COHOMOLOGY OF $\mathcal{M}_3$ AND $\mathcal{M}_3^1$

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Abstract. We give a description of the moduli space of pointed smooth quartic curves in terms of the root system E_7, and use this to determine the rational cohomology (with its mixed Hodge structure) of the moduli spaces $\mathcal{M}_3$ and $\mathcal{M}_3^1$ of smooth (pointed) curves of genus three.

0. Introduction

In this paper we determine the Poincaré polynomials of the moduli spaces $\mathcal{M}_3$ and $\mathcal{M}_3^1$ of smooth (pointed) genus three curves and of certain natural strata therein. Our approach is based on the well-known fact that the moduli space of smooth quartic curves $\mathcal{Q}$ (which is simply the complement of the hyperelliptic locus in $\mathcal{M}_3$) is also the moduli space of (smooth) Del Pezzo surfaces of degree two. This allows us to find simple descriptions of the strata of the moduli space $\mathcal{Q}^1$ of *pointed* smooth quartic curves (C, p) in terms of root systems. For instance, we may define strata in $\mathcal{Q}^1$ according to the intersection behaviour of C and the tangent line L through p. There are four cases: L can meet C in 3 distinct points (the general situation), or L is a genuine bitangent (two points of intersection with C), or L and C meet in p with a contact of order 3 resp. 4. These define strata $\mathcal{Q}^{\rm ord}$, $\mathcal{Q}^{\rm btg}$, $\mathcal{Q}^{\rm flx}$, $\mathcal{Q}^{\rm hflx}$. It turns out that each of these strata can be described in terms of a Weyl group (of type E_7 or E_6) acting on a torus T of adjoint type or its projectivized Lie algebra $P(\mathrm{Lie}(T))$. The toroidal cases correspond to $\mathcal{Q}^{\rm ord}$, $\mathcal{Q}^{\rm btg}$ and the projective cases to $\mathcal{Q}^{\rm flx}$ and $\mathcal{Q}^{\rm hflx}$. As we point out at the end of this introduction, the latter two are intimately related to the miniversal deformations of the plane curve singularities that also bear the names E_7 and E_6. The relationship between $\mathcal{Q}^{\rm btg}$ and the adjoint E_6-torus had already been observed by us in [9]. Our first main result is theorem (1.20) which tells us how these strata fit together. It contains a description of $\mathcal{Q}^1$ solely in terms of a root system of type E_7.

We compute the Poincaré polynomials of these strata and related varieties. Actually we obtain finer information. According to Deligne [4], a cohomology group of an algebraic variety carries a mixed Hodge structure. The varieties encountered here (such as $\mathcal{M}_3$ and $\mathcal{M}_3^1$) turn out to have particularly simple Hodge structures: in every degree their cohomology is pure of type (k, k) for a certain k. It is therefore convenient to express our results in terms of what we propose to call *Poincaré-Serre polynomials*: this is a polynomial in two variables t, u attached to the cohomology

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of an algebraic variety (or more generally, to a graded mixed Hodge structure) by letting the coefficient of $t^k u^l$ be the dimension of the weight l subquotient in degree k. For $u = 1$ this reduces to the Poincaré polynomial, and if we substitute $t = -1$, then the coefficients of the resulting polynomial are the weighted euler characteristics which were apparently first used by Serre in a characteristic p setting. We can now state our two other main results:

(4.7)*The moduli space of smooth quartic curves resp. of smooth genus three curves has Poincaré-Serre polynomial* $1 + t^6u^{12}$ *resp.* $1 + t^2u^2 + t^6u^{12}$.

(4.10)*The Poincaré-Serre polynomial of the moduli space of pointed smooth quartic curves resp. of pointed smooth genus three curves is equal to* $1 + t^2u^2 + t^6u^{12} + t^7u^{12} + 2t^8u^{14}$ *resp.* $1 + 2t^2u^2 + t^4u^4 + t^6u^{12} + t^7u^{12} + 2t^8u^{14}$.

The proof of (4.7) uses the fact, due to Harer and Zagier [8], that the euler characteristic of $\mathcal{M}_3$ equals 3, but the proof of (4.10) is independent of this. According to (a generalization of) the Hodge conjecture, a class of degree $2k$ is algebraic if and only if it is of type (k, k). So (4.7) predicts that the class of $\mathcal{M}_3$ in degree two is algebraic, whereas the one in degree six is not. This is in agreement with the computation by Faber [7] of the Chow groups of $\mathcal{M}_3$. We also observe that (4.10) implies that the euler characteristic of $\mathcal{M}_3^1$ is 6, which checks with another result of Harer and Zagier.

Our proof also allows the computation of the rational cohomology of the above moduli spaces with level two structure, even as representation spaces of $Sp(6, F_2)$, but we have not pursued this. Our method generalizes well to other Del Pezzo surfaces, but this will possibly be the subject matter of another paper.

We briefly indicate the contents of each section.

Section 1 contains the description of the four strata in terms of root data. At the end we explain how these strata fit together, so that we obtain a description of $\mathcal{M}_3^1$ in terms of the root system E_7. We also show that $\mathcal{M}_3^1$ is rational. (It is still not known whether $\mathcal{M}_3$ is rational.)

Sections 2 and 3 are different in character and are independent of section 1. The toroidal cases alluded to above lead us to study arrangements of hypertori and the cohomology of their complements. For this purpose we found it convenient to set up things in a more general manner in order to have a unified treatment of affine-linear and toroidal arrangements. It turns out that the most effective approach is then by the methods of sheaf theory; even in the well-studied case of hyperplane arrangements it yields probably the fastest proof of the basic results of Brieskorn. The general discussion is carried out in section 2, whereas in section 3 the special features of arrangements coming from root systems are examined.

In the final section 4 we apply the methods developed in section 3 to compute various Poincaré-Serre polynomials.

For the computation of the Poincaré polynomial of $\mathcal{M}_3$ it is possible to avoid Del Pezzo surfaces altogether if one is willing to use some facts about the plane curve singularities of type E_6 and E_7 instead. Since that proof essentially consists of properly arranging known results, and uses less in the way of algebraic geometry, we may as well outline it here. Consider the moduli space $\mathcal{Q}^{\overline{\mathrm{flx}}}$ of pairs (C, L) where $C$is a smooth quartic curve in $\mathbf{P}^2$ and L is a line with a point of contact of order

≥ 3. It is clear that this moduli space is the disjoint union of $\mathcal{Q}^{\text{flx}}$ and $\mathcal{Q}^{\text{hflx}}$, and that the forgetful map $\mathcal{Q}^{\overline{\text{flx}}} \to \mathcal{Q}$ is finite. Consider the miniversal deformations of the plane curve singularities E_6, E_7:

$$E_6 \quad C(u) : x^3 + y^4 + u_2xy^2 + u_5xy + u_6y^2 + u_8x + u_9y + u_{12} = 0,$$
$$E_7 \quad C(v) : x^3 + xy^3 + v_2xy^2 + v_3x^2 + v_4xy + v_5y^2 + v_6x + v_7y + v_9 = 0.$$

If we give the u- and v-parameters weights equal the their subscript, then the spaces $U \cong \mathbb{C}^6$ resp. $V \cong \mathbb{C}^7$ for which they are coordinates acquire a $\mathbb{C}^*$-action. If we let $\mathbb{C}^*$ act on xy-space by giving x weight 4 resp. 3 and y weight 3 resp. 2, then the two equations become weighted homogeneous. This implies that curves in the same $\mathbb{C}^*$-orbit are projectively equivalent. Every curve $C(u)$ is smooth at infinity, and meets the line at infinity, L_∞, in a smooth point with contact of order 4. A similar statement holds for the curves $C(v)$, except that they meet L_∞ in two distinct points with contact of order 3 and 1. So if $U' \subset U$ resp. $V' \subset V$ parametrize the smooth curves, then we have natural maps $U'/\mathbb{C}^* \to \mathcal{Q}^{\text{hflx}}$ and $V'/\mathbb{C}^* \to \mathcal{Q}^{\text{flx}}$. It is easily verified that these maps are isomorphisms.

The Poincaré polynomials of the spaces U' and V' have been determined by Brieskorn [2]: they are $1+t$ resp. $1+t+t^6+t^7$. Those of their $\mathbb{C}^*$-orbit spaces are obtained by dividing these by the Poincaré polynomial of $\mathbb{C}^*$ (which is $1+t$). It follows that the Poincaré polynomial of $\mathcal{Q}^{\text{hflx}}$ resp. $\mathcal{Q}^{\text{flx}}$ is equal to 1 resp. $1+t^6$. The Gysin sequence for rational cohomology may be applied to the pair $(\mathcal{Q}^{\overline{\text{flx}}}, \mathcal{Q}^{\text{hflx}})$ (see lemma (4.1)), and it shows that the Poincaré polynomial of $\mathcal{Q}^{\overline{\text{flx}}}$ is equal to $1+t^2+t^6$. Hence the Poincaré polynomial of $\mathcal{Q}$ is termwise bounded by $1+t^2+t^6$. We regard $\mathcal{Q}$ as the complement of the hyperelliptic locus $\mathcal{H}_3$ in $\mathcal{M}_3$. One verifies that the Poincaré polynomial of $\mathcal{H}_3$ is constant equal to 1 (see lemma (4.6)). Another Gysin sequence argument shows that then the Poincaré polynomial of $\mathcal{M}_3$ must be termwise dominated by $1+2t^2+t^6$. If we combine this with the fact that the second betti number of $\mathcal{M}_3$ is equal to 1 (Harer) and the fact that its euler characteristic is 3 (Harer and Zagier [8]), it follows that the Poincaré polynomial of $\mathcal{M}_3$ must be equal to $1+t^2+t^6$.

This paper was conceived after the conference and its contents bears no relation to the talk I gave there.

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Notation. We shall denote the Poincaré polynomial, resp. Poincaré-Serre polynomial, of a variety Z by $P(Z)$ resp. $PS(Z)$.

If a reflection group W acts on a space X, then as a rule D_X denotes the union of the fixed point sets of the reflections in W, and X' denotes its complement in X.

1. The Del Pezzo model for $\mathcal{Q}^1$

In this section we will describe a model for the moduli space of pointed genus three curves, based on the theory of Del Pezzo surfaces. We begin by recalling some classical material which is set out in more detail in Manin [10], Demazure[5] and Dolgachev and Ortland [6].

(1.1) Let us start out with a smooth plane curve $C \subset \mathbf{P}^2$ of degree 4. It is known that such a curve has 28 double tangents. Let $X \to \mathbf{P}^2$ be the double cover which ramifies along C. This is a Del Pezzo surface of degree two. To be more precise, the pre-image of a line is the polar divisor of a meromorphic 2-form on X without zeroes (such a divisor is said to be *anti-canonical*) and its self-intersection is 2. Every anti-canonical curve on X is thus obtained and the projection $X \to \mathbf{P}^2$ can therefore be thought of as the map defined by the anti-canonical system on X. So this map and the involution of X defined by it are (up to a linear transformation) intrinsic to X.

