

Canonical and BRST-Quantization of Constrained Systems

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ABSTRACT. We discuss the possibility of generalizing the Dirac condition $\Phi_i|\Psi_{phys}\rangle = 0$ for Hamiltonian systems with non-abelian first-class constraints and introduce a distinction between constrained systems of a)-type and b)-type, according to the "classical" or "quantum" nature of their constraints. The equivalence between BRST quantum theory and a canonical quantization à la Dirac is investigated, and different definitions of quantum observables and states in the two approaches are set into correspondence. We find that only little of the algebraic structure of the classical BRST symmetry of observables can be preserved in the quantum theory.

1. Generalizing the Dirac condition

This paper deals with some aspects of the Hilbert space-based quantization of (finite-dimensional) systems subject to a set of first-class constraints. Part of its content, especially from the general discussion in this section, is also relevant to the infinite-dimensional case.

The main results, concerning the equivalence of the BRST and the usual canonical quantization à la Dirac and the quantization of the full classical BRST symmetry, are presented in §2. We analyze the usual BRST prescriptions for constructing quantum observables and physical states by decomposing them into Grassmann-odd and -even parts and comparing the latter with their usual definitions in a canonical setting without ghosts. We show that only little of the BRST symmetry of classical (extended) observables can be preserved in the quantum theory. In particular, no interpretation emerges for the higher-order coefficients of BRST quantum observables and wave functions, and the unphysical indefinite-metric "Hilbert space" structure and hermiticity assignments to ghost operators appear as mere artefacts of the "formal" quantization procedure. We conjecture

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that the definition $[\hat{\Omega}, \hat{A}_{ext}] = 0$ for quantum observables \hat{A}_{ext} is in general more restrictive than corresponding definitions in the canonical formulation, but for the special case where the constraints arise from a group action on phase space, and under the general assumptions made, the two approaches are essentially equivalent.

In this section, we discuss the issue of generalizing the Dirac condition ((D2) below) to project out physical quantum states, of which the BRST condition ((B2) below) is just one example. Moreover we use the opportunity to introduce a distinction between “classical” and “quantum constraints” which we think is conceptually useful, and emphasize the need for some “physicality criteria”.

The classical data we start from are a $2n$ -dimensional phase space \mathcal{S} and its associated function space $C^\infty(\mathcal{S})$, together with a set of m constraint functions $\Phi_i \in C^\infty(\mathcal{S})$, $i = 1, \dots, m$, which restrict the classically allowed states of the system to points $s \in \mathcal{S}$ satisfying $\Phi_i(s) = 0$, $\forall i$, and which furthermore give rise to a degeneracy (“gauge equivalence”) between sets of points describing the same physical state of the system. As first-class constraints, they obey Poisson-bracket relations of the form

$$\{\Phi_i, \Phi_j\} = C_{ij}{}^k \Phi_k, \quad (1.1)$$

where $C_{ij}{}^k \in C^\infty(\mathcal{S})$. Examples of such systems and their field theoretic analogues abound in physics, and both classical and quantum structures have been studied extensively (overall references are [Sun], [HRT], [GiTy]). The first systematic and foundational studies of the classical aspects were made by Dirac [Dir1] in 1950 (see also [Sun] for more references). A corresponding quantum theory was laid down by Dirac in his famous “Lectures on Quantum Mechanics” [Dir2], which till today lies at the heart of most attempts to canonically quantize theories with constraints, although some of its limitations were already clear to its author.

It is surprisingly hard to adapt the usual set of rules, relating physical properties of a quantum mechanical system to certain mathematical structures (to be found in any text book on quantum mechanics), to the case of systems with (first-class) constraints. The main question seems to be: “What are observables?”, in particular, “Are the constraints themselves observables (which must be represented by self-adjoint operators on some Hilbert space)?”. We would like to suggest that before attempting to answer these and related questions, some more *physical* input is needed, which depends on the system under consideration, namely, which of the following two categories it belongs to:

- a) the system is for some a priori reason confined to a state in which it can be said to *possess* exactly the classical values of the constraints, $\Phi_i = 0$, and/or strictly obey the gauge symmetry associated with the Φ_i ;
- b) the constraints are obeyed only in some appropriate statistical average or semiclassical limit; the system is in principle subject to “quantum tunnelling” off the classical constraint surface defined by $\Phi_i = 0$, and/or the classical

symmetry can be broken at the quantum level.

In case a) one would try to treat the constraints "as classically as possible" in order to strictly enforce them. Ideally the classically reduced system, i.e. the true physical degrees of freedom, should be quantized directly. If that is not possible, one could adopt a quantization procedure for the unreduced system which in case where a quantization for both the reduced and the unreduced system is available, leads to the commutativity of constraining and quantizing. Such commutativity is not expected to occur in case b), where the degrees of freedom associated with the constraints are themselves subject to quantization. For example, such quantum behaviour of the constraints may manifest itself in a non-zero probability to find the system in a classically forbidden region. Still the quantization must account for the fact that the classical constraints and their associated gauge symmetry cannot be violated to an *arbitrary* extent. In the example just used this may translate into the statement that "the probability of finding the system outside a certain neighbourhood of the constraint surface $\Phi_i = 0$ is practically zero". An estimate of the size of such a neighbourhood must again be related to the properties of a specific system under consideration and may be translatable into a condition on the spectrum of the corresponding operators $\hat{\Phi}_i$.

Clearly the distinction between cases a) and b) is not a philosophical one (as suggested, for instance, in [Kun]), because they will in general lead to different quantum theories for a given classical system, and hence different predictions which are in principle subject to experimental verification. In practice it may be difficult to decide between the two possibilities, or a system may possess constraints of both a)- and b)-type which cannot be properly disentangled. There may not be sufficiently many physical data available, in particular when the constrained system is given by an abstract mathematical model with little or no relation to any physical realization. If the constraints are just the result of a previous enlargement of phase space, which was made on purely mathematical grounds, say, to obtain a structure that is suitable as a starting point for a specific quantization scheme, one would expect them to be of a)-type. If, on the other hand, the system was enlarged in order to realize some fundamental physical principle (like, for example, Lorentz covariance), then the ensuing constraints may be of b)-type.

