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OPTIMAL EXPANSIONS IN NON-INTEGER BASES

KARMA DAJANI, MARTIJN DE VRIES, VILMOS KOMORNIK, AND PAOLA LORETI

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Abstract. For a given positive integer m, let $A = \{0, 1, \ldots, m\}$ and $q \in (m, m+1)$. A sequence $(c_i) = c_1c_2\ldots$ consisting of elements in A is called an expansion of x if $\sum_{i=1}^{\infty} c_i q^{-i} = x$. It is known that almost every x belonging to the interval $[0, m/(q-1)]$ has uncountably many expansions. In this paper we study the existence of expansions (d_i) of x satisfying the inequalities $\sum_{i=1}^{n} d_i q^{-i} \geq \sum_{i=1}^{n} c_i q^{-i}$, $n = 1, 2, \ldots$, for each expansion (c_i) of x.

1. Introduction

Let $x \in [0, 1)$. The decimal expansion

$$x = \frac{b_1}{10} + \frac{b_2}{10^2} + \frac{b_3}{10^3} + \cdots ,$$

where we choose a finite expansion whenever it is possible, has a well-known "each-step" optimality property: for each $k = 1, 2, \ldots$, among all finite sequences $c_1 \ldots c_k$ of integers with $0 \leq c_i \leq 9$ for $i = 1, \ldots, k$, satisfying the inequality $\sum_{i=1}^{k} c_i 10^{-i} \leq x$, the sum $\sum_{i=1}^{k} b_i 10^{-i}$ is the closest to x. An analogous property holds for expansions in all integer bases $2, 3, \ldots$.

In his celebrated paper [16], Rényi generalized these expansions to arbitrary real bases $q > 1$ as follows. If $b_1, \ldots, b_{n-1}$ have already been defined for some $n \geq 1$ (no condition for $n = 1$), then let b_n be the largest integer satisfying the inequality

$$\frac{b_1}{q} + \cdots + \frac{b_n}{q^n} \leq x.$$

One may readily verify that

$$\sum_{i=1}^{\infty} \frac{b_i}{q^i} = x;$$

it is called the *greedy* expansion of x in base q.

The purpose of this paper is to show that the natural analogue of the above optimality property fails for most non-integer bases, but it still holds for a particular

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countable set of bases, the smallest of them being the golden ratio $q = (1+\sqrt{5})/2 \approx 1.618$. Before formulating our result precisely we will first introduce expansions of real numbers with respect to a more general set of digits.

Given a real number $q > 1$ and a finite *alphabet* or *digit set* $A = \{a_0, \dots, a_m\}$ consisting of real numbers satisfying $a_0 < \cdots < a_m$, by an *expansion* of x (in *base* q with respect to A) we mean a sequence (c_i) of *digits* $c_i \in A$ satisfying

$$\sum_{i=1}^{\infty} \frac{c_i}{q^i} = x. \tag{1.1}$$

Pedicini [15] proved the following basic result on the existence of such expansions.

Proposition 1.1. *Each* $x \in J_{A,q} := [a_0/(q-1), a_m/(q-1)]$ *has an expansion if and only if*

$$\max_{1 \le j \le m} (a_j - a_{j-1}) \le \frac{a_m - a_0}{q-1}. \tag{1.2}$$

For the convenience of the reader we provide an elementary proof of this proposition. Observe that (c_i) is an expansion of x in base q with respect to A if and only if $(c_i - a_0) = (c_1 - a_0)(c_2 - a_0)\dots$ is an expansion of $x - a_0/(q-1)$ in base q with respect to the alphabet $\{0, a_1 - a_0, \dots, a_m - a_0\}$. Moreover, the inequality (1.2) holds if and only if the same inequality holds with $a_j - a_0$ in place of a_j, $0 \le j \le m$. Hence we may (and will) assume in the rest of this paper that $a_0 = 0$.

Proof of Proposition 1.1. First assume that the inequality (1.2) holds. We define recursively a sequence (b_i) with digits b_i belonging to A by applying the following *greedy algorithm*: if for some integer $n \in \mathbb{N} := \{1, 2, \dots\}$ the digits b_i have already been defined for all $1 \le i < n$ (no condition for $n = 1$), then let b_n be the largest digit in A satisfying the inequality $\sum_{i=1}^{n} b_i q^{-i} \le x$. Note that this algorithm is well defined for each $x \ge 0$. We show that (b_i) is an expansion of x for each x belonging to $J_{A,q}$.

If $x = a_m/(q-1)$, then the greedy algorithm provides $b_i = a_m$ for all $i \ge 1$, whence (b_i) is indeed an expansion of x.