The pre-image of a double tangent consists of two exceptional curves, and this yields all the $2 \times 28 = 56$ exceptional curves on X. If $E_1, \dots, E_7$ are disjoint exceptional curves on X, then contracting them yields a projective plane. Any anti-canonical curve D on X will meet $E_1, \dots, E_7$ transversally and will project onto a cubic curve in the projective plane passing through the images $P_1, \dots, P_7$ of $E_1, \dots, E_7$. Conversely, the strict transform of every cubic through these points is anti-canonical. This implies that the linear system of cubic curves through $P_1, \dots, P_7$ cannot have any fixed point after blowing up $P_1, \dots, P_7$. This condition translates into: no three of the P_i's are on a line and no six are on a quadric. We then say that the points $P_1, \dots, P_7$ are *in general position*. If $L \subset \mathbf{P}^2$ is a line tangent to C which is not a double tangent, then its pre-image $\tilde{L}$ in X is a rational curve with a node, a rational curve with a cusp, or consists of two smooth rational curves meeting each other in a tacnodal singularity, depending on whether C and L have a point of contact of order 2, 3 or 4.

We can also go in the opposite direction: if we start out from seven points in a projective plane which are in general position in the above sense, then blowing them up yields a Del Pezzo surface of degree two and thereby a smooth quartic curve in a (different) projective plane. A singular cubic curve passing through these seven points yields a tangent to the quartic. There are many ways of choosing seven disjoint exceptional curves on X; in fact, as we will recall below, there is a large group (a Weyl group of type E_7) which permutes transitively the possible choices. Distinct choices lead to configurations of seven points in a projective plane which are in general not projectively equivalent. The relationship between these point configurations is rather subtle and still not fully understood. If however, we also fix a tangent line of C, then the point configurations come with a singular cubic curve through them and, as we will see, the relations between these algebro-geometric data are a lot easier to describe.

(1.2) Let L be the free $\mathbb{Z}$-module with generators $e_1, \dots, e_7, l$, equipped with the symmetric bilinear form for which this is an orthogonal basis and $e_i.e_i = -1$ and $l.l = 1$. Let $k := 3l - e_1 - \cdots - e_7$. Notice that $k.k = 2$ and that the orthogonal complement Q of k in L is negative definite. The elements $\alpha \in Q$ with $\alpha.\alpha = -2$ form a root system R of type E_7; a basis for this root system is $\alpha_1 := e_1 - e_2, \dots, \alpha_6 := e_6 - e_7, \alpha_7 := l - e_1 - e_2 - e_3$. The positive roots with respect to this basis are the elements $e_i - e_j$ with $i < j$, $l - e_i - e_j - e_k$ with $i < j < k$ and $2l - e_1 - e_2 \cdots - \hat{e}_i \cdots - e_7$ with $1 \leq i \leq 7$.

Denote by W the corresponding Weyl group. This is also the group of automorphisms of the lattice Q.

(1.3) Choose seven (ordered) points $P_1, \dots, P_7$ in a projective plane which are in general position in the sense that no three are collinear and no six lie on a conic, so that the surface $X := X(P_1, \dots, P_7) \to \mathbf{P}^2$ obtained by blowing up these points is a Del Pezzo surface of degree 2. If $E_1, \dots, E_7$ are the corresponding exceptional curves, then we have an isomorphism of lattices $\phi : L \cong \mathrm{Pic}(X)$ under which e_i corresponds to the class of E_i and l to the class of the pre-image of a line in $\mathbf{P}^2$. The strict transform of a cubic curve passing through the seven points is an anti-canonical curve whose class corresponds to k. The classes of the (56) exceptional curves correspond to: e_i, $1 \leq i \leq 7$ (7), the class $l - e_i - e_j$ of the strict transform of the line P_iP_j, $1 \leq i < j \leq 7$ (21), the class $k - l + e_i + e_j$ of the strict transform of the conic through five of the seven points, $1 \leq i < j \leq 7$ (21), the class $k - e_i$ of the cubic that passes through the seven points and has a double point in P_i, $1 \leq i \leq 7$ (7). They are precisely the elements $e \in L$ with $-e.e = e.k = 1$. We denote that set by $\mathcal{E}$. It is an orbit of W.

(1.4) For us X is the primary object and we will regard the collection exceptional curves $E_1, \dots, E_7$ as an additional structure on X, called a *marking*. We noticed that a marking determines an isomorphism $\phi : L \cong \mathrm{Pic}(X)$ which sends k onto the anti-canonical class. The converse is also true. Hence the possible markings are permuted simply transitively by the stabilizer of k in the orthogonal group of L. This stabilizer is just W. The anti-canonical system maps X to a projective plane and this map is a double cover of that projective plane ramifying along a smooth quartic curve C. So X admits a canonical involution whos fixed point set is C. This involution fixes of course the anti-canonical class, and acts on its orthogonal complement as minus the identity. For reasons that become clear in a moment, we will denote this orthogonal complement by $\mathrm{Pic}^0(X)$. Giving a marking of X up to the canonical involution is equivalent to giving a level two structure on C. To make this somewhat more precise, let us give names to the relevant moduli spaces:

- The moduli space of Del Pezzo (marked) surfaces of degree two is denoted by $\mathcal{DP}_2$ (resp. $\widetilde{\mathcal{DP}}_2$). The forgetful map $\widetilde{\mathcal{DP}}_2 \to \mathcal{DP}_2$ is a Galois covering with covering group $W_+ := W/\{\pm\mathbf{1}\}$.
- The moduli space $\mathcal{Q}(2)$ of smooth quartic curves with level two structure. The forgetful map $\mathcal{Q}(2) \to \mathcal{Q}$ is a Galois covering with covering group $Sp(6, \mathbb{F}_2)$.

The following result is due to Van Geemen, see [6].

Proposition (1.5). *There is an isomorphism of* $W_+ \cong Sp(6, \mathbb{F}_2)$ *and a compatible equivariant isomorphism* $\widetilde{\mathcal{DP}}_2 \to \mathcal{Q}(2)$.

(1.6) We will rather be concerned with moduli spaces parametrizing more structure. We introduce:

- $\mathcal{DP}_2(\mathrm{sing})$: the moduli space of pairs (X, p) where X is a Del Pezzo surface of degree two and $p \in X$ is such that there is an anti-canonical curve K on X which has a singular point at p. (This K is then unique.) We identify this space with the moduli space of pointed smooth quartic curves $\mathcal{Q}^1$.

- $\mathcal{DP}_2(\text{node})$: the Zariski open subset of $\mathcal{DP}_2(\text{sing})$ parametrizing pairs (X,p) for which K has an ordinary double point at p. We identify this with the Zariski open subset $\mathcal{Q}^{\text{ord}}$ of $\mathcal{Q}^1$.
- $\mathcal{DP}_2(\text{cusp})$: the locus in $\mathcal{DP}_2(\text{sing})$ parametrizing pairs (X,p) where K has a cusp at p. We identify this with the subvariety $\mathcal{Q}^{\text{flx}}$ of $\mathcal{Q}^1$.
- $\mathcal{DP}_2(\text{nnode})$: the locus in $\mathcal{DP}_2(\text{sing})$ parametrizing pairs (X,p) such that K is a union of two exceptional curves meeting transversally in two distinct points, one of which is p. We identify this with the subvariety $\mathcal{Q}^{\text{btg}}$ of $\mathcal{Q}^1$.
- $\mathcal{DP}_2(\text{tacn})$: the locus in $\mathcal{DP}_2(\text{sing})$ parametrizing pairs (X,p) such that K which is a union of two exceptional curves meeting in p with multiplicity two (and nowhere else). We identify this with the subvariety $\mathcal{Q}^{\text{hflx}}$ of $\mathcal{Q}^1$.

We have similar moduli spaces of these objects equipped with marking (resp. level two structure). So we find for instance that there is an equivariant isomorphism $\widetilde{\mathcal{DP}}_2(\text{sing}) \to \mathcal{Q}^1(2)$.

We notice that $\mathcal{DP}_2(\text{cusp})$ and $\mathcal{DP}_2(\text{nnode})$ are of codimension one, while the subvariety $\mathcal{DP}_2(\text{tacn})$ is of codimension two and is the intersection of the closures of the two other strata. We shall first obtain explicit models of these strata separately, and then indicate how they fit together. We will also use

- $\mathcal{Q}_{\text{btg}}$: the moduli space of pairs (C,L) consisting of a smooth quartic curve with a genuine bitangent L. Clearly, there is a natural map $\mathcal{Q}^{\text{btg}} \to \mathcal{Q}_{\text{btg}}$ of degree two.

(1.7) Let X be a Del Pezzo surface of degree 2, and let K be an anti-canonical curve on X which is a irreducible rational curve with a node p. We recall that $\text{Pic}^0(K) \cong \mathbb{C}^*$, and that the obvious map $K_{\text{reg}} \to \text{Pic}^1(K)$ is an isomorphism, so that K_{reg} is a $\text{Pic}^0(K)$-torsor. The restriction homomorphism $\text{Pic}(X) \to \text{Pic}(K)$ induces a homomorphism

$$\chi : \text{Pic}^0(X) \to \text{Pic}^0(K).$$

Notice that $\text{Hom}(\text{Pic}^0(X), \text{Pic}^0(K))$ is an algebraic torus and that the Weyl group of X acts on it via $\text{Pic}^0(X)$.

Let T denote the algebraic torus $\text{Hom}(Q, \mathbb{C}^*)$. It comes with a natural action of the Weyl group W. There are exactly two group isomorphisms between $\text{Pic}^0(K)$ and $\mathbb{C}^*$ which are mutually inverse. Since W contains minus the identity, it follows that there is a natural isomorphism between the Weyl group-orbit spaces of $\text{Hom}(\text{Pic}^0(X), \text{Pic}^0(K))$ and T. So χ determines an element $m(X,p)$ of $W\backslash T$. A marking ϕ of X determines a lift $m(X,p,\phi) \in \{\pm\mathbf{1}\}\backslash T$. If we compose ϕ with the automorphism of $\text{Pic}(X)$ induced by the canonical involution i, then $m(X,p,\phi)$ does not change since i acts on $\text{Pic}^0(X)$ as minus the identity.

The invariant $m(X,p,\phi)$ defines a W_+-equivariant morphism

$$\tilde{m} : \mathcal{Q}^{\text{ord}}(2) \cong \widetilde{\mathcal{DP}}_2(\text{node}) \to \{\pm\mathbf{1}\}\backslash T.$$

Every root $\alpha \in R$ determines a character $\chi \in T \mapsto \chi(\alpha) \in \mathbb{C}^*$. The kernel of this character is exactly the fixed point locus of the reflection in W defined by the root α. Let D_T denote the union of these fixed point hypertori and put $T' := T - D_T$.

