG flow* construction should work by choosing correlated hypersurfaces Σ_1 and Σ_2 in the UV and IR universes for each scale Λ as shown in Fig. 1. These hypersurfaces should be given by the two equations:

$$\Sigma_1 := u^{\mathrm{UV}} = u^{\mathrm{UV}}(\Lambda, x), \quad \Sigma_2 := u^{\mathrm{IR}} = u^{\mathrm{IR}}(\Lambda, x). \tag{3.36}$$

Furthermore, we can invoke new hypersurface coordinates via diffeomorphisms defined on each hypersurface

$$x'^{\mathrm{UV,IR}} = x'^{\mathrm{UV,IR}}(\Lambda, x). \tag{3.37}$$

Choosing these $2(d+1)$ functions we may be able to define the two hypersurfaces Σ_1 and Σ_2 and also hypersurface coordinates in the UV and IR universes such that *there exists a reference metric $\gamma_{\mu\nu}(\Lambda)$ at each Λ with respect to which the induced metrics $\gamma^{(i)}_{\mu\nu}$ and $\gamma^{(2)}_{\mu\nu}$ on Σ_1 and Σ_2 respectively will be correlated with the general coupling rules* (2.31) *so that the existence of a conserved energy-momentum tensor of the full system at each scale taking the form* (2.32) *in the effective background $\gamma_{\mu\nu}(\Lambda)$ is ensured. The latter follows from the coupling rules because diffeomorphism invariance of the classical gravity equations in each Universe implies that the Brown-York stress tensors (renormalised by covariant counterterms) on each hypersurface is conserved in the background metrics $\gamma^{(1)}_{\mu\nu}(\Lambda)$ and $\gamma^{(2)}_{\mu\nu}(\Lambda)$ respectively.* [25]

This highly efficient RG flow construction is clearly possible only for $d \leq 4$ because otherwise with the $2(d+1)$ functions specifying the hypersurfaces and hypersurface coordinates we may not be able to solve for the right background metric $\gamma_{\mu\nu}(\Lambda)$ which has $d(d+1)/2$ independent components. Furthermore, *the effective hard-soft coupling constants featuring in the general coupling rules* (2.31) *should not only be scale but also be state-dependent except for the case $\Lambda = \infty$.* At $\Lambda = \infty$, the hypersurfaces Σ_1 and Σ_2 are the conformal boundaries of the UV and IR universes respectively. Here the hard-soft couplings remain same as in the vacuum state and indeed these are used to then construct all excited states of the theory as mentioned above. Furthermore, at $\Lambda = \infty$, $\gamma_{\mu\nu}(\Lambda)$ simply coincides with the background metric on which the full system and it's energy-momentum tensor lives by construction. It is not clear if such a RG flow perspective makes sense for $d > 4$, i.e. in bi-AdS spaces with more than 5 dimensions.

[25] Note that actually we also need to define a reference effective source $J^{\mathrm{ext}}(\Lambda)$ along with the reference metric $\gamma_{\mu\nu}(\Lambda)$ and consider modified coupling rules (3.31a)-(3.31d) between $\gamma^{(i)}$s and $J^{(i)}$s on the two hypersurfaces (3.37). We have kept this implicit in this discussion to avoid over-cluttering of words. Also note that $J^{\mathrm{ext}}(\Lambda)$ need not vanish at finite Λ even when it vanishes at $\Lambda = \infty$.

The highly efficient RG flow perspective gives a very coherent view of the full biholographic construction. In particular, by construction it breaks the apparent independent $(d+1)$−diffeomorphism invariance of the two Universes into only one kind of $(d+1)$−diffeomorphism invariance. The invariance of the conservation equation for the full energy-momentum in a reference metric background gives d−constraints. An additional Hamiltonian constraint arises naturally in order to form a first class constraint system. These $(d+1)$−constraints result in having $(d+1)$−diffeomorphism symmetry instead of twice the number.

