1.B *Nuclear Physics* A127 (1969) 161—168;

SOLUTION OF THE FADDEEV EQUATIONS FOR THE TRITON PROBLEM USING LOCAL TWO-PARTICLE INTERACTIONS

R. A. MALFLIET and J. A. TJON
Institute for Theoretical Physics, University of Utrecht, Utrecht, The Netherlands

Received 9 December 1968

Abstract: A simple method is described to compute exactly the binding energy (B.E.) of the ground state of three nucleons in the framework of the Faddeev equations. The two-body potentials thereby used are of the local central Yukawa type.

The effect of including repulsion in the two-body forces is also studied and it is found to be considerable. As a result of this calculation a value of 8.4 MeV is obtained for the B.E. of triton.

1. Introduction

In the past few years several exact calculations of the bound state energy of three nucleons have been made with separable potentials within the framework of the Faddeev equations [1]). In particular one has studied the effect of repulsive and tensor-like interactions on the binding energy of the triton. As a result it has been possible to obtain a fair agreement with experiment [2]).

So far, very little has been done in the case of local potentials because of the numerical complexity in solving the Faddeev equations [3]). For realistic triton calculations using local potentials, the only results which have been obtained are by variational method, which of course only gives an upper bound [4]). Recently, Humberston, Hall and Osborn [5]) have given in addition to variational upper bounds also lower bounds for simple potentials which are very close to each other. However, it is doubtful whether these bounds will also lie as close together for the case of more complicated potentials (i.e. with repulsive and tensor forces included), so that the uncertainty of a variational calculation will probably remain.

In this paper we present some results obtained by solving the Faddeev equations with local Yukawa potentials including also "soft core" repulsion. The technique for solving the equations is based on a method which has also been used by Schwartz [5]) for the calculation of bound states in the two-particle case. It essentially makes use of the property that the perturbation series starts to diverge at an energy equal to the bound state energy of the ground state.

In sect. 2 the Faddeev equations are written down for three nucleons in the case of pure two-body central forces. It is thereby assumed that isospin invariance is not violated and furthermore that only singlet and triplet s-waves contribute in the two particle channels.

Sect. 3 deals with the two-body potentials which have been used. Also a comparison is made of their phase shifts up to 300 MeV laboratory energy with the experimental ones. In sect. 4 a detailed account is given of the method used to determine the three-particle bound state. Finally, in the last section the results are given and discussed.

2. The three-body problem

The Faddeev equations for three nucleons can be reduced to a system of three equations in two continuous variables using the angular momentum decomposition given by Ahmadzadeh and Tjon [7]). These equations are furthermore decoupled since we are dealing with identical particles. Keeping only the s-wave part of the two body interaction, we obtain the following equation (with total angular momentum zero)

$$\psi(p, q, \beta, s) = \varphi(p, q, \beta, s) - \frac{4\pi}{q\sqrt{3}} \sum_{\beta'} F(\beta|\beta') \times \int_0^\infty \mathrm{d}(q'^2) \int_{L(q,q')}^{U(q,q')} \mathrm{d}(p'^2) \frac{t_\beta(p, \bar{p}; s-q^2)}{p'^2+q'^2-s} \psi(p', q', \beta', s), \tag{2.1}$$

with

$$\varphi(p, q, \beta, s) = t_\beta(p, p''; s-q^2) \frac{2}{q''} \delta(q''^2-q^2) \delta_{\beta'', \beta}. \tag{2.2}$$

Here p and q describe the magnitudes of the relative momenta which in terms of the particle momenta $\boldsymbol{k}_i$ ($i = 1, 2, 3$) are given by

$$\boldsymbol{p} = \frac{1}{2\sqrt{M}}(\boldsymbol{k}_2 - \boldsymbol{k}_3),$$

$$\boldsymbol{q} = \frac{1}{2\sqrt{3M}}[(\boldsymbol{k}_2 + \boldsymbol{k}_3) - 2\boldsymbol{k}_1].$$