If $0 \le x < a_m/(q-1)$, then there exists an index n such that $b_n < a_m$. If $b_n < a_m$ for infinitely many n, then for each such n we have

$$0 \le x - \sum_{i=1}^{n} \frac{b_i}{q^i} < \frac{\max_{1 \le j \le m}(a_j - a_{j-1})}{q^n}.$$

Letting $n \to \infty$, we see that (b_i) is an expansion of x. Next we show that there cannot be finitely many n such that $b_n < a_m$. Indeed, if there were a last index n with $b_n = a_j < a_m$, then

$$\left(\sum_{i=1}^{n} \frac{b_i}{q^i}\right) + \sum_{i=n+1}^{\infty} \frac{a_m}{q^i} \le x < \left(\sum_{i=1}^{n} \frac{b_i}{q^i}\right) + \frac{a_{j+1} - a_j}{q^n}$$

or equivalently

$$\frac{a_m}{q-1} < a_{j+1} - a_j,$$

contradicting (1.2).

Finally, if condition (1.2) does not hold and $a_\ell - a_{\ell-1} > a_m/(q-1)$ for some $\ell \in \{1, \dots, m\}$, then none of the numbers belonging to the non-empty interval

$$\left(\frac{a_{\ell-1}}{q} + \sum_{i=2}^{\infty} \frac{a_m}{q^i}, \frac{a_\ell}{q} \right) \subset J_{A,q}$$

has an expansion. □

The proof of Proposition 1.1 shows that if (1.2) holds, then each $x \in J_{A,q}$ has a lexicographically largest expansion $(b_i(x, A, q))$, which we call the *greedy expansion* of x. The *normalized errors* of an arbitrary expansion (c_i) of x are defined by

$$\theta_n((c_i)) := q^n \left(x - \sum_{i=1}^{n} \frac{c_i}{q^i} \right), \quad n \in \mathbb{N}.$$

We call an expansion (d_i) of x *optimal* if $\theta_n((d_i)) \le \theta_n((c_i))$ for each $n \in \mathbb{N}$ and each expansion (c_i) of x. It follows from the definitions that only the greedy expansion of a number $x \in J_{A,q}$ can be optimal. The following example shows that the greedy expansion of a number $x \in J_{A,q}$ is not always optimal. Other examples can be found in [3].

Example 1.2. Let $A = \{0, 1\}$ and $1 < q < (1 + \sqrt{5})/2$. The sequence $(c_i) := 011(0)^\infty$ is clearly an expansion of $x := q^{-2} + q^{-3}$. Applying the greedy algorithm we find that the first three digits of the greedy expansion $(b_i) = (b_i(x, A, q))$ of x equal 100. Hence $\theta_3((b_i)) > \theta_3((c_i)) = 0$.

Let $A = \{0, 1, \dots, m\}$ and $q \in (m, m + 1)$ for some positive integer m. Proposition 1.1 implies that in this case each $x \in J_{A,q}$ has an expansion. Let P be the set consisting of those bases $q \in (m, m + 1)$ which satisfy one of the equalities

$$1 = \frac{m}{q} + \cdots + \frac{m}{q^n} + \frac{p}{q^{n+1}}, \quad n \in \mathbb{N} \text{ and } p \in \{1, \dots, m\}.$$

We have the following dichotomy:

Theorem 1.3.

(i) *If $q \in P$, then each $x \in J_{A,q}$ has an optimal expansion.*
(ii) *If $q \in (m, m + 1) \setminus P$, then the set of numbers $x \in J_{A,q}$ with an optimal expansion is nowhere dense and has Lebesgue measure zero.*

In Section 2 we compare greedy expansions with respect to different alphabets. This gives us a characterization of optimal expansions which is essential to our proof of Theorem 1.3 in Section 3. In Section 4 we briefly discuss optimal expansions of real numbers in negative integer bases.

2. Greedy expansions

Consider an alphabet $A = \{a_0, a_1, \dots, a_m\}$ $(0 = a_0 < \cdots < a_m)$ and a base q satisfying the condition (1.2) as in the preceding section. Let the *greedy transformation* $T : J_{A,q} \to J_{A,q}$ corresponding to (A, q) be given by

$$T(x) := \begin{cases} qx - a_j & \text{if } x \in C(a_j) := \left[\frac{a_j}{q}, \frac{a_{j+1}}{q}\right), \ 0 \le j < m, \\ qx - a_m & \text{if } x \in C(a_m) := \left[\frac{a_m}{q}, \frac{a_m}{q-1}\right]. \end{cases}$$

Observe that $b_i(x, A, q) = a_j$ if and only if $T^{i-1}(x) \in C(a_j)$, $i \ge 1$.

