Massive Skyrmions in quantum Hall ferromagnets

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We apply the theory of elasticity to study the effects of Skyrmion mass on lattice dynamics in quantum Hall systems. We find that massive Skyrme lattices behave like a Wigner crystal in the presence of a uniform perpendicular magnetic field. We make a comparison with the microscopic Hartree-Fock results to characterize the mass of quantum Hall skyrmions at $\nu=1$ and investigate how the low temperature phase of Skyrme lattices may be affected by the Skyrmion mass.

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I. INTRODUCTION

In the last few years topological spin textures in the quantum Hall effect have received considerable attention.[1–12] The existence of Skyrmions can be anticipated within the framework of the Chern-Simon-Landau-Ginsburg mean field theory,[2] i.e., integrating out the charge fluctuations of the composite bosons yields an effective model for the Chern-Simon gauge field. The transport properties of the Skyrmions can be extracted through considering the fluctuating Chern-Simon gauge field, which can be derived by expanding the effective action about its minimum energy solution. One of the leading terms of this expansion is the Maxwell action of the Chern-Simon fluctuating field, e.g., $(m^*/\overline{\rho})\Sigma_{k,\omega}|\mathbf{J}^s|^2$. m^* is the electron effective mass in a host semiconductor, $\overline{\rho}=1/(2\pi l_0^2)$ is the average electron density at $\nu=1$, and l_0 is the magnetic length.

Although a Skyrmion mass is physically reasonable, in the usual minimal field theories[9,10] Skyrmions are considered as massless objects. From the microscopic point of view, e.g., a microscopic Hartree-Fock approximation,[5,12] it is not certain if the skyrmions are massive.[15] The resolution of this ambiguity between the microscopic Hartree-Fock approximation and the Landau-Ginsburg-Chern-Simon theory (considered in this paper), is an open question.

In this paper we apply the theory of elasticity[16] to investigate the effect of Skyrmion mass on the thermodynamic properties of a Skyrmion crystal. We derive the collective mode dispersion relations for the Skyrme lattices at long wavelengths. We then make a comparison with the massless microscopic Hartree-Fock calculations to reconcile the prediction of the Chern-Simon theory with the massless models.[12] In addition, we suggest how the mass of Skyrmions may be characterized within the microscopic picture. We also study the stability of the Skyrme lattices at low temperature and show that the low temperature phases of these Skyrme lattices are *not* affected by including a mass term in the effective action, unless the mass is sufficiently large. We show that the mass of Skyrmions is suppressed by decreasing the Zeeman energy, indicating that Skyrmions are massless at zero Zeeman energy.

II. SKYRMION MASS

If Skyrmions have mass, this will affect a host of properties, from their tunneling through a constriction to the thermodynamics of a lattice of skyrmions. The rationale for their having mass is that in the starting point for many calculations, the Chern-Simons Lagrangian of Lee and Kane, the electrons have mass. After standard manipulations (introduction of a CP^1 field, changing variables to $\mathbf{m}$, the local spin field) one has a continuum theory in which gradients in the spin texture become associated with the charge distribution. If a Skyrmion moves slowly across the system, this corresponds to the motion of one quasiparticle. It seems reasonable that the motion of such a texture will involve inertial terms. Questions similar in spirit have arisen in determining the mass of vortices.[13,14]

Recalling the duality relation between the topological three-current of Skyrmions and the Chern-Simon gauge field,[7,8] $J^s_\mu=(\nu/2\pi)\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda$, one might anticipate a second derivative term in time will appear in the effective action. After integrating out the statistical gauge field one obtains in the limit of low frequency and long wavelength[7]

$$S_E[\mathbf{m}]=\frac{1}{2}\sum_{\mathbf{k},\omega}\left(V(k)|J_0^s|^2+\frac{m^*}{\overline{\rho}}|\mathbf{J}^s|^2+i\alpha\mathbf{A}^{(0)}(-k)\cdot\mathbf{J}^s(k)-\frac{2\pi\alpha}{k^2}J_0^s(-k)\,\hat{\mathbf{z}}\cdot\mathbf{k}\times\mathbf{J}^s(k)\right)+S_{NL\sigma M}+S_z+S_{\rm Hopf}, \tag{1}$$

where the last three terms are the nonlinear sigma action, the Zeeman contribution, and a term in the action that guarantees that the Skyrmions obey Fermi statistics. The first term is electrostatic, and involves the Fourier transform of an effective interaction potential $V(k)$ including screening by fluctuations in the texture, the second contains the kinetic energy, and the third reflects that the Skyrmion experiences the original boson as a magnetic flux tube. The constant α is an odd integer and arises from a Chern-Simon term that makes the system fermionic.

