Nonlinear coupling between scissors modes of a Bose-Einstein condensate

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We explore the nonlinear coupling of the three scissors modes of an anisotropic Bose-Einstein condensate. We show that only when the frequency of one of the scissors modes is twice the frequency of another scissors mode, these two modes can be resonantly coupled and a down-conversion can occur. We perform the calculation variationally using a gaussian trial wave function. This enables us to obtain simple analytical results that describe the oscillation and resonance behavior of the two coupled modes.

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I. INTRODUCTION

Similar to monopole and quadrupole breathing modes of a gaseous Bose-Einstein condensate [1–7], scissors modes were first studied theoretically [8,9] and subsequently observed experimentally [10]. The scissors modes are, however, rather special since they directly manifest the superfluid behavior of these atomic gases. Moreover, recent experimental studies appear to show a resonance behavior between two coupled scissors modes [11]. From a theoretical point of view this is interesting because a linear-response calculation can account neither for the coupling nor for the resonance behavior [8,9]. Therefore, a first step towards an explanation of these experimental observations is to perform a calculation that goes beyond linear response theory and accurately takes into account the mean-field interaction that couples the scissors and breathing modes. In this paper we present a simple variational method for calculating the frequencies of these various modes and their couplings beyond the linear-response. We perform our calculation at zero temperature and therefore do not consider the damping rates of the scissors modes [12,13].

The main idea behind our method is to use a time-dependent Gaussian ansatz for the ground-state wave function to derive the equations of motion of the breathing modes and the scissors modes. Then we expand the resulting equations of motion in deviations from equilibrium. In first order, i.e., linear response, we recover the expected uncoupled set of equations [8,9]. The second-order calculation produces a set of coupled equations that show that we need to consider all three scissors modes in order to get a nonzero coupling. At higher orders we, however, find that we can restrict ourselves to two modes to get a nonlinear coupling. Furthermore, we actually find under certain conditions a resonance behavior between these two modes.

The layout of the paper is as follows. First, we rederive in Sec. II the frequencies of the scissors modes in the linear-response limit. In Sec. III we extend the calculation first to second, and then also to higher orders, which ultimately lead to a resonant coupling. In Sec. IV we solve the equations of motion analytically near the resonance using an envelope function approach. In Sec. V we end with a discussion of our results.

II. FREQUENCIES OF THE SCISSORS MODES

We start by considering a Bose-Einstein condensate trapped by the following harmonic potential

$$V(\mathbf{r})=\frac{1}{2}m(\omega_x^2x^2+\omega_y^2y^2+\omega_z^2z^2), \tag{1}$$

where ω_x, ω_y, and ω_z are the angular frequencies of the trap, and m is the atomic mass. A scissors mode in a Bose-Einstein condensate is associated with an irrotational flow with a velocity field of the form $\mathbf{v}(\mathbf{r})\propto\boldsymbol{\nabla}(xy)$, if the motion is taking place in the xy plane [8]. Similar expressions hold for the two other Cartesian planes. These kind of modes can be excited by a sudden rotation of the equilibrium axes of the trap. To such a perturbation the condensate will respond by oscillating around the new equilibrium axes. For example, to excite a scissors mode in the xy plane, we rotate the x and y axes of the trap slightly around the z axis. If the angle of rotation is sufficiently small, the scissors mode can be approximated by a simple oscillation of the condensate around the new equilibrium axes. On the other hand, if the axes change through a large angle this method excites the $m=2$ quadrupole mode, where m labels the projection of the angular momentum along the axis of symmetry. The maximum angle for which the scissors mode is defined increases with deformation of the trap [14].

To account for all three scissors modes in the three Cartesian planes we employ the following trial function for the condensate order parameter

$$\psi(\mathbf{r},t)=A(t)\exp[-b_x(t)x^2-b_y(t)y^2-b_z(t)z^2 -c_{xy}(t)xy-c_{xz}(t)xz-c_{yz}(t)yz], \tag{2}$$

where b_i and c_{ij}, are complex time-dependent variational parameters and

$$A=\frac{\sqrt{N}2^{1/4}}{\pi^{3/4}}\sqrt[4]{c_{xy,r}c_{xz,r}c_{yz,r}+4b_{x,r}b_{y,r}b_{z,r}-(b_{z,r}c_{xy,r}^2+b_{y,r}c_{xz,r}^2+b_{x,r}c_{yz,r}^2)}. \quad (3)$$

