Cooper-pair formation in trapped atomic Fermi gases

M. Houbiers and H. T. C. Stoof

Institute for Theoretical Physics, University of Utrecht, Princetonplein 5, 3584 CC Utrecht, The Netherlands

(Received 12 October 1998)

We apply the Schwinger-Keldysh formalism to study the nonequilibrium dynamics of the BCS transition to the superfluid state in trapped atomic ^{6}Li. We find that the Fokker-Planck equation for the probability distribution of the order parameter is, sufficiently close to the critical temperature, identical to the equation that describes the switching on of a single-mode laser. [S1050-2947(99)08502-9]

PACS number(s): 03.75.Fi, 32.80.Pj, 42.50.Vk, 67.40.$-$w

I. INTRODUCTION

After the successful experiments to trap and cool gases of the bosonic alkali-metal atoms ^{87}Rb, ^{7}Li, and ^{23}Na [1–3] below the critical temperature for Bose-Einstein condensation (BEC), the field of atomic physics has attracted a lot of interest. Besides the ongoing experiments on the properties of the Bose condensate in these gases, several groups have started experiments in order to observe also quantum degeneracy effects in fermionic gases. A particularly promising candidate in this respect is the fermionic isotope ^{6}Li which is predicted to make a BCS transition to a superfluid state at relatively high temperatures. Indeed, the triplet s-wave scattering length a_s of this atom is enormously large and negative [4], resulting in a critical temperature of the same order of those of the BEC experiments [5]. A requirement for this relatively high critical temperature is that a *mixture* of two different hyperfine levels of the ^{6}Li atom must be trapped. Depending on the precise hyperfine states trapped, this can be done in a magnetic or optical trap. In either case, a study of the dynamics of this BCS phase transition offers the exciting opportunity to observe the evolution of a spontaneous breaking of symmetry under almost ideal conditions.

In the case of a magnetically trapped gas mixture, a drawback of the large s-wave scattering length is the corresponding large decay rate and resulting short lifetime of the gas [5,6]. Fortunately, this decay can be suppressed by applying a magnetic bias field. For example, the lifetime of a gas with a typical density of 2×10^{12} cm^{-3} atoms per hyperfine state, is of the order of 1 s in a magnetic bias field of about 7 T, whereas it is of the order of only 0.2 ms at 0.1 T. Evidently, the lifetime of the gas must be longer than the time scale on which the formation of Cooper pairs occurs. In previous work [5], we made on the basis of mean-field theory the rough estimate that the nucleation time is of $O(\hbar/k_B T_c)$. For a homogeneous mixture with a density of 2×10^{12} cm^{-3} in each hyperfine level, the critical temperature was calculated to be 37 nK, resulting in a nucleation time of about 0.2 ms. Since low magnetic fields are experimentally much more convenient, it is also from this point of view sensible to make a precise calculation of the nucleation time of the BCS transition. The aim of the present paper is therefore to determine the dynamics of the BCS phase transition, from which, as a byproduct, the time scale for the formation of Cooper pairs can be extracted.

The paper is organized as follows. In Sec. II we present the appropriate quantum kinetic theory for a trapped Fermi gas with effectively attractive interactions. In particular, we first derive in Sec. II A the nonlinear Langevin equation that describes the long wavelength dynamics of the BCS order parameter. Next, we also incorporate in Sec. II B the effects of an external trapping potential, by making use of the fact that for the experimental conditions of interest the density of the atoms varies only slowly over the extent of the Cooper-pair wave function. In Sec. II C we then derive the desired Fokker-Planck equation with which we can study the time evolution of the gas during the BCS phase transition. We finally discuss our results in Sec. III and end with some conclusions.

II. QUANTUM KINETIC THEORY

The dynamics of the phase transition can be described by using a path-integral representation of the Schwinger-Keldysh formalism [7,8]. This can, for instance, be understood from an analogy in quantum mechanics, where the probability amplitude for a particle moving from a position $\mathbf{x}_0$ at time t_0 to another position $\mathbf{x}$ at time t is given by the path integral

$$\langle \mathbf{x},t | \mathbf{x}_0,t_0 \rangle = \int d[\mathbf{x}] \exp\left(\frac{i}{\hbar} S[\mathbf{x}] \right), \qquad (1)$$

with $S[\mathbf{x}] = \int_{t_0}^{t} dt' L(\mathbf{x}, d\mathbf{x}/dt')$ the action and L the appropriate Lagrangian of the particle. The analogy with a second-order phase transition is obtained when we consider the particle to move in the ''Mexican-hat'' potential $V(\mathbf{x}) = -\alpha \mathbf{x}^2 + \beta \mathbf{x}^4/4$. The quantity of interest is then the probability distribution for being at position $\mathbf{x}$ at time t, i.e.,

$$P[\mathbf{x};t] = \int d\mathbf{x}_0 \langle \mathbf{x},t | \mathbf{x}_0,t_0 \rangle \langle \mathbf{x}_0,t_0 | \mathbf{x},t \rangle P[\mathbf{x}_0;t_0]. \qquad (2)$$