One more attractive feature of the highly efficient RG flow construction is that one can take the point of view that spacetime emerges from the endpoint of the RG flow corresponding to the horizon of the emergent geometry rather than from the boundary. Imposing that the end point of the RG flow under an universal rescaling of the scale and time coordinate (corresponding to zooming in the long time and near horizon limits of the dual spacetime) can be mapped to a fixed point with a few parameters, we obtain bounds for the first order flows of effective physical observables near the end point such that at the boundary they take the necessary physical values which ensures absence of naked singularities in the dual spacetime [53, 54, 262]. This has been explicitly demonstrated in the context of the hydrodynamic limit of the dynamics in the dual quantum system specially. Taking such a point of view is natural in the bi-holographic context, because it is only at the matching hypersurface ($u = 0$) of the two Universes corresponding to the endpoint of the highly efficient RG flow the two Universes physically overlap and share common data. Therefore, the two Universes naturally emerge from the $u = 0$ hypersurface. In the future, we will like to investigate the RG flow reconstruction of bi-holography and also investigate if one can define c−functions for such RG flows.

4 Concluding remarks

4.1 How to proceed in the case of QCD?

The bi-holographic construction provides an illuminating illustration of how the semi-holographic framework can be derived from first principles, particularly regarding how some simple consistency rules can be used to determine the parameters of the IR holographic theory and the hard-soft couplings in terms of the parameters of the UV theory. Let us discuss briefly how the steps of the construction of bi-holography can be generalised to the case of the semi-holographic framework for QCD.

Firstly, in the case of bi-holography the IR Universe was necessary to cure the edge singularity of the UV Universe, and the smoothness of the gluing between the two Universes was a key principle that determined the parameters of the gravitational theory of the IR Universe as well. In the case of QCD, an analogous issue is the cure the non-Borel resummability of the perturbation series, i.e. we need the non-perturbative (holographic) physics to cancel the *renormalon* Borel poles of perturbation theory that lie on the positive real axis and control large order behaviour of the perturbation series in the large N limit [245, 263]. It is known that each such renormalon pole that appears in the perturbative calculations of the operator product expansion (OPE) of a product of two gauge-invariant operators can be cancelled by invoking a non-perturbative condensate of an appropriate gauge-invariant operator with the right mass dimension [243, 245] as originally observed by Parisi. The further the renormalon pole is from the origin, the larger the mass dimension of the operator whose condensate cancels this pole should be. Furthermore, *the non-perturbative dependence of this condensate on* Λ_{QCD}*, or equivalently on the perturbative strong coupling constant is completely determined by pQCD (in particular by the location of the corresponding Borel pole that gets cancelled).*

This observation of Parisi can be transformed into a physical mechanism via semi-holography. In particular the non-perturbative condensate of a given operator with given mass dimension should be reproduced by the dynamics of the dual holographic bulk field. Since the condensate is determined by perturbation theory, the holographic gravitational theory should be *designed* appropriately in order to reproduce the right behaviour as a function of the confinement scale. Furthermore, the gravitational boundary condition determined by the hard-soft coupling(s) with the corresponding operator of the perturbative sector appearing in the perturbative expansion of the OPE must also be specified in an appropriate way. Such a *designer gravity* approach for designing a holographic gravitational theory and its boundary conditions in order to reproduce right behaviour of the dual condensates has been studied in [264–266]. This approach can be adapted to the semi-holographic construction to determine the holographic theory dual to the non-perturbative sector, and also the hard-soft couplings between the perturbative and the non-perturbative sectors as functions of Λ_{QCD}.

It is clear that the construction of this semi-holographic framework should be far more complicated than in the bi-holographic case. Multiple number of non-perturbative condensates, i.e. operators of the emergent holographic theory should couple to each gauge-invariant operator of the perturbative sector. However, we can proceed systematically by considering the cancellation of perturbative Borel poles in closer proximity to the origin for which we would require non-perturbative condensates of lower mass dimensions only.