This value of the prefactor $A(t)$ guarantees the normalization of the square of the wave function $\psi(\mathbf{r},t)$ to the total number of condensed atoms N. Here $b_{i,r}$ and $c_{ij,r}$ are the real parts of b_i and c_{ij}, respectively. The first set of parameters, b_i, give rise to the well-studied breathing modes which, for axially symmetric traps, are called the monopole and quadrupole modes depending on the value of m being equal to 0 or 2, respectively. The parameters c_{ij}, on the other hand, determine the three scissors modes. The equations of motion for these variational parameters can be derived from the Lagrangian

$$L[\psi,\psi^*]=\frac{1}{2}i\hbar\int d\mathbf{r}\left(\psi^*(\mathbf{r},t)\frac{\partial\psi(\mathbf{r},t)}{\partial t}-\psi(\mathbf{r},t)\frac{\partial\psi^*(\mathbf{r},t)}{\partial t}\right)-E[\psi,\psi^*], \quad (4)$$

where $E[\psi,\psi^*]$ is the usual Gross-Pitaevskii energy functional given by

$$E[\psi,\psi^*]=\int d\mathbf{r}\left[\frac{\hbar^2}{2m}|\boldsymbol{\nabla}\psi(\mathbf{r},t)|^2+V(\mathbf{r})|\psi(\mathbf{r},t)|^2+\frac{1}{2}T^{2B}|\psi(\mathbf{r},t)|^4-\mu|\psi(\mathbf{r},t)|^2\right]. \quad (5)$$

Here T^{2B} is the two-body T matrix, which for the atomic Bose-Einstein condensates of interest is related to the s wave scattering length a through $T^{2B}=4\pi a\hbar^2/m$.

Inserting our trial wave function into the Lagrangian and scaling frequencies with $\overline{\omega}=(\omega_x\omega_y\omega_z)^{1/3}$ and lengths with $\overline{a}=\sqrt{\hbar/m\overline{\omega}}$, it takes the dimensionless form

$$\begin{aligned}L[b,c]/N=&(\alpha_x\dot{b}_{x,i}+\alpha_y\dot{b}_{y,i}+\alpha_z\dot{b}_{z,i})/Q-\frac{1}{2}[\alpha_x(4|b_x|^2\\&+|c_{xy}|^2+|c_{xz}|^2)+\alpha_y(4|b_y|^2+|c_{xy}|^2+|c_{yz}|^2)\\&+\alpha_z(4|b_z|^2+|c_{xz}|^2+|c_{yz}|^2)]/Q-\frac{1}{2}[\alpha_x\omega_x^2\\&+\alpha_y\omega_y^2+\alpha_z\omega_z^2]/Q-\frac{1}{2\sqrt{\pi}}\gamma\sqrt{Q},\end{aligned} \quad (6)$$

where $Q=2\pi^3A^4/N^2, \alpha_x=4b_{y,r}b_{z,r}-c_{yz,r}^2, \alpha_y=4b_{x,r}b_{z,r}-c_{xz,r}^2, \alpha_z=4b_{x,r}b_{y,r}-c_{xy,r}^2$, and the dot corresponds to a time derivative. In addition $\gamma=Na/\overline{a}$ is the dimensionless parameter that represents the strength of the mean-field interaction. Minimizing the Lagrangian with respect to the 12 variational parameters, we get a set of 12 coupled equations of motion. The resulting equations of motion are rather lengthy and complicated. A significant simplification takes place if we expand these equations in the deviation of the variational parameters from their equilibrium values, i.e., in $\delta b_i(t)=\exp(-i\omega t)(b_i-b_i^{(0)})=\exp(-i\omega t)\delta b_i$ and $\delta c_{ij}(t)=\exp(-i\omega t)(c_{ij}-c_{ij}^{(0)})=\exp(-i\omega t)\delta c_{ij}$, where ω is the yet unknown eigenfrequency of the modes, and $b_i^{(0)}$ and $c_{ij}^{(0)}$ denote the equilibrium values of b_i and c_{ij}, respectively. These latter values can simply be obtained by setting the time derivatives in the equations of motion to zero. For large condensates in the so-called Thomas-Fermi regime we can ignore contributions from the kinetic energy [15] and the equilibrium variational parameters take the simple form

$$b_i^{(0)}=\left(\frac{\sqrt{\pi}}{8\gamma}\right)^{2/5}\omega_i^2,$$
$$c_{ij}^{(0)}=0. \quad (7)$$

It is required that $\gamma\gg1$ for the last equation to be valid. To first order in the deviations, the equations of motion read simply

$$\mathbf{M}\cdot\mathbf{P}=0, \quad (8)$$

where the vector $\mathbf{P}=(\delta b_{x,r},\delta b_{y,r},\delta b_{z,r},\delta b_{x,i},\delta b_{y,i},\delta b_{z,i},\delta c_{xy,r},\delta c_{xz,r},\delta c_{yz,r},\delta c_{xy,i},\delta c_{xz,i},\delta c_{yz,i})$ contains all the possible fluctuations, and the matrix $\mathbf{M}$ is given by

$$\mathbf{M}=\begin{pmatrix}\mathbf{M}^{\text{breathing}} & 0\\ 0 & \mathbf{M}^{\text{scissors}}\end{pmatrix}, \quad (9)$$

where $\mathbf{M}^{\text{breathing}}$ and $\mathbf{M}^{\text{scissors}}$ are given explicitly in Appendix. It is clear from the last equation that to linear order the breathing modes and scissors modes are uncoupled. The dispersion relation of these modes can be obtained by setting the determinant of $\mathbf{M}$ to zero. This results in