The Skyrmion current density is related to the spin texture via

$$J_\lambda^s = \frac{-\nu}{8\pi}\epsilon_{\lambda\mu\nu}(\partial_\mu \mathbf{m}\times\partial_\nu \mathbf{m})\cdot\mathbf{m},$$

where the indices run over time and two spatial dimensions. The zeroth component is the topological charge density of the texture, which is proportional to the quasiparticle number density $\rho(r)\equiv J_0^s$. If we consider a single Skyrmion texture that moves uniformly, $J_\lambda^s = J_\lambda^s(\mathbf{x}-\mathbf{v}t)$, then it is straightforward to show that the relation between the Skyrmionic current and their charge density satisfies the usual charge conservation law, i.e., $\mathbf{J}^s=\rho\mathbf{v}$. Then the kinetic term simplifies to

$$\frac{1}{2}\sum_{\mathbf{k}}\frac{m^*}{\rho}|\mathbf{J}^s(\mathbf{k})|^2=\frac{1}{2}M_0 v^2,$$

which yields a transport mass for the Skyrmions $M_0 =(m^*/\overline{\rho})\int d^2\mathbf{r}\rho^2(\mathbf{r})/(2\pi)^2$.

This mass is derived from kinetic considerations. It is possible that the correct ''mass'' to be calculated will depend upon exactly what is being measured in a given experiment.

III. COLLECTIVE MODES

The long range order of the Skyrme crystal is determined by the repulsive Coulomb interaction, and the topological, XY interaction of the hedgehog fields. An antiferromagnetic ordering between the single Skyrmions, within a square lattice minimizes the topological interaction. This can be realized after mapping the topological hedgehog fields of the charge 1 Skyrmions onto a system of classical dipoles. However, a triangular lattice is favored by the Coulomb interaction, similar to the Wigner crystals. When the Zeeman energy is small enough, the Skyrmions can pair up into charge two Skyrmions and lower the total energy of the lattice. In contrast to single Skyrmions where their topological hedgehog fields are analogous to a system of classical dipoles, the charge 2 Skyrmions mimic a system of classical quadrapoles. This favors triangular lattice ordering of charge 2 Skyrmions, i.e., a bi-Skyrmion lattice.

The low-lying collective modes of a Skyrme lattice consist of phonons and spin waves.[9,10,12] The dispersion relation of these collective modes can be obtained by adding a dynamical term to the effective Hamiltonian. This Hamiltonian has been derived in Ref. 9 by a first principles calculation of a nonlinear σ model, assuming that a specific Skyrmion is localized and well separated from other Skyrmions, i.e., they interact weakly. More precisely, we assume that we can divide the two-dimensional configuration space into N regions (N is the number of Skyrmions) such that a given Skyrmion exists in the region where other Skyrmions are close to their vacuum. This condition enables us to linearize the interaction potential energy among the isolated-Skyrmions. This assumption may break down if the size of Skyrmions λ becomes comparable with the distance between them. In this case the next to linear terms in the potential energy may be significant, and one should take them into account. Roughly speaking, this happens if $R\leqslant 2\lambda$ (R is the separation between two Skyrmions). Here the Skyrmion size λ is defined as the radius at which the spin lies in the XY plane. The relevant potential energy functional of the excitations is obtained by introducing the lattice of equilibrium positions of the Skyrmions $R_{i\alpha}$ in an initial lattice ansatz (α is the Cartesian component of the position of the ith Skyrmion), the displacement field $u_\alpha(\mathbf{R}_i)$, and the orientation field of the Skyrmions $\theta(\mathbf{R}_i)$. The result is then

$$\begin{aligned}E[\mathbf{u},\theta]=&\sum_{i\neq j}V_0[|\mathbf{R}_i+\mathbf{u}(\mathbf{R}_i)-\mathbf{R}_j-\mathbf{u}(\mathbf{R}_j)|]\\&+\sum_{\langle ij\rangle}J[|\mathbf{R}_i+\mathbf{u}(\mathbf{R}_i)-\mathbf{R}_j-\mathbf{u}(\mathbf{R}_j)|]\\&\times\cos[\chi(\mathbf{R}_i)+\theta(\mathbf{R}_i)-\chi(\mathbf{R}_j)-\theta(\mathbf{R}_j)].\end{aligned}\tag{2}$$