For any fixed positive integer k, the equation (1.1) can be rewritten in the form

$$\frac{d_1}{q^k} + \frac{d_2}{q^{2k}} + \cdots = x$$

by setting

$$d_i := \sum_{j=0}^{k-1} c_{ik-j} q^j, \quad i = 1, 2, \ldots.$$

In other words, every expansion in base q with respect to the alphabet A can also be considered as an expansion in base q^k with respect to the alphabet

$$A_k := \left\{ c_1 q^{k-1} + \cdots + c_k \ : \ c_1, \ldots, c_k \in A \right\}.^1$$

(For $k = 1$ this reduces to the original case.) In particular we have

$$J_{A_k, q^k} = J_{A,q}$$

for every k. We may therefore compare the greedy transformation T_k corresponding to (A_k, q^k) with the k-th iteration T^k of the map T corresponding to (A, q). It is easily seen that $T_k(x) \le T^k(x)$ for each $x \in J_{A,q}$, but in general we do not have equality here.

Given (A, q) and a positive integer k, we denote by $S_{A,q,k}$ the set of sequences $(c_1, \ldots, c_k) \in A^k$ satisfying the following condition: if $(d_1, \ldots, d_k) \in A^k$ and $(d_1, \ldots, d_k) > (c_1, \ldots, c_k)$, then

$$\sum_{i=1}^k \frac{d_i}{q^i} \ne \sum_{i=1}^k \frac{c_i}{q^i}.$$

For each $x \in J_{A,q}$, the sequence $b_1(x, A, q) \ldots b_k(x, A, q) 0^\infty$ is the greedy expansion in base q with respect to A of the number

$$\sum_{i=1}^k \frac{b_i(x, A, q)}{q^i}$$

as follows from the definition of the greedy algorithm. Hence

$$S_{A,q,k} \supset \{(b_1(x, A, q), \ldots, b_k(x, A, q)) : x \in J_{A,q}\}.$$

Let the injective map $f : S_{A,q,k} \to J_{A,q}$ be given by

$$f((c_1, \ldots, c_k)) = \frac{c_1}{q} + \cdots + \frac{c_k}{q^k}, \quad (c_1, \ldots, c_k) \in S_{A,q,k}. \tag{2.1}$$

Proposition 2.1. *The following statements are equivalent.*

(i) *The map f is increasing.*
(ii) $T_k = T^k$.
(iii) $S_{A,q,k} = \{(b_1(x, A, q), \ldots, b_k(x, A, q)) : x \in J_{A,q}\}$.

Proof. (i) $\Rightarrow$ (ii). Given any $x \in J_{A,q}$, let $(c_1, \ldots, c_k)$ be the lexicographically largest sequence in A^k satisfying

$$s := \frac{c_1}{q} + \cdots + \frac{c_k}{q^k} \le x.$$

Then $(c_1, \ldots, c_k) \in S_{A,q,k}$, and (i) implies that $T_k(x) = q^k(x - s)$. On the other hand, we also have $T^k(x) = q^k(x - s)$ by definition of the greedy expansion.

[1] Other aspects of expansions with respect to alphabets of the form A_k are studied in [4], [11].

(ii) ⇒ (iii). Assume that $(c_1,\dots,c_k)\in S_{A,q,k}$, and let

$$x' := \sum_{i=1}^{k}\frac{c_i}{q^i}.$$

If we had $(c_1,\dots,c_k)\notin\{(b_1(x,A,q),\dots,b_k(x,A,q)) : x\in J_{A,q}\}$, then there would exist an index $m>k$ such that $b_m(x',A,q)\neq 0$, whence $T_k(x')=0<T^k(x')$, contradicting (ii).

(iii) ⇒ (i). As already observed, the sequence $b_1(x,A,q)\dots b_k(x,A,q)0^\infty$ is the greedy expansion of the number

$$\sum_{i=1}^{k}\frac{b_i(x,A,q)}{q^i}.$$

It remains to note that $x<y$ if and only if $(b_i(x,A,q))<(b_i(y,A,q))$ for numbers x and y belonging to $J_{A,q}$. □

Remarks 2.2.

(i) Observe that the maps T_k and T^k are continuous from the right. Hence if $T_k\neq T^k$, then the maps T_k and T^k differ on a whole interval.