$$(\omega^2-\Omega_{xy}^2)(\omega^2-\Omega_{xz}^2)(\omega^2-\Omega_{yz}^2)\times(\omega^6-3\omega_a^2\omega^4+8\omega_b^4\omega^2-20\omega_c^6)=0, \quad (10)$$

where

$$\Omega_{xy}=\sqrt{\omega_x^2+\omega_y^2},\quad \Omega_{xz}=\sqrt{\omega_x^2+\omega_z^2},$$
$$\Omega_{yz}=\sqrt{\omega_y^2+\omega_z^2},\quad \omega_a^2=\omega_x^2+\omega_y^2+\omega_z^2,$$
$$\omega_b^4=\omega_x^2\omega_y^2+\omega_x^2\omega_z^2+\omega_y^2\omega_z^2,$$

and

$$\omega_c^6=(\omega_x\omega_y\omega_z)^2.$$

The zeros of the first three factors in the left-hand side of the last equation give the frequencies of the scissors modes,

,whereas the zeros of the sixth-order polynomial give the frequencies of the breathing modes.

So far we have not produced a new result, since these frequencies have been calculated previously [8,9,16]. The use of a Gaussian variational approach to calculate the frequencies of the breathing modes was first presented in Ref. [7] and the scissors modes frequencies were first calculated using this method in Ref. [9]. The most important part of this paper is, therefore, contained in the next section, where we consider also the nonlinear effects produced by the mean-field interaction.

III. BEYOND LINEAR RESPONSE

In this section we consider the equations of motion for the variational parameters by taking into account several higher-order terms in the deviations δb_i and δc_{ij}. We have calculated these equations analytically up to second order, but they turn out to be rather lengthy and contain terms that couple the breathing modes and the scissors modes. As we discuss below, in the present experiments with axially symmetric traps the coupling of the scissors modes with the quadrupole mode is always of importance, but we leave the treatment of this more complicated situation to future work. For simplicity, therefore, we focus here on triaxial traps, in which case we can ignore the breathing modes and simply put $\delta b_i = 0$ in the full equations of motion. The neglect of the breathing modes is then justified for our purposes because for these traps the degeneracy between the quadrupole modes and the scissors modes is lifted. Consequently, if two scissors modes are resonantly coupled the quadrupole modes will be off resonance.

Up to second order, the remaining six first-order equations for $\delta c_{xy,r} \ldots \delta c_{yz,i}$ can be reduced to three second-order equations for $\delta c_{xy,r}$, $\delta c_{xz,r}$, and $\delta c_{yz,r}$ by eliminating the imaginary parts. We find in detail

$$\delta\ddot{c}_{xy,r} + \Omega_{xy}^2 \delta c_{xy,r} + \left(\frac{64}{\pi}\right)^{1/5} \gamma^{2/5} \left(\frac{\omega_x \omega_y}{\omega_z^4}\right)^{2/5} \times \left[-\Omega_{xy}^2 \delta c_{xz,r} \delta c_{yz,r} + 2\left(\frac{\omega_x\omega_y}{\Omega_{xz}\Omega_{yz}}\right)^2 \delta\dot{c}_{xz,r} \delta\dot{c}_{yz,r}\right] = 0, \tag{11}$$

$$\delta\ddot{c}_{xz,r} + \Omega_{xz}^2 \delta c_{xz,r} + \left(\frac{64}{\pi}\right)^{1/5} \gamma^{2/5} \left(\frac{\omega_x \omega_z}{\omega_y^4}\right)^{2/5} \times \left[-\Omega_{xz}^2 \delta c_{xy,r} \delta c_{yz,r} + 2\left(\frac{\omega_x\omega_z}{\Omega_{xy}\Omega_{yz}}\right)^2 \delta\dot{c}_{xy,r} \delta\dot{c}_{yz,r}\right] = 0, \tag{12}$$

and

$$\delta\ddot{c}_{yz,r} + \Omega_{yz}^2 \delta c_{yz,r} + \left(\frac{64}{\pi}\right)^{1/5} \gamma^{2/5} \left(\frac{\omega_y \omega_z}{\omega_x^4}\right)^{2/5} \times \left[-\Omega_{yz}^2 \delta c_{xy,r} \delta c_{xz,r} + 2\left(\frac{\omega_y\omega_z}{\Omega_{xy}\Omega_{xz}}\right)^2 \delta\dot{c}_{xy,r} \delta\dot{c}_{xz,r}\right] = 0. \tag{13}$$

These equations show that if we ignore the second-order terms we recover our previous three uncoupled scissors modes with frequencies Ω_{xy}, Ω_{xz}, and Ω_{yz}. It is interesting to observe that the coupling terms couple the three scissors modes such that if only one mode is initially excited then it will never couple to the two other modes. We believe that this important result is not an artifact of the Gaussian approximation, but also holds for an exact calculation using the Bogoliubov theory.