Using Eq. (1) and absorbing the initial probability distribution $P[\mathbf{x}_0;t_0]$ into the definition of the integration measure, it thus equals a path integral over the periodic paths $\mathbf{x}(t')$ where t' runs first backward in time from t to t_0 and then forward in time from t_0 to t, i.e., over a Schwinger-Keldysh contour $\mathcal{C}$. Moreover, the action for this path integral is

$$S[\mathbf{x}] = \int_{\mathcal{C}} dt' \left\{ \frac{1}{2} m \left(\frac{d\mathbf{x}}{dt'} \right)^2 + \alpha \mathbf{x}^2 - \frac{1}{4} \beta \mathbf{x}^4 \right\}. \qquad (3)$$

The equation of motion for the probability distribution can now be obtained from this path integral by noting that the path $\mathbf{x}(t')$ defined on the Schwinger-Keldysh contour can be decomposed according to

$$\mathbf{x}(t') = \mathbf{x}_{\mathrm{cl}}(t') \pm \frac{1}{2} \boldsymbol{\xi}(t')$$

into a classical path $\mathbf{x}_{\mathrm{cl}}(t')$ and fluctuations $\boldsymbol{\xi}(t')$ both defined on the real time axis. The fluctuations can subsequently be integrated out and one arrives in this manner at an effective theory $S^{\mathrm{eff}}[\mathbf{x}_{\mathrm{cl}}]$ for the classical path only. Because we still have to integrate over all the classical paths, the equation of motion for the probability $P[\mathbf{x};t]$ then follows by applying the usual quantization procedure to this effective action. How this program works out for the BCS transition in a trapped atomic Fermi gas is the topic of this paper [9].

A. Cooper-pair wave function

As mentioned, we here study the dynamics of the BCS phase transition. Analogous to the simple quantum mechanical description for the position probability of a particle, we thus need to derive the time dependence of the probability distribution for the order parameter Δ which describes the superfluid phase [10–12]. The starting point is the probability $P[\psi,\psi^*;t]$ for a gas of atoms in two hyperfine states denoted by $|\alpha\rangle = |\uparrow\rangle, |\downarrow\rangle$,

$$P[\psi,\psi^*;t] = \int d[\psi]d[\psi^*] \exp\left(\frac{i}{\hbar} S[\psi,\psi^*]\right), \quad (4)$$

where the functional integral is over the complex fermionic fields $\psi_\alpha(\mathbf{x},t)$ that are defined on the Schwinger-Keldysh contour $\mathcal{C}$ with $t_0 \to -\infty$. Physically, the latter limit ensures that we study only the ''universal'' long-time behavior of the gas that is most relevant for our purposes. The action $S[\psi,\psi^*]$ is given by

$$S[\psi,\psi^*] = \sum_\alpha \int_{\mathcal{C}} dt \int d\mathbf{x}\, \psi_\alpha^*(\mathbf{x},t)\left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2 \mathbf{\nabla}^2}{2m} + \mu_\alpha\right.$$
$$\left. - \frac{V_0}{2}\psi_{-\alpha}^*(\mathbf{x},t)\psi_{-\alpha}(\mathbf{x},t)\right)\psi_\alpha(\mathbf{x},t), \quad (5)$$

where $V_0 < 0$ represents a local attractive interatomic interaction, which should not be confused with the two-body T matrix $4\pi a_s \hbar^2/m$ to avoid a double counting of the effects of the interaction [5]. We now introduce by means of a Hubbard-Stratonovich transformation the complex order parameter $\Delta(\mathbf{x},t)$. This amounts to adding to the action $S[\psi,\psi^*]$ the complete square

$$\frac{1}{V_0}\int_{\mathcal{C}} dt \int d\mathbf{x}|\Delta(\mathbf{x},t) - V_0\psi_\downarrow(\mathbf{x},t)\psi_\uparrow(\mathbf{x},t)|^2,$$