In a given quantization scheme, assumptions about the nature of the constraints are often made only implicitly, which may lead to confusion when comparing different methods or interpreting their results. In §2 below we will leave open the possibility that the quantum structures associated with the unreduced system (for example, the scalar product, states that can be prepared, the constraint operators etc.) *do* have some physical significance, and in this sense we are concerned with systems of the b)-type. For instance, we are interested in whether anything can be learned about the unreduced system by BRST-extending it. If, on the other hand, one takes the view point that quantum structures of the

unreduced system do not have any physical significance (i.e. it is of type a) according to our classification), the choice of possible “quantizations” of this system is greatly enlarged. The only physical requirement in this case is the internal consistency of and the absence of unphysical features from the final, reduced quantum system.

So far we have not spelled out what precise mathematical structures should be set into correspondence with systems of type a) and b). Unfortunately no “best quantization method” has emerged as yet for general classical Hamiltonian systems (not even for systems without constraints), for a variety of technical and conceptual reasons, one out of many being that the distinction introduced above is rarely addressed explicitly. The guide-line for any general (presumably non-unique) quantization scheme can only be a minimal set of physical requirements on the expectation values of the constrained theory. For example, one may use the condition

$$(D1) \quad \langle \Psi' | \hat{\Phi}_i | \Psi \rangle = 0, \quad \forall i$$

for any pair of physical wave functions $|\Psi\rangle, |\Psi'\rangle \in \mathcal{H}_{ph} \subset \mathcal{H}$, which we will call the “weak Dirac condition on states”. (We are using Dirac’s bra- and ket-notation.) It presupposes the existence of a scalar product on the Hilbert space \mathcal{H} , with $|\Psi\rangle, |\Psi'\rangle \in \mathcal{H}$, and the $\hat{\Phi}_i$ to be sufficiently well-behaved (usually self-adjoint) operators on \mathcal{H} in order for (D1) to be well defined. However, in certain situations (D1) may be too restrictive, and we may want to allow some of the eigenvalues $\langle \Psi | \hat{\Phi}_i | \Psi \rangle$, with $|\Psi\rangle \in \mathcal{H}_{ph}$, to be non-zero and of order \hbar , say. Alternatively, if the constraints are not regarded as observables in the usual sense, one may not even assign operator status to them and construct the physical Hilbert space in some other way. Another drawback of condition (D1) is that it is not very practicable: using it to find all of \mathcal{H}_{ph} requires knowledge of the entire spectrum of the $\hat{\Phi}_i$; moreover, \mathcal{H}_{ph} may not be unique. In any case, a condition on states like (D1) always has to be supplemented by a definition of physical observables. Part of §2 of this paper is devoted to the analysis of such pairs of conditions on states and observables, which are in some sense dual to each other.

The condition that is almost universally adopted to project out physical wave functions is not (D1), but

$$(D2) \quad \hat{\Phi}_i | \Psi \rangle = 0, \quad \forall i,$$

which we will call the “strong Dirac condition on states”. If the $\hat{\Phi}_i$ are self-adjoint operators and $|\Psi\rangle \in \mathcal{H}$, then for given $|\Psi\rangle$, (D2) implies (D1); however, often (D2) is interpreted by physicists in a much looser sense, namely as a set of

differential equations on elements $|\Psi\rangle$ of some linear function space \mathcal{L} , given by differential operators $\hat{\Phi}_i$ acting on \mathcal{L} . Ignoring for the moment the question whether and how such a procedure can be made physically meaningful, it has certainly contributed to the popularity of (D2). This condition is also the one given by Dirac in [Dir2]. Its inherent problem is that it is mainly geared towards the case of linear phase spaces \mathcal{S} and a set of abelian constraints $\{\Phi_i\}$, as we will illustrate in the following.

The condition (D2) is rather restrictive, and the indications are that its limitations will become more pronounced in cases where the phase space is not a vector space and/or the constraint algebra is non-abelian, as often happens in interesting physical applications. Use of the strong Dirac condition in such systems may lead to unexpected results. For example, in [Lo2] a $U(1)$ -gauge model with a non-trivial phase space is quantized using group-theoretical methods. The quantization of the unreduced system is non-unique and application of the Dirac condition $\hat{\Phi}|\Psi\rangle = 0$ turns out to be representation-dependent, and yields as physical Hilbert spaces either the zero vector, a single irreducible or an infinitely reducible representation of the reduced classical system.

Another well-known problem associated with (D2) concerns the spectra of the constraint operators $\hat{\Phi}_i$: once a quantization for the unreduced system (with Hilbert space \mathcal{H}) has been found, the value zero may lie in the discrete or the continuous part of the spectrum or even outside the spectrum of $\hat{\Phi}_i$. In the latter case the physical Hilbert space \mathcal{H}_{ph} is trivial, and for zero lying in the continuous part, wave functions projected out by the Dirac condition (D2) have zero-measure with respect to the scalar product on \mathcal{H} , i.e. the scalar product on \mathcal{H} does not project down to \mathcal{H}_{ph} . For this reason in some quantization approaches the imposition of a scalar product and thus a Hilbert space structure on \mathcal{H}_{ph} is left to the very end. For example, in Ashtekar's quantization program [Ash] the scalar product is selected by requiring self-adjointness of a set of basic physical observables. Duval et al. [DET], [DEGST] obtain the scalar product on \mathcal{H}_{ph} from the quantization of the classically reduced system.