The variable s stands for the total energy of the three-particle systems. Furthermore we have defined

$$\bar{p}^2 = p'^2 + q'^2 - q^2,$$
$$U(q, q') = \tfrac{1}{3}(2q + q')^2,$$
$$L(q, q') = \tfrac{1}{3}(2q - q')^2,$$

and have used an abbreviated notation β for the spin-isospin dependence. The factor $F(\beta|\beta')$ is the spin-isospin recoupling coefficient plus an additional sign factor depending on the statistics

$$F(S, I, s, i|S', I', s', i') = \varepsilon_S \langle (s_1 s_2) s s_3 S | (s_1 s_3) s' s_2 S' \rangle \times \langle (i_1 i_2) i i_3 I | (i_1 i_3) i' i_2 I' \rangle \delta_{I, I'} \delta_{S, S'}, \tag{2.3}$$

$$\varepsilon_S = +1 \quad \text{for bosons}$$
$$= -1 \quad \text{for fermions}$$

where s_1 and s_2 are coupled together to form s which in turn is coupled to s_3 to form the total spin S, and similarly for the isospin dependence. The expressions between brackets are, in the notation of Edmonds [8])

$$\langle (j_1 j_2) j j_3 J | (j_1 j_3) j' j_2 J \rangle = (-1)^{j_2+j_3+j+j'} [(2j+1)(2j'+1)]^{\frac{1}{2}} \begin{Bmatrix} j_3 & J & j \\ j_2 & j_1 & j' \end{Bmatrix}. \quad (2.4)$$

The two body T-matrix $t_\beta(p, p''; z)$ obeys the usual Lippmann-Schwinger equation

$$t_\beta(p, p''; z) = V_\beta(p, p'') - 4\pi \int_0^\infty p'^2 \mathrm{d}p' \frac{V_\beta(p, p')}{p'^2 - z} t_\beta(p', p''; z), \quad (2.5)$$

and the on-energy shell T-matrix is related to the phase shift $\delta_\beta(p)$ as follows:

$$-2\pi^2 t_\beta(p, p; p^2) = \frac{1}{p} \sin \delta_\beta(p) \mathrm{e}^{i\delta_\beta(p)}. \quad (2.6)$$

Since the triton is a pure total spin $\frac{1}{2}$, total isospin $\frac{1}{2}$ state, eq. (2.1) becomes a coupled two-channel equation for this case.

3. The two-body potentials

In this section we describe the various sets of potentials which are used in the three-body problem. As a first step towards a realistic calculation we have used for the nucleon-nucleon interaction in the $^1S_0$ channel a superposition of two Yukawa potentials one of which is repulsive, i.e.

$$V(r) = -\lambda_A \frac{\mathrm{e}^{-\mu_A r}}{r} + \lambda_R \frac{\mathrm{e}^{-\mu_R r}}{r}, \quad (3.1)$$

with $\mu_R = 2\mu_A$. The potential constants were determined from a fit to the low energy parameters and the phase shifts up to 300 MeV lab energy. In order to get the triplet potential we only varied λ_A of the singlet potential in such a way that the binding energy of the deuteron is reproduced. The results of the $^1S_0$ and $^3S_1$ phase shifts are given in figs. 1 and 2 respectively. For comparison we have also plotted the Yale phase shifts [9]), and the phase shifts of pure attractive Yukawa potentials (i.e. $\lambda_R = 0$) the constants of which are obtained by fitting only the low-energy parameters. (For the triplet potential this is achieved by adjusting the constants to obtain the appropriate binding energy and scattering length). We see from the figures that the potentials with repulsion give a much better agreement with the experimental phase shifts. In table 1 are presented the potentials together with their low-energy parameters.