For higher-order couplings the last conclusion is no longer true. Two modes can then be coupled, even when the third is not involved in the dynamics. From now on, therefore, we assume without loss of generality that only the δc_{xy} and δc_{xz} modes are excited, while $\delta c_{yz} = 0$ always. Moreover, to investigate the possibility of a resonant coupling between these two modes, we have considered coupling terms up to ninth order. Similar to the second-order case explained above, the equations of motion can thus be expressed as

$$\delta\ddot{c}_{xy,r} + \Omega_{xy}^2 \delta c_{xy,r} + \sum_{j+k+l+m \leq 9} \alpha_{jklm} (\delta c_{xy,r})^j \times (\delta\dot{c}_{xy,r})^k (\delta c_{xz,r})^l (\delta\dot{c}_{xz,r})^m = 0, \tag{14}$$

$$\delta\ddot{c}_{xz,r} + \Omega_{xz}^2 \delta c_{xz,r} + \sum_{j+k+l+m \leq 9} \beta_{jklm} (\delta c_{xz,r})^j \times (\delta\dot{c}_{xz,r})^k (\delta c_{xy,r})^l (\delta\dot{c}_{xy,r})^m = 0, \tag{15}$$

where $j,k,l,m = 0,1,2,3$ and the sum $j+k+l+m$ does not excced 9, which is the order up to which we have chosen to expand the equations of motion. The coefficients α_{jklm}, β_{jklm} are given in terms of the trap parameters $\omega_x, \omega_y, \omega_z$, and the interaction parameter γ. A resonance between these two modes takes place when the frequency of the coupling terms is equal to the frequency of the zeroth-order term, i.e., the first two terms. The frequency of each coupling term in the above summations is determined by substituting for δc_{ij} and $\delta\dot{c}_{ij}$ their zeroth-order solutions. Therefore, the latter frequencies will be a certain linear combination of the zeroth-order scissors-modes frequencies. Imposing the above resonance condition on each coupling term thus results in a relation between the two scissors-modes frequencies Ω_{xy} and Ω_{xz}. Inspecting all coupling terms up to the ninth order, we found a very small number of terms for which the relation between Ω_{xy} and Ω_{xz} can be satisfied by real values of ω_x, ω_y, and ω_z. Ultimately, we find only a resonance when either

$$\Omega_{xy} = \Omega_{xz} \tag{16}$$

or

$$2\Omega_{xy}=\Omega_{xz} \tag{17}$$

is satisfied. The first resonance condition leads in the experimentally relevant axially symmetric case, where $\omega_x=\omega_y=\omega_z/\sqrt{\lambda}$, to a value for the anisotropy ratio λ that is equal to 1. This clearly corresponds to a spherically symmetric condensate. Since in this case the scissors modes are degenerate with the quadrupole breathing modes, which we have ignored here, we focus from now on only on the second resonance condition.

It is interesting to mention that the second resonance condition is exactly the same as the one observed experimentally [11]. For a resonance of this kind the resonant coupling terms turn out to be of seventh order, and the equations of motion in that case read

$$\begin{aligned}\delta\ddot{c}_{xy,r}+\Omega_{xz}^2\delta c_{xy,r}+\beta(\delta c_{xy,r})^3(\delta\dot{c}_{xy,r})^2(\delta c_{xz,r})^2\\+\eta(\delta c_{xy,r})^5(\delta\dot{c}_{xz,r})^2=0,\end{aligned} \tag{18}$$

$$\delta\ddot{c}_{xz,r}+\Omega_{xz}^2\delta c_{xz,r}+\alpha(\delta c_{xz,r})^3(\delta c_{xy,r})^2(\delta\dot{c}_{xy,r})^2=0, \tag{19}$$

if we neglect all nonresonant terms. Here α,β, and η are functions of ω_x, ω_y, ω_z, and γ that are given explicitly in the Appendix. It is important to note here that our neglect of the nonresonant terms is justified when we are close enough to resonance. This is similar to the rotating-wave approximation known from quantum optics. We see that the coupling terms indeed lead to the above-mentioned resonance condition, by inserting in them the zeroth-order solutions, i.e., $\delta c_{xy,r}\propto\exp(-i\Omega_{xy}t)$ and $\delta c_{xz,r}\propto\exp(i\Omega_{xz}t)$. For example, the coupling terms in Eq. (18) have a total frequency of $2\Omega_{xz}-5\Omega_{xy}$. Separating out $\delta c_{xy,r}\propto\exp(-i\Omega_{xy}t)$ as a prefactor for the whole equation, the coupling term oscillates thus as $2\Delta=4\Omega_{xy}-2\Omega_{xz}$. Therefore, a resonance takes place when $\Delta=0$, i.e., when the condition in Eq. (17) is met. Similarly, the coupling term of Eq. (19) is also oscillating with a frequency of 2Δ.