which after integration over the order parameter only leads to a multiplication of Eq. (4) by 1. As a result the action becomes quadratic in the fermionic fields $\psi_\alpha(\mathbf{x},t)$, which can immediately be integrated out [11]. We thus arrive at an effective field theory for the order parameter $\Delta(\mathbf{x},t)$ that equals

$$S[\Delta,\Delta^*] = -i\hbar\, \mathrm{Tr}[\ln G^{-1}] + \int_{\mathcal{C}} dt \int d\mathbf{x} \frac{|\Delta(\mathbf{x},t)|^2}{V_0}, \quad (6)$$

where the trace runs over the spin, time, and spatial variables, and the Green's-function matrix is defined as $G(\mathbf{x},t;\mathbf{x}',t') \equiv -i\langle T_{\mathcal{C}}[\psi(\mathbf{x},t)\psi^\dagger(\mathbf{x}',t')]\rangle$. The spinor $\psi(\mathbf{x},t)$ is given by $\psi(\mathbf{x},t) \equiv (\psi_\downarrow(\mathbf{x},t), \psi_\uparrow^*(\mathbf{x},t))$ and $T_{\mathcal{C}}$ is the time ordering operator on the Schwinger-Keldysh contour $\mathcal{C}$, taking into account the anticommuting nature of the fields.

Around the critical temperature T_c, the order parameter Δ is small compared to $k_B T$, and Eq. (6) can be expanded in powers of Δ, by expanding the Green's function $G(\mathbf{x},t;\mathbf{x}',t')$. To include to lowest order all relevant physics, it suffices to take into account only the terms proportional to $|\Delta|^2$ and $|\Delta|^4$. To calculate these quadratic and quartic parts of the effective action for Δ explicitly, we must remember that the time integrals within these expressions are over the Schwinger-Keldysh contour, and must be rewritten as time integrals over the real axis $-\infty < t' < t$ only. The zeroth-order Green's function $G_0(\mathbf{x},t;\mathbf{x}',t')$ defined on the contour must then be decomposed into so-called retarded, advanced, and Keldysh components. Furthermore, the order parameter $\Delta(\mathbf{x},t)$ on the contour can be decomposed in a classical part $\phi(\mathbf{x},t)$ and fluctuations $\xi(\mathbf{x},t)$ defined on the real time axis as

$$\Delta(\mathbf{x},t_\pm) = \phi(\mathbf{x},t) \pm \frac{1}{2}\xi(\mathbf{x},t), \quad (7)$$

where the upper and lower sign refer to the chronological and antichronological branch of the Schwinger-Keldysh contour, respectively. The important point here is that the fluctuations allow the classical part of the order parameter to evolve from zero to a nonzero value when the system is driven through the phase transition, and we can thus indeed describe the dynamics of the classical field $\phi(\mathbf{x},t)$ around the critical temperature by means of this Schwinger-Keldysh method. In contrast, this cannot be achieved within the framework of mean-field theory, where the order parameter always remains zero if it is zero initially.

Applying subsequently a gradient expansion, which is justified because it turns out that the spatial variations in the order parameter are small on the size of the Cooper pairs [5], the quadratic plus quartic part of the effective action Eq. (6) up to quadratic order in the fluctuations becomes

$$S[\phi,\phi^*,\xi,\xi^*] = \int d\mathbf{x} \int_{-\infty}^{t} dt' \left\{ \phi(\mathbf{x},t')L^{(R)}(\mathbf{x},t')\xi^*(\mathbf{x},t') \right.$$
$$+ \xi(\mathbf{x},t')L^{(A)}(\mathbf{x},t')\phi^*(\mathbf{x},t')$$
$$\left. + \frac{1}{2}\xi(\mathbf{x},t')L^{(K)}(\mathbf{x},t')\xi^*(\mathbf{x},t') \right\}, \quad (8)$$

where to lowest nonvanishing order in $\mathbf{\nabla}$ and in $\partial/\partial t$ the retarded and advanced factors are well known [10–12],

$$L^{(R,A)}(\mathbf{x},t) = \alpha + \gamma\mathbf{\nabla}^2 - \beta|\phi(\mathbf{x},t)|^2 \pm N_0 \frac{\pi}{8} \frac{\hbar}{k_B T} \frac{\partial}{\partial t}, \quad (9)$$

and the Keldysh part is given by

$$L^{(K)}(\mathbf{x},t)=iN_0\frac{\pi}{2}. \qquad (10)$$

Here we considered only the optimal case $\mu_\uparrow=\mu_\downarrow$ and used particle-hole symmetry around the Fermi level of the system. The constants α, β, and γ are given by

$$\alpha=N_0\left(1-\frac{T}{T_c}\right), \qquad (11)$$

$$\beta=N_0\frac{7}{8}\frac{\zeta(3)}{(\pi k_B T)^2}, \qquad (12)$$

and

$$\gamma=N_0\frac{7}{8}\frac{\zeta(3)}{6\pi^2}\left(\frac{\hbar v_F}{k_B T}\right)^2. \qquad (13)$$