(ii) If $T_k\neq T^k$, then $T_n\neq T^n$ for all $n\geq k$. In order to prove this, it is sufficient to show that $T_{k+1}\neq T^{k+1}$. By Proposition 2.1 there exist two sequences $(b_1,\dots,b_k)$, $(c_1,\dots,c_k)$ both belonging to $S_{A,q,k}$ such that $(b_1,\dots,b_k)<(c_1,\dots,c_k)$ and

$$\sum_{i=1}^{k}\frac{b_i}{q^i}>\sum_{i=1}^{k}\frac{c_i}{q^i}.$$

Note that the sequences $(a_m,b_1,\dots,b_k)$ and $(a_m,c_1,\dots,c_k)$ both belong to $S_{A,q,k+1}$, and

$$\frac{a_m}{q}+\sum_{i=1}^{k}\frac{b_i}{q^{i+1}}>\frac{a_m}{q}+\sum_{i=1}^{k}\frac{c_i}{q^{i+1}}.$$

Applying Proposition 2.1 once more, we reach the desired conclusion.

3. Proof of Theorem 1.3

Let m be a given positive integer. Throughout this section we consider expansions with respect to the alphabet $A=\{0,1,\dots,m\}$ in a base q belonging to $(m,m+1)$. For any integers $n\geq 1$ and $0\leq p\leq m$ we denote by $q_{m,n,p}$ the positive solution of the equation

$$1=\frac{m}{q}+\dots+\frac{m}{q^n}+\frac{p}{q^{n+1}}.$$

We have

$$m=q_{m,1,0}<\dots<q_{m,1,m}=q_{m,2,0}<\dots<q_{m,2,m}=q_{m,3,0}<\cdots$$

and

$$q_{m,n,p}\to\ m+1\quad\text{if}\quad n\to\infty.$$

Recall that the set P introduced in Section 1 consists of the numbers $q_{m,n,p}$ with $n\geq 1$ and $1\leq p\leq m$.

Proposition 3.1. *Let $n\geq 1$ and $1\leq p\leq m$.*

(i) *If* $q = q_{m,n,p}$*, then* $T_k = T^k$ *for all* $k \ge 1$.
(ii) *If* $q_{m,n,p-1} < q < q_{m,n,p}$*, then* $T_k = T^k$ *if and only if* $k \le n+1$.
(iii) *If* $q \in (m, m+1) \setminus P$*, then there exists a positive integer* $k = k(q)$ *such that the maps* T_k *and* T^k *differ on an interval contained in* $[0,1)$.

Proof. (i) By Proposition 2.1 it is sufficient to prove that if

$$(c_1, \dots, c_k), (d_1, \dots, d_k) \in S_{A,q,k} \quad \text{and} \quad (c_1, \dots, c_k) > (d_1, \dots, d_k),$$

then

$$\sum_{i=1}^{k} \frac{c_i}{q^i} > \sum_{i=1}^{k} \frac{d_i}{q^i}. \tag{3.1}$$

Let j be the first index such that $c_j > d_j$. Since $q = q_{m,n,p}$, the elements of $S_{A,q,k}$ do not contain any block of the form $am^n b$ with $a < m$ and $b \ge p$. Indeed, the sum corresponding to such a block is the same as the sum corresponding to the lexicographically larger block $(a+1)0^n(b-p)$. Therefore, since $d_j < m$, a block of the form $m^n b$ with $b \ge p$ cannot occur in $(d_{j+1}, \dots, d_k)$. This implies that if $d_{\ell+1} \dots d_{\ell+n+1}$ is a block of length $n+1$ that is contained in $(d_{j+1}, \dots, d_k)$, then

$$\begin{aligned}\sum_{i=1}^{n+1} \frac{d_{\ell+i}}{q^i} &\le \max\left\{\frac{m}{q} + \dots + \frac{m}{q^{n-1}} + \frac{m-1}{q^n} + \frac{m}{q^{n+1}}, \frac{m}{q} + \dots + \frac{m}{q^n} + \frac{p-1}{q^{n+1}}\right\} \\ &= \frac{m}{q} + \dots + \frac{m}{q^n} + \frac{p-1}{q^{n+1}}.\end{aligned}$$

Therefore

$$\sum_{i=j+1}^{k} \frac{d_i}{q^i} < \frac{1}{q^j} \sum_{k=0}^{\infty} \left(\frac{1}{q^{n+1}}\right)^k \left(\frac{m}{q} + \dots + \frac{m}{q^n} + \frac{p-1}{q^{n+1}}\right) = \frac{1}{q^j},$$

which implies (3.1).

(ii) It follows from our assumption on q that

$$\frac{m}{q^2} + \dots + \frac{m}{q^{n+1}} + \frac{p-1}{q^{n+2}} < \frac{1}{q} < \frac{m}{q^2} + \dots + \frac{m}{q^{n+1}} + \frac{p}{q^{n+2}}. \tag{3.2}$$

First we show that $T_k = T^k$ for every $k \le n+1$. Let $(c_1, \dots, c_k)$ and $(d_1, \dots, d_k)$ be sequences in A^k satisfying $(c_1, \dots, c_k) > (d_1, \dots, d_k)$, and let j be the smallest positive integer such that $c_j > d_j$. Then we have

$$\begin{aligned}\sum_{i=1}^{k} \frac{c_i - d_i}{q^i} &\ge \frac{1}{q^{j-1}}\left(\frac{1}{q} - \frac{m}{q^2} - \dots - \frac{m}{q^{k+1-j}}\right) \\ &\ge \frac{1}{q^{j-1}}\left(\frac{1}{q} - \frac{m}{q^2} - \dots - \frac{m}{q^{n+1}}\right) \\ &> 0\end{aligned}$$

by using (3.2) in the last step.