In the context of a BRST-extended version of geometric quantization, a modification of (D2), first appearing in [KoSt], was elaborated upon in [Tuy1], [DET]. It is given by

$$\hat{\Phi}_i|\Psi\rangle = \frac{i}{2}C_{ji}{}^k|\Psi\rangle, \quad (1.3)$$

for the case that the constraints are related to a free and proper Lie group action on phase space. This result is obtained in an attempt to achieve commutativity between constraining and quantizing in systems of type a) within a geometric quantization approach. For so-called non-unimodular groups the right-hand side of (1.3) is non-vanishing and thus turns the invariance condition (D2) into an equivariance condition. It is easy to check that (1.3) is consistent with the quantum analogues $[\hat{\Phi}_i, \hat{\Phi}_j] = i\hbar C_{ij}{}^k\hat{\Phi}_k$ of equations (1.1). However, (1.3) clearly violates the condition (D1), and so far has not been given a physical

interpretation. It would be interesting to know whether conditions similar to (1.3) occur also in other quantization schemes.

Let us at this point mention another slightly “strange” feature of the Dirac condition (D2): for non-abelian constraints, wave functions $|\Psi\rangle$ solving (D2) are required to be *simultaneous eigenvectors (with zero eigenvalues) of a whole set of self-adjoint, non-commuting operators $\hat{\Phi}_i$* . This does not usually happen in any physical quantum system, and is another indication that the Φ_i are somewhat special observables, if at all. In a group-theoretical quantization approach, Aldaya et al. [ALN] have therefore considered applying only *part* of the operators $\hat{\Phi}_i$ via conditions (D2) on physical wave functions (in the language of geometric quantization, this amounts to introducing a polarization on the set of constraints).

For non-trivial phase spaces and non-abelian constraints one is therefore interested in other possible modifications or generalizations of the strong Dirac condition, nevertheless compatible with some minimal physical requirements, which are important if one wishes to interpret them in the context of systems of type b). There is no a priori reason why the reduced quantum structure should necessarily come from *invariant* structures of the unreduced system. In the case of a non-abelian Lie group G acting on \mathcal{S} (and giving rise to the constraints $\Phi_i = 0$), a natural generalization of (D2) would be to consider an *equivariance* condition, with physical wave functions transforming according to some G -representation, of which we will encounter an example below in §2.5. Another well-known modification of the strong Dirac condition is given by

$$\hat{\Omega}|\Psi\rangle = 0 \tag{1.4}$$

in a Hamiltonian BRST-approach, with a formally self-adjoint BRST operator $\hat{\Omega}$ to project out “extended” physical wave functions (depending on both even and odd variables). Interestingly enough, the *classical* BRST condition $\{\Omega, A\} = 0$ can be interpreted as an equivariance condition on functions $A \in \mathcal{S}_{ext}$, the BRST-extended classical phase space [Lo1], which suggests looking for similar structures at the quantum level.

Of course there is no unique way of setting up a canonical BRST-quantization, just as there is no unique quantization without using additional “ghost” variables. On the BRST-quantum theory some rigorous work has been done for the case of a non-abelian Lie algebra of constraints (without making use of an abelianization of constraints which is possible only locally) [KoSt], [ALN], [DET], [DEGST], [Tuy2]. This is in contrast with the vast number of results that is by now available on the (co)homological structure of the classical BRST construction, and for much more general assumptions about the nature of the constraints Φ_i (see, for example, the monograph by Henneaux [Hen2] or the comprehensive treatise by Hübschmann [Hüb] and references therein). The results of the present work concern mainly the quantization of this non-abelian case.

2. Equivalence of BRST and Dirac quantization

The classical BRST structure of first-class systems, in particular, the meaning of the classical BRST symmetry of extended observables $A_{ext} \in C^\infty(\mathcal{S}_{ext})$,

$$A_{ext} \rightarrow A_{ext} + \delta K := A_{ext} + \{\Omega, K\}, \quad (2.0.1)$$

is well understood. In a cohomological language, (2.0.1) corresponds to the addition of a coboundary term within the zeroth cohomology of the Koszul complex [McM], [Hüb], defined by $\{\Omega, A_{ext}\} = 0$ and characterizing classical physical observables.

We want to investigate whether the central role of this “cohomological symmetry” can be preserved in the quantization. At the classical level the ghosts and antighosts of the BRST formalism do not have an immediate physical interpretation, but just serve to encode certain (symmetry) features of gauge systems in an elegant and compact way. Not even the Grassmann character of the additional ghost variables is absolutely essential, as exemplified in recent attempts of Duval et al. [DEGST] to “bosonize” the BRST formalism. One is relatively free in the quantization of the ghost structure, the only requirement being that after elimination of these “extra degrees of freedom” the resulting physical quantities should satisfy some minimal physical criteria, like, for instance, (D1).

In the present work, rather than starting with a mathematical definition of a BRST-quantization, we will investigate to what extent it is possible to preserve the classical BRST symmetry at the quantum level, and analyze which restrictions criteria of physicality pose on the BRST quantum structure, i.e. on the extended quantum observables, the extended wave functions and the scalar product including ghosts. We will make use of a result in [Lo1], where it is shown that the algebra of classical extended observables can be understood as the space of C^∞ -functions on a supermanifold \mathcal{S}_{ext} . This supermanifold is a vector bundle over the phase space \mathcal{S} , with “ghost fibres”. The important observation is that this bundle is trivial and possesses a canonical trivialization, which allows us to quantize the ordinary phase space part and the ghost part separately. The total Hilbert space \mathcal{H}_{tot} of the theory is thus the tensor product of the Hilbert space \mathcal{H} associated with the unextended system and the Hilbert space \mathcal{H}_{gh} associated with the ghost variables, $\mathcal{H}_{tot} = \mathcal{H} \otimes \mathcal{H}_{gh}$.

In view of the discussion in the previous section, we will consider the simplest non-trivial case which may require a modification of the usual Dirac condition (D2). This is the case where the first-class constraints Φ_i form a finite-dimensional non-abelian gauge Lie algebra \mathfrak{g} and arise from a free symplectic action of a corresponding connected Lie group G on \mathcal{S} , whence the $C_{ij}{}^k$ in expression (1.1) are the structure constants of \mathfrak{g} . Although this case is much too restrictive from the point of view of physical applications, it will be sufficient to illustrate most of the points made below. We will not consider the possibility of transforming to an open gauge algebra [Hen1] by a canonical transformation on the extended phase space.