Furthermore, $N_0=mk_F/(2\pi^2\hbar^2)$ denotes the density of states at the Fermi level for a single spin state, the critical temperature obeys $k_B T_c\simeq 5/3\epsilon_F\exp(-1/\lambda)$ [5] with $\lambda=2k_F|a_s|/\pi$, and v_F is the Fermi velocity of the system.

The last term in the right-hand side of Eq. (8) describes the quadratic fluctuations. If one would neglect them, and integrate out the (in that case linear) fluctuations, one would obtain a dissipative nonlinear Schrödinger equation for the classical part $\phi(\mathbf{x},t)$ of the order parameter which would also arise from the BCS equilibrium theory. Of course, for our nonequilibrium purposes, the quadratic fluctuations are important and have the effect of adding noise to the dissipative nonlinear Schrödinger equation. In general, the Fourier transforms of Eqs. (9) and (10), i.e., $L^{(R,A)}(\mathbf{k},\omega)$ and $L^{(K)}(\mathbf{k},\omega)$, are therefore related through the famous fluctuation-dissipation theorem [13]

$$L^{(K)}(\mathbf{k},\omega)=\pm 2i[2N^B(\hbar\omega)+1]\mathrm{Im}[L^{(R),(A)}(\mathbf{k},\omega)], \quad (14)$$

where $N^B(\hbar\omega)$ denotes the Bose distribution evaluated at $\hbar\omega$. At low frequencies it indeed gives $L^{(K)}(\mathbf{k},\omega)=\pm 2i(2k_B T/\hbar\omega)\mathrm{Im}[L^{(R),(A)}(\mathbf{k},\omega)]$, in agreement with the above results. Notice that if we expand the quadratic part of the action $S[\Delta,\Delta^*]$ in Eq. (6) up to lowest non-vanishing order in ∇ and $\partial/\partial t$, we only have to expand the quartic part of $S[\Delta,\Delta^*]$ to zeroth order, i.e., we only need to consider the local contributions proportional to $\phi(\mathbf{x},t)\xi^*(\mathbf{x},t)|\phi(\mathbf{x},t)|^2$ and its complex conjugate. Since the factor β has no imaginary part, we conclude that it is indeed consistent to neglect Keldysh-like terms that would in principle also arise from the quartic part of the action $S[\Delta,\Delta^*]$.

B. Trapped Fermi gases

In order to arrive at an equation of motion for the probability distribution of the classical value $\phi(\mathbf{x},t)$ of the order parameter, we first integrate out the fluctuations $\xi(\mathbf{x},t)$ from the action. Before doing so, however, we need to realize that in real experiments, the gas will be trapped in an external potential $V(\mathbf{x})$ generally in the shape of a three-dimensional (3D) isotropic harmonic oscillator $V(\mathbf{x})=m\omega_0^2\mathbf{x}^2/2$. As a result, the coefficients α, β, and γ in Eqs. (11)–(13) now depend on $\mathbf{x}$, since the Fermi energy $\mu_\alpha(\mathbf{x})=\mu_\alpha-V(\mathbf{x})$

$-(4\pi a_s\hbar^2/m)n_{-\alpha}(\mathbf{x})$ and therefore the densities $n_\alpha(\mathbf{x})$ and the critical temperature $T_c(\mathbf{x})$ are position dependent. Hence, we can make use of the following expansion in the effective action Eq. (8) $\phi(\mathbf{x},t)=\Sigma_n a_n(t)\phi_n(\mathbf{x})$ and $\xi(\mathbf{x},t)=\Sigma_n b_n(t)\phi_n(\mathbf{x})$, where the normalized wave functions $\phi_n(\mathbf{x})$ are solutions to Schrödinger's equation [14]:

$$-[\alpha(\mathbf{x})+\gamma(\mathbf{x})\nabla^2]\phi_n(\mathbf{x})=N_0(\mathbf{x})\varepsilon_n\phi_n(\mathbf{x}). \qquad (15)$$

It is well known that close to T_c, the spatial region where the order parameter becomes nonzero is small compared to the region over which the density changes [5,14]. So, we can take $N_0(\mathbf{x})\simeq N_0(\mathbf{0})$, $v_F(\mathbf{x})\simeq v_F(\mathbf{0})$, but we must expand $T_c(\mathbf{x})$ around $\mathbf{x}=\mathbf{0}$, since it depends exponentially on the coupling constant $\lambda(\mathbf{x})$.