The Borel poles of the (appropriately resummed) perturbation theory can shift at scales intrinsic to a non-trivial state (as for instance the temperature). This naturally implies that we need to invoke state-dependence in the running of the hard-soft couplings with the scale. The bi-holographic construction further indicates that we need to do field-redefinitions in the holographic gravitational theory which could be state-dependent although its parameters should be not vary with the state. We need to understand these issues better in the future. However, the arguments presented above indicate that the semi-holographic framework for QCD can indeed be constructed systematically in the large N limit.

4.2 Possible applications of bi-holography

In this chapter, we have invoked bi-holography to illustrate semi-holography. However, it should be worthwhile to pursue the biholographic framework and its applications on its own right. In particular, the bi-holographic framework gives rise to a consistent bi-metric gravitational theory as mentioned before. Such a construction when invoked in the context of positive (instead of negative) cosmological constant, can perhaps also be relevant for shedding light on the origin of dark matter (in the form of matter in the ghost Universe which gives the second covering of the full spacetime). It has also been pointed out in the literature that the possibility that baryonic matter and dark matter can live in different effective metrics can explain late time acceleration of the Universe without invoking the cosmological constant [267]. The visible Universe with baryonic matter, and the coexisting ghost Universe with dark matter and the second metric should be joined at the beginning of time such that each can cure the other's initial-time singularity. It might be interesting to pursue such a cosmological model.

Finally, bi-holography can have applications which are more wide ranging than holography. In particular the effective metrics on which the UV and IR sectors live can have different topologies from the original background metric[26] which should be determined dynamically. This can then serve as examples of theories with hidden topological phases which cannot be captured by local order parameters, and admitting simple geometric descriptions. With such applications in view, it should be interesting to study bi-holographic RG flows, and also thermal and non-equilibrium dynamics in bi-holography.

We conclude with the final remark that we must also pursue if bi-holography and semi-holography can be embedded in the string theoretic framework. This direction of research may extend the horizons of string theory, and may also lead to a deeper and more enriching understanding of the field-theoretic implications of bi-holography and semi-holography.

[26] This is specially relevant when the tensorial hard-soft couplings $\gamma_1 \neq 0$ and/or $\gamma_2 \neq 0$. In non-relativistic bi-holography, we will not be able to set these to zero even in the limit $\Lambda \to \infty$ in order to construct the vacuum state.

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A summary for my curious friend

FOR as long as we can remember, humans have asked questions. Often about the universe(s?) we find ourselves in. We want to know the 'why', the 'what' and the 'how'. This thesis is not ambitious enough to answer any of these questions.

As disappointing as that may be for you, my reader, it is the aim of this thesis to set these questions in a certain context. Using the 'why' as guiding direction, we answer aspects of the 'what' and the 'how'. We concern ourselves with the mathematical and physical descriptions of the most fundamental of interactions that take place in nature, as I write. And in fact, even as you read.

These interactions range from those occurring between the smallest building blocks we know, of nature, to those between the largest astrophysical objects we have come to learn of. The Standard Model of particle physics (built on Quantum Field Theory) has been remarkably successful in describing the small. It has been extremely well tested and experimental evidence continues to pour in, in support of its validity as a good description of the fundamental interactions between the smallest of particles we are aware of, in nature. The massive objects, on the other hand, are extremely well described by Einstein's theory of General Relativity. Putting the small and the large together, however, has turned out to be a considerable challenge. Most glaringly when it came to the study of black holes; these are solutions to Einstein's theory which have been confirmed to exist in nature, albeit through indirect detection.