Here the topological hedgehog interaction for a single-Skyrmion lattice is given by $J(R)=c^2\tilde{g}/(4\pi^2 l_0^2)K_0(\kappa R)$ where $\tilde{g}=ge^2/(2\epsilon l_0)$ is the Zeeman energy, g is the effective gyromagnetic ratio, $\kappa^2=\tilde{g}/(2\pi l_0^2\rho_s)$, ρ_s is the spin stiffness, and $K_0(x)$ is the modified Bessel function. In addition, c is a constant that can be obtained from the asymptotic form of an isolated Skyrmion and equals $30.4 l_0$.[9] For a bi-Skyrmion lattice $J(R)=-c^2\tilde{g}^2/(8\pi^3 l_0^4\rho_s)K_0(\kappa R)$ and $c=79 l_0^2$. One should note that c, and therefore the topological interaction between Skyrmions $J(R)$, depend on the local form of the Skyrmions.

A. Phonons

We find the spectrum of the phonons, using the standard technique of expanding the energy functional about its minima, i.e., $\mathbf{u}=\mathbf{0}$, $\theta=0$, assuming the orientational field of Skyrmions is frozen out. For the single-Skyrmion case $J(R)$ is positive; hence $\chi_i-\chi_j=\pi$ is the lowest energy state of the topological XY interaction. For the bi-Skyrmion case $J(R)$ is negative and $\chi_i-\chi_j=0$ is the lowest energy state. An estimate on the total energy of skyrmions shows that the bi-Skyrmion configuration is only likely if the Zeeman energy is very small ($\leqslant 10^{-5}$). We therefore do not consider this configuration for the rest of this paper. Expanding the potential energy in terms of the displacement fields (up to quadratic order terms) gives

$$E[\mathbf{u}]=E_{\mathrm{classic}}+\frac{1}{2}\sum_{\mathbf{k}\in\mathrm{BZ}}\sum_{\alpha\beta}\mathbf{u}_\alpha^*(\mathbf{k})D_{\alpha\beta}(\mathbf{k})\mathbf{u}_\beta(\mathbf{k}),\tag{3}$$

with E_{classic} the classical ground state energy of the Skyrme lattice and $D_{\alpha\beta}(\mathbf{k})$ the dynamical matrix

$$\begin{aligned}D_{\alpha\beta}(\mathbf{k})=&\frac{2\pi}{k}\frac{e^2}{\epsilon a_c}k_\alpha k_\beta+a_c[(\mu+\lambda)k_\alpha k_\beta+\mu k^2\delta_{\alpha\beta}\\&+\gamma k_x k_y(1-\delta_{\alpha\beta})+O(k^4)].\end{aligned}\tag{4}$$

The first term comes from the electrostatic interactions of a Wigner crystal,[17] and the others arise from standard two-dimensional (2D) elasticity theory. The quantity a_c is the area of a unit cell, while μ,λ, and γ are the conventional Lamé coefficients for a square lattice.[16]

Since the effective interaction consists of two terms, i.e., the direct and exchange interactions, the Lamé coefficients can be expressed by means of $\mu=\mu_0+\mu_1$, $\lambda=\lambda_0+\lambda_1$, and $\gamma=\gamma_0+\gamma_1$. Here 0 and 1 are labels associated with the direct and the exchange Coulomb energy. Expanding the exchange term (second term) in Eq. (2) up to quadratic order in the displacement field, and for the single-skyrmionic square lattices with only nearest neighbor exchange interactions, we find

$$\mu_1=\frac{c^2\tilde{g}}{2\pi^2 a_c}\sqrt{\frac{\pi}{2}}\,\frac{0.5+\kappa R}{(\kappa R)^{1/2}}e^{-\kappa R}, \tag{5a}$$

$$\lambda_1=-\frac{c^2\tilde{g}}{2\pi^2 a_c}\sqrt{\frac{\pi}{2}}\,\frac{7/4+3\kappa R+(\kappa R)^2}{(\kappa R)^{1/2}}e^{-\kappa R}, \tag{5b}$$

$$\gamma_1=\frac{c^2\tilde{g}}{2\pi^2 a_c}\sqrt{\frac{\pi}{2}}\,\frac{5/4+2\kappa R+(\kappa R)^2}{(\kappa R)^{1/2}}e^{-\kappa R}. \tag{5c}$$