In order to study the effect of a core on the three-particle system we have, following Tabakin [10]), also computed the scattering length and bound state energy belonging to the average of the singlet and triplet potentials I and III. Subsequently a single Yukawa interaction was determined from these low-energy parameters. These potentials (V and VI) are also given in table 1. In these cases where for simplicity it is as-

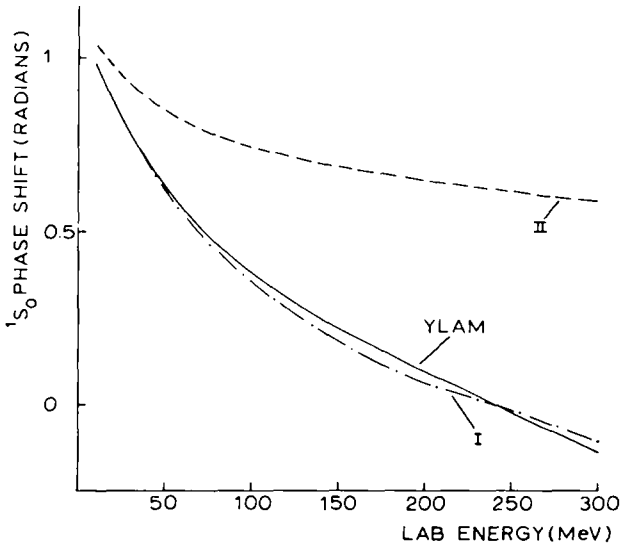

Fig. 1. The $^1S_0$ phase shifts for potentials I and II with parameters given in table 1, compared with the $^1S_0$ Yale phase shift analysis K_0 of YLAM.

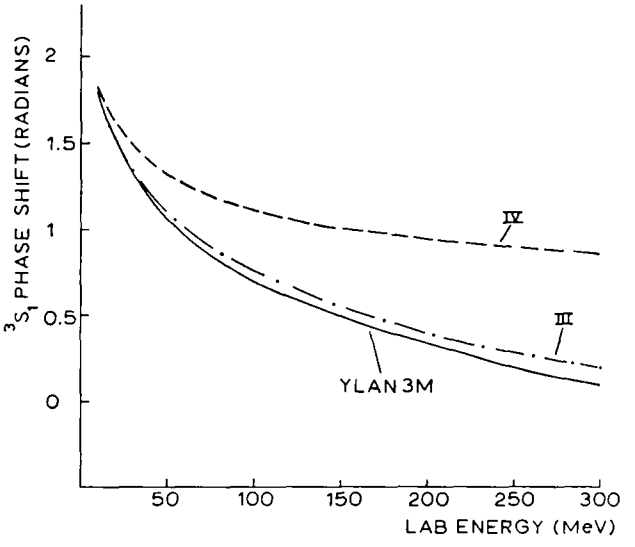

Fig. 2. The $^3S_1$ phase shifts for potentials III and IV with parameters given in table 1, compared with the $^3S_1$ Yale phase shift analysis $^3\mathcal{O}_1^S$ of YLAN 3M.

sumed that the singlet and triplet forces may be replaced by an average effective potential which is identical in both channels, the three-particle problem simplifies to the solution of a one channel equation being eq. (2.1) with effectively taking $F(\beta|\beta') = = \delta_{\beta,\beta'}$, i.e. the nucleons can be treated in these cases as identical spinless bosons.

TABLE 1
The sets of two-body potentials and their corresponding low-energy parameters

Potential no.	λ_A	μ_A (fm^{-1})	γ_R	μ_R (fm^{-1})	a (fm)	r (fm)	E_D (MeV)
I	2.64	1.55	7.39	3.11	−23.3	2.8	
II	0.266	0.809	0		−23.3	2.8	
III	3.22	1.55	7.39	3.11	5.45	1.8	2.23
IV	0.330	0.633	0		5.45	1.8	2.23
V	2.93	1.55	7.39	3.11	12.1	2.2	0.35
VI	0.298	0.723	0		12.1	2.2	0.35

4. Method of solution

One procedure to find the binding energy of the three-particle system is to calculate the Fredholm determinant. However, even in a one-channel problem the numerical complications to overcome are quite large. We shall now describe a different method for determining the binding energy of the ground state. It is far simpler and can even be used for the case of more channels.