Due to a remark following the proof of Proposition 2.1 it remains to show that $T_{n+2} \ne T^{n+2}$. The sequence 10^{n+1} clearly belongs to $S_{A,q,n+2}$. In order to show that $0m^n p$ belongs to $S_{A,q,n+2}$ as well, we must prove that

$$\sum_{i=1}^{n+2} \frac{c_i}{q^i} \ne \frac{m}{q^2} + \dots + \frac{m}{q^{n+1}} + \frac{p}{q^{n+2}}$$

for every sequence $c_1 \dots c_{n+2} \in A^{n+2}$ satisfying $c_1 \dots c_{n+2} > 0m^n p$.

If $c_1 = 0$, this is clear. If $c_1 \dots c_{n+2} = 10^{n+1}$, then

$$\sum_{i=1}^{n+2} \frac{c_i}{q^i} = \frac{1}{q} < \frac{m}{q^2} + \cdots + \frac{m}{q^{n+1}} + \frac{p}{q^{n+2}} \tag{3.3}$$

by (3.2). In the remaining cases we have $c_1 \geq 1$ and $c_1 + \cdots + c_{n+2} \geq 2$, so that

$$\sum_{i=1}^{n+2} \frac{c_i}{q^i} \geq \frac{1}{q} + \frac{1}{q^{n+2}} > \frac{m}{q^2} + \cdots + \frac{m}{q^{n+1}} + \frac{p}{q^{n+2}} \tag{3.4}$$

by (3.2) again.

Since $10^{n+1}, 0m^n p \in S_{A,q,n+2}$ and $10^{n+1} > 0m^n p$, the inequality (3.3) shows that the map (2.1) with $k = n+2$ is not increasing.

(iii) As in part (ii), suppose that $q_{m,n,p-1} < q < q_{m,n,p}$ for some $n, p \geq 1$. It follows from (3.3) and (3.4) that if x belongs to the non-empty interval

$$D := \left[\frac{m}{q^2} + \cdots + \frac{m}{q^{n+1}} + \frac{p}{q^{n+2}}, \frac{1}{q} + \frac{1}{q^{n+2}}\right),$$

then

$$\sum_{i=1}^{n+2} \frac{b_i(x,A,q)}{q^i} = \frac{1}{q} < \frac{m}{q^2} + \cdots + \frac{m}{q^{n+1}} + \frac{p}{q^{n+2}} = \frac{b_1(x, A_{n+2}, q^{n+2})}{q^{n+2}},$$

i.e.,

$$T_{n+2}(x) = q^{n+2}\left(x - \frac{m}{q^2} - \cdots - \frac{m}{q^{n+1}} - \frac{p}{q^{n+2}}\right) < q^{n+2}\left(x - \frac{1}{q}\right) = T^{n+2}(x).$$

If $(m,n,p) \neq (1,1,1)$, then the interval D is contained in $[0,1)$. If $(m,n,p) = (1,1,1)$ and $1 > q^{-2} + q^{-3}$, then $D \cap [0,1)$ is non-empty. Therefore, also in this case the maps T_{n+2} and T^{n+2} differ on an interval contained in $[0,1)$. It remains to consider those values of q that satisfy $1 \leq q^{-2} + q^{-3}$.

If $1 \leq q^{-2} + q^{-3}$, then let $\ell \geq 3$ be the (unique) positive integer satisfying

$$\frac{1}{q^\ell} + \frac{1}{q^{\ell+1}} < 1 \leq \frac{1}{q^{\ell-1}} + \frac{1}{q^\ell}. \tag{3.5}$$

If the latter inequality in (3.5) is strict, then for each x belonging to the non-empty interval

$$\left[\frac{1}{q^\ell} + \frac{1}{q^{\ell+1}}, \min\left\{1, \frac{1}{q} + \frac{1}{q^{\ell+1}}\right\}\right),$$

we have $b_1(x,A,q) \dots b_{\ell+1}(x,A,q) = 10^\ell$ and

$$T_{\ell+1}(x) \leq q^{\ell+1}\left(x - \frac{1}{q^\ell} - \frac{1}{q^{\ell+1}}\right) < q^{\ell+1}\left(x - \frac{1}{q}\right) = T^{\ell+1}(x).$$