Proposition (1.8). *The morphism $\tilde{m}$ maps $\mathcal{Q}^{\mathrm{ord}}(2) \cong \widetilde{\mathcal{D}P}_2(node)$ isomorphically and W_+-equivariantly onto $\{\pm\mathbf{1}\}\backslash T'$ and hence induces an isomorphism*

$$\mathcal{Q}^{\mathrm{ord}} \cong W\backslash T'.$$

Proof. We will construct its inverse. Let $\chi \in T'$. This means that χ is a character of Q which does not take the value 1 on any root. Choose an abstract rational curve K with a node, an isomorphism $\mathbb{C}^* \cong \mathrm{Pic}^0(K)$ and $P_1 \in K_{\mathrm{reg}}$. Define $P_2, \ldots, P_7 \in K_{\mathrm{reg}}$ by $(P_{i+1}) = \chi(e_{i+1} - e_i) + (P_i)$, $i = 1, \ldots, 6$. The linear system of degree 3 on K defined by $\chi(l - e_1 - e_2 - e_3) + (P_1) + (P_2) + (P_3)$ determines an embedding of K in a projective plane. The condition that the images of the seven points be in general position amounts to: $P_i \neq P_j$ if $i \neq j$, i.e., $\chi(e_i - e_j) \neq 1$, no three points are on a line, i.e., $\chi(l - e_i - e_j - e_k) \neq 1$ if i, j, k are distinct, no six are on a conic, i.e., for $1 \leq i \leq 6$, $\chi(2l - e_1 \cdots - \hat{e}_i \cdots - e_6) \neq 1$. But this is just saying that $\chi(\alpha) \neq 1$ for all roots α.

Corollary (1.9). *The moduli space $\mathcal{M}_3^1$ of pointed genus three curves is rational.*

Proof. The previous proposition implies that $\mathcal{M}_3^1$ and $W\backslash T$ are birationally equivalent. So it is enough to prove that $W\backslash T$ is rational. Let P be the weight lattice of R, i.e., the set of elements of $Q \otimes \mathbb{Q}$ which have integral inner product with all elements of Q. This lattice contains Q as a sublattice of index two. So the algebraic torus $S := \mathrm{Hom}(P, \mathbb{C}^*)$ covers T twice and the corresponding involution of S is given by translation over the isomorphism $\chi_0 : P/Q \cong \{1, -1\}$. If $\lambda_i \in P$ is defined by $\lambda_i.\alpha_j = \delta_{i,j}$, then define a W-invariant function $f_i : S \to \mathbb{C}$ by

$$f_i(\chi) := \sum_{\lambda \in W\lambda_i} \chi(\lambda).$$

Together these functions define a mapping $f : W\backslash S \to \mathbb{C}^7$. According to the exponential invariant theory of root systems [1, Ch. 6] this is an isomorphism. Translation over χ_0 multiplies f_i with $\chi_0(\lambda_i)$, and so the involution induces in $\mathbb{C}^7$ a linear involution (whose eigenvalues 1 resp. -1 have multiplicity 4 resp. 3). The quotient of $\mathbb{C}^7$ by such an involution is rational. Since that quotient is isomorphic to $W\backslash T$, the proposition follows.

(1.10) Our aim is to give an explicit description of $\widetilde{\mathcal{D}\mathcal{P}}_2(\mathrm{sing}) \cong \mathcal{Q}^1(2)$ as an extension of $W\backslash T'$. We shall first give similar descriptions of the three missing strata.

In the discussion (1.7) we replace K by a rational curve with a cusp. Then $\mathrm{Pic}^0(K) \cong \mathbb{C}$, and $\mathrm{Pic}^1(K) \cong K_{\mathrm{reg}}$. Put $V := \mathrm{Hom}(Q, \mathbb{C})$. Each root $\alpha \in R$ defines a (reflection) hyperplane in V; we let $D_V \subset V$ denote the union of these hyperplanes and put $V' := V - D_V$. This corresponds to an open subset $\mathbf{P}(V)' \subset \mathbf{P}(V)$ of the associated projective space.

If K lies as an anti-canonical curve on a marked Del Pezzo surface (X, ϕ) of degree 2, then an isomorphism $\mathrm{Pic}(K) \cong \mathbb{C}$ identifies $\chi : \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(K)$ with an element of V. As in the previous case we find that this element lies in V'. Since the isomorphism is determined up to a scalar multiplication, only its image $m(X, p, \phi)$ in $\mathbf{P}(V)'$ is well-defined. The same argument as before shows:

Proposition (1.11). *The invariant* $m(X,p,\phi) \in \mathbf{P}(V)'$ *defines an equivariant isomorphism*

$$\tilde{m} : \mathcal{Q}^{\mathrm{flx}}(2) \cong \widetilde{\mathcal{DP}}_2(cusp) \to \mathbf{P}(V)'$$

and hence an isomorphism

$$m : \mathcal{Q}^{\mathrm{flx}} \cong W\backslash\mathbf{P}(V)'.$$

The last isomorphism is known in a somewhat different guise in singularity theory: there the right-hand side is interpreted as a $\mathbb{C}^*$-orbit space of the punctured base of a miniversal deformation of a E_7 plane curve singularity.

(1.12) We now address the case where we are given a point $p \in X$ that lies on an anti-canonical curve K consisting of two rational curves E, E' intersecting transversally in two distinct points. These two curves are exceptional and are interchanged by the canonical involution i. If $\mathrm{Pic}^0(X,E) \subset \mathrm{Pic}(X)$ denotes the orthogonal complement of the classes of E and E' in $\mathrm{Pic}(X)$, then we have again a restriction homomorphism $\chi : \mathrm{Pic}^0(X,E) \to \mathrm{Pic}^0(K)$. We have $\mathrm{Pic}^1(K) \cong K_{\mathrm{reg}}$ as before and now $\mathrm{Pic}^0(K) \cong \mathbb{C}^* \times \mathbb{C}^*$. We fix this last isomorphism by requiring that the first resp. second factor comes from E resp. E', and that $(t,t') \in \mathbb{C}^* \times \mathbb{C}^*$ moves a point of $K_{\mathrm{reg}} \cap E$ resp. $K_{\mathrm{reg}} \cap E'$ to p if t resp. t' tends to 0. So replacement of p by the other singular point of K inverts the isomorphism.

Since $\chi(\alpha)^{-1} = \chi(-\alpha) = \chi i^*(\alpha) = i^*\chi(\alpha)$, we see that the image of χ is contained in the subgroup $\mathrm{Pic}^0(K)^-$ of i-anti-invariant elements. We identify $\mathrm{Pic}^0(K)^-$ with $\cong \mathbb{C}^*$ by means of the parametrization $s \mapsto (s, s^{-1})$. Notice that the involution i inverts this isomorphism. Suppose that X is equipped with a marking ϕ. Then E and E' define elements $e, e' \in L$ such that $e + e' = k$. Denote the orthogonal complement of this pair in L by $Q(e)$. Then $R(e) := R \cap Q(e)$ is a subroot system of type E_6 which spans $Q(e)$. (To see this, recall that W acts transitively on the exceptional classes, so that we may assume that $e = e_7$. Then $\alpha_1, \dots, \alpha_5, \alpha_7$ is a root basis of $R(e)$.) Hence χ plus the choice of E determines a well-defined element of the torus $T(e) := \mathrm{Hom}(Q(e), \mathbb{C}^*)$. The other choice of irreducible component E' yields the opposite element of $T(e')$ under the equality $T(e') = T(e)$. So we really get an element

$$m(X,p,\phi) \in \{\pm\mathbf{1}\}\backslash \coprod_{e\in\mathcal{E}} T(e),$$

where $-\mathbf{1}$ is the involution that sends $m \in T(e)$ to $-m \in T(i(e))$.

As before we find that $\chi(\alpha) \neq 1$ for all roots $\alpha \in R(e)$. If $T(e)' \subset T(e)$ denotes the open subset defined by this property, then we have:

Proposition (1.13). *The invariant* $m(X,p,\phi)$ *defines a* W_+*-equivariant isomorphism*

$$\mathcal{Q}^{\mathrm{btg}}(2) \cong \widetilde{\mathcal{DP}}_2(nnode) \to \{\pm\mathbf{1}\}\backslash \coprod_{e\in\mathcal{E}} T(e)',$$

and hence induces for any $e \in \mathcal{E}$ *an isomorphism*

$$m : \mathcal{Q}^{\mathrm{btg}} \cong W_e\backslash T(e)'.$$

Moreover, interchanging the singular points corresponds on the right-hand side to applying inversion in each component $T(e)'$, so that we also get an isomorphism

$$m : \mathcal{Q}_{\mathrm{btg}} \cong W_e.\{\pm\mathbf{1}\}\backslash T(e)'.$$

Proof. We outline the construction of the inverse mapping in case $e = e_7$. Omitting the roots involving e_7 from the root basis, resp. system of positive roots for Q in (1.2), gives a root basis, resp. system of positive roots for $Q(e)$. Now let be given $\chi \in \mathrm{Hom}((Q(e), \mathbb{C}^*)'$. Choose an abstract curve K made up of two smooth rational curves E, E' which intersect in two ordinary double points, call one of these intersection points p and identify $\mathrm{Pic}^0(K)$ with $\mathbb{C}^* \times \mathbb{C}^*$ as above. Use the one parameter subgroup $s \mapsto (s, s^{-1})$ of $\mathbb{C}^* \times \mathbb{C}^*$ to identify χ with an element of $\mathrm{Hom}(Q(e), \mathrm{Pic}^0(K))$. Choose $P_1 \in E \cap K_{\mathrm{reg}}$ arbitrary and let $P_2, \dots, P_6 \in E \cap K_{\mathrm{reg}}$ be defined by the condition that $(P_{i+1}) = \chi(e_{i+1} - e_i) + (P_i)$, $i = 1, \dots, 5$. Map K to a projective plane by means of the linear system $\chi(l - e_1 - e_2 - e_3) + (P_1) + (P_2) + (P_3)$. This collapses E' to a point so that K is mapped onto a cubic with a node. The images of $P_1, \dots, P_6, E'$ are in general position precisely if χ does not take the value 1 on a root of $Q(e)$.

(1.14) Finally we do the case where $p \in X$ lies on an anti-canonical curve K having a tacnode at p. Then K consists of two exceptional curves E, E' which are interchanged by the canonical involution i and which intersect in a single point p (with multiplicity 2). The anti-invariant part of $\mathrm{Pic}^0(K)$ is isomorphic to $\mathbb{C}$, so that if ϕ is a marking of X, we end up with an invariant $m(X, p, \phi) \in \mathbf{P}(V(e))$, where $V(e) := \mathrm{Hom}(Q(e), \mathbb{C})$. We find in a similar fashion:

Proposition (1.15). *The invariant $m(X, p, \phi)$ defines a W_+-equivariant isomorphism*

$$\mathcal{Q}^{\mathrm{hflx}} \cong \widetilde{\mathcal{DP}}_2(tacn) \to \{\pm\mathbf{1}\}\backslash \coprod_{e \in \mathcal{E}} \mathbf{P}(V(e))'$$

and hence for any $e \in \mathcal{E}$ an isomorphism

$$\mathcal{Q}^{\mathrm{hflx}} \cong W_e\backslash \mathbf{P}(V(e))'.$$

The last isomorphism has an interpretation similar to that in the case $\mathcal{Q}^{\mathrm{flx}}$; the E_7 plane curve singularity is here a E_6 plane curve singularity.