IV. SOLUTION OF THE EQUATIONS OF MOTION NEAR RESONANCE

Sufficiently close to resonance we can write δc_{xy} and δc_{xz} as a product of two functions. One of them describes the slow envelope and the other the fast oscillation with the uncoupled scissors-mode frequency. In particular, we have

$$\delta c_{xy}(t)=g(t)\exp(i\Omega_{xy}t) \tag{20}$$

and

$$\delta c_{xz}(t)=f(t)\exp(-i\Omega_{xz}t), \tag{21}$$

where $g(t)$ and $f(t)$ are the slowly varying envelope functions. Substituting these expressions into Eqs. (18) and (19), ignoring second-order time derivatives of $f(t)$ and $g(t)$, and then eliminating $g(t)$, we obtain the following equation for $f(t)$

$$-(i\ddot{f}+2\Delta\dot{f})f+i\varepsilon\dot{f}^2=0, \tag{22}$$

where

$$\varepsilon=\frac{3\alpha\Omega_{xy}^3-4\beta\Omega_{xy}^2\Omega_{xz}-4\eta\Omega_{xz}^3}{\alpha\Omega_{xy}^3}. \tag{23}$$

This equation has a solution of the form

$$f(t)=[C_1+C_2\exp(2i\Delta t)]^{1/(1-\varepsilon)}, \tag{24}$$

where C_1 and C_2 are two constants of integration that are determined by the initial conditions. Note that the relevant quantity here is $|f(t)|$, which represents the actual envelope of the oscillation and is given by

$$|f(t)|=[C_1^2+C_2^2+2C_1C_2\cos(2\Delta t)]^{1/[2(1-\varepsilon)]}. \tag{25}$$

In first instance we might think that the real part of $f(t)$ is the relevant quantity. However, in Eqs. (20) and (21) we should in principle have taken the real part of the right-hand side. If we do that we automatically are lead to the condition that $|f(t)|$ is the envelope of the oscillation.

For definiteness sake let us take the initial conditions $f(0)=f_r(0)\equiv f_0$ and $\dot{f}(0)=\dot{f}_i(0)\equiv\dot{f}_0$, where $f_r(t)$ and $f_i(t)$ are the real and imaginary parts of $f(t)$, respectively. Physically, this set of initial conditions corresponds to exciting the scissors modes in the xz and the xy planes simultaneously. This should be performed experimentally by initially rotating the condensate in the xz plane by an angle θ_0 and around the z axis by an angle ϕ_0 and then releasing the condensate. The initial angles θ_0 and ϕ_0 are related to the constants f_0 and $\dot{f}_0$ by

$$f_0=2|b_x^{(0)}-b_z^{(0)}|\cos\theta_0\sin\theta_0, \tag{26}$$

$$\begin{aligned}\dot{f}_0=\frac{\alpha\Omega_{xy}^2}{\Omega_{xz}}(2|b_x^{(0)}-b_z^{(0)}|\cos\theta_0\sin\theta_0)^3|b_x^{(0)}-b_y^{(0)}|\\\times\cos\phi_0\sin\phi_0.\end{aligned} \tag{27}$$

With these initial conditions the constants C_1 and C_2 are given by

$$C_1=f_0^{1-\varepsilon}+\frac{\dot{f}_0f_0^{-\varepsilon}(1-\varepsilon)}{2\Delta}, \tag{28}$$

$$C_2=-\frac{\dot{f}_0f_0^{-\varepsilon}(1-\varepsilon)}{2\Delta}. \tag{29}$$

Using these expressions and the experimental parameters from Ref. [10], we give in Fig. 1 the real part of $\delta c_{xz}(t)$. This clearly shows how the energy is being exchanged between the two modes.

An interesting property of Eq. (22) is that exactly on resonance, i.e., $\Delta=0$, its solution becomes nonoscillatory. Indeed we find that in this case the solution is