Black holes Classically, a black hole is a massive 'hole'—expected to be found at the center of every galaxy in our universe—into which one could only fall in and never escape. A few decades ago, however, it was discovered that black holes radiate heat; one could escape after all, but not before being turned into heat. And in fact, that laws of black holes could be reformulated as laws of thermodynamics! A box of gas, for instance, can be characterized by only a few parameters like temperature, volume, density and mass. Nevertheless, we know that the box contains many tiny particles with small masses colliding against each other to generate the total mass and temperature. Through these tiny particles, thermodynamic systems exhibit a quantum statistical structure. And therefore, so must black holes.

These indicatives spurred extensive research on the so-called 'information paradox'.

The premise of which is that in a collapse of large amounts of matter forming a black hole, a small number of physical parameters (like mass, charge, angular momentum, etc.) characterize the black hole. However, it was found that these black holes radiate as black bodies. That is, the heat coming from them does not appear to remember what the black hole was made of. One formed by a collapsing shell of tables and another by a shell of chairs could not be told apart! In stark contrast, quantum mechanics postulates that no information can be lost in physical processes. There began the most fierce of competitions: between General theory of Relativity and Quantum Mechanics. For this reason, a microscopic study of black holes has received significant attention. There are two obvious aspects that need scrutiny: the static and the dynamical. The former unveils how big a black hole is and 'how many tiny particles' it takes to produce a massive one, for instance. While the latter is concerned with how these tiny particles interact with each other to produce mysterious dynamics that solve the information paradox.

Static aspects of black hole physics are often easiest to study when there is sufficient symmetry in the game. Think of a balloon. Imagine we said a balloon was approximately spherical. That would leave a child the freedom to poke it a little, see how it responds and play around with it. Imagine we said that the balloon must be exactly spherical all the time. Any touch is going to distort its shape (even if only ever so slightly) and so that does not leave the child any room for play. Nevertheless, it is when nobody touches the balloon that it is easiest to study! That is when we know exactly how to tell its shape; it is spherical. Knowing just the radius of the sphere, we would know its precise surface area and even the volume of air contained inside the balloon. In fact, demanding spherical symmetry and fixing one additional parameter (say the size of the balloon) allows us to completely determine how it behaves with time: exactly nothing would change and the balloon would be a thousand years later just as it is now. Of course, the most fun might be to pierce the balloon with a pin and see it explode! But that is arbitrarily far from any symmetric process—unfortunately rendering it too difficult to write down, say, an evolution equation for.

On account of similar logic, the more symmetric they are, the easier the black holes are to study. The completely static ones often possess more symmetry and are called 'supersymmetric black holes'. They are stable; in fact more so than the balloon. A little poke does not disturb a supersymmetric black hole. The ones that may be disturbed a little while still allowing us some control are often non-supersymmetric. But they tend to retain most of their character upon disturbing them by, say, throwing a particle in; these are called 'large semi-classical black holes' in technical jargon. To burst the balloon is to watch a black hole evaporate entirely. In fact, the reverse process of creating a small black hole is rather exciting too. But again, such fun does not come easy. And in this thesis, we either let the black holes be or poke them a little, ever so slightly. For good or for bad, we will neither create nor destroy them.

String Theory has offered a spectacular framework to study static aspects of highly symmetric black holes. It postulates that the many vibrations of many many little strings turn out to produce large black holes! In the case where black holes are static and supersymmetric, we use string theory to attempt a complete and precise understanding of the 'tiny particles' that render them with a mass; this is presented in Chapter II. Further on, in Chapter III with a view towards what happens when we throw particles into them, when the black holes are large and semi-classical, we still aim at understanding their mass, but admittedly with less precision. As patience catches up with us on how unwieldy studying dynamics is, within a controlled string theoretic setting, we then turn the plate upside down to move to our interest in understanding black holes that are closer to those found in nature. These black holes, being far from supersymmetric, have rich dynamics which we study with the help of quantum mechanics. We show, in Chapter IV, that black holes behave very much like some of the simplest quantum mechanical models that are often studied at an undergraduate level! These two contrasting approaches may neatly converge to help us understand the complete underlying story in the long-run. Conveniently enough for the impatient, however, an intermediate probe has emerged in recent decades, via a study of 'strongly coupled gauge theories'.