(1.16) We now show how the target spaces of the various maps $\tilde{m}$ fit together. It will be convenient to combine the four types of strata into two groups: we let $\mathcal{DP}_2(\mathrm{irr})$ be the union of node and cusp strata and let $\mathcal{DP}_2(\mathrm{red})$ be the union of the node^2 and tacnode strata. Notice that the former is open in $\mathcal{DP}_2(\mathrm{sing})$ and that the latter is its complement. We denote the subvariety of $\mathcal{Q}^1$ corresponding to $\mathcal{DP}_2(\mathrm{red})$ by $\mathcal{Q}^{\overline{\mathrm{btg}}}$. The variety $\mathcal{Q}_{\overline{\mathrm{btg}}}$ is similarly defined.

Let $\tilde{T} \to T$ be the blow-up of the origin of T. If we identify the tangent space of T at O with V, then the exceptional divisor gets identified with the projective space $\mathbf{P}(V)$. Let $D_{\tilde{T}}$ be the strict transform of D_T and put $\tilde{T}' := \tilde{T} - D_{\tilde{T}}$, so that $\mathbf{P}(V)' = \mathbf{P}(V) - D_{\tilde{T}}$. We regard T' and $\mathbf{P}(V)'$ as subvarieties of $\tilde{T}'$. Notice that $\tilde{T}'$ is their disjoint union.

Proposition (1.17). *The maps $\tilde{m}$ define a W_+-equivariant isomorphism*

$$\mathcal{Q}^1(2) - \mathcal{Q}^{\overline{\mathrm{btg}}}(2) \cong \widetilde{\mathcal{D}\mathcal{P}}_2(irr) \to \{\pm\mathbf{1}\}\backslash\tilde{T}'$$

and hence induce an isomorphism

$$\mathcal{Q}^1 - \mathcal{Q}^{\overline{\mathrm{btg}}} \cong W\backslash\tilde{T}'.$$

Proof. The map $\tilde{m}$ is clearly W_+-equivariant. Think of the inverse of the isomorphism (1.8) as a birational map from $\pm\{\mathbf{1}\}\backslash\tilde{T}'$ to $\widetilde{\mathcal{D}\mathcal{P}}_2(\mathrm{irr})$. We must show that this map is actually a morphism and that its restriction to the added stratum gives the inverse of the isomorphism (1.11). (This will do, since target and domain are normal.) For this in turn, it suffices to prove that for every smooth germ of an algebraic curve in $\widetilde{\mathcal{D}\mathcal{P}}_2(\mathrm{irr})$ whose generic point is in $\widetilde{\mathcal{D}\mathcal{P}}_2(\mathrm{node})$ and whose special point is in $\widetilde{\mathcal{D}\mathcal{P}}_2(\mathrm{cusp})$, the restriction of $\tilde{m}$ to that curve is a morphism. Take such a curve and represent it by a parametrized curve

$$u : t \in \Delta \mapsto (K(t); P_1(t), \dots, P_7(t)),$$

where $(P_1(t), \dots, P_7(t))$ are points of $\mathbf{P}^2$ in general position and $K(t)$ is a cubic curve passing simply through these points such that for $t \neq 0$, $K(t)$ is a rational irreducible curve with a node, whereas $K(0)$ is a rational irreducible curve with a cusp. A natural way of trivializing the Pic ard groups over Δ^* is as follows: Choose a generating section $t \mapsto \omega(t)$ of the relative dualizing sheaf. So for every $t \in \Delta$, $\omega(t)$ is a holomorphic nowhere vanishing differential on $K(t)_{\mathrm{reg}}$ which extends meromorphically over the normalization $\hat{K}(t)$ of $K(t)$. The family $K(t)$ is trivial over Δ^*, and hence the two points of $\hat{K}(t)$ lying over the singular point are given by two sections $P_0(t)$ and $P_\infty(t)$. For $t \neq 0$, $\omega(t)$ has simple poles at these two points. A straightforward local computation shows that $f(t) := (\mathrm{Res}_{P_0(t)}\omega(t))^{-1}$ has a zero at $t = 0$. For $t \neq 0$, an identification $\mathrm{Pic}^0(K(t)) \cong \mathbb{C}^*$ is given by a kind of Abel-Jacobi map:

$$(Q) - (P) \mapsto \exp(f(t) \int_P^Q \omega(t)\ ),$$

wheras $\mathrm{Pic}^0(K(0))$ may be identified with the additive group of $\mathbb{C}$ via

$$(Q) - (P) \mapsto \int_P^Q \omega(0).$$

If $P(t), Q(t) \in K(t)$ are sections with $P(0), Q(0) \in K(0)_{\mathrm{reg}}$, then there is a canonical homotopy class $\gamma(t)$ of paths in $K(t)$ from $P(t)$ to $Q(t)$, so that $\int_{\gamma(t)} \omega(t)$ is a well-defined analytic function of $t \in \Delta$. From this we see that the image of $Q(t) - P(t)$ under the above Abel-Jacobi map goes to 1 as t goes to 0. In particular, for $t \to 0$, $\chi(t) \in T$ tends to the identity element of T. On the other hand, $f(t)^{-1}\log\chi(t)$ tends to an element of V which represents $\tilde{m}(u(0))$, as $t \to 0$. Hence $\tilde{m}u$ is a morphism.

We do the same thing for cases "nnode" and "tacn": Let $\tilde{T}(e) \to T(e)$ be the blow-up of the origin of $T(e)$ and use the obvious notation; so for instance $\tilde{T}(e)'$ is the disjoint union of $\tilde{T}'$ and $\mathbf{P}(V(e))'$. We have:

Proposition (1.18). *The maps $\tilde{m}$ make up a W_+-equivariant isomorphism*

$$\mathcal{Q}^{\overline{\text{btg}}}(2) \cong \widetilde{\mathcal{DP}}_2(red) \to \{\pm\mathbf{1}\}\backslash \coprod_{e\in\mathcal{E}} \tilde{T}(e)'$$

and induce for every $e \in \mathcal{E}$ isomorphisms

$$\mathcal{Q}^{\overline{\text{btg}}} \cong W_e\backslash\tilde{T}(e)', \qquad \mathcal{Q}_{\overline{\text{btg}}} \cong W_e.\{\pm\mathbf{1}\}\backslash\tilde{T}(e)'.$$

The proof is similar to (1.17), and we therefore omit it.

(1.19) Although we will not need it in what follows, we complete the picture by describing how the two groups of strata fit together.

Taking the inner product with an element $e \in \mathcal{E}$ defines an element of the weight lattice $\mathrm{Hom}(Q, \mathbb{Z})$ and hence a one-parameter subgroup p_e of T. Such a one parameter subgroup determines an affine torus embedding $T \subset T_e$ with $T_e - T = T(e)$. The union $T_\mathcal{E}$ of these torus embeddings (with the T_e's glued along T) is an open subvariety of the familiar complete torus embedding defined by the decomposition of $\mathrm{Hom}(Q, \mathbb{R})$ into Weyl chambers. Notice that the closure $\mathbf{P}_e$ of the image of p_e in $T_\mathcal{E}$ is a projective line. Blowing up the origin of $T_\mathcal{E}$ makes (the strict transforms of) the $\mathbf{P}_e$'s disjoint. Blowing up once more along these strict transforms yields a modification $\tilde{T}_\mathcal{E} \to T_\mathcal{E}$. The exceptional divisors of the second blow-up are of the form $\mathbf{P}(V(e)) \times \mathbf{P}_e$ with normal bundle the external tensor product of the tautological bundles (of degree -1). Such a divisor can be analytically collapsed onto the $\mathbf{P}(V(e))$-factor, and this defines a map $\tilde{T}_\mathcal{E} \to \hat{T}_\mathcal{E}$. The theorem below implies that this contraction can be performed algebraically. The space $\hat{T}'_\mathcal{E}$ is then the disjoint union of T', $\mathbf{P}(V)'$, the $T(e)'$'s and the $\mathbf{P}(V(e))'$'s.

Theorem (1.20). *The union of the maps $\tilde{m}$,*

$$\tilde{m} : \mathcal{Q}^1(2) \cong \widetilde{\mathcal{DP}}_2(sing) \to \{\pm\mathbf{1}\}\backslash\hat{T}'_\mathcal{E},$$

make up a W_+-equivariant isomorphism and hence induce an isomorphism

$$m : \mathcal{Q}^1 \to W\backslash\hat{T}'_\mathcal{E}.$$

Proof. We proceed as in the proof of (1.17) and show that the restriction of m to any smooth germ of an algebraic curve in $\widetilde{\mathcal{DP}}_2(\text{sing})$ is a morphism. In view of (1.17) and (1.18), we only need to consider the case when the generic point of the curve maps to $\widetilde{\mathcal{DP}}_2(\text{irr})$ and the special point maps to $\widetilde{\mathcal{DP}}_2(\text{red})$. Assume such a curve is given. Its generic point is either in $\widetilde{\mathcal{DP}}_2(\text{node})$ or in $\widetilde{\mathcal{DP}}_2(\text{cusp})$. Suppose for definiteness that the former holds and represent the germ by a parametrized curve

$$u : t \in \Delta \mapsto (K(t); P_1(t), \ldots, P_7(t)),$$

where $(P_1(t), \ldots, P_7(t))$ are points of $\mathbf{P}^2$ in general position and $K(t)$ is an irreducible cubic curve through them such that the first six are its the smooth locus.

The special point $u(0)$ maps to $\mathcal{D}P_2$(nnode) or $\mathcal{D}\mathcal{P}_2$(tacn) depending on whether $K(0)$ has a an ordinary double point or a cusp at $P_7(0)$.

In the first case, the corresponding $\chi(t) \in \mathrm{Hom}(Q, \mathbb{C}^*)$ has the property that

$$\lim_{t\to 0} \chi(t)(\alpha) = \chi(0)(\alpha) \quad for\ \alpha \in Q(e_7),$$

whereas $\chi(t)(P_6(t) - P_7(t))$ tends to 0 or ∞. It follows that $\tilde{m}u$ is a morphism.

In the second case we trivialize the Picard groups over Δ^* as in (1.17) and find an analytic function $f : \Delta \to \mathbb{C}$ with the property that $f(0) = 0$ and

$$\lim_{t\to 0} f(t)^{-1} \log \chi(t)(\alpha) = \chi(0)(\alpha) \quad for\ \alpha \in Q(e_7).$$

So $\tilde{m}u$ is in this case a morphism as well.

Remark. It would be interesting to "complete" this isomorphism with the hyperelliptic locus and the boundary of the Deligne-Mumford compactification. Even more of a challenge is to find a description of the fibration of $\{\pm\mathbf{1}\}\backslash\hat{T}'_{\mathcal{E}}$ by quartic curves entirely in terms of the root system.