We also find that the contributions of the direct interaction to the elastic constants in units of $(e^2/\epsilon l_0 a_c)\sqrt{|1-\nu|/2\pi}$ are $\mu_0=-0.146\,289$, $\lambda_0=-0.536\,199$, and $\gamma_0=-1.560\,224$ (we take the nearest neighbor separation between the Skyrmions $R=l_0\sqrt{2\pi/|\nu-1|}$ as the square lattice constant and therefore $a_c=R^2$). Note that $\gamma=0$ for triangular lattices.

The time-derivative part in the action of the displacement field $\mathbf{u}$ consists of the Wess-Zumino term and the kinetic energy of the Skyrmions, which are first and second derivatives in (imaginary) time, respectively. Combining these with our previous results gives the following effective Euclidean action for the low energy spectrum of the displacement field:

$$\begin{aligned} S_{\text{eff}}[\mathbf{u}]=\frac{1}{2}\sum_{\alpha\beta}\sum_{\mathbf{k}\in\text{BZ}}\int_0^{\hbar\beta}d\tau[&-i\varepsilon_{\alpha\beta}m^*\omega_c u_\alpha^*(\mathbf{k},\tau)\partial_\tau u_\beta(\mathbf{k},\tau)\\ &-M_0\delta_{\alpha\beta}u_\alpha^*(\mathbf{k},\tau)\partial_\tau^2 u_\beta(\mathbf{k},\tau)\\ &+u_\alpha^*(\mathbf{k},\tau)D_{\alpha\beta}(\mathbf{k})u_\alpha(\mathbf{k},\tau)], \end{aligned} \tag{6}$$

where ω_c is the cyclotron frequency of the electron.The dependence of the Skyrmion mass on the Zeeman energy can be obtained by solving the nonlinear differential equation for the spin texture of a single Skyrmion[6] and is shown in Fig. 1. Note that, unlike the total number of spins participating in the Skyrmion, the Skyrmion mass *decreases* with g. This arises because the mass is proportional to the square of the Skyrmion density, and as g decreases the Skyrmion becomes more spread out. Expanding the time dependence of the complex field $\mathbf{u}(\mathbf{k},\tau)$ in terms of the bosonic Matsubara frequencies $\omega_n=2\pi n/\beta$ leads to the fluctuation matrix

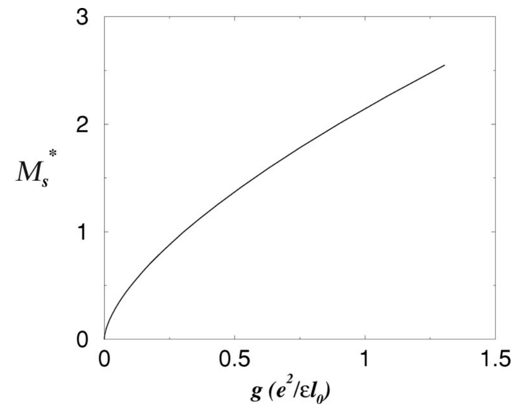

FIG. 1. The Skyrmion mass $[M_s^*\equiv 2\pi M_0/m^*=\int d^2\mathbf{r}\rho^2(\mathbf{r})]$ is plotted as a function of the g factor. The Landau level filling factor is ν and m^* is the effective mass of the electron in the semiconductor.

$$\mathcal{S}_n(\mathbf{k})=\begin{pmatrix} M_0\omega_n^2+D_{xx}(\mathbf{k}) & m^*\omega_c\omega_n+D_{xy}(\mathbf{k})\\ -m^*\omega_c\omega_n+D_{yx}(\mathbf{k}) & M_0\omega_n^2+D_{yy}(\mathbf{k})\end{pmatrix}. \tag{7}$$

The action in Eq. (6) is similar to the action of a Wigner crystal in the presence of a magnetic field. Although this is as expected, it should be noted that the magnetic field interaction is here exactly recovered by the topological Wess-Zumino term.