$$f(t)=(C_3+C_4t)^{1/(1-\varepsilon)}, \tag{30}$$

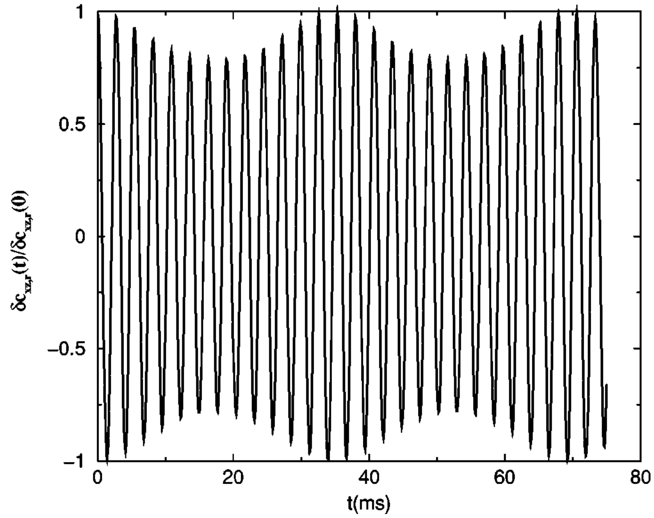

FIG. 1. The real part of δc_{xz} showing two kinds of oscillation. The one with the larger frequency corresponds to the unperturbed scissors-mode oscillation with frequency Ω_{xz}. The slower oscillation is due to the mean-field coupling between the scissors mode in the xz plane and the xy plane. The frequency of this oscillation is $2\Delta=2|2\Omega_{xy}-\Omega xz|$. The Bose-Einstein condensate parameters used to make this plot are $\omega_x=\omega_y=\omega_z/\sqrt{\lambda}=128$ Hz. As in the case of Ref. [10], $\lambda=2.54$ and $N=10^4$ of ^{87}Rb atoms. The initial conditions are $\theta_0=20\times\pi/180$ and $\phi_0=0.03\times\pi/180$. The latter was taken small to show that already such a small perturbation in the xz scissors mode is sufficient to initiate a substantial coupling between this mode and the xy mode.

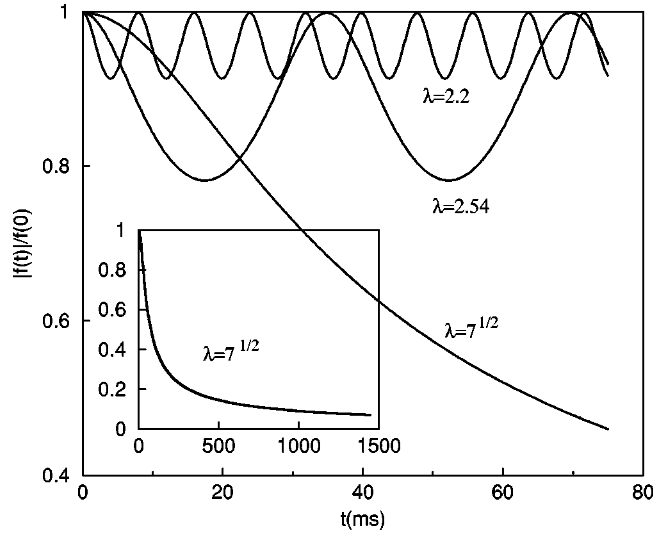

FIG. 2. The envelope $|f(t)|$ that determines the energy transferred between the two coupled scissors modes. This set of curves shows that when approaching the resonance condition $\Delta=|2\Omega_{xy}-\Omega_{xz}|=0$, the frequency of the curve decreases until it becomes equal to zero on resonance. The inset shows this last behavior for much larger times. For axially symmetric traps the resonance takes place at $\lambda=\sqrt{7}$. The parameters used for these plots are the same as those mentioned in Fig. 1.

where again C_3 and C_4 are constants that are determined by the initial conditions. For the above initial conditions this is a decreasing function in time. Physically, this means that, unlike the case of an off-resonant oscillation, it takes an infinite time for the energy that is transferred from the scissors mode in the xz plane to the scissors mode in the xy plane to get back to the mode in the xz plane. In Fig. 2 we show this resonance behavior. Finally, we can show that in the limit $\Delta\to 0$ the oscillatory solution given in Eq. (24) reduces to the nonoscillatory one at resonance given by Eq. (30).

V. SUMMARY AND CONCLUSION

We have explored the role of the mean-field interaction in coupling the three scissors modes of a Bose-Einstein condensate. A variational approach with a Gaussian trial wave function, that contains a number of variational parameters describing the scissors modes, provides a relatively simple way in which we can extract the main features of this coupling. To first order in the deviations in the variational parameters from their equilibrium values we reproduce the correct frequencies of the scissors modes. To second order we show that it is not possible to have two modes that are coupled if the third mode is not involved in the dynamics. Instead, all three modes need to be involved for nonlinear dynamics to occur.

At higher orders we find that it is possible to consider only two modes. In this case we find a resonance behavior if $2\Omega_{xy}=\Omega_{xz}$ or $\Omega_{xy}=\Omega_{xz}$. Up to the ninth order in the deviations, these are the only two cases of resonance that we have found. Close to resonance the equations of motion have been solved exactly using an envelope approach. The resulting dynamics is similar to a beating between two modes with a beating frequency $2\Delta=|2\Omega_{xy}-\Omega_{xz}|$. We notice that the observed resonance behavior occurs exactly for the same condition that we have obtained, namely, $2\Omega_{xy}=\Omega_{xz}$ [11]. However, the dynamics we find here is different from that found by Hodby *et al.*, presumably because the resonance behavior shown in their paper is observed with axially symmetric traps [11]. The fact that we did not find any other resonance condition up to the ninth order, indicates that the coupling terms that lead to this resonance are also responsible for the experimental resonance in the triaxial case.