2. Arrangements of divisors

(2.1) Let M be a connected complex manifold of dimension m and let D be a reduced divisor D on M. Assume that D is *arrangement-like*, i.e., can locally be given as a product of linear functions. For simplicity we make the additional assumption that the irreducible components of D are smooth. This is so in the three examples which are our main concern:

(1) M is an affine space A and the irreducible components of D are affine-linear hyperplanes. This is the case that has been studied most, see for instance Orlik and Solomon [11].
(2) M is a projective space P and the irreducible components of D are projective hyperplanes.
(3) M is an algebraic torus T and the irreducible components of D are hypertori, i.e., (translated) subtori of codimension one.

Let $\mathcal{S}$ be the collection of irreducible components of intersections of irreducible components of D. We include M in $\mathcal{S}$ (as an intersection with empty index set). Notice that every member of $\mathcal{S}$ is smooth.

For $S \in \mathcal{S}$, we denote the inclusion of S in M by i_S. We further put $M' := M - D$, and we will denote the inclusion of M' in M by j.

We shall describe a complex of sheaves on M that represents the full direct image $Rj_*\mathbb{Z}_{M'}$. We do this by means of an inductive procedure. First a simple lemma.

Lemma (2.2). *There is a covariant functor $S \in \mathcal{S} \mapsto E_S$ from the partial ordered set $\mathcal{S}$ to the category of abelian groups satisfying*

(1) $E_M = \mathbb{Z}$,
(2) *for every $S \in \mathcal{S}$ of codimension $k \geq 1$, the sequence of homomorphisms*

$$0 \to E_S \to \bigoplus_{\substack{S' \supset S \\ \operatorname{codim} S' = k-1}} E_{S'} \to \cdots \to \bigoplus_{\substack{S' \supset S \\ \operatorname{codim} S' = 1}} E_{S'} \to E_M \to 0$$

is an exact complex.

This functor is unique up to unique isomorphism. Moreover, E_S is free and its rank $\epsilon(S)$ is given by the inductive formula

$$\epsilon(M)=1, \quad \textit{and if } S\neq M, \textit{ then } \sum_{S'\supset S}(-1)^{\operatorname{codim} S'}\epsilon(S')=0.$$

Proof. The first part is easy and is left to the reader. The second part follows from the fact that the euler characteristic of the displayed complex must be zero.

The $\mathbb{Z}$-modules E_S were introduced by Orlik and Solomon [11].

(2.3) If $\mathcal{F}$ is a sheaf on $S\in\mathcal{S}$, then the tensor product of $i_{S!}\mathcal{F}$ with the complex of (2.2) gives an exact complex of sheaves on M. We may apply this to $i_S^!\mathcal{I}$, where $\mathcal{I}$ is a sheaf on M. If $\mathcal{I}$ has the property that any local section with support in the union of the $S\in\mathcal{S}$ of codimension k is a sum of sections with support in a single $S\in\mathcal{S}$ of codimension k, then it follows that

$$\to \bigoplus_{\operatorname{codim} S=k} i_{S!}i_S^!\mathcal{I}\otimes E_S\to\cdots\to\bigoplus_{\operatorname{codim} S=1} i_{S!}i_S^!\mathcal{I}\otimes E_S\to i_{D!}i_D^!\mathcal{I}\to 0$$

is exact. Let us take for $\mathcal{I}$ the Godement resolution $\mathcal{I}^\bullet$ of $\mathbb{Z}_M$; it clearly posseses this property. If we combine this with the standard exact sequence

$$0\to i_{D!}i_D^!\mathcal{I}^\bullet\to\mathcal{I}^\bullet\to j_*j^*\mathcal{I}^\bullet\to 0,$$

we find that $j_*j^*\mathcal{I}^\bullet$ is injectively resolved by the double complex

$$\to \bigoplus_{\operatorname{codim} S=k} i_{S!}i_S^!\mathcal{I}^\bullet\otimes E_S\to\cdots\to\bigoplus_{\operatorname{codim} S=1} i_{S!}i_S^!\mathcal{I}^\bullet\otimes E_S\to \mathcal{I}^\bullet\to 0.$$

Now $j^*\mathcal{I}^\bullet$ is the Godement resolution of $\mathbb{Z}_{M'}$, and the Thom isomorphism shows that $i_S^!\mathcal{I}^\bullet$ is quasi-isomorphic to $\mathbb{Z}_S[-2\operatorname{codim} S](-\operatorname{codim} S)$. So a cohomological grading gives the spectral sequence

$$E_1^{-p,q}=\bigoplus_{\operatorname{codim} S=p} H^{q-2p}(S)\otimes E_S(-p)\Rightarrow H^{q-p}(M'). \tag{2.3-1}$$

In the algebraic setting, this is a spectral sequence of mixed Hodge structures.

(2.4) Let us see what this spectral sequence yields in our three examples.

(2.4.1) In the affine-linear case, the members of $\mathcal{S}$ are affine-linear subspaces and are therefore acyclic. In particular, the spectral sequence degenerates and we recover the result of Brieskorn [2] and Orlik and Solomon [11] that states that $H^k(A')$ is free of rank $\sum_{S\in\mathcal{S},\operatorname{codim} S=k}\epsilon(S)$. In particular, the Poincaré polynomial of A' is

$$P(A')(t)=\sum_{S\in\mathcal{S}}\epsilon(S)t^{\operatorname{codim} S}.$$

This also shows that in the general case $E_S(-p)$ can be interpreted as the cohomology in degree $\operatorname{codim} S$ of the intersection of a small spherical neighborhood of a point of S with M'. If H is a hyperplane in D defined by an affine-linear form f_H, then the logarithmic differential $\omega_H := df_H/(2\pi i f_H)$ only depends on H (not on f_H), and according to Brieskorn [2] the $\mathbb{Z}$-subalgebra of the DeRham complex of A' generated by these forms maps isomorphically onto $H^\bullet(A')$. It is clear that the cohomology in degree k is pure of type (k,k), so that $PS(A')(t,u) = P(M')(tu^2)$.

(2.4.2) In the projective case we can regard one of the irreducible components as a hyperplane at infinity and this reduces the situation to the previous case. Perhaps a better approach is the following. Let V be the vector space such that P is its associated projective space, and let $D_V \subset V$ be the union of linear hyperplanes corresponding to D. Then V' is a $\mathbb{C}^*$-bundle over P'; this $\mathbb{C}^*$-bundle is trivial if $D \neq \emptyset$. Assuming that this is the case, we see that we have short exact sequences

$$0 \to H^k(P') \to H^k(V') \to H^{k-1}(P')(-1) \to 0$$

and that

$$P(P')(t) = \frac{P(V')(t)}{1+t}; \quad PS(P')(t,u) = P(V')(tu^2).$$

Notice that $H^\bullet(P')$ is the subalgebra of $H^\bullet(V')$ generated by the differences of logarithmic differentials $\omega_H - \omega_{H'}$.

(2.4.3) As we already noticed, the spectral sequence (2.3-1) is a spectral sequence of mixed Hodge structures. Every $S \in \mathcal{S}$ is an algebraic torus, so its cohomology is the exterior algebra of $H^1(S)$, and $H^1(S)$ is pure of type $(1,1)$. Hence $E_1^{-p,q}$ is pure of weight $2(q-p)$. Since the differentials must respect the weight, the spectral sequence with rational coefficients degenerates at the E_1-term. In particular, $H^k(T')$ is pure of Tate type (k,k). It also follows that the Poincaré polynomial of T' is equal to

$$P(T')(t) = \sum_{S\in\mathcal{S}} \epsilon(S) t^{\operatorname{codim} S}(1+t)^{\dim S} = (1+t)^m \sum_{S\in\mathcal{S}} \epsilon(S)\Big(\frac{t}{1+t}\Big)^{\operatorname{codim} S},$$

and that $PS(T')(t,u) = P(T')(tu^2)$. It is not difficult to prove that the whole complex cohomology is generated by the logarithmic differentials (these include the translation invariant differentials!). So the $\mathbb{C}$-algebra generated by these forms maps surjectively to $H^\bullet(T';\mathbb{C})$. According to Deligne [4] this map is also injective.

(2.5) Assume that G is a finite group operating on M which preserves D. The spectral sequence then becomes a spectral sequence of G-modules. The stabilizer $N_G(S)$ of S acts on any sum $\oplus_{T\supset S} H^k(T) \otimes E_T$, where the sum is taken over all strata $T \in \mathcal{S}$ of a given codimension which contain S. If $S \neq M$, then it follows from lemma (2.2) that the virtual representation

$$\bigoplus_{T\supset S} (-1)^{\operatorname{codim} S_1} E_T$$

is zero. This allows us in principle to compute the character of the representation of $N_G(S)$ on E_S.

The G-invariants yield a spectral sequence $(E_1^{-p,q})^G \Rightarrow H^{q-p}(M')^G$. After tensoring with $\mathbb{Q}$, this last group becomes isomorphic to $H^{q-p}(G\backslash M';\mathbb{Q})$. So if $\mathcal{S}_0$ is a system of orbit representatives for the G-action in $\mathcal{S}$, then we have a spectral sequence

$$(2.5\text{-}1)\qquad \bigoplus_{\operatorname{codim} S=p; S\in\mathcal{S}_0} (H^{q-2p}(S;\mathbb{Q})\otimes E_S)^{N_G(S)}(-p) \Rightarrow H^{q-p}(G\backslash M';\mathbb{Q}).$$

It degenerates in the affine and toroidal cases.

3. Arrangements of divisors attached to root sytems

In this section we focus our attention on arrangement-like divisors that come from root systems.

(3.1) Let W be a finite reflection group of rank l acting effectively in a complex vector space V, and let $D\subset V$ be the union of reflection hyperplanes. If $1=m_1\leq m_2\leq m_3\leq\cdots\leq m_l$ are the exponents of W, then according to a formula of Solomon and Brieskorn,

$$P(V')(t)=\Pi_{i=1}^l(1+m_it).$$

We denote the space of W-invariants in $E_{\{0\}}\otimes\mathbb{Q}\cong H^l(V';\mathbb{Q})$ by $L(W)$. So if $(W_i)_i$ are the irreducible components of W, then $L(W)$ is the tensor product of the $L(W_i)$'s. For later purposes we observe that minus the identity acts trivially on the modules E_S; if $\operatorname{codim} S=1$ this is clear, and the general case easily follows from this. In particular, minus the identity acts trivially on $L(W)$.

Choose a fundamental chamber C for W and identify its codimension one faces with the vertex set of its Coxeter diagram $\operatorname{Cox}(W)$. Then every set X of vertices of $\operatorname{Cox}(W)$ determines an intersection of reflection hyperplanes $S(X)$ of W, and in this way we meet every W-orbit in $\mathcal{S}$. If $\mathcal{X}$ is a collection of vertex subsets of $\operatorname{Cox}(W)$ such that the corresponding subset of $\mathcal{S}$ is a system of representatives of W-orbits, then it follows from (2.5) that

$$H^p(W\backslash V';\mathbb{Q})\cong\bigoplus_{X\in\mathcal{X},|X|=p} E_{S(X)}^{N_W(S(X))}\otimes\mathbb{Q}\cong\bigoplus_{X\in\mathcal{X},|X|=p} L(W_X)^{N_W(X)}.$$

This reduces the computation of the rational cohomology of $W\backslash V'$ to that of the spaces $L(W')$ (for all reflection groups W') as a representation of the automorphism group of $\operatorname{Cox}(W')$. The way this is done is indicated below. Here we only notice that in case W is of type A_1, $E_{\{0\}}=\mathbb{Z}$ so that $L(W)$ is canonically isomorphic to $\mathbb{Q}$. If W is of type $(A_1)^k$ then the permutation group on the irreducible components of W acts on $L(W)$ according to the sign character.