Considering again only the optimal case that $\mu_\downarrow=\mu_\uparrow$ and using that the density in each hyperfine level around $\mathbf{x}=\mathbf{0}$ can be approximated by $n_\alpha(\mathbf{x})=n_t(\mathbf{0})[1-(\mathbf{x}/R)^2]/2$, where $n_t(\mathbf{0})$ is the total gas density in the center of the trap, it is found from the zero temperature expression for the respective chemical potentials that R obeys

$$R^2=l^4\left(4\pi n_t(\mathbf{0})a_s+2\left(\frac{\pi^2}{\sqrt{3}}\right)^{2/3}[n_t(\mathbf{0})]^{2/3}\right), \qquad (16)$$

with $l=\sqrt{\hbar/m\omega_0}$. From this expression one can immediately see that the effect of the mean-field interaction in the gas is to contract the gas cloud to the center of the trap. Expanding $T_c(\mathbf{x})$ around $\mathbf{x}=\mathbf{0}$, one arrives at [14]

$$\alpha(\mathbf{x})=N_0(\mathbf{0})\left\{\ln\left(\frac{T_c(\mathbf{0})}{T}\frac{\epsilon_F(\mathbf{x})}{\epsilon_F(\mathbf{0})}\right)-\frac{1}{\lambda(\mathbf{0})}\left(\frac{N_0(\mathbf{0})}{N_0(\mathbf{x})}-1\right)\right\} \qquad (17)$$

and Eq. (15) turns out to be just the Schrödinger equation for a 3D harmonic oscillator with effective frequency

$$\omega_\Delta=4\pi\sqrt{\frac{2\lambda+1}{7\lambda\zeta(3)}}\frac{k_B T}{mv_F(\mathbf{0})R}\equiv\frac{\hbar}{ml_\Delta^2}. \qquad (18)$$

The eigenvalues are therefore given by

$$\varepsilon_n=\frac{2m}{\hbar^2}\frac{\gamma(\mathbf{0})}{N_0(\mathbf{0})}\left(n+\frac{3}{2}\right)\hbar\omega_\Delta-\ln\left(\frac{T_c(\mathbf{0})}{T}\right). \qquad (19)$$

Notice that depending on the temperature, the eigenvalues ε_n become unstable, i.e., with decreasing temperature they can become negative one by one. The temperature where ε_0 becomes zero, is the critical temperature T_c of the inhomogeneous system. It is clear from Eq. (19) that due to finite size effects, this critical temperature T_c is *lower* than the homogeneous critical temperature $T_c(\mathbf{0})$ corresponding to the density in the center of the trap as first pointed out by Baranov and Petrov [14].

C. Fokker-Planck equation

Now it is important to realize that there is always a region sufficiently close to the critical temperature T_c of the system, where it is a good approximation to take into account only one single mode

$$\phi_0(\mathbf{x}) = \left(\frac{1}{\sqrt{\pi}l_\Delta}\right)^{3/2} \exp\left(-\frac{\mathbf{x}^2}{2l_\Delta^2}\right) \qquad (20)$$

in the expansion for ϕ and ξ because the higher modes ϕ_n remain stable and their influence thus remains small. Furthermore, as will be shown explicitly by a numerical example later on, the coupling between modes in the nonlinear terms of Eqs. (8) and (9) can be neglected sufficiently close to the critical temperature. Substituting everything into the action $S[\phi,\phi^*,\xi,\xi^*]$ in Eq. (8), we first perform the integral over $\mathbf{x}$, and subsequently integrate out the fluctuations that are now solely represented by the fields $b_0(t)$ and $b_0^*(t)$. From the resulting action the effective Hamiltonian for the classical fields $a_0(t)$ and $a_0^*(t)$ can be obtained by introducing their conjugated momenta in the standard way. We find

$$H_\Delta = \frac{8ik_BT}{\pi N_0(\mathbf{0})}\left\{2k_BT\frac{\partial^2}{\partial a_0 \partial a_0^*} + \frac{\partial}{\partial a_0}[N_0(\mathbf{0})\varepsilon_0 + \beta'|a_0|^2]a_0\right.$$
$$\left. + \frac{\partial}{\partial a_0^*}[N_0(\mathbf{0})\varepsilon_0 + \beta'|a_0|^2]a_0^*\right\}, \qquad (21)$$

where $\beta' = \beta\int d\mathbf{x}|\phi_0(\mathbf{x})|^4 = \beta/(2\pi l_\Delta^2)^{3/2}$.