Strongly coupled gauge theories Some of the smallest particles that we know nature is built out of, are quarks. They interact with each other via the Strong force. Of the many remarkable successes of 20^{th} century theoretical physics, the theory of Quantum Chromodynamics (QCD) explaining these interactions is among the finest. This theory was discovered to possess stunning features. At very short distances, quarks behave like free particles, almost oblivious to each other. At large distances, however, they behave rather collectively and can hardly do without each other! Much like the modern world, one might wonder: up close, we are often oblivious to our fellow shoppers in the supermarket but when viewed from far above, we appear to shop collectively as a group in the same building. Experimentally tested descriptions for both extremes have since been successfully developed. However, an understanding of this theory at all intermediate distances has proven to be a big challenge. Studying the limit of large number of particles (called the large N limit) has opened up an unprecedented set of tools to study QCD-like theories, even if in a simplified setting. It paved way for an entirely new way of studying gravitational dynamics via field theoretic tools and vice-versa. It was discovered that some QCD (like) theories often describe the physics of gravity at large distances. In fact, some of the most incisive contributions to our modern understanding of the microscopics of black hole physics are owed to the study of strongly coupled gauge theories in the large N limit. This field of research goes by the name of gauge-gravity duality or holographic duality. An obvious shortcoming of this field of study as it stands today is that the gravitational universes we are able to explore are far from the real world. Nevertheless, it provides a fantastic framework to ask tractable conceptual questions.

One such outstanding question is, how or where does gravity emerge from? As it turns out, this question is intricately linked to understand the gauge theories in various regimes, not just at very short and large distances. In view of addressing this question, in Chapter V, we construct a toy theory that captures physics corresponding to two different, interacting gauge theories, one describing very short-distance effects while the other, very long-range physics. An artist's illustration of this theory may be seen in Fig. V.1 where the two gravitational theories live on both ends of the 'tube'. They interact where the red and blue universes meet! We see that asking why gravitational physics can emerge out of field theories enables us to address how this happens in an illustrative toy-model.

Samenvatting

SINDS mensenheugenis hebben we vragen gesteld. Vaak gaan deze vragen over het heelal waarin we ons bevinden. We willen het 'waarom', het 'wat' en het 'hoe' doorgronden. Dit proefschrift is niet ambitieus genoeg om deze vragen te beantwoorden.

Zo teleurstellend als dit wellicht is voor jou, mijn lezer, is het doel van dit proefschrift om deze vragen in een bepaalde context te zetten. Met behulp van het 'waarom' als leidraad beantwoorden we aspecten van het 'wat' en het 'hoe'. We houden ons bezig met de wiskundige en natuurkundige beschrijvingen van de meest fundamentele processen die plaatsvinden in de natuur, altijd en overal, ook terwijl ik dit schrijf en wanneer u het leest.

Deze variëren van de wisselwerkingen die zich voordoen tussen de kleinste bouwstenen die in de natuur bekend zijn tot die tussen de grootste astrofysische objecten waar we van weten. Het Standaard Model van de deeltjesfysica is een theorie die is gebaseerd op kwantumveldentheorie en opmerkelijk succesvol is in het beschrijven van het allerkleinste. Het is reeds zeer nauwkeurig getest, en er komt nog altijd meer experimenteel bewijs binnen ter ondersteuning van haar geldigheid als een goede beschrijving van de fundamentele wisselwerkingen tussen de kleinste deeltjes waar we ons bewust van zijn in de natuur. De allergrootste, zeer massieve voorwerpen worden daarentegen zeer goed beschreven door Einstein's algemene relativiteitstheorie, die eveneens wordt ondersteund door een hoop experimenteel bewijs. Het verenigen van de theorieën van het allerkleinste en het allergrootste is echter een geweldige theoretische uitdaging gebleken. Dit geldt al helemaal als het gaat om zwarte gaten, astrofysische objecten die worden voorspeld door de theorie van Einstein en ook daadwerkelijk zijn gevonden in het heelal, zij het via indirecte waarnemingen.