The inhomogeneous term in eq. (2.1) with the δ-function can easily be gotten rid of by writing down the equation for the difference $F = \psi - \varphi$. One obtains an integral equation without δ-functions which can formally be written as

$$F(s) = F_0(s) - K(s)F(s). \tag{4.1}$$

We now for convenience introduce a complex variable λ by replacing in eq. (4.1) the kernel $K(s)$ by $\lambda K(s)$. Of course the solution $F(s)$ will also become a function of λ. Since $K(s)$ is of the Hilbert-Schmidt type we know that the matrix elements of $F(s)$ will be meromorphic functions of the variable λ, with poles at the characteristic values λ_α of the integral equation. This means that we may write $F(s)$ as

$$F(s) = \sum_\alpha \frac{B_\alpha(s)}{\lambda_\alpha(s) - \lambda} + R(\lambda, s), \tag{4.2}$$

where $R(\lambda, s)$ is an entire function of λ. Expanding now $F(s)$ and $R(\lambda, s)$ in a power series of λ

$$\begin{aligned} F(s) &= \sum_{n=0}^{\infty} \lambda^n F_n(s), \\ R(\lambda, s) &= \sum_{n=0}^{\infty} \lambda^n R_n(s), \end{aligned} \tag{4.3}$$

where $F_n(s)$ can simply be found from the iterative solution of eq. (4.1) we obtain the following relation between the coefficients

$$F_n(s) = \sum_\alpha \frac{B_\alpha(s)}{\lambda_\alpha^{n+1}(s)} + R_n(s). \tag{4.4}$$

Taking the ratios F_{n+1}/F_n and remembering that $R(\lambda, s)$ is an entire function in λ, one easily finds

$$\lim_{n\to\infty} \frac{F_{n+1}(s)}{F_n(s)} = \frac{1}{\lambda_0(s)},$$

where $\lambda_0(s)$ is, in magnitude, the smallest characteristic value of eq. (4.1).

In order to find the binding energy of the ground state of the three-particle system we only have to determine the energy s for which the set of ratios $r_n = F_{n+1}/F_n$ will tend to one in the limit $n \to \infty$. In the cases we have considered here it turns out that r_n converges very rapidly to a constant value. Even for the two-channel case we only needed ten iterations of eq. (4.1) to find r_∞.

The integral equation (4.1) was iterated numerically on a CDC-3200 computer. The integrations were done using a Gaussian quadrature routine with suitable point distributions. To do the integration over p' in eq. (2.1) we used a mesh point distribution which depends on the variable lower and upper limits of integration. The functions present in the integrand have thereby been calculated by simple linear interpolation between the values of the functions given at a fixed point distribution in p-space. Increasing the number of integration points in each variable from 8 to 16 gave a maximal change of about 1 % in the set of ratios r_n. As a result the accuracy of the binding energy is estimated to be better than $1\frac{1}{2}$ %.

As a check for the program we solved by matrix inversion the equations for a separable potential in the reduced form of a one dimensional integral equation and compared the results obtained with those from the two dimensional equation. The results were identical. As another check we used the local Yukawa interaction studied by Osborn [3]) and recalculated the position of the three-body ground state as a function of the coupling constant. We found an excellent agreement with the curves reported by Humberston *et al.* [5]). Hence also no "collapse" phenomenon for large coupling constant was found.