The equation of motion, or Fokker-Planck equation, for the probability distribution P is now given by Schrödinger's equation

$$i\hbar\frac{\partial}{\partial t}P[a_0,a_0^*;t] = H_\Delta P[a_0,a_0^*;t], \qquad (22)$$

which has an equilibrium solution

$$P[a_0,a_0^*;\infty] \propto \exp\left(-\frac{F(a_0,a_0^*)}{k_BT}\right), \qquad (23)$$

where the Landau free energy $F(a_0,a_0^*) = N_0(\mathbf{0})\varepsilon_0|a_0|^2 + \beta'|a_0|^4/2$. Since $\beta' > 0$, this free energy has the shape of a parabolic well for $T > T_c$ where $\varepsilon_0 > 0$, whereas it becomes Mexican-hat shaped below the critical temperature where $\varepsilon_0 < 0$. It is important to note from the first term in the right-hand side of Eq. (21) that the above Fokker-Planck equation has a positive and diagonal diffusion matrix. This is caused by the fact that we are allowed to neglect memory effects in the effective action for the order parameter, and that the probability distribution $P[a_0,a_0^*;t]$ corresponds to a Wigner representation of the density matrix. In particular, the absence of off-diagonal terms in the diffusion matrix is not a result of the neglect of any quantum effects, as one might be tempted to think on the basis of similar Fokker-Planck equations used in quantum optics.

The dynamics of the phase transition can most easily be extracted if we introduce dimensionless variables I, θ, τ according to

$$a_0 = \left(\frac{k_BT}{2\beta'}\right)^{1/4}\sqrt{I}\exp i\theta, \qquad (24)$$

$$t = \frac{\pi N_0(\mathbf{0})\hbar}{\sqrt{32(k_BT)^3\beta'}}\tau, \qquad (25)$$

and also the constant

$$a = -\sqrt{\frac{2\beta'}{k_BT}}\frac{N_0(\mathbf{0})\varepsilon_0}{\beta'}. \qquad (26)$$

The Fokker-Planck equation in Eq. (22) can then be rewritten in the form

$$\frac{\partial}{\partial\tau}P = \left\{\frac{\partial^2}{\partial I^2}4I + \frac{1}{I}\frac{\partial^2}{\partial\theta^2} + \frac{\partial}{\partial I}[2(I-a)I-4]\right\}P, \qquad (27)$$

which is exactly the Fokker-Planck equation associated with the single-mode laser [15]. In that case the constant a represents the "pump parameter," and I is equivalent to the "laser intensity."

The second term on the right-hand side of Eq. (27) gives rise to phase diffusion [16], but when the system is initially in a state which does not depend on θ, it will remain phase independent, and we can omit this term from the Fokker-Planck equation. The stationary solution in terms of the dimensionless variables becomes

$$P_{st}[I;a] = \mathcal{N}(a)\exp\left(-\frac{(I-a)^2}{4}\right), \qquad (28)$$

where $\mathcal{N}(a)$ is a normalization factor. This is a Gaussian distribution centered around $I=a$, where $a<0$ for $T>T_c$, and $a>0$ for $T<T_c$. Note, however, that I only assumes positive values, so even above the critical temperature, the average $\langle I\rangle$ is larger than zero.