The contribution of the zero-point energy of the phonons to the total energy of the Skyrme lattices may be obtained by integrating out the quadratic fluctuations in $\mathbf{u}$. It turns out that $E=E_{\text{classic}}+E_{\text{flu}}$ where

$$\begin{aligned} E_{\text{flu}}=\lim_{\beta\to\infty}\frac{1}{\beta}\sum_n\sum_{\mathbf{k}\in\text{BZ}}\ &\ln\{\beta^2\hbar^2[\omega_-(\mathbf{k})+i\omega_n]\\ &\times[\omega_+(\mathbf{k})-i\omega_n]\} \end{aligned} \tag{8}$$

and $\omega_\pm(\mathbf{k})$ are the phonon frequencies

$$\omega_\pm^2(\mathbf{k})=\frac{1}{2M_0^2}\{m^{*2}\omega_c^2+M_0(D_{xx}+D_{yy})\pm\sqrt{[m^{*2}\omega_c^2+M_0(D_{xx}+D_{yy})]^2-4M_0^2(D_{xx}D_{yy}-D_{xy}D_{yx})}\}, \tag{9}$$

which have been obtained by substituting the analytical continuation $i\omega_n\to\omega(\mathbf{k})+i\delta$ into the fluctuation determinant. The dispersion relation of the phonons consists of a gapped mode, as well as a gapless mode. For the former, the gap starts at the cyclotron frequency $\omega(\mathbf{k}\to\mathbf{0})=(m^*/M_0)\omega_c$ and at long wavelengths obeys

$$\omega_+^2(\mathbf{k})=\left(\frac{m^*}{M_0}\omega_c\right)^2+\frac{2\pi e^2}{\epsilon a_c M_0}k+a_c\frac{3\mu+\lambda}{M_0}k^2+O(k^3), \tag{10}$$

where φ is the azimuthal angle of the wave vector $\mathbf{k}$ in the XY plane. The gapless mode on the other hand obeys

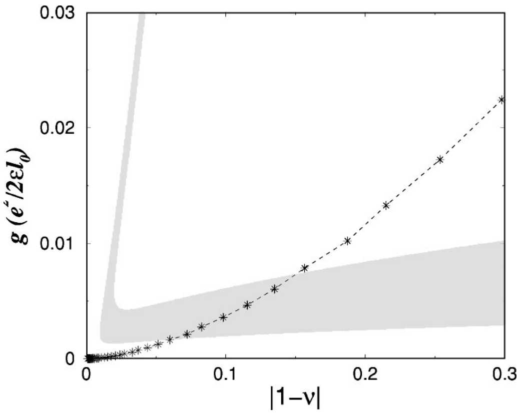

FIG. 2. The phase diagram for massive (and massless) Skyrmions is shown. The stable region of the square Skyrmion lattices is represented by the points (dark region). The triangular lattice is stable in the white region. For $|1-\nu|\leqslant 0.01$, the square Skyrme lattices are unstable. The curve of $R=2\lambda$ is shown. The overlap between Skyrmions is significant below this curve and the topological interaction becomes strong.

$$\omega_-^2(\mathbf{k})=\frac{2\pi e^2}{\epsilon(m^*\omega_c)^2}\left(\mu-\frac{\gamma}{2}\sin^2 2\varphi\right)k^3+O(k^4). \quad (11)$$