2.1. The classical BRST symmetry.

Let us first give a brief summary of the main features of the classical BRST treatment and the origin of the BRST symmetry. The main ingredient in the formalism is the BRST generator Ω , which in the present case is given by

$$\Omega = \eta^i \Phi_i - \frac{1}{2} C_{ij}{}^k \eta^i \eta^j P_k, \quad (2.1.1)$$

where the m ghost-antighost pairs (η^i, P_i) , $i = 1, \dots, m$, are basic anticommuting variables (hence $(\eta^i)^2 = 0$, $(P_i)^2 = 0$), satisfying super-Poisson bracket relations

$$\begin{aligned} \{\eta^i, \eta^j\} &= \{\eta^j, \eta^i\} = 0 \\ \{P_i, P_j\} &= \{P_j, P_i\} = 0 \\ \{\eta^i, P_j\} &= \{P_j, \eta^i\} = \delta^{ij}. \end{aligned} \quad (2.1.2)$$

We introduce a \mathbb{Z} -grading, the so-called ghost number, on polynomials of the η 's and P 's. Define $gh(\eta^i) = 1$, $gh(P_i) = -1$, and the extensions to polynomials to be additive, e.g. $gh(\eta^i P_j) = gh(\eta^i) + gh(P_j)$ etc.. Ω has ghost number one and is nilpotent, $\{\Omega, \Omega\} = 0$. The gauge invariance of physical observables is encoded in the condition on extended observables $\{\Omega, A_{ext}\} = 0$, with $A_{ext} \in C^\infty(\mathcal{S}) \otimes \Lambda(\mathfrak{g} \oplus \mathfrak{g}^*)$ [KoSt], [Lo1], denoting a general extended function of total ghost number zero,

$$A_{ext} = A + A_i{}^j \eta^i P_j + A_{ij}{}^{kl} \eta^i \eta^j P_k P_l + \dots, \quad (2.1.3)$$

the dots standing for terms containing six or more ghosts and anti-ghosts. Because of

$$\begin{aligned} \{\Omega, A_{ext}\} &= [\{\Phi_i, A\} - A_i{}^j \Phi_j] \eta^i + [\{\Phi_i, A_j{}^k\} - \frac{1}{2} C_{ij}{}^m A_m{}^k + \\ &\quad + C_{mj}{}^k A_i{}^m - 2A_{ij}{}^{km} \Phi_m] \eta^i \eta^j P_k + \text{higher order ghost terms}, \end{aligned} \quad (2.1.4)$$

this yields the conditions

$$\begin{aligned} i) \quad & \{\Phi_i, A\} - A_i{}^j \Phi_j = 0 \\ ii) \quad & \{\Phi_{[i}, A_{j]}{}^k\} - \frac{1}{2} C_{[ij]}{}^m A_m{}^k + C_{m[j}{}^k A_i]{}^m - 2A_{[ij]}{}^{km} \Phi_m = 0 \\ & \text{etc.,} \end{aligned} \quad (2.1.5)$$

on the even coefficient functions $A, A_i{}^j, \dots$, with the square brackets indicating antisymmetrization. The first one is the usual classical condition for A to be an observable [Dir2], which is equivalent to the statement that the Hamiltonian vector field associated with A is tangential to the constraint surface.

In the classical theory two observables A and $A' = A + \lambda^i(s) \Phi_i$ are physically equivalent since they describe the same function on the constraint surface (defined by $\Phi_i \equiv 0$). i.e. we can add arbitrary phase space-dependent linear

combinations of the constraints Φ_i to any observable without affecting physical results.

The BRST-invariant extended observable corresponding to A is of the form (2.1.3) with coefficients satisfying (2.1.5). The BRST observable corresponding to A' is

$$A'_{ext} = A + \lambda^i \Phi_i + [A_i^j - \{\lambda^j, \Phi_i\} - \lambda^k C_{ki}^j] \eta^i P_j + \dots \tag{2.1.6}$$

One can now check that $\Delta A_{ext} = A'_{ext} - A_{ext} = \{\lambda^i P_i, \Omega\}$ holds if the constraints form a true Lie algebra with structure constants C_{ij}^k , as we are assuming. However, since we want A_{ext} and A'_{ext} to describe the same physical observable, their difference must be equivalent to the zero-observable. According to [Hen1], one has to regard two BRST-invariant extended observables as physically equivalent if they differ by a term $\{K_{ext}, \Omega\}$, where K_{ext} is an arbitrary function with $gh(K_{ext}) = -1$, i.e.

$$K_{ext} = K^i P_i + K_i^{jk} \eta^i P_j P_k + K_{ij}^{klm} \eta^i \eta^j P_k P_l P_m + \dots, \tag{2.1.7}$$

with arbitrary phase space-dependent coefficients K^i, K_i^{jk} etc.. The freedom of adding a term $\delta K := \{K_{ext}, \Omega\}$ to a BRST-observable without affecting physical results is often referred to as the classical BRST symmetry. Direct computation gives

$$\{K_{ext}, \Omega\} = \{\Omega, K_{ext}\} = K^i \Phi_i + [\{\Phi_i, K^j\} + C_{ik}^j K^k + 2K_i^{jk} \Phi_k] \eta^i P_j + \dots, \tag{2.1.8}$$

hence we have encoded the original ‘‘symmetry’’ $A \rightarrow A + \lambda^i \Phi_i$, which appears as the first term in the ghost series.

2.2. Canonical approach: quantum observables and physical wave functions.

Before discussing the BRST quantum theory we will describe four different ways of defining quantum observables \hat{A} according to some orthodox canonical approach à la Dirac and check their compatibility with the weak and strong Dirac conditions on physical quantum states, (D1) and (D2). They will be needed later for comparison with the BRST approach. We assume that the gauge algebra (1.1) is quantized without anomaly, i.e. the quantum commutators are given by

$$[\hat{\Phi}_i, \hat{\Phi}_j] = i\hbar C_{ij}^k \hat{\Phi}_k. \tag{2.2.1}$$

The only physical requirement we impose is that

$$\langle \Psi'_{ph} | [\hat{\Phi}_i, \hat{A}] | \Psi_{ph} \rangle = 0, \quad \forall i, \tag{2.2.2}$$

for observables \hat{A} and $|\Psi_{ph}\rangle, |\Psi'_{ph}\rangle \in \mathcal{H}_{ph}$.