(3.2) In the previous example we denote by P the projective space of V and we let D_P be the divisor corresponding to D. Since V' is a trivial $\mathbb{C}^*$ bundle over $P':=P-D_P$,

$$P(P')(t)=(1+t)^{-1}P(V')(t)=\Pi_{i=2}^l(1+m_it),$$

and similarly

$$P(W\backslash P')(t)=(1+t)^{-1}P(W\backslash V')(t).$$

This may also help us to represent $L(W)$ (inductively) by logarithmic forms of degree l on V'. Suppose that W is irreducible and that $l \geq 1$. For every reflection hyperplane H let $\omega_H := df_H/(2\pi i f_H)$ be the logarithmic differential defined in (2.4). Then $\omega := \sum_H \omega_H$ represents a generator of the W-invariant part of $H^1(V';\mathbb{Q})$. Multiplication by ω maps $H^{l-1}(V')$ onto $H^l(V')$, and this map is W-equivariant. So this induces a surjection

$$\bigoplus_{X\in\mathcal{X},|X|=l-1} L(W_X)^{N_W(X)} \to L(W).$$

This, or a similar program, has been carried out by Brieskorn [2]. For irreducible W he finds that $L(W)$ is one-dimensional in case W is of type A_1, C_n, D_{even}, E_7, E_8, F_4, H_3, H_4, $I_2(\text{even})$ and is trivial in all other cases. So $\dim L(W) \leq 1$ always. Since we shall need to know how the symmetries of the Coxeter diagram of D_l act on $L(W(D_l))$, we will follow this procedure in that case, assuming that we already know that $L(W) = 0$ if W is of type A_k, $k \geq 2$. We use the standard convention that $D_1 = A_1, D_2 = (A_1)^2, D_3 = A_3$.

Lemma (3.3). *Assume that W is a Coxeter group of type D_l. For odd l, $L(W) = 0$. For even l, $L(W)$ is one-dimensional and a generator is the class of*

$$\zeta := \sum_{w\in W} w^*(\omega_{H_1}\wedge\cdots\wedge\omega_{H_l}),$$

where $H_1,\ldots,H_l$ be mutually orthogonal reflection hyperplanes. Moreover, an automorphism of $\mathrm{Cox}(W)$ that interchanges two branches acts as minus the identity on $L(W)$.

Proof. For $l = 1, 2, 3$ this is clear. Suppose $l \geq 4$ and assume by induction that we have proved the assertion in degree $< l$. The induction hypothesis implies that subdiagrams X of $\mathrm{Cox}(D_l)$ with $l-1$ nodes that have $L(W_X) \neq 0$ occur for even l only, and they are all of type $D_{l-2}\oplus A_1$. It already follows that $L(W) = 0$ if l is odd. Assume now l even. For $l > 4$ there is precisely one subdiagram of type $D_{l-2}\oplus A_1$; for $l = 4$, there are three, but they all belong to the same W-orbit. So in either case there is only one such X in $\mathcal{X}$ and hence $\dim L(W) \leq 1$. We show that ζ represents a nonzero class. First notice that the hyperplane H_1 spanned by X is a reflection hyperplane of a reflection s of W (the reflection w.r.t. the highest root). This implies that the W-stabilizer of $S(X)$ is just $\langle s\rangle \times W_X$. Now choose $l-1$ mutually orthogonal reflection hyperplanes $H_2,\ldots,H_l$ of W_X. Our induction hypothesis implies that $\zeta_X := \sum_{w\in W_X} w^*(\omega_{H_2}\wedge\cdots\wedge\omega_{H_l})$ is a generator of $L(W_X)$. So $\omega_{H_1}\wedge\zeta_X$ is a generator of $L(\langle s\rangle\times W_X)$.

Let $p \in P$ be the point defined by X, B be a spherical neighborhood of p that does not meet any reflection hyperplane associated to W_X, and U the preimage of B in $V - \{0\}$. Then $w^*(\omega_{H_1}\wedge\cdots\wedge\omega_{H_l})|U\cap V'$ is exact unless $w^*S(X) = S(X)$. Hence $\zeta|U\cap V'$ is cohomologous to

$$\sum_{w\in\langle s\rangle\times W_X} w^*(\omega_{H_1}\wedge\cdots\wedge\omega_{H_l}) = 2\omega_{H_1}\wedge\zeta_X,$$

and the latter represents a nonzero class. Therefore, ζ represents a nonzero class.

Finally, if $g \in GL(V)$ leaves the chamber C invariant and induces an automorphism of $\mathrm{Cox}(D_l)$ that interchanges two branches (components if $l = 2$), then it is easy to find a g-invariant set of mutually orthogal reflection planes $H_1, \ldots, H_l$ such that g induces the transposition of H_{l-1} and H_l. This implies that g acts on $L(W)$ as minus the identity.

(3.4) Let R be a reduced irreducible root system, and let T be the algebraic torus whose character group is the (root) lattice spanned by R. So T is the tensor product of the lattice $P^\vee$ of dual weights and $\mathbb{C}^*$. It comes with an action of W. We take for D the union of the fixed point hypertori of the reflections in W. We wish to find a system of representatives of the W-orbits in $\mathcal{S}$. For this, we follow the discussion in Bourbaki [1, Ch. 6, no 2.3]. Let $\mathfrak{h}$ denote the *real* vector space spanned by the dual root system $R^\vee$, so that $\mathfrak{h}/P^\vee$ can be identified with the maximal compact torus T_c in T. The affine transformation group of $\mathfrak{h}$ generated by W and the translations in $P^\vee$ is a semi-direct product $W.P^\vee$, and a fundamental domain of $W.P^\vee$ in $\mathfrak{h}$ will map isomorphically onto a fundamental domain of W in T_c.

Consider the somewhat smaller group $W.Q^\vee$, where $Q^\vee$ is the lattice spanned by the coroots. It is known Bourbaki [1, Ch. 6, no 2] that $W.Q^\vee$ acts as a reflection group on $\mathfrak{h}$ and that a fundamental domain (in the strict sense) of this action is the simplex C defined by the affine-linear inequalities $\alpha_1 \geq 0, \ldots, \alpha_l \geq 0$ and $\tilde\alpha \leq 1$, where $\tilde\alpha$ is the highest root. It is customary to write α_0 for $1 - \tilde\alpha$. We regard $\alpha_0, \ldots, \alpha_l$ as the set of nodes of the completed Dynkin $\widehat{\mathrm{Dyn}}(R)$ diagram of R. The faces of C are now in bijective (incidence-reversing) correspondence with the proper subsets of $\alpha_0, \ldots, \alpha_l$. Let us denote by $\mathcal{H}$ the collection of affine-linear subspaces of $\mathfrak{h}$ that are intersections of reflection hyperplanes.

Since $W.Q^\vee$ is normal in $W.P^\vee$ there is an induced action of the semi-direct product $P^\vee/Q^\vee$ on C. This realizes this group as an automorphism group of $\widehat{\mathrm{Dyn}}(R)$. It also follows that $W.P^\vee$ preserves the union of reflection hyperplanes of $W.Q^\vee$. So the complexification of this union is the pre-image of D under the covering $\mathfrak{h} \otimes \mathbb{C} \to T$. It follows that the W-orbits in $\mathcal{S}$ and the $W.P^\vee$-orbits in $\mathcal{H}$ are in bijective correspondence. So for every proper subset X of $\widehat{\mathrm{Dyn}}(R)$ we find a member $S(X)$ of $\mathcal{S}$ and in this way we hit every W-orbit in $\mathcal{S}$ (usually more than once). We select a collection $\mathcal{X}$ of proper subsets of $\widehat{\mathrm{Dyn}}(R)$ such that the collection $\{S(X) : X \in \mathcal{X}\}$ is a system of representatives of W-orbits in $\mathcal{S}$.

According to the previous section, the rational cohomology of $W\backslash T'$ is isomorphic to

$$\bigoplus_{|X|=p, X \in \mathcal{X}} (H^\bullet(S(X); \mathbb{Q}) \otimes E_{S(X)})^{N_W(S(X))}[p].$$

Let us consider the contribution from X in more detail. Let $W_X \subset W$ resp. $W^X \subset W$ denote the subgroup of W generated by the reflections $s(\alpha)$ with $\alpha \in X$ resp. $\alpha \perp X$ (where $s(\alpha_0) = s(\tilde\alpha)$). If $w \in W$ fixes every element of X, then $w \in W^X$. So if we put

$$W(X) := N_W(S(X))/(W_X \times W^X)$$

then $W(X)$ acts effectively on X as a group of graph automorphisms.

We may identify $(E_{S(X)}\otimes\mathbb{Q})^{W_X}$ with $L(W_X)$. Since W_X acts trivially on $S(X)$, we can, in the above expression, replace $E_{S(X)}\otimes\mathbb{Q}$ by $L(W_X)$. In particular, we only get a contribution if W_X is a product of reflection groups of the type listed in (3.2). If $\mathfrak{h}_X\subset\mathfrak{h}$ denotes the subspace of $\mathfrak{h}$ common zeroes of the roots in X, then $H^\bullet(S;\mathbb{Q})$ may be identified with the exterior algebra of the rational vector space $\mathfrak{h}^*_{X,\mathbb{Q}}$. It is known that a reflection group has no invariants in the exterior algebra of its tautological representation, except in degree zero [1, Ch. 5 , exerc. 5.3]. So if $\mathfrak{a}_X$ is fixed point set of W^X in $\mathfrak{h}_X$, then

$$(H^\bullet(S(X);\mathbb{Q})\otimes E_{S(X)})^{W^X\times W_X}=\wedge^\bullet\mathfrak{a}^*_{X,\mathbb{Q}}\otimes L(W_X)[|X|].$$

It then remains to find the space of invariants of the group $W(X)$ in the latter space. Fortunately, the space $\mathfrak{a}_X$ tends to be small.

(3.5) Let $\tilde T\to T$ be the blow up of the identity element of T, and let $D_{\tilde T}$ be the strict transform of D. We regard T' as an open subvariety of $\tilde T'=\tilde T-D_{\tilde T}$. Denote by V the tangent space of T at the origin, by $\mathbf{P}(V)$ the corresponding projective space. We identify the latter with the exceptional divisor in $\tilde T$. So $\tilde T'$ is the disjoint union of T' and $\mathbf{P}(V)':=\mathbf{P}(V)\cap\tilde T'$.