Furthermore, it is important to note here that, quite generally, the down-conversion process from one excitation quantum into two excitation quanta with half the energy each, i.e., the so-called Beliaev damping [17], vanishes for the scissors modes. This is so because of the negative parity of the scissors modes. In terms of fluctuations of the anihilation operator $\hat{\psi}(\mathbf{r},t)$ given by $\hat{\varphi}(\mathbf{r},t)=\hat{\psi}(\mathbf{r},t)-\psi(\mathbf{r},t)$, the Beliaev damping process is accounted for by an interaction term proportional to $\int d\mathbf{r}\psi(\mathbf{r},t)\hat{\varphi}^{\dagger}(\mathbf{r},t)\hat{\varphi}^{\dagger}(\mathbf{r},t)\hat{\varphi}(\mathbf{r},t)$. Here $\psi(\mathbf{r},t)\equiv\langle\hat{\psi}(\mathbf{r},t)\rangle$ is again the condensate wave function. Using our variational Gaussian wave function given by Eq. (2), we clearly see that with only two scissors modes present this integral vanishes, since the integrand is an odd function. We believe that this result is independent of our trial wave function and also true within an exact approach [18]. In our case the seventh-order coupling terms in Eqs. (18) and (19) correspond to a quadratic collisional damping process, i.e., a collisional process for which the amplitude is quadratic in

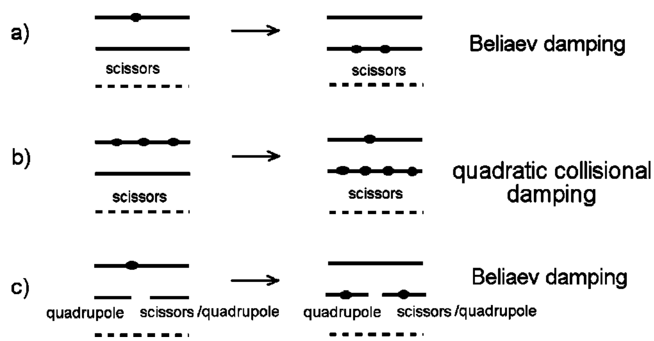

FIG. 3. A schematic figure representing three down-conversion processes. (a) Beliaev damping where one excitation decays into two excitations with half the frequency. It is shown in the text that for two scissors modes this process vanishes. (b) Quadratic collisional damping in which three excitations decay into one excitation of the same frequency and four excitations with half the frequency. According to the present work this is the first nonvanishing down-conversion process. (c) Beliaev damping due to down-conversion from one scissors mode into another scissors mode and a degenerate quadrupole mode, or into two quadrupole modes. This is the situation that corresponds to the most detailed experiments of Ref. [11], but the matrix elements of these processes again vanish.

the interaction, where three excitation quanta decay into four excitation quanta with half the frequency and one excitation quantum with the same frequency. This is shown schematically in Fig. 3. This makes sense physically since this is the lowest-order nonvanishing process if Beliaev damping is forbidden and we are forced to apply the interaction term proportional to $\int d\mathbf{r}\hat{\varphi}^{\dagger}(\mathbf{r},t)\hat{\varphi}^{\dagger}(\mathbf{r},t)\hat{\varphi}(\mathbf{r},t)\hat{\varphi}(\mathbf{r},t)$ twice to accomplish the down conversion. Another point worth recalling here is that in the experiments the resonance is studied in most detail for an axially symmetric trap with an anisotropy ratio close to $\lambda=\sqrt{7}$, when the xy scissors mode is degenerate with the quadrupole mode, which we have excluded in the present calculation. In that case we may argue that one scissors-mode quantum can decay into either another scissors-mode quantum and a quadrupole-mode quantum or into two quadrupole-mode quanta. These are in principle also Beliaev damping processes, which are, however, again forbidden for similar reasons as before. Nevertheless, a good understanding of these experiments requires an analysis that includes not only the xy and the xz scissors modes but also the quadrupole mode, because of the degeneracy that occurs in this case. However, our theory should be directly applicable to the experiments with a triaxial trap. We hope that in the future more detailed experiments of this kind will also be performed, to make a direct comparison between our theory and experiment possible.

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APPENDIX: COUPLING MATRICES AND COEFFICIENTS

The matrices $\mathbf{M}^{\text{breathing}}$ and $\mathbf{M}^{\text{scissors}}$ are given by