If the latter inequality in (3.5) is in fact an equality, then we consider the non-empty interval

$$\left[\frac{1}{q^{\ell-1}} + \frac{1}{q^{\ell+1}}, \min\left\{1, \frac{1}{q} + \frac{1}{q^{\ell+1}}\right\}\right).$$

For each x belonging to this interval we have $b_1(x,A,q) \dots b_{\ell+1}(x,A,q) = 10^\ell$ and

$$T_{\ell+1}(x) \leq q^{\ell+1}\left(x - \frac{1}{q^{\ell-1}} - \frac{1}{q^{\ell+1}}\right) < q^{\ell+1}\left(x - \frac{1}{q}\right) = T^{\ell+1}(x).$$

For each $q \in (m, m+1) \setminus P$ we now have constructed an interval $I \subset [0,1)$ and a positive integer k such that $T_k < T^k$ on I. □

Remarks 3.2.

(i) It follows from the above proof that if $q_{m,n,p-1} < q < q_{m,n,p}$ $(n, p \geq 1)$ and $(m, n, p) \neq (1,1,1)$, then one may take $k = n+2$ in the statement of Proposition 3.1(iii).

(ii) If $T_k(x) \neq T^k(x)$ for some $x \in [0,1)$, then the first digit of any expansion of xq^{-1} in base q with respect to A must be zero, whence

$$T_{k+1}\left(\frac{x}{q}\right) = T_k(x) < T^k(x) = T^{k+1}\left(\frac{x}{q}\right).$$

Hence if $T_k \neq T^k$ on a subinterval of $[0,1)$, then $T_n \neq T^n$ on a subinterval of $[0,1)$ for each integer $n \geq k$.

Proof of Theorem 1.3. (i) Let $q \in P$. Note that the greedy expansion of $x \in J_{A,q}$ is optimal if and only if $T_k(x) = T^k(x)$ for each $k \geq 1$. Hence each $x \in J_{A,q}$ has an optimal expansion by Proposition 3.1(i).

(ii) Let $q \in (m, m+1) \setminus P$. It is well known (see, e.g., [14], [16]) that the map T is ergodic with respect to a unique normalized absolutely continuous T-invariant measure μ with a density which is positive on the interval $[0,1)$. According to Proposition 3.1(iii) there exists an interval $I \subset [0,1)$ and a number $k = k(q)$ such that $T_k < T^k$ on I. An application of Birkhoff's ergodic theorem yields that for almost every $x \in [0,1)$ there exists a positive integer $\ell = \ell(x)$ such that $T^\ell(x) \in I$. For each such x the greedy expansion of x is not optimal because the greedy expansion $b_{\ell+1}(x, A, q)b_{\ell+2}(x, A, q)\ldots$ of $T^\ell(x)$ is not optimal. Since the map T is non-singular[2] and since for each $x \in [1, m/(q-1))$ there exists a positive integer $n = n(x)$ such that $T^n(x) \in [0,1)$, we may conclude that x has no optimal expansion for almost every $x \in J_{A,q}$.

It remains to show that the set of numbers with an optimal expansion is nowhere dense. We call an expansion (d_i) of a number $x \in J_{A,q}$ *infinite* if $d_n > 0$ for infinitely many $n \in \mathbb{N}$. Otherwise it is called *finite*. Let $x \in J_{A,q}$ be a number with no optimal and no finite expansion, and let $(b_i) = (b_i(x, A, q))$. Then there exists an expansion (c_i) of x and a number $n \in \mathbb{N}$ such that the inequalities

$$\sum_{i=1}^{n} \frac{b_i}{q^i} < \sum_{i=1}^{n} \frac{c_i}{q^i} < x$$

hold. Hence the number x belongs to the interior of the interval

$$E := \left[\sum_{i=1}^{n} \frac{c_i}{q^i}, \left(\sum_{i=1}^{n} \frac{c_i}{q^i}\right) + \sum_{i=n+1}^{\infty} \frac{m}{q^i}\right].$$

It follows from Proposition 1.1 that the set E consists precisely of those numbers in $J_{A,q}$ that have an expansion starting with $c_1 \ldots c_n$. Since (b_i) is infinite by hypothesis, there exists a number $\delta = \delta(x) > 0$ such that $(x - \delta, x + \delta) \subset E$ and such that the greedy expansion of each number belonging to $(x - \delta, x + \delta)$ starts with $b_1 \ldots b_n$ (this follows for instance from Lemmas 3.1 and 3.2 in [5]). Hence none of the numbers in $(x - \delta, x + \delta)$ has an optimal expansion. Denoting by $\mathcal{O}_q$ the set of numbers in $J_{A,q}$ with an optimal expansion and its closure by $\overline{\mathcal{O}_q}$, we may