(C1) Define quantum observables \hat{A} by $[\hat{\Phi}_i, \hat{A}] = 0, \forall i$.

This definition is consistent since the commutator of two observables \hat{A} and \hat{B} is again an observable:

$$[\hat{\Phi}_i, [\hat{A}, \hat{B}]] = [[\hat{\Phi}_i, \hat{A}], \hat{B}] + [\hat{A}, [\hat{\Phi}_i, \hat{B}]] = 0. \quad (2.2.3)$$

Both (D1) and (D2) are compatible with (C1). Using (D2), the observables map physical wave functions into physical ones, using (D1), this does not necessarily happen, but nevertheless condition (2.2.2) is fulfilled.

(C2) Define quantum observables \hat{A} by $[\hat{\Phi}_i, \hat{A}] = i\hbar \hat{A}_i^j \hat{\Phi}_j$, $\forall i$.

This means we had $\{\Phi_i, A\} = A_i^j \Phi_j$ classically and the quantization prescription for $\hat{\Phi}_i$ and \hat{A} is such that all operators \hat{A}_i^j appear on the left of the quantum constraints $\hat{\Phi}_i$. This is exactly Dirac's requirement on quantum observables [Dir2]. It is consistent since for the commutator of two observables \hat{A} and \hat{B} , with $[\hat{\Phi}_i, \hat{A}] = i\hbar \hat{A}_i^j \hat{\Phi}_j$ and $[\hat{\Phi}_i, \hat{B}] = i\hbar \hat{B}_i^j \hat{\Phi}_j$, we obtain

$$[\hat{\Phi}_i, [\hat{A}, \hat{B}]] = i\hbar ([\hat{A}, \hat{B}_i^j] - [\hat{B}, \hat{A}_i^j] - i\hbar \hat{B}_i^k \hat{A}_k^j + i\hbar \hat{A}_i^k \hat{B}_k^j) \hat{\Phi}_j =: i\hbar \hat{C}_i^j \hat{\Phi}_j. \quad (2.2.4)$$

The matrix elements evaluated on physical wave functions become

$$\langle \Psi'_{ph} | [\hat{\Phi}_i, \hat{A}] | \Psi_{ph} \rangle = i\hbar \langle \Psi'_{ph} | \hat{A}_i^j \hat{\Phi}_j | \Psi_{ph} \rangle, \quad (2.2.5)$$

which according to (2.2.2) is required to vanish. Since the \hat{A}_i^j on the right-hand side are arbitrary operators, this will in general only be fulfilled if we define physical wave functions by the strong Dirac condition (D2).

(C3) Define quantum observables \hat{A} by $[\hat{\Phi}_i, \hat{A}] | \Psi_{ph} \rangle = 0$, $\forall i$.

Using (D2), this reduces to $\hat{\Phi}_i \hat{A} | \Psi_{ph} \rangle = 0$, i.e. the operator \hat{A} maps physical states into physical states. For the commutator of two observables we have

$$[\hat{\Phi}_i, [\hat{A}, \hat{B}]] | \Psi_{ph} \rangle = (\hat{\Phi}_i \hat{A} \hat{B} - \hat{\Phi}_i \hat{B} \hat{A}) | \Psi_{ph} \rangle = 0. \quad (2.2.6)$$

(Since both \hat{A} and \hat{B} map \mathcal{H}_{ph} into itself, so does $[\hat{A}, \hat{B}]$.) Using (D1), we find for the commutators of two observables

$$[\hat{\Phi}_i, [\hat{A}, \hat{B}]] | \Psi_{ph} \rangle = ([\hat{\Phi}_i, \hat{A}] \hat{B} - [\hat{\Phi}_i, \hat{B}] \hat{A}) | \Psi_{ph} \rangle, \quad (2.2.7)$$

which would vanish if the quantum observables mapped physical states into physical states. This however is not the case since in general $\langle \Psi'_{ph} | \hat{\Phi}_i \hat{A} | \Psi_{ph} \rangle \neq 0$, i.e. a state $\hat{A} | \Psi_{ph} \rangle$ is not necessarily physical. Hence the commutator of two observables is not necessarily an observable, and (C3) and (D1) are in general not compatible.

(C4) Define quantum observables \hat{A} by $\langle \Psi'_{ph} | [\hat{\Phi}_i, \hat{A}] | \Psi_{ph} \rangle = 0$, $\forall i$.

Using (D2), we find that every operator \hat{A} is an observable, and so is every commutator $[\hat{A}, \hat{B}]$. Using (D1), the above condition on \hat{A} becomes non-trivial, but again this leads to an inconsistency if we consider the commutator of two such observables:

$$\langle \Psi'_{ph} | [\hat{\Phi}_i, [\hat{A}, \hat{B}]] | \Psi_{ph} \rangle = \langle \Psi'_{ph} | [\hat{\Phi}_i, \hat{A}] \hat{B} - [\hat{\Phi}_i, \hat{B}] \hat{A} | \Psi_{ph} \rangle, \quad (2.2.8)$$

and there is no reason why this should vanish.

Of the four conditions on quantum observables, (C1) is the strongest and (C4) the weakest. Using (D2) on wave functions, an observable satisfying (C2) always satisfies (C3), but the same is not true if we use (D1). We have seen that the conditions (C2), (C3) and (C4) are in general only compatible with the strong Dirac condition (D2) on physical states. Still this does not mean that in particular cases we may not be able to use the weak Dirac condition. In a given classical system, only a limited subset of physical observables can be quantized, and it may happen that these observables are such that conditions like $[\hat{\Phi}_i, [\hat{A}, \hat{B}]] |\Psi_{ph}\rangle = 0$ or $\langle \Psi'_{ph} | [\hat{\Phi}_i, [\hat{A}, \hat{B}]] |\Psi_{ph}\rangle = 0$ are fulfilled.