$$\mathbf{M}^{\text{breathing}} = \begin{pmatrix}
-\frac{3\gamma^{2/5}\omega_x^{2/5}\omega_y^{2/5}\omega_z^{2/5}}{2^{4/5}\pi^{1/5}} & -\frac{\gamma^{2/5}\omega_x^{12/5}\omega_z^{2/5}}{2^{4/5}\pi^{1/5}\omega_y^{8/5}} & -\frac{\gamma^{2/5}\omega_x^{12/5}\omega_y^{2/5}}{2^{4/5}\pi^{1/5}\omega_z^{8/5}} & i\omega & 0 & 0 \\
-\frac{\gamma^{2/5}\omega_x^{12/5}\omega_z^{2/5}}{2^{4/5}\pi^{1/5}\omega_y^{8/5}} & -\frac{3\gamma^{2/5}\omega_x^{22/5}\omega_z^{2/5}}{2^{4/5}\pi^{1/5}\omega_y^{18/5}} & -\frac{\gamma^{2/5}\omega_x^{22/5}}{2^{4/5}\pi^{1/5}\omega_y^{8/5}\omega_z^{8/5}} & 0 & \frac{i\omega\omega_x^4}{\omega_y^4} & 0 \\
-\frac{\gamma^{2/5}\omega_x^{12/5}\omega_y^{2/5}}{2^{4/5}\pi^{1/5}\omega_z^{8/5}} & -\frac{\gamma^{2/5}\omega_x^{22/5}}{2^{4/5}\pi^{1/5}\omega_y^{8/5}\omega_z^{8/5}} & -\frac{3\gamma^{2/5}\omega_x^{22/5}\omega_y^{2/5}}{2^{4/5}\pi^{1/5}\omega_z^{18/5}} & 0 & 0 & \frac{i\omega\omega_x^4}{\omega_z^4} \\
-i\omega & 0 & 0 & -\frac{2^{4/5}\pi^{1/5}\omega_x^{8/5}}{\gamma^{2/5}\omega_y^{2/5}\omega_z^{2/5}} & 0 & 0 \\
0 & \frac{-i\omega\omega_x^4}{\omega_y^4} & 0 & 0 & -\frac{2^{4/5}\pi^{1/5}\omega_x^{18/5}}{\gamma^{2/5}\omega_y^{12/5}\omega_z^{2/5}} & 0 \\
0 & 0 & \frac{-i\omega\omega_x^4}{\omega_z^4} & 0 & 0 & -\frac{2^{4/5}\pi^{1/5}\omega_x^{18/5}}{\gamma^{2/5}\omega_y^{2/5}\omega_z^{12/5}}
\end{pmatrix} \tag{A1}$$

and

$$
\mathbf{M}^{\text{scissors}} = \begin{pmatrix}
-\frac{\gamma^{2/5}\omega_x^{12/5}\omega_z^{2/5}}{2^{4/5}\pi^{1/5}\omega_y^{8/5}} & 0 & 0 & \frac{i\omega\omega_x^2}{2\omega_y^2} & 0 & 0 \\
0 & -\frac{\gamma^{2/5}\omega_x^{12/5}\omega_y^{2/5}}{2^{4/5}\pi^{1/5}\omega_z^{8/5}} & 0 & 0 & \frac{i\omega\omega_x^2}{2\omega_z^2} & 0 \\
0 & 0 & -\frac{\gamma^{2/5}\omega_x^{22/5}}{2^{4/5}\pi^{1/5}\omega_y^{8/5}\omega_z^{8/5}} & 0 & 0 & \frac{i\omega\omega_x^4}{2\omega_y^2\omega_z^2} \\
\frac{-i\omega\omega_x^2}{2\omega_y^2} & 0 & 0 & \frac{-\pi^{1/5}\omega_x^{8/5}\omega_x^2+\omega_y^2}{2(2\gamma^{2/5}\omega_y^{12/5}\omega_z^{2/5})} & 0 & 0 \\
0 & \frac{-i\omega\omega_x^2}{2\omega_z^2} & 0 & 0 & \frac{-\pi^{1/5}\omega_x^{8/5}\omega_x^2+\omega_z^2}{2(2\gamma^{2/5}\omega_y^{2/5}\omega_z^{12/5})} & 0 \\
0 & 0 & \frac{-i\omega\omega_x^4}{2\omega_y^2\omega_z^2} & 0 & 0 & \frac{-\pi^{1/5}\omega_x^{18/5}\omega_y^2+\omega_z^2}{2(2\gamma^{2/5}\omega_y^{12/5}\omega_z^{12/5})}
\end{pmatrix}. \tag{A2}
$$

The coupling coefficients α,β, and η, appearing first in Eqs. (18) and (19), are given by

$$
\alpha = 2\,2^{1/5}\gamma^{12/5}[3\omega_x^6+4\omega_y^4\omega_z^2-2\omega_y^2\omega_z^4+\omega_x^4(3\omega_y^2+5\omega_z^2) + 2\omega_x^2(2\,\omega_y^4+\omega_z^4)]\times(\pi^{6/5}\omega_x^4\omega_y^2\Omega_{xy}^4\Omega_{xz}^4)^{-1}, \tag{A3}
$$

$$
\beta = -2\,2^{1/5}\gamma^{12/5}[3\omega_x^8-2\omega_x^2\omega_y^4\omega_z^2-4\omega_y^6\omega_z^2+2\omega_y^4\omega_z^4 + \omega_x^6(7\omega_y^2+2\omega_z^2)+\omega_x^4(6\omega_y^4-\omega_y^2\omega_z^2+\omega_z^4)] \times(\pi^{6/5}\omega_x^4\omega_y^2\Omega_{xy}^4\Omega_{xz}^4)^{-1}, \tag{A4}
$$

and

$$
\eta = \frac{2\,2^{1/5}\gamma^{12/5}(\Omega_{xy}^2)^{12/5}}{\pi^{6/5}\omega_x^6\omega_y^4\Omega_{xz}^4}, \tag{A5}
$$

respectively.

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