Zwarte gaten Klassiek, dus niet kwantummechanisch, gezien is een zwart gat een massief 'gat'—waarvan men verwacht dat zich er een bevindt in het midden van elk sterrenstelsel in ons universum—waar men alleen in kan vallen en zelfs licht nooit uit kan ontsnappen. Enkele tientallen jaren geleden voorspelden natuurkundigen echter dat zwarte gaten toch warmte uitstralen; men kon toch uit een zwart gat ontsnappen maar helaas niet voordat men is omgezet in warmte. Bovendien werd ontdekt dat de natuurkundige wetten die zwarte gaten beschrijven ook kunnen worden geformuleerd als de wetten van de thermodynamica! Laten we kort bij die laatste stilstaan om te begrijpen wat dit impliceert. Een gas in een doos kan eenvoudig worden gekenmerkt door een klein aantal parameters zoals de temperatuur, volume, dichtheid en massa. Niettemin weten we dat de doos in feite op 'microscopisch' niveau vele kleine kwantummechanische deeltjes bevat, met kleine massa's die tegen elkaar botsen en zo de totale massa, temperatuur en andere statistische eigenschappen van het gas voortbrengen. Meer algemeen hebben thermodynamische systemen die uit kleine deeltjes bestaan eveneens een kwantum-statistische structuur. Als we dit toepassen op de ideeën van Hawking dan moeten zwarte gaten ook een kwantum-statistische structuur vertonen.

Deze theoretische aanwijzingen hebben geleid tot uitgebreid onderzoek naar de zogenaamde 'informatieparadox'. Het uitgangspunt daarvan is dat in de vorming van een zwart gat een klein aantal fysieke parameters (zoals de massa, lading, impulsmoment, etc.) volstaan om het zwarte gat te karakteriseren. Er werd vastgesteld dat deze zwarte gaten warmte uitstralen als zwarte lichamen in de thermodynamica. Dat wil zeggen dat de warmte van zwarte gaten niet lijkt te herinneren van welke ingewikkelde combinatie van ingrediënten het zwarte gat oorspronkelijk gemaakt is. Het is dus niet mogelijk om een zwart gat dat gevormd is door een ineenstortende schil van tafels te onderscheiden van een die ontstaan is uit een schil van stoelen! Dit staat in schril contrast met de kwantummechanica, die stelt dat er geen informatie verloren kan gaan in natuurkundige processen. Deze paradox leidde tot een felle 'oorlog' tussen aanhangers van Einstein's algemene relativiteitstheorie en die van de kwantummechanica. Daarom heeft de microscopische studie van zwarte gaten veel aandacht gekregen. Er zijn twee duidelijke aspecten die toetsing nodig hebben: de statische en de dynamische. De eerste onthult bijvoorbeeld hoe groot een zwart gat is en 'hoeveel microtoestanden' nodig zijn om één massief zwart gat te produceren, terwijl de laatste betrekking heeft op hoe deze kleine deeltjes samen de mysterieuze dynamiek produceren die de informatieparadox oplost.