This means that a stronger condition on quantum observables implies a weaker condition on physical states and vice versa. Note that approaches to the quantization of constrained systems rarely make a statement about the existence of quantum observables, and that particularly for examples involving non-abelian constraints it is often hard to find more than a very few (see [Lo3], [DET] for examples). This highlights a more general problem that always occurs when quantization of a smaller “subsystem” is to be obtained from the quantization of an extended system. Suppose, for example, we are given some (not necessarily constrained) physical system with a non-linear phase space \mathcal{S} which we do not know how to quantize. Then we may be able to embed \mathcal{S} into some higher-dimensional vector space V (for example, if \mathcal{S} is a G -manifold for some compact Lie group G , general mathematical theorems [Mos] ensure the existence of an equivariant embedding of \mathcal{S} in some V). However, even if we can find an appropriate space V , the problem of quantizing is by no means trivialized, but has merely been translated into the problem of finding observables which map the subspace \mathcal{S} into itself. Related difficulties occur for all types of phase space extensions, for which the BRST method is one example.

2.3. Quantization of the ghosts.

In the BRST quantum theory, our aim will be to preserve as much as possible of the classical (super-)Poisson bracket algebra of extended functions as commutator algebra of (pseudo-)selfadjoint quantum operators. In the quantization the super-Poisson bracket relations (2.1.2) of the basic ghost variables become basic anticommutation relations

$$\begin{aligned} [\hat{\eta}^i, \hat{\eta}^j] &= [\hat{\eta}^j, \hat{\eta}^i] = 0 \\ [\hat{P}_i, \hat{P}_j] &= [\hat{P}_j, \hat{P}_i] = 0 \\ [\hat{\eta}^i, \hat{P}_j] &= [\hat{P}_j, \hat{\eta}^i] = i\hbar \delta^i_j. \end{aligned} \tag{2.3.1}$$

Mathematically these are usually identified as the anticommutators of the Clifford algebra associated with the vector space $V = \mathfrak{g} \oplus \mathfrak{g}^*$, the direct sum of the Lie algebra \mathfrak{g} and its dual \mathfrak{g}^* , spanned by the ghosts $\eta^i \in \mathfrak{g}^*$ and the antighosts $P_i \in \mathfrak{g}$, $i = 1, \dots, m$, with the scalar product $(\eta^i, P_j) = \frac{1}{2} \delta^i_j$ and all other components vanishing [KoSt], [DET]. We can define a Clifford map Γ from the

vector space V into its associated Clifford algebra $C(V)$ satisfying

$$[\Gamma(v), \Gamma(v')] = \Gamma(v)\Gamma(v') + \Gamma(v')\Gamma(v) = 2(v, v')\mathbb{1}_{C(V)}, \quad (2.3.2)$$

where $v, v' \in V$ and $\mathbb{1}_{C(V)}$ is the unit in $C(V)$. In terms of the basis of $\mathfrak{g} \oplus \mathfrak{g}^*$ these relations read

$$\begin{aligned} [\Gamma(\eta^i), \Gamma(\eta^j)] &= 0 \\ [\Gamma(P_i), \Gamma(P_j)] &= 0 \\ [\Gamma(\eta^i), \Gamma(P_j)] &= 2(\eta^i, P_j) = \delta^i_j. \end{aligned} \quad (2.3.3)$$

which, apart from an inessential factor, reproduce exactly (2.3.1). This Clifford algebra is usually called $C(m, m)$ since in an orthonormal basis for $\mathfrak{g} \oplus \mathfrak{g}^*$ the scalar product has the signature (m, m) . The dimension of $C(m, m)$ is 2^{2m} , it is isomorphic to the algebra of real $2^m \times 2^m$ -matrices, and it has a unique irreducible representation. The super-Poisson algebra morphism between (2.1.2) and (2.3.1) can be extended to higher polynomials in the ghost variables of sufficiently low degree [DET].

We will choose as basis of ghost wave functions the 2^m independent polynomials in the m ghosts, i.e. $|1\rangle, |\eta^1\rangle, \dots, |\eta^m\rangle, |\eta^1\eta^2\rangle, \dots, |\eta^1 \dots \eta^m\rangle$, where the ghosts are supposed to be anticommuting as usual, i.e. $|\eta^i\eta^j\rangle = -|\eta^j\eta^i\rangle$ etc.. The operator versions of the classical ghost functions η and P are written symbolically as

$$\hat{\eta}^i = \eta^i \quad \text{and} \quad \hat{P}_i = i\hbar \frac{\partial}{\partial \eta^i}, \quad (2.3.4)$$

and their action on the basic wave functions is, as usual, given by

$$\begin{aligned} \hat{\eta}^i |1\rangle &= |\eta^i\rangle, \quad \hat{\eta}^i |\eta^j\rangle = |\eta^i\eta^j\rangle, \quad \text{etc.} \\ \hat{P}_i |1\rangle &= 0, \quad \hat{P}_i |\eta^j\rangle = i\hbar \delta^j_i |1\rangle, \quad \text{etc.} \end{aligned} \quad (2.3.5)$$

A general BRST-extended wave function will be written as

$$\begin{aligned} |\Psi^{ext}\rangle_{tot} &= |\Psi + \Psi_i \eta^i + \Psi_{ij} \eta^i \eta^j + \dots\rangle_{tot} \equiv \\ &\equiv |\Psi\rangle |1\rangle_{gh} + |\Psi_i\rangle |\eta^i\rangle_{gh} + |\Psi_{ij}\rangle |\eta^i \eta^j\rangle_{gh} + \dots, \end{aligned} \quad (2.3.6)$$

where we have decomposed the scalar product on \mathcal{H}_{tot} into a canonical and a ghost part and introduced the explicit notation $\langle | \rangle_{tot} = \langle | \rangle \langle | \rangle_{gh}$. The even wave functions from \mathcal{H} appearing as coefficients in this expansion carry totally antisymmetric indices. Note that at this stage the only natural scalar product we can define on the representation space of the ghost wave functions is the one induced by the scalar product on V . However, since the subspace of V spanned by the η^i is maximally isotropic, this scalar product vanishes identically. Since we want to explore the possibility of having the (canonical) scalar product on physical wave functions induced by that on \mathcal{H}_{tot} , we will *not* use this trivial scalar product. Rather we will leave the scalar product on the ghost sector

undefined and determine some restrictions on $\langle \cdot | \cdot \rangle_{gh}$ derived from physical requirements.