Within a low Zeeman energy limit, where the mass of the Skyrmion is small, the gapped mode goes toward the higher energies and the effect of the mass of Skyrmions becomes less significant. At this limit the prediction of the Chern-Simon theory approaches the current prediction of the microscopic Hartree-Fock approximation. However, for higher Zeeman energies, it is not clear if the prediction of the microscopic Hartree-Fock approximation leads to a single gapless mode or if it can support the gapped mode and subsequently the possibility of the existence of a nonzero Skyrmion mass too.[15] The square lattice can be unstable against lattice fluctuations. Moreover, the contribution of the exchange interaction to the Lamé coefficients falls off exponentially (much faster than the contribution from the direct interaction) and when $\nu\rightarrow 1$ a triangular Wigner crystal configuration is more favorable than a square lattice. A similar situation arises when either $\tilde{g}\rightarrow 0$ or $\tilde{g}\rightarrow\infty$. Our numerical studies show that this instability can be seen for values of the filling fraction and g factors that are shown in Fig. 2. The curve of $R=2\lambda$ is also plotted. The overlap between Skyrmions is significant below this curve and the topological interaction becomes strong. The assumption of weakly interacting Skyrmions may fail within the region below this curve, and one should take the next to linear topological XY terms into account. As shown in Fig. 2, for $|1-\nu|\leqslant 0.01$, and for a certain direction of $\mathbf{k}$, the gapless mode becomes imaginary, implying an instability of the square Skyrme lattice that is again consistent with the microscopic Hartree-Fock models.[12] The stable region of the square lattice is characterized by the dark area and the stable region of the triangular lattices is shown by the white area.[20] It is seen that the triangular lattice reappears when the Zeeman energy is small enough. Without Coulomb interactions a lattice configuration is always unstable against the attractive interaction between Skyrmions, i.e., a single Skyrmion with topological charge N is the global minimum of the Skyrmionic energy functional.

As seen in Fig. 1, the quantum Hall Skyrmions are massless at $g=0$. For the massless case the gapped mode goes to infinity and there is just one gapless mode,

$$[m\rightarrow m^*\omega_c\omega(\mathbf{k})]^2=\mu\left(\frac{2\pi e^2}{\epsilon k}+a_c^2[\lambda+2\mu]\right)k^4 -2\gamma\left(\frac{2\pi e^2}{\epsilon k}+a_c^2\left[\mu+\lambda+\frac{1}{2}\gamma\right]\right)k_x^2k_y^2+O(k^6), \quad (12)$$

which is identical to the gapless mode of the massive Skyrmions. In fact, the long wavelength power-law behavior of the gapless mode is totally unaffected by the mass of the Skyrmions. For both the massless and massive theories, we thus find that $\omega\propto k^{3/2}$ at long wavelengths, which is consistent with the microscopic Hartree-Fock calculations.[12] Experimentally, the mass of the Skyrmions may not be probed *directly* by phonon excitations unless we excite the gapped modes (the cyclotron modes), e.g., by optical measurements.[19] Furthermore, the mass of the Skyrmions cannot change the melting point of the 2D Skyrmions, since T_m is specified by the elastic constants of the Skyrme lattices (the 2D melting point just depends on the interaction energy between Skyrmions). Returning to the evaluation of the zero-point energy of the phonons, the dominant contribution at low temperatures is thus obtained as $E_{\rm flu}=\Sigma_k\hbar\omega_-(\mathbf{k})$. Similarly, the average square displacement of the Skyrmions is given by

$$\langle u^2\rangle=l_0^4\sum_{\mathbf{k}\in \mathrm{BZ}}\frac{D_{xx}(\mathbf{k})+D_{yy}(\mathbf{k})}{\hbar\omega_-(\mathbf{k})}\{2n_B[\beta\hbar\omega_-(\mathbf{k})]+1\}, \quad (13)$$

where $n_B(x)$ is the Bose-Einstein distribution function.

B. Magnons

Next we consider the spin waves of the Skyrme lattices by expanding the energy functional in Eq. (2) also in the orientation field θ. Up to quadratic order, it gives the XY energy contribution

$$E[\mathbf{u},\theta]=E[\mathbf{u}]+\frac{1}{2}K_{XY}\int d^2\mathbf{r}|\boldsymbol{\nabla}\theta(\mathbf{r})|^2, \quad (14)$$

where $K_{XY}=J(\kappa R)$ is the effective stiffness associated with gradients in the Skyrmion orientations. As shown in Ref. 9, the effective action for the spin waves also contains a mass term (Λ_0 is the moment of inertia) and we obtain finally

$$S_{\rm eff}[\theta]=\int_0^{\hbar\beta}d\tau\int d^2\mathbf{r}\left(\frac{\Delta M}{a_c}\partial_\tau\theta+\frac{\Lambda_0}{2a_c}(\partial_\tau\theta)^2 +\frac{K_{XY}}{2}|\boldsymbol{\nabla}\theta|^2\right). \quad (15)$$