III. RESULTS AND CONCLUSIONS

We model the phase transition by changing the constant a suddenly from a negative value $a_0<0$ to a positive value $a>0$ at $\tau=0$. The reasoning behind this simplified description is as follows. In principle the pump parameter a is a dynamical quantity whose value follows from a kinetic equation for the distribution function of the atoms in the gas. However, the process of evaporative cooling that is most likely to be used in actual experiments with spin-polarized atomic ^{6}Li to cool the gas to the desired low temperatures is very fast due to the large value of the s-wave scattering length and corresponding large cross section for elastic collisions. As a consequence, it is an excellent first approximation to assume that the "translational" degrees of freedom of the atoms in the gas are always in thermal equilibrium, but that the order parameter is not and needs more time to adapt. In this picture of the phase transition the pump parameter will then indeed change almost instantaneously for the time scales of interest on which the probability distribution $P[I;\tau]$ evolves from the stationary solution $P_{st}[I;a_0]$ at $\tau=0$ to the stationary solution $P_{st}[I;a]$ at sufficiently large τ. We have therefore solved Eq. (27) numerically for $a_0=-2$ and $a=+4$ using the methods developed in Ref. [15]. The evolution of the probability distribution $P[I;\tau]$ is plotted for several times in Fig. 1. It is clear that after a dimensionless time of $O(1)$, the order parameter of the phase transition has reached its stationary value. This is shown explicitly in the inset of Fig. 1, where the average $\langle I\rangle(\tau)$ and the variance $\langle(I-\langle I\rangle)^2\rangle(\tau)$ are plotted. It should be noted that Fig. 1 differs from similar
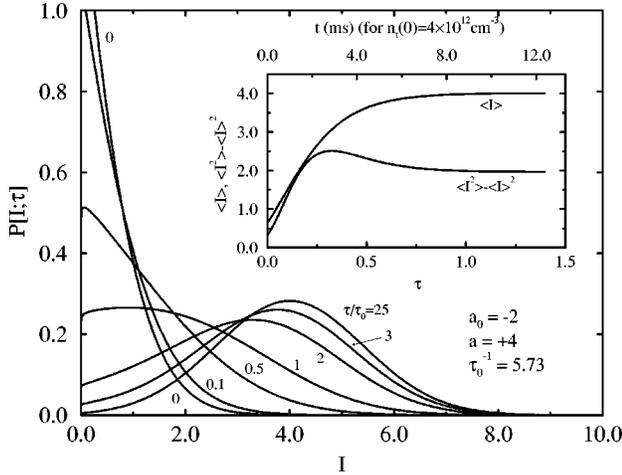
FIG. 1. Evolution of the probability distribution for the order parameter in normalized units. The inset shows the average value $\langle I \rangle$, and its variance as a function of time.

plots known from laser theory, because of the different initial condition that is relevant here.

In order to see to what extent the approximations we have made are valid, we finish the paper with an explicit numerical example. For a gas mixture with density $n_\alpha(\mathbf{0}) = 2 \times 10^{12}$ cm^{-3}, or equivalently $n_t(\mathbf{0}) = 4 \times 10^{12}$ cm^{-3}, one has for the homogeneous critical temperature $T_c(\mathbf{0}) \simeq 37$ nK [5]. If this gas is trapped in a 3D isotropic harmonic oscillator with $\hbar \omega_0 / k_B = 7.1$ nK [2], the extent of the cloud is $R \simeq 11l$ at $T = 30$ nK, whereas $l_\Delta \simeq 2l$ is then indeed much smaller. The critical temperature T_c according to Eq. (19) is $T_c \simeq 31$ nK. The second mode of Eq. (19) becomes unstable only at $T \simeq 26$ nK. At this temperature the pump parameter $a \simeq 13$, which gives a first upper bound on the validity of our theory. However, a more stringent one is that it is only a good approximation to take one single mode into account if we can also neglect the nonlinear coupling in the Hamiltonian. We must then require that the interaction $\beta |\phi(\mathbf{0})|^2$ is small compared to $N_0(\mathbf{0}) \hbar \omega_\Delta$. This is true only for $T > 29.3$ nK, from which it follows that the approximations are valid if $a \lesssim 4.3$. For $T = 29.3$ nK, the order parameter $\Delta(\mathbf{0}) / k_B T \simeq 1$.

In Fig. 1 we therefore have chosen $a = 4$, and from the inset of this figure we conclude that for a total density $n_t(\mathbf{0}) = 4 \times 10^{12}$ cm^{-3}, one unit of dimensionless time τ corresponds to 8.9 ms, which is indeed much larger than the time between two thermalizing collisions that turns out to be about 0.03 ms for these densities. In addition, it is also much larger than the estimate of $\hbar / k_B T_c(\mathbf{0}) = 0.2$ ms in our previous work [5], because Eq. (25) clearly shows that the time

scale depends also on several other relevant parameters. From the general theory of phase-ordering kinetics [17], we expect the time scale for the formation of the Cooper pairs to be proportional to

$$\frac{\hbar}{k_B T_c(\mathbf{0})} \left(\frac{l_\Delta}{\xi(\mathbf{0})} \right)^z,$$

with $\xi(\mathbf{0}) \equiv \hbar v_F(\mathbf{0}) / 2\pi k_B T_c(\mathbf{0})$ the typical extent of the Cooper pairs and z the appropriate dynamical exponent. Comparing with our result

$$\frac{\pi N_0(\mathbf{0}) \hbar}{\sqrt{32 [k_B T_c(\mathbf{0})]^3 \beta'}} = \frac{1}{\sqrt{7 \zeta(3)}} \left(\frac{\pi}{2} \right)^{5/4} \frac{\hbar}{k_B T_c(\mathbf{0})} k_F(\mathbf{0}) \xi(\mathbf{0})$$
$$\times \left(\frac{l_\Delta}{\xi(\mathbf{0})} \right)^{3/2}, \tag{29}$$