Lemma (3.6). *The sequence*

$$0\to H^\bullet(\tilde T')\to H^\bullet(T')\xrightarrow{\mathrm{Res}}H^\bullet(P')(-1)[1]\to 0$$

is exact and W-equivariant. In particular, $H^k(\tilde T')$ carries a pure Hodge structure of Tate type (k,k). Furthermore,

$$P(W\backslash\tilde T')(t)=P(W\backslash T')(t)-\frac{t}{1+t}P(W\backslash V')(t),$$
$$P(W.\{\pm\mathbf{1}\}\backslash\tilde T')(t)=P(W.\{\pm\mathbf{1}\}\backslash T')(t)-\frac{t}{1+t}P(W\backslash V')(t).$$

The cohomology in degree k is Tate of weight $2k$, so the Poincaré-Serre polynomials are obtained by substitution of tu^2 in the corresponding Poincaré polynomials.

Proof. The sequences stem from the long exact sequence of the pair $(\tilde T',T')$ and the Thom isomorphism. So for the first assertion it is enough to show that the map $H^k(T')\to H^{k+1}(\tilde T',T')\cong H^{k-1}(P')$ is surjective. If B is a small convex neighborhood of the identity of T, then we can factor this map as $H^k(T')\to H^k(B')\to H^{k-1}(P')$. The first map is onto for $k=1$. Since $H^\bullet(B')$ is generated in degree one, it follows that this is so for all k. The second map is surjective because B' is topologically a trivial punctured disk bundle over P'.

The last assertions are a consequence of the first one and the formula in (2.4.2).

4. Computations

We shall apply the methods of the previous section to the cases E_6 and E_7. They will enable us to determine (among other things) the Poincaré-Serre polynomial of the moduli spaces $\mathcal{M}_3$ and $\mathcal{M}_3^1$. We begin with a lemma that will be used frequently.

Lemma (4.1). *Let X be an algebraic variety of pure dimension, $Y \subset X$ a hypersurface, and assume that both are rational homology manifolds. Then we have a Gysin exact cohomology sequence of mixed Hodge structures*

$$\to H^{k-2}(Y;\mathbb{Q})(-1) \to H^k(X;\mathbb{Q}) \to H^k(X-Y;\mathbb{Q}) \to H^{k-1}(Y;\mathbb{Q})(-1) \to \ .$$

Proof. The assumption implies that the local cohomology sheaf $\mathcal{H}^k_Y(X;\mathbb{Q})$ vanishes for $k \neq 2$, and can be identified with $i_*\mathbb{Q}_X(-1)$ for $k=2$. The lemma follows from this.

Lemma (4.2). *The Poincaré-Serre polynomials of $\mathcal{Q}^{\mathrm{flx}}$, $\mathcal{Q}^{\mathrm{hflx}}$, $\overline{\mathcal{Q}^{\mathrm{flx}}}$ are respectively $1+t^6u^{12}$, 1, $1+t^2u^2+t^6u^{12}$.*

Proof. According to the table in Brieskorn [2] the Poincaré polynomial of $W\backslash V'$ is for W of type E_6 resp. E_7 equal to $1+t$ resp. $1+t+t^6+t^7$. Hence the Poincaré polynomial of the corresponding projectivized space is by (2.4.2) equal to 1 resp. $1+t^6$. Since we have identifications of these spaces with $\mathcal{Q}^{\mathrm{flx}}$ resp. $\mathcal{Q}^{\mathrm{hflx}}$, the first two assertions follow.

The subvariety $\mathcal{Q}^{\mathrm{hflx}}$ of $\overline{\mathcal{Q}^{\mathrm{flx}}}$ satisfies the hypothesis of (4.1). The Gysin sequence splits into short exact sequences, showing that

$$PS(\overline{\mathcal{Q}^{\mathrm{flx}}})(t,u) = t^2u^2 PS(\mathcal{Q}^{\mathrm{hflx}})(t,u) + PS(\mathcal{Q}^{\mathrm{flx}})(t,u).$$

This implies the last formula.

Corollary (4.3). *The Poincaré polynomial of $\mathcal{Q}$ is termwise bounded by $1+t^2+t^6$.*

Proof. The forgetful map $\overline{\mathcal{Q}^{\mathrm{flx}}} \to \mathcal{Q}$ is finite, and so the Poincaré polynomial of $\mathcal{Q}$ is termwise bounded by the one of $\overline{\mathcal{Q}^{\mathrm{flx}}}$ which is $1+t^2+t^6$.

As indicated in the introduction, we can at this point easily derive that the Poincaré-Serre polynomial of $\mathcal{M}_3$ is $1+t^2u^2+t^6u^{12}$ if we make use of the fact that $\mathcal{M}_3$ has second betti number 1 (due to Harer) and euler characteristic 3 (Harer and Zagier [8]). We will follow however a slightly different path that does not use Harer's computation of $b_2(\mathcal{M}_3)$.

(4.4) *Determination of the Poincaré polynomials of $W\backslash T'$ and $W.\{\pm\mathbf{1}\}\backslash T'$ in case R is of type E_6.*

We label the fundamental roots $\alpha_1,\dots,\alpha_6$ such that in the completed E_6 Dynkin diagram $(\alpha_1,\dots,\alpha_5)$ and $(\alpha_0,\alpha_6,\alpha_3)$ are strings. The only subsets X of $\widehat{\mathrm{Dyn}}(E_6)$ that may contribute to the Poincaré polynomial of $W\backslash T'$ are those of type A_1^k, $k=1,2,3,4$ and D_4. It is easily verified that such subsets belong to the same W-orbit if they are of the same type.

Case 1: X is of type A_1, say $X=\{\alpha_0\}$.

Then W^X is equal to the reflection group generated by $s(\alpha_1),\dots,s(\alpha_5)$ and $\mathfrak{a}_X$ is trivial. Hence $\wedge^\bullet\mathfrak{a}_X \otimes L(W_X) \cong L(W_X)$ is one-dimensional. It is easy to see that $L(W_X)$ has a canonical generator, so $W(X)$ and $-\mathbf{1}$ act trivially on it. We conclude that X contributes t to each Poincaré polynomial.

Case 2: X is of type $(A_1)^2$, say $X=\{\alpha_0,\alpha_5\}$.

Then W^X equals the reflection group generated by $s(\alpha_1), s(\alpha_2), s(\alpha_3)$, and so $\mathfrak{a}_X$ is one-dimensional. There is a $w \in W$ which permutes α_0 and α_6, and any such a w acts as minus the identity on $L(W_X)$. A small computation shows that w acts as minus the identity on $\mathfrak{a}_X$. Hence there are no $W(X)$-invariants in $\wedge^\bullet \mathfrak{a}_X \otimes L(W_X)$. Therefore X does not contribute in either case.

Case 3: X is of type $(A_1)^3$, say $X = \{\alpha_0, \alpha_1, \alpha_5\}$.

The only positive root orthogonal to X is α_3, and hence $\dim \mathfrak{a}_X = 2$. We determine $W(X)$ and its action on $\mathfrak{a}_X$. The roots α_1 and α_5 belong to the A_5-system orthogonal to α_0. Now for any pair of orthogonal roots in an A_5-system there is always a Weyl group element of this system that interchanges these roots. So there is a $w \in W$ that leaves α_0 fixed and interchanges α_1 and α_5. Since these roots are transitively permuted by $W(X)$, it follows that $W(X)$ maps (isomorphically) onto the full permutation group $\mathcal{S}_X$ of X. The elements of $W(X)$ that induce the identity on $L(W_X)$ are those that induce an even permutation of X. The group $W(X)$ acts on $\mathfrak{a}_X$ as a reflection group of type A_2. From this it follows that the space of $W(X)$-invariants $\wedge^\bullet \mathfrak{a}_X \otimes L(W_X)$ is equal to the one-dimensional space $\wedge^2 \mathfrak{a}_X \otimes L(W_X)$. So X contributes t^5 to $P(W \backslash T')$. Since $-\mathbf{1}$ acts as minus the identity in $\mathfrak{a}_X$, but as the identity on $L(W_X))$, it acts as the identity on $\wedge^2 \mathfrak{a}_X \otimes L(W_X)$, and so we get the same contribution to $P(W.\{\pm\mathbf{1}\} \backslash T')$

The last two cases are not much different from the previous case.

Case 4: X is of type $(A_1)^4$, so $X = \{\alpha_0, \alpha_1, \alpha_3, \alpha_5\}$.

There are no roots orthogonal to X, and so $\mathfrak{a}_X = \mathfrak{h}_X$ is of dimension two.

A similar argument shows that $W(X)$ maps isomorphically onto the full permutation group $\mathcal{S}_X$ of X. The elements of $W(X)$ that act as the identity on $L(W_X)$ are precisely the ones that induce an even permutation of X. Consider the representation of $W(X) \cong \mathcal{S}_4$ on $\mathfrak{a}_X$. We saw in the previous case that the restriction of this representation to the stabilizer of α_3 ($\cong \mathcal{S}_3$) has image a reflection group of type A_2. Up to isomorphism, there is only one such irreducible representation of $\mathcal{S}_4$ and the image of this representation is the same as its restriction to $\mathcal{S}_3$. Clearly, each even element will go to an even element. As before, $-\mathbf{1}$ acts as the identity on $L(W_X)$.

It follows that the space of $W(X)$-invariants and the space of $W(X).\{\pm\mathbf{1}\}$-invariants of $\wedge^\bullet \mathfrak{a}_X \otimes L(W_X)$ are both equal to $\wedge^2 \mathfrak{a}_X \otimes L(W_X)$, so that we get a contribution t^6 in either case.

Case 5: X is of type D_4, so $X = \{\alpha_2, \alpha_3, \alpha_4, \alpha_6\}$.

We have that $\mathfrak{a}_X = \mathfrak{h}_X$ is of dimension two, the group $W(X)$ maps isomorphically onto the symmetry group $\mathrm{Aut}(X) \cong \mathcal{S}_3$ of X as a subgraph of $\widehat{\mathrm{Dyn}}(R)$ and $W(X)$ induces in $\mathfrak{h}_X$ a reflection group of type A_2. According to lemma (3.3) this group acts on $L(W_X)$ according to the sign character.Since $-\mathbf{1}$ acts on $L(W_X)$ as the identity, the space of $W(X)$-invariants in $\wedge^\bullet \mathfrak{a}_X \otimes L(W_X)$ is also the space of $W(X).\{\pm\mathbf{1}\}$-invariants, and equal to $\wedge^2 \mathfrak{a}_X \otimes L(W_X)$. Hence we get a contribution t^6 in either case.

Conclusion. The Poincaré polynomial of both $W \backslash T'$ and $W.\{\pm\mathbf{1}\} \backslash T'$ is equal to $1 + t + t^5 + 2t^6$.