The first term in Eq. (15) is the usual (dynamical) Berry's phase of a quantum Hall ferromagnet, where ΔM is the average change in the total magnetization induced by a single Skyrmion texture. Note that in principle there is also a contribution from the Hopf term in the effective action for the quantum Hall ferromagnet, which at the quantum level ensures that the Skyrmion obeys the correct spin-statistics relation.[18] Since both these terms are total derivatives, however, the equation of motion is not affected by these terms and the long wavelength dispersion relation that follows from action (15) turns out to be $\omega(k)=c_s k$, where $c_s=\sqrt{a_c K_{XY}/\Lambda_0}$ is the velocity of the spin waves. The contribution from these fluctuations to the total energy of the crystal is again $E_{fl\theta}=\Sigma_k \hbar\omega(\mathbf{k})$, and the mean square value of the associated fluctuations is

$$\langle\theta^2\rangle=\hbar\sum_{\mathbf{k}\in \mathrm{BZ}}\frac{1}{\Lambda_0\omega(\mathbf{k})}\{2n_B[\beta\hbar\omega(\mathbf{k})]+1\}. \qquad (16)$$

The coupling between the displacement and the orientational fields ($\mathbf{u}$ and θ) turns out to be the next to leading order terms and they are therefore negligible for our purposes. These terms lead to interactions between the phonons and spin waves, which we do not consider here. As a result we find for the zero-point energy of the phonons and spin waves simply $E_{fl}=E_{\mathrm{flu}}+E_{fl\theta}$. From Eq. (13), we find $\langle u^2\rangle\sim|1-\nu|R^2/5$ at zero temperature. We also find $\langle\theta^2\rangle=\sqrt{2U/K_{XY}}$ from Eq. (16) where $U=1/(2\Lambda_0)$. For small values of K_{XY} and/or large values of U the quantum fluctuations are severe that the disordered phase can emerge. For very small Zeeman energy, where a phase transition from a square single-Skyrmion lattice into a triangular bi-Skyrmion lattice can be observed,[12,9] we obtain $\langle\theta^2\rangle\sim\tilde{g}^{0.46}$. At the limit of small $\tilde{g}$, fluctuations are negligible and the long-range order of the bi-Skyrmion lattices is not influenced by the quantum fluctuations.

IV. CONCLUSION

In this paper we have studied a system of two-dimensional quantum Hall Skyrmions, starting from a Chern-Simon-Landau-Ginsburg mean field theory. A Maxwell term can be obtained through the gradient expansion of the Chern-Simon action around its minimum energy solution. This term is responsible for generating the Skyrmionic inertial mass. Away from $\nu=1$ Skyrmions stay in a crystal form. The long range order of the crystal depends on the Landau level filling factor and the Zeeman energy. Optical phonons that are out of phase fluctuations of the Skyrmion lattices are gapped since Skyrmions carry an inertial mass. Therefore quantum Hall skyrmions behave like a Wigner crystal in the presence of an external magnetic field. The inertial mass of the Skyrmions vanishes at zero Zeeman energy. In this situation the optical phonons are highly gapped, and they become inaccessible. We finish this paper with a final comment. At $T=0$, and far from $\nu=1$, the zero-point quantum fluctuations of the phonons destroy the long range order of Skyrmion crystals. This is the high density limit of Skyrmions where the Coulomb interaction is screened from $1/r$ to $\ln(1/r)$ by the Chern-Simon fluctuations. In this limit, the Skyrmions behave like a gas of classical particles, i.e., they are crystallized under high pressures. This allows the possibility of observing the reentrance of the solid phase, followed by the disorder (liquid) phase at $T=0$ when ν is far enough from 1.

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[19] According to the Kohn theorem [Walter Kohn, Phys. Rev. **123**, 1242 (1961)], the disorder plays a crucial role in observing the gapped mode by optical measurements. Otherwise, it is impossible to probe the mass of Skyrmions by making optical measurements.
[20] In this paper we assume the triangular lattice is the global minimum of the Skyrmionic energy functional when the topological XY interaction is negligible in comparison with the Coulomb interaction. However, other possible lattice types (like rectangular lattices) are not overruled by symmetry; we leave them for further investigations.