Statische aspecten van zwarte gaten zijn vaak het gemakkelijkst te bestuderen als er voldoende symmetrie in het spel is. Denk aan een ballon. Stel dat we zeggen dat een ballon ongeveer, maar niet precies, bolvormig is. Dat zou een kind de vrijheid laten om het een beetje te porren, te zien hoe het reageert en ermee te spelen. Stel nu dat we zeggen dat de ballon de hele tijd precies bolvormig moet zijn. Elke aanraking verandert de vorm, al is het maar een heel klein beetje, dus dit laat het kind geen ruimte om ermee te spelen. Toch is de tweede situatie, wanneer niemand de ballon aanraakt, het gemakkelijkst te bestuderen! Dat is zo omdat we precies weten hoe we de vorm kunnen beschrijven: het is exact bolvormig. Wanneer we de straal van de bol weten dan zouden we zijn precieze oppervlakte kennen en zelfs de luchtinhoud. In feite stelt de veeleisende sferische symmetrie, samen met de vaststelling van een extra parameter (zeg de omvang van de ballon), ons in staat om volledig te bepalen hoe de ballon zich gedraagt in de tijd: er zou helemaal niets veranderen, en de ballon zou een duizend jaar later net zo zijn als het nu is. Natuurlijk zou het het leukste zijn om in de ballon te prikken met een speld en het te zien ontploffen! Maar dat is helaas zeer ver van alle symmetrische processen—waardoor het te moeilijk is om te beschrijven wat er dan precies gebeurt.

Uit zulke redeneringen volgt dat hoe meer symmetrie zwarte gaten hebben des te makkelijker het is om ze te bestuderen. De compleet statische zwarte gaten beschikken vaak over meer symmetrie en heten ‘supersymmetrische zwarte gaten’. In tegenstelling tot de perfect bolvormige ballon zijn zulke geïdealiseerde zwarte gaten nog steeds erg interessant omdat ze ons een inkijkje bieden in de informatieparadox. Ze zijn erg stabiel; in feite nog meer dan de ballon. Een kleine por verstoort een supersymmetrisch zwarte gat niet. De gaten die we nog steeds kunnen beïnvloeden zonder de controle te verliezen zijn vaak niet-supersymmetrisch. Maar ze hebben de neiging om het grootste deel van hun eigenschappen te behouden als we er, laten we zeggen, een deeltje in werpen. Deze gaten worden ‘grote semiklassieke zwarte gaten’ genoemd in vakjargon. Het analogon van het laten knallen van de ballon is hier het kijken naar het geheel verdampen van een zwart gat. In feite is het omgekeerde proces, van het creëren van een klein zwart gat, ook best wel spannend. Maar nogmaals: zulke leuke dingen zijn erg lastig om te bestuderen. In dit proefschrift laten we de zwarte gaten ofwel met rust of we porren ze een heel klein beetje: we zullen ze helaas niet maken of vernietigen.

Snaartheorie biedt een spectaculair kader om statische aspecten van zeer symmetrische zwarte gaten te bestuderen. Het veronderstelt dat grote zwarte gaten op microscopische schaal kunnen worden beschreven door vele kleine trillende snaren. In het geval dat de zwarte gaten statisch en supersymmetrisch zijn gebruiken we snaartheorie om een compleet en nauwkeurig inzicht te proberen te geven van de microscopische bouwstenen, de ‘kleine deeltjes’, die zwarte gaten hun massa geven; dit wordt gepresenteerd in Hoofdstuk II. Later, in Hoofdstuk III, waarin we bestuderen wat er gebeurt als we deeltjes in grote en semiklassieke zwarte gaten gooien, zijn we eveneens gericht op het begrijpen van hun massa, zij het op een minder precieze manier. Het is echter nog steeds een zeer ingewikkeld vraagstuk om de dynamiek van grote semiklassieke zwarte gaten te doorgronden binnen het zeer gecontroleerde snaartheoretisch kader. Omdat we hier te ongeduldig voor zijn verleggen we vervolgens onze focus naar het begrijpen van meer realistische zwarte gaten die meer lijken op astrofysische zwarte gaten. Deze zwarte gaten, die verre van supersymmetrische zijn, hebben een rijke dynamiek die we bestuderen met de hulp van de kwantummechanica. In Hoofdstuk IV laten we zien dat zulke zwarte gaten erg lijken op de meest eenvoudige kwantummechanische systemen, zoals de situaties die studenten al in hun eerste jaren van hun studie natuurkunde bestuderen! Deze twee contrasterende benaderingen kunnen elkaar mooi aanvullen en ons zo helpen op de lange termijn het volledige onderliggende verhaal te begrijpen. Gelukkig voor de ongeduldige is er in de afgelopen decennia echter een tussenliggende route gekomen door middel van de studie van de ‘sterk gekoppelde ijktheorieën’.