[2]Non-singularity of T means that $T^{-1}(B)$ is a null set whenever $B \subset J_{A,q}$ is a null set.

thus conclude that numbers belonging to $\overline{\mathcal{O}_q} \setminus \mathcal{O}_q$ have a finite expansion, whence $\overline{\mathcal{O}_q} \setminus \mathcal{O}_q$ is at most countable. This implies in particular that the set $\overline{\mathcal{O}_q}$ is also a null set and therefore has no interior points. □

For each positive integer k, the map T_k is also ergodic with respect to a unique normalized absolutely continuous T_k-invariant measure μ_k as follows from Theorem 4 in [13]. Since $T_1 = T$, the measure μ introduced in the proof of Theorem 1.3 equals μ_1. Methods to construct an explicit formula for (a version of) the density of the measure μ_k can be found in [12] (see also [9], [2]).

Corollary 3.3. *$q \in P$ if and only if $\mu_1 = \mu_k$ for each $k \geq 1$.*

Proof. Proposition 3.1(i) implies that $\mu_1 = \mu_2 = \cdots$ if q belongs to P. Conversely, suppose that $q \in (m, m+1) \setminus P$ and let $I \subset [0,1)$ be an interval such that $T_k < T^k$ on I for some positive integer k. Since the maps T_k and T^k are continuous from the right, there exists a subinterval $J \subset I$ and a number $t > 0$ such that $T_k < t < T^k$ on J. Note that $T^{-k}([0,t)) \subset T_k^{-1}([0,t))$ because $T_k \leq T^k$ on $J_{A,q}$. If we had $\mu_k = \mu_1$, then μ_1 would also be T_k-invariant, whence

$$0 = \mu_1\left(T_k^{-1}[0,t)\right) - \mu_1\left(T^{-k}[0,t)\right) \geq \mu_1(J),$$

which contradicts the fact that the density of μ_1 is positive on the interval $[0,1)$. □

Remarks 3.4.

(i) For *each* $q \in (m, m+1)$, almost every $x \in J_{A,q}$ has uncountably many expansions (see [17], [1]). It follows from Theorem 1.3(i) that a number with an optimal expansion may have uncountably many expansions. We do not know whether the greedy expansion of a number with at most countably many expansions is always optimal.

(ii) It has been shown in [8] (see also [5], [6]) that if $q \in (m, m+1)$ is close enough to $m+1$, then the set $\mathcal{U}_q$ of numbers in $J_{A,q}$ with a unique expansion is uncountable. Moreover, the Hausdorff dimension of $\mathcal{U}_q$ tends to one if $q \to m+1$. Since a unique expansion is clearly optimal, the same properties hold for the set of numbers belonging to $J_{A,q}$ with an optimal expansion.

(iii) Let $\mathcal{U}$ be the set of bases $q \in (m, m+1)$ such that the number $1 \in J_{A,q}$ has a unique expansion. The set $\mathcal{U}$ has been extensively studied in [7], [10], [5]. For instance it has been shown in [5] that $\mathcal{U}_q$ is closed if and only if $q \in (m, m+1) \setminus \overline{\mathcal{U}}$, where $\overline{\mathcal{U}}$ is the closure of $\mathcal{U}$. It follows from the proof of Theorem 1.3 in [5] that each number x belonging to the closure $\overline{\mathcal{U}_q}$ of the set $\mathcal{U}_q$ has an optimal expansion for each $q \in (m, m+1)$. We conclude this section with an example showing that the set $\mathcal{O}_q$ of numbers with an optimal expansion properly contains $\overline{\mathcal{U}_q}$ for all $q \in (m, m+1)$.

Example 3.5. Fix $q \in (m, m+1)$. It is well known that each number $x \in J_{A,q} \setminus \{0\}$ has a lexicographically largest infinite expansion $(a_i(x))$ which coincides with its greedy expansion if and only if the latter is infinite. If the greedy expansion $(b_i(x))$ of a number $x \in J_{A,q} \setminus \{0\}$ is finite and $b_n(x)$ is its last non-zero element, then $(a_i(x)) = b_1(x) \ldots b_{n-1}(x)(b_n(x) - 1)a_1(1)a_2(1) \ldots$. For convenience we set $(a_i(0)) := 0^\infty$. It is shown in [5] that $\overline{\mathcal{U}_q} \subset \mathcal{V}_q$, where $\mathcal{V}_q$ is the set of numbers $x \in J_{A,q}$ such that

$$(m - a_{n+1}(x))(m - a_{n+2}(x)) \ldots \leq a_1(1)a_2(1) \ldots \quad \text{whenever} \quad a_n(x) > 0.$$

Let k be the largest positive integer satisfying the inequality $\sum_{i=1}^{k} mq^{-i} < 1$, and consider the number

$$x := \frac{1}{q} + \frac{1}{q^{k+2}}.$$

The greedy expansion $(b_i(x))$ of x is clearly given by 10^k10^∞. Our choice of k implies that $(b_i(x))$ is optimal. However, the number x does not belong to $\mathcal{V}_q$ because $a_1(x)\dots a_{k+2}(x) = 10^{k+1}$ and $a_1(1)\dots a_{k+1}(1) = m^k c$ with $c < m$.