2.4. BRST quantization and the scalar product.

We will give a short summary of the main features of the standard quantum BRST formulation [Hen1].

- a) Main ingredient is the quantum BRST operator $\hat{\Omega}$. It has ghost number one, $[\hat{N}, \hat{\Omega}] = \hat{\Omega}$, with the ghost number operator $\hat{N} := \frac{1}{i\hbar} \hat{\eta}^i \hat{P}_i$, and is nilpotent, $\frac{1}{2}[\hat{\Omega}, \hat{\Omega}] = \hat{\Omega}^2 = 0$.
- b) Quantum BRST-observables \hat{A}_{ext} have ghost number zero, $[\hat{N}, \hat{A}_{ext}] = 0$, and are BRST-invariant, i.e. $[\hat{\Omega}, \hat{A}_{ext}] = 0$.
- c) Two extended observables \hat{A}_{ext} and $\hat{A}_{ext} + [\hat{K}_{ext}, \hat{\Omega}]$ are physically equivalent for arbitrary operators \hat{K}_{ext} of ghost number minus one, $[\hat{N}, \hat{K}_{ext}] = -\hat{K}_{ext}$. As usual we take this to mean

$$\langle \Psi_{ph}^{ext} | [\hat{K}_{ext}, \hat{\Omega}] | \Psi_{ph}^{ext} \rangle = 0, \tag{2.4.1}$$

where $|\Psi_{ph}^{ext}\rangle$ and $|\Psi_{ph}^{ext}\rangle$ are elements of some physical subspace of the Hilbert space \mathcal{H}_{tot} . Formally this is guaranteed by defining physical functions by the condition $\hat{\Omega}|\Psi_{ph}^{ext}\rangle = 0$ and requiring $\hat{\Omega}$ to be skew-selfadjoint with respect to the scalar product $\langle \cdot | \cdot \rangle_{tot}$. Under these assumptions any state of the form $|\Psi^{ext}\rangle = \hat{\Omega}|\chi^{ext}\rangle$ is physical because of the nilpotency of $\hat{\Omega}$. This leads to the BRST quantum cohomology of physical states (wave functions annihilated by $\hat{\Omega}$ modulo functions of the form $\hat{\Omega}|\chi^{ext}\rangle$). Similarly, due to the nilpotency of $\hat{\Omega}$, we have the BRST quantum cohomology of physical observables (BRST-invariant observables modulo observables of the form $[\hat{K}_{ext}, \hat{\Omega}]$).

Note that the characteristic properties of the BRST operator $\hat{\Omega}$, $\hat{\Omega}^2 = 0$ and $\hat{\Omega}^\dagger = \hat{\Omega}$ require that $\hat{\Omega}$ must be an operator on an indefinite inner product space, as is stated by the following theorem (see [Bog]):

If for any zero-norm vector χ orthogonal to itself, $(\chi, \chi) = 0$, in an inner product space E , there exists some vector ϕ not orthogonal to it, $(\chi, \phi) \neq 0$, then the inner product is necessarily indefinite (i.e. we necessarily have both positive and negative elements in E).

In our case a state $\hat{\Omega}\Psi^{ext}$ has always zero norm:

$$\|\hat{\Omega}\Psi^{ext}\|^2 = \langle \hat{\Omega}\Psi^{ext} | \hat{\Omega}\Psi^{ext} \rangle = \langle \Psi^{ext} | \hat{\Omega}^2\Psi^{ext} \rangle = 0 \tag{2.4.2}$$

This means that, unless all states of the form $\hat{\Omega}|\Psi^{ext}\rangle$ vanish identically, we are dealing with an indefinite metric ‘‘Hilbert space’’. Roughly speaking this is an infinite-dimensional indefinite-metric space with respect to which $\hat{\Omega}$ is a (pseudo-)hermitian operator. (Pseudo-) hermiticity is still a well-defined concept

in an indefinite-metric space, the hermitian conjugate \hat{T}^\dagger of an operator \hat{T} being defined by $(\chi, \hat{T}^\dagger \phi) := (\phi, \hat{T} \chi)^*$.

Finally we need a prescription of how to recover genuinely physical quantities from BRST-extended objects, i.e. how to get rid of the ghosts at the end of the quantization. Important questions in this context are

- What is the meaning of the BRST quantum cohomologies?
- Which part of a BRST wave function and a BRST observable is to be regarded as physical? Is there a natural meaning one can assign to higher-order coefficients in both wave functions and observables?
- What is the physical significance of the indefinite scalar product? How can we recover the usual, positive-definite scalar product on physical, ghost-free states?

The precise answers to these questions may of course depend on the particular BRST quantization chosen. The classical higher-order BRST cohomology contains information about the gauge system, for example, the topology of the gauge orbits (see [HeTe],[Fig] and references therein), and it would be interesting if analogous statements could be made in the quantum theory. Note that the BRST cohomology of classical observables “splits” into two quantum cohomologies, one of the quantum observables and one of the quantum states. One expects questions about the Hilbert space structure of the BRST-extended theory to be less straightforward to answer, because the Hilbert space structure of the theory has, unlike the algebraic structure of the quantum operators, no close classical analogue. This is illustrated by the discussion above where we modelled the quantum algebra of observables as closely as possible on the classical super-Poisson bracket algebra, with the condition $\hat{\Omega}|\Psi_{ph}^{ext}\rangle = 0$ emerging only as a *consequence* of these considerations.