we therefore conclude that $z = 3/2$. Due to the fact that $l_\Delta \gg \xi(\mathbf{0})$, the nucleation time is thus considerably larger than the rough estimate of $O(\hbar / k_B T_c(\mathbf{0}))$, which a posteriori also justifies our neglect of memory effects. In addition, also the factor $k_F(\mathbf{0}) \xi(\mathbf{0}) \gg 1$. The appearance of this additional factor can be understood by noting that the average binding energy for the Cooper pairs is of $O(k_B T_c(\mathbf{0}))$ and that the number of one-particle states involved in the formation of a Cooper pair is of $O(N_0(\mathbf{0}) k_B T_c(\mathbf{0}) \xi^3(\mathbf{0})) = O([k_F(\mathbf{0}) \xi(\mathbf{0})]^2)$. Applying the usual arguments of statistical mechanics, we then conclude that the spread in binding energy is of order $O(k_B T_c(\mathbf{0}) / k_F(\mathbf{0}) \xi(\mathbf{0}))$, leading to an intrinsic time scale for the formation of Cooper pairs of $O(\hbar k_F(\mathbf{0}) \xi(\mathbf{0}) / k_B T_c(\mathbf{0}))$.

In principle, we expect to be able to reduce the nucleation time by increasing the final value of the pump parameter a, but a detailed analysis of this situation requires inclusion of more modes in the theory. In a future publication we will come back to this multimode regime. However, within the context of the present theory, we conclude that for a lifetime of the gas of the order of the nucleation time, one needs a magnetic bias field of about 0.7 T. A better option, therefore, appears to trap and cool the lowest two hyperfine states in an optical trap, which was recently shown to be possible for bosons [18]. In any case, we believe that this paper, in combination with Ref. [5], shows that there are at least no theoretical limitations to achieve a BCS transition in spin-polarized atomic ^{6}Li.

ACKNOWLEDGMENTS

[1] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Science **269**, 198 (1995).

[2] C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet, Phys. Rev. Lett. **75**, 1687 (1995); C. C. Bradley, C. A. Sackett, and R. G. Hulet, ibid. **78**, 985 (1997).

[3] K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten,

D. S. Durfee, D. M. Kurn, and W. Ketterle, Phys. Rev. Lett. **75**, 3969 (1995).

[4] E. R. I. Abraham, W. I. McAlexander, J. M. Gerton, R. G. Hulet, R. Coté, and A. Dalgarno, Phys. Rev. A **55**, R3299 (1997).

[5] H. T. C. Stoof, M. Houbiers, C. A. Sackett, and R. G. Hulet,

Phys. Rev. Lett. **76**, 10 (1996); M. Houbiers, R. Ferwerda, H. T. C. Stoof, W. I. McAlexander, C. A. Sackett, and R. G. Hulet, Phys. Rev. A **56**, 4864 (1997).

[6] M. Houbiers, H. T. C. Stoof, W. I. McAlexander, and R. G. Hulet, Phys. Rev. A **57**, R1497 (1998).

[7] J. Schwinger, J. Math. Phys. **2**, 407 (1961).

[8] L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1515 (1964) [Sov. Phys. JETP **20**, 1018 (1965)].

[9] For the problem of Bose-Einstein condensation in trapped Bose gases see H. T. C. Stoof, J. Low Temp. Phys. **114**, Nos. 1/2 (1999).

[10] E. Abrahams and T. Tsuneto, Phys. Rev. **152**, 419 (1966).

[11] H. Kleinert, Fortschr. Phys. **26**, 565 (1978).

[12] H. T. C. Stoof, Phys. Rev. B **47**, 7979 (1993).

[13] D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions* (Benjamin, Reading, MA, 1975).

[14] M. A. Baranov and D. S. Petrov, Phys. Rev. A **58**, R801 (1998).

[15] H. Risken, *The Fokker-Planck Equation* (Springer-Verlag, Berlin, 1989).

[16] M. Lewenstein and L. You, Phys. Rev. Lett. **77**, 3489 (1996).

[17] A. J. Bray, Adv. Phys. **43**, 357 (1994).

[18] D. M. Stamper-Kurn, M. R. Andrews, A. P. Chikkatur, S. Inouye, H. J. Miesner, J. Stenger, and W. Ketterle, Phys. Rev. Lett. **80**, 2027 (1998).