5. Results and discussion

In table 2 we present the results obtained for the triton binding energy, using the various potentials discussed in sect. 3. The first aspect we have studied is the effect of

TABLE 2
Three-nucleon binding energies for different sets of potentials

Potential	E_T (MeV)
average V	7.3
average VI	10.6
triplet III singlet I	8.3
triplet IV singlet I	8.4
triplet IV singlet II	12.1

introducing repulsion in the interaction and whether the use of an "effective" interaction, as proposed by Blatt and Weisskopf [11]), is good enough to reproduce the experimental value. For this purpose we have done the same model calculation as Tabakin did [10]) for separable potentials. Using an average of triplet and singlet forces (potential no. V) which contains a "soft-core" repulsion, we obtained 7.3 MeV for the binding energy compared with 10.6 MeV for a purely attractive potential (potential no. VI) which was adjusted to the same low-energy data. If we compare this with the variation of about 0.5 MeV found by Tabakin using separable interactions with or without core, we see that the effect of a repulsion is more substantial in the local case. Furthermore, our average potential V produces smaller binding energy of 7.3 MeV, compared with the value of 8.8 MeV in the separable case.

We now turn to the result obtained for a more realistic triton calculation using the potentials III or IV for the triplet interaction and potentials I or II for the singlet interaction. As one can see from the numbers quoted in table 2 the effect of a core in the singlet state is important, while including also repulsion in the triplet force has almost no effect on the binding energy. In the separable case, according to Mitra [2]), the inclusion of a tensor force in the triplet interaction changes the binding energy by 0.3 MeV, if one starts with a purely attractive triplet interaction and a singlet interaction which contains repulsion. Assuming that this will also be the case for local interactions one would obtain with the tensor force included 8.7 MeV for the binding energy of the triton which should be compared with the experimental value of 8.48 MeV. It should therefore be interesting to study the effect of local tensor forces on the three-body system.

Finally, if we compare our result for the triton binding energy viz. 8.4 MeV with the 8.9 MeV obtained by Schrenk and Mitra [2]) using separable potentials adjusted to the same data as our potentials I and IV, we see that nonlocal potentials tend to give overbinding with respect to the experimental value, while our local potentials give some underbinding.

This investigation is part of the research program of the "Stichting voor Fundamenteel Onderzoek der Materie" and was made possible by financial support from the "Nederlandse Organisatie voor Zuiver Wetenschappelijk Onderzoek". We are greatly indebted for the computational facilities offered by the "Stichting voor Fundamenteel Onderzoek der Materie" at the Zeeman laboratory in Amsterdam.

References

1) L. D. Faddeev, ZhETF **39** (1960) 1459; (JETP (Sov. Phys.) **12** (1961) 1014);
L. D. Faddeev, Mathematical aspects of the three-body problem in the quantum scattering theory, (Translation: Daniel Davey and Co., New York, 1963)
2) L. M. Delves and A. C. Phillips, The present status of the nuclear three-body problem, to be published in Rev. Mod. Phys.

3) J. S. Ball and D. Y. Wong, Phys. Rev. **169** (1968) 1362;
T. A. Osborn, SLAC Report no. 79; Ph.D. thesis, Stanford (1968)
4) References are found in ref. [3]) and H. P. Noyes and H. Fieldeldey, Calculations of the three-nucleon low-energy parameters, SLAC Report no. 409 (1968)
5) J. W. Humberston, R. L. Hall and T. A. Osborn, Phys. Lett. **27B** (1968) 195
6) C. Schwartz, J. Comp. Phys. **1** (1966) 21
7) A. Ahmadzadeh and J. A. Tjon, Phys. Rev. **139** (1965) B1085
8) A. R. Edmonds, Angular momentum in quantum mechanics, (Princeton University Press, New Jersey, 1957)
9) G. Breit and R. D. Haracz in High energy physics, vol. I, edited by E. H. S. Burhop (Academic Press, New York, 1967)
10) F. Tabakin, Phys. Rev. **137** (1965) B75
11) J. M. Blatt and V. F. Weisskopf, Theoretical nuclear physics (John Wiley, New York, 1952) p. 195