4. Optimal expansions in negative bases

Given a positive integer m and a real number $m < q \le m+1$, by an expansion of a real number x in base $-q$ we mean a sequence $(c_i) = c_1c_2\dots$ of integers $c_i \in A := \{0, 1, \dots, m\}$ satisfying

$$\sum_{i=1}^{\infty} \frac{c_i}{(-q)^i} = x.$$

One easily verifies that (c_i) is an expansion of a real number x in base $-q$ if and only if $(c_i') := (m - c_1, c_2, m - c_3, c_4, \dots)$ is an expansion of $x' := x + mq/(q^2-1)$ in base q (with respect to A). It follows from Proposition 1.1 that each x belonging to the interval

$$J_{A,-q} := \left[\frac{-mq}{q^2-1}, \frac{m}{q^2-1}\right]$$

has an expansion in base $-q$.

Definition 4.1. An expansion (d_i) of x in base $-q$ is *optimal* if for any other expansion (c_i) of x in base $-q$ we have

$$\left| x - \sum_{i=1}^{n} \frac{d_i}{(-q)^i} \right| \le \left| x - \sum_{i=1}^{n} \frac{c_i}{(-q)^i} \right|$$

for all $n = 1, 2, \dots$.

We only consider here expansions in negative integer bases $-2, -3, \dots$. While in positive integer bases the greedy expansion is always optimal, in negative integer bases there are infinitely many numbers with no optimal expansion:

Proposition 4.2. *In negative integer bases only the unique expansions are optimal.*

Proof. Let $q = m+1$ for some positive integer m. If $x \in J_{A,-q}$ has no unique expansion in base $-q$, then x has exactly two expansions (c_i) and (d_i) in base $-q$ because (c_i') and (d_i') are the only expansions of x' in base q. Moreover, there exists a positive integer k such that $c_i' = d_i'$ for $1 \le i \le k-1$ and such that the sequences $(c_k', c_{k+1}', \dots)$ and $(d_k', d_{k+1}', \dots)$ are equal to $(p+1)0^\infty$ or pm^∞ for some $p \in \{0, \dots, m-1\}$. If necessary, interchange (c_i) and (d_i) so that $(c_i') > (d_i')$, and let n be a positive integer such that $2n \ge k$. Then

$$x = \left(\sum_{i=1}^{2n} \frac{c_i}{(-q)^i}\right) - \sum_{i=n}^{\infty} \frac{m}{q^{2i+1}} = \left(\sum_{i=1}^{2n} \frac{d_i}{(-q)^i}\right) + \sum_{i=n}^{\infty} \frac{m}{q^{2i+2}},$$

whence

$$\left| x - \sum_{i=1}^{2n+1} \frac{c_i}{(-q)^i} \right| = \frac{1}{q} \left| x - \sum_{i=1}^{2n+1} \frac{d_i}{(-q)^i} \right| < \left| x - \sum_{i=1}^{2n+1} \frac{d_i}{(-q)^i} \right|$$

and

$$\left|x - \sum_{i=1}^{2n} \frac{d_i}{(-q)^i}\right| = \frac{1}{q}\left|x - \sum_{i=1}^{2n} \frac{c_i}{(-q)^i}\right| < \left|x - \sum_{i=1}^{2n} \frac{c_i}{(-q)^i}\right|$$

so that the expansions (c_i) and (d_i) are not optimal. □

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Department of Mathematics, Utrecht University, 3508 TA Utrecht, The Netherlands

E-mail address: k.dajani1@uu.nl

Tussen de Grachten 213, 1381 DZ Weesp, The Netherlands

E-mail address: martijndevries0@gmail.com

Département de Mathématique, Université de Strasbourg, 7 rue René Descartes, 67084, Strasbourg Cedex, France

E-mail address: vilmos.komornik@math.unistra.fr

Dipartimento di Scienze di Base e Applicate per l'Ingegneria, Sezione di Matematica, Sapienza Università di Roma, Via A. Scarpa 16, 00161 Roma, Italy

E-mail address: loreti@dmmm.uniroma1.it