Corollary (4.5). *The subvariety $\mathcal{Q}^{\rm btg}$ of $\mathcal{Q}^1$, resp. its closure $\mathcal{Q}^{\overline{\rm btg}}$ in $\mathcal{Q}^1$, has all its cohomology in degree k pure of type (k,k), and its Poincaré polynomial is $1+t+t^5+2t^6$, resp. $1+t^5+2t^6$. The same assertion holds for the subvariety $\mathcal{Q}_{\rm btg}$ of $\mathcal{Q}$, resp. its closure $\mathcal{Q}_{\overline{\rm btg}}$ of $\mathcal{Q}$.*

Proof. It follows from (1.18) that $\mathcal{Q}^{\rm btg}$ resp. $\mathcal{Q}^{\overline{\rm btg}}$ can be identified with the W-orbit space of T' resp. $\tilde{T}'$ (where the root system is of type E_6). According to the table in Brieskorn [2], the Poincaré polynomial of $W\backslash V'$ is equal to $1+t$. The first part of the claim then follows from lemma (3.6) and the above computation. The second part follows in the same way.

In order to finish our computation of the Poincaré-Serre polynomial of $\mathcal{M}_3$ we need to deal with the hyperelliptic locus.

Lemma (4.6). *Let n be an integer ≥ 3. Then the moduli space Σ_n of n-element subsets of $\mathbf{P}^1$ (taken modulo projective equivalence) has no rational homology in nonzero degree.*

Proof. Let $\tilde{\Sigma}_n \to \Sigma_n$ be the $\mathcal{S}_n$-covering obtained by ordering the elements. We can represent a point of $\tilde{\Sigma}_n$ by an $(n-1)$-tuple $(z_1,\dots,z_{n-1})$ in $\mathbb{C}$ with $\sum_i z_i = 0$ and the z_i's of course distinct (the nth point is ∞). This $(n-1)$-tuple is unique up to scalar multiplication. Now the hyperplane V of $\mathbb{C}^{n-1}$ defined by $\sum_i z_i = 0$ is the natural representation space of $\mathcal{S}_{n-1}$ as a reflection group (of type A_{n-2}). We just proved that $\tilde{\Sigma}_n$ can be identified with $\mathbf{P}(V)'$. This identification is clearly $\mathcal{S}_{n-1}$-equivariant, and so we have an unramified covering $\mathcal{S}_{n-1}\backslash\mathbf{P}(V)' \to \Sigma_n$ (of degree n). According to Brieskorn [2] the Poincaré polynomial of $\mathcal{S}_{n-1}\backslash V'$ equals $1+t$, and so by (3.2) the Poincaré polynomial of $\mathcal{S}_{n-1}\backslash\mathbf{P}(V)'$ reduces to 1. Hence the same is true for Σ_n.

Since the moduli space $\mathcal{H}_g$ of smooth hyperelliptic curves of genus g may be identified with Σ_{2g+2}, we find that $\mathcal{H}_g$ is acyclic for rational homology.

Theorem (4.7). *The moduli space $\mathcal{Q}$, resp. $\mathcal{M}_3$, has Poincaré-Serre polynomial $1+t^6u^{12}$ resp. $1+t^2u^2+t^6u^{12}$.*

Proof. Since the forgetful map $\mathcal{Q}_{\overline{\rm btg}} \to \mathcal{Q}$ is finite, the Poincaré polynomial of $\mathcal{Q}$ is termwise bounded by $1+t^5+2t^6$. According to (4.3) it is also termwise bounded by $1+t^2+t^6$. Hence it is termwise bounded by $1+t^6$. The pair $(\mathcal{M}_3,\mathcal{H}_3)$ satisfies the hypothesis of lemma (4.1) and the Gysin sequence of this pair shows that the Poincaré polynomial of $\mathcal{M}_3$ is equal to the Poincaré polynomial of $\mathcal{Q}$ plus t^2, that is, either $1+t^2+t^6$ or $1+t^2$. According to Harer and Zagier [8], the euler characteristic of $\mathcal{M}_3$ equals 3, and so the first case holds. It is clear that the class in degree 2 resp. 6 has weight 2 resp. 12.

Next we calculate the Poincaré-Serre polynomial of $\mathcal{M}_3^1$.

(4.8) *Determination of the Poincaré polynomial of $W\backslash T'$ in case R is of type E_7.* In this case $P^\vee/Q^\vee$ is of order two and can be identified with the automorphism group of $\widehat{\rm Dyn}(R)$. The types of subdiagrams X that may contribute to the Poincaré polynomial are $(A_1)^k$, $k=1,\dots,5$, D_4, D_6 and E_7. We shall list a representative collection of subsets X of $\widehat{\rm Dyn}(R)$ of these types. It turns out that in each case the root system R^X of roots orthogonal to X spans the orthogonal complement of X. So we always have that $\mathfrak{a}_X = 0$ and hence that $(\wedge^\bullet \mathfrak{a}_X \otimes L(W_X))^{W(X)} = L(W_X)^{W(X)}$.

We label the fundamental roots $\alpha_1, \dots, \alpha_7$ such that in the completed E_7 Dynkin diagram, $(\alpha_0, \dots, \alpha_6)$ and (α_7, α_3) are strings.

Case 1: X of type A_1. Then R^X is of type D_6 and $W(X)$ acts trivially on $L(W_X)$. So X contributes t.

Case 2: X of type $(A_1)^2$, say $X = \{\alpha_0, \alpha_2\}$. Then R^X is type $(A_1)^5$. There is a transformation in the Weyl group that permutes the two elements of X. Such a transformation acts as minus the identity on $L(W_X)$, and so X does not contribute.

Case 3: There are two subcases here which can be distinguished by the type of R^X

Subcase 3a: $X = \{\alpha_0, \alpha_2, \alpha_7\}$. Then R^X is of type D_4. There is a $w \in W_{\{\alpha_0, \alpha_1, \alpha_2\}}$ which interchanges α_0 and α_2. Such a transformation fixes α_7 and acts on $L(W_X)$ as minus the identity. Hence X does not contribute.

Subcase 3b: $X = \{\alpha_0, \alpha_2, \alpha_6\}$. Then R^X is of type $(A_1)^4$. The Weyl group element w of the previous case fixes α_6, and acts as minus the identity on $L(W_X)$. So this X does not contribute either.

Case 4: X of type $(A_1)^4$. Here too, there are two subcases, which are dual to (3a) and (3b): in case (4a) W_X is contained in a Weyl group of type D_4, while in case (4b) this is not so:

Subcase 4a: $X = \{\alpha_0, \alpha_2, \alpha_4, \alpha_6\}$

Subcase 4b: $X = \{\alpha_0, \alpha_2, \alpha_4, \alpha_7\}$

In either case R^X is of type $(A_1)^3$. The Weyl group element used in case 3 serves the same purpose here and we find that neither case contributes.

Case 5: X of type $(A_1)^5$, so $X = \{\alpha_0, \alpha_2, \alpha_4, \alpha_6, \alpha_7\}$. Then the system R^X is of type $(A_1)^2$. Arguing as before, we find that X does not contribute.

Case 6: X is of type D_4, so $X = \{\alpha_2, \alpha_3, \alpha_4, \alpha_7\}$. The system R^X is of type $(A_1)^3$. This case is dual to case 3a. The group $W(X)$ is the automorphism group X. If $w \in W$ is the element that interchanges α_2 and α_4 and fixes the other elements of X, then w induces minus the identity in $L(W_X))$. We therefore get no contribution from X.

Case 7: X is of type D_6, say $X = \{\alpha_2, \dots, \alpha_7\}$. This case is dual to case (1). We have $R^X = \{\pm\alpha_0\}$. Since $W(X)$ is trivial, X contributes t^6 to the Poincaré polynomial.

Case 8: X is of type E_7. Clearly $W(X)$ is trivial, so this case contributes t^7.

Conclusion. The Poincaré polynomial of $W\backslash T'$ is $1 + t + t^6 + t^7$ and the Poincaré polynomial of $W\backslash\tilde{T}'$ is $1 + t^6$.

Corollary (4.9). *The space $\mathcal{Q}^1 - \mathcal{Q}^{\overline{\rm btg}}$ has the property that its cohomology in degree k is pure of type (k, k). Its Poincaré polynomial is equal to $1 + t^6$.*

Proof. By (1.17) this moduli space can be identified with the W-orbit space of $\tilde{T}'$, where R is of type E_7. Now apply the preceding conclusion.

Theorem (4.10). *The Poincaré-Serre polynomial of the moduli space of pointed smooth quartic curves $\mathcal{Q}^1$ resp. pointed smooth genus three curves $\mathcal{M}_3^1$ is equal to $1+t^2u^2+t^6u^{12}+t^7u^{12}+2t^8u^{14}$ resp. $1+2t^2u^2+t^4u^4+t^6u^{12}+t^7u^{12}+2t^8u^{14}$.*

Proof. Lemma (4.1) applies to the pair $(\mathcal{Q}^1, \overline{\mathcal{Q}^{\mathrm{btg}}})$. It follows from (4.5) and 4.9) that the associated Gysin sequence breaks up in short exact sequences and that the Poincaré-Serre polynomial of $\mathcal{Q}^1$ is equal to $PS(\mathcal{Q}^1 - \overline{\mathcal{Q}^{\mathrm{btg}}}) + t^2u^2PS(\overline{\mathcal{Q}^{\mathrm{btg}}})$. This proves the first part of the theorem. The complement of $\mathcal{Q}^1$ in $\mathcal{M}_3^1$ is the hyperelliptic locus $\mathcal{H}_3^1$. The forgetful map $\mathcal{H}_3^1 \to \mathcal{H}_3$ is a fibration by projective lines. Its spectral sequence degenerates, and so the inclusion of a fibre ($\cong \mathbf{P}^1$) induces an isomorphism on rational cohomology. Hence $PS(\mathcal{H}_3^1)(t,u) = 1 + t^2u^2$. The second part follows from another application of (4.1).

Corollary (4.11). *The cohomology of the variation of Hodge structure over $\mathcal{M}_3$ defined by the first direct image of $\mathcal{M}_3^1 \to \mathcal{M}_3$ is pure Tate in every degree, and its Poincaré-Serre polynomial is equal to $t^6u^{12}+t^7u^{14}$.*

Proof. The forgetful map $\mathcal{M}_3^1 \to \mathcal{M}_3$ is proper and smooth modulo quotient singularities. According to a theorem of Deligne [3], the associated Leray spectral sequence degenerates at the E_2-term over $\mathbb{Q}$. It is in fact a spectral sequence of Hodge structures; this is a special case of a theorem of M. Saito [12], but in the present case (where we deal with a map which is projective and essentially smooth) it can probably also be proved using the methods of Deligne. The three possibly nonzero subquotients in degree k are $H^k(\mathcal{M}_3;\mathbb{Q})$, $H^{k-1}(\mathcal{M}_3;\mathbf{E})$, and $H^{k-2}(\mathcal{M}_3;\mathbb{Q})(-1)$, where $\mathbf{E}$ denotes the first direct image. So the Poincaré-Serre polynomial of $H^\bullet(\mathcal{M}_3;\mathbf{E})$ is equal to

$$t^{-1}\big(PS(\mathcal{M}_3^1)(t,u) - (1+t^2u^2)PS(\mathcal{M}_3)(t,u)\big) = t^6u^{12}+t^7u^{14}.$$

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