Sterk gekoppelde ijktheorieën Quarks maken deel uit van de kleinst bekende deeltjes waaruit de natuur is opgebouwd. Ze communiceren met elkaar via de sterke kernkracht. Een van de mooiste theorieën van de vele opmerkelijke successen van de theoretische natuurkunde in de twintigste eeuw is de kwantumchromodynamica (quantum chromodynamics, QCD) voor het beschrijven van deze wisselwerkingen. Deze theorie bleek verbluffende eigenschappen te bezitten. Op zeer korte afstanden gedragen quarks zich vrije deeltjes die zich bijna niet bewust zijn van elkaar. Op grotere afstanden gedragen ze zich echter meer collectief en kunnen ze nauwelijks zonder elkaar! Men zou dit kunnen vergelijken met de moderne wereld: als individuen, op kleine schaal, zijn we ons vaak niet bewust van de andere mensen die winkelen in de supermarkt, maar van verder bovenaf gezien lijken we collectief te winkelen als één groep in hetzelfde gebouw. Er zijn inmiddels succesvolle beschrijvingen van deze twee uitersten van QCD ontwikkeld en experimenteel getoetst. Maar het begrip van QCD op gematigde afstanden, tussen de uitersten, blijkt nog steeds een grote uitdaging te zijn. Het bestuderen van de limiet van een groot aantal deeltjes, de grote-N limiet genaamd, heeft echter een ongekende gereedschapsset gegeven om QCD-achtige theorieën te bestuderen, zij het in een vereenvoudigde omgeving. Verrassend genoeg heeft dit ook een geheel nieuwe manier mogelijk gemaakt om de dynamische aspecten van zwaartekracht te bestuderen via veldtheoretische methodes, en omgekeerd. Er werd ontdekt dat sommige QCD-achtige theorieën vaak de natuurkunde van de zwaartekracht op grote afstand beschrijven. In feite zijn we sommige van de meest inzichtelijke bijdragen aan ons moderne begrip van de microscopische aspecten van zwarte gaten verschuldigd aan de studie van sterk gekoppelde ijktheorieën in de grote-N limiet. Dit onderzoeksgebied staat bekend als de 'ijktheorie-zwaartekracht dualiteit' of 'holografische dualiteit'. Een duidelijke tekortkoming van dit vakgebied in zijn huidige vorm is dat de gravitationele universa die we in staat zijn te verkennen ver van de echte wereld af staan. Toch biedt het een fantastisch kader om interessante conceptuele vragen te stellen en beantwoorden.

Een voorbeeld van zo een uitstekende vraag is hoe of waar de zwaartekracht vandaan komt. Het blijkt dat deze vraag onlosmakelijk verbonden is met het begrijpen van ijktheorieën op gematigde afstanden, en niet alleen in de uitersten van zeer korte en lange afstanden. Met het oog op de aanpak van deze vraag construeren we in hoofdstuk V een stuk 'speelgoedtheorie' dat de bijpassende natuurkunde die overeenkomt vangt aan twee verschillende, interactie onderling ijktheorieën in zich draagt: een beschrijving van zeer-korte-afstand effecten en de zeer-lange-afstand fysica. Een illustratie van deze theorie kan worden gezien in Fig. V.1 waar de twee zwaartekracht(ijk)theorieën leven aan beide uiteinden van de 'buis'. De theorieën communiceren met elkaar waar de rode en blauwe universa elkaar ontmoeten! We zien dat de vraag 'waarom en hoe kan de natuurkunde van de zwaartekracht ontstaan uit veldentheorie?' ons in staat stelt om na te gaan hoe dit precies gebeurt in een illustratief speelgoedmodel.

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