In the next paragraphs we will decompose the basic relations defining the BRST quantum theory into their Grassmann-even and -odd parts, to understand which conditions they impose on the even, ghost-free sector. We will also consider alternative definitions of BRST observables and physical states, and compare the results with the canonical scheme. Of course, operator ordering ambiguities appear in the BRST quantum theory, both for ordinary and ghost quantities. Two expressions $\eta^i P_j$ and $P_j \eta^i$ differ in the quantum theory by a term of order \hbar . For reasons explained below we will adopt the convention that in any expression containing ghosts all the P 's should stand to the right of all the η 's (the so-called normal ordering).

2.5. The quantum BRST operator and physical BRST wave functions.

The quantum BRST operator is defined by

$$\hat{\Omega} = \hat{\eta}^i \hat{\Phi}_i - \frac{1}{2} C_{ij}{}^k \hat{\eta}^i \hat{\eta}^j \hat{P}_k. \quad (2.5.1)$$

It is straightforward to verify its nilpotency, $[\hat{\Omega}, \hat{\Omega}] = 0$. There are two obvious

ways of defining physical BRST wave functions $|\Psi_{ph}^{ext}\rangle \in \mathcal{H}_{tot,ph} \subset \mathcal{H}_{tot}$, imitating the canonical approach (c.f. §2.2), the “weak BRST condition” on matrix elements,

$$(B1) \quad \langle \Psi_{ph}^{\prime ext} | \hat{\Omega} | \Psi_{ph}^{ext} \rangle_{tot} = 0,$$

and the “strong BRST condition”,

$$(B2) \quad \hat{\Omega} | \Psi_{ph}^{ext} \rangle_{tot} = 0.$$

We will analyze under what circumstances one can make these statements meaningful. Defining physical wave functions by (B2) and using equations (2.3.5) and (2.3.6) yields the following set of equations (using the linear independence of the ghost wave functions)

$$\begin{aligned} i) \quad & \hat{\Phi}_i | \Psi_{ph} \rangle = 0 \\ ii) \quad & \hat{\Phi}_{[i} | \Psi_{ph,j]k} \rangle > -\frac{1}{2} i\hbar C_{[ij}{}^k | \Psi_{ph,k} \rangle = 0 \\ iii) \quad & \hat{\Phi}_{[i} | \Psi_{ph,jk]l} \rangle > -\frac{1}{2} i\hbar C_{[ij}{}^m | \Psi_{ph,|m|k]l} \rangle = 0 \\ & etc. \end{aligned} \tag{2.5.2}$$

Note that we could have chosen a different factor ordering for the trilinear ghost term in (2.5.1). In this case one obtains additional contributions $\sim i\hbar C_{ij}{}^j$ to all equations in (2.5.2), in particular, equation i) modulo constant factors turns into condition (1.3). For the b)-type systems under discussion this leads to an *imaginary* contribution of order \hbar in (D1). Since we do not know of any physical interpretation of this fact, we will stick to the factor ordering of equation (2.5.1).

The first equation in (2.5.2) is exactly the strong Dirac condition (D2) on physical states, the other equations are equivariance conditions which tell us that the higher-order physical wave functions are not invariant but transform according to some tensor representation of the gauge group under the action of the quantum constraints. Note that these conditions do not mix wave functions with different number of “ghost indices”, and that it is therefore consistent to set coefficients $|\Psi_{ph}^{ij\dots}\rangle$ with the same number of ghost indices to zero.

Similarly, in order to make any sense of the weak BRST condition (B1) (in the context of systems of type b)), we will require it to contain (at least) the weak Dirac condition on states, (D1) above, i.e. for arbitrary $|\Psi^{ext}\rangle$, $|\Psi^{\prime ext}\rangle$, $\langle \Psi^{\prime ext} | \hat{\Omega} | \Psi^{ext} \rangle_{tot} = 0$ should imply $\langle \Psi^{\prime} | \hat{\Phi}_i | \Psi \rangle = 0$, $\forall i$. This leads to the following conditions

$$\begin{aligned}
i) & \quad \langle 1 | \eta^i \rangle_{gh} \neq 0 \\
ii) & \quad \langle \Psi' | \hat{\Phi}_i \delta^k_j - \frac{1}{2} i \hbar C_{ij}{}^k | \Psi_k \rangle \langle 1 | \eta^i \eta^j \rangle_{gh} = 0 \\
iii) & \quad \langle \Psi'_i | \hat{\Phi}_j | \Psi \rangle \langle \eta^i | \eta^j \rangle_{gh} = 0 \\
iv) & \quad \langle \Psi'_i | \hat{\Phi}_j \delta^k_m - \frac{1}{2} i \hbar C_{jm}{}^k | \Psi_k \rangle \langle \eta^i | \eta^j \eta^m \rangle_{gh} = 0 \\
v) & \quad \langle \Psi'_i | \hat{\Phi}_j \delta^k_m - \frac{1}{2} i \hbar C_{jm}{}^k | \Psi_{kl} \rangle \langle 1 | \eta^j \eta^m \eta^l \rangle_{gh} = 0
\end{aligned} \tag{2.5.3}$$

etc.

There are three different solutions:

- 1) $\langle 1 | \eta^i \rangle \neq 0$ and all higher-order coefficients $|\Psi_i \rangle$, $|\Psi_{ij} \rangle$ etc. vanish.
- 2) $\langle 1 | \eta^i \rangle \neq 0$ and all products of ghost wave functions containing more than one ghost vanish, i.e. $\langle \eta^i | \eta^j \rangle_{gh} = 0$, $\langle 1 | \eta^i \eta^j \rangle_{gh} = 0$ etc.
- 3) $\langle 1 | \eta^i \rangle \neq 0$ and some combination of 1) and 2). For example, we could assume that $|\Psi \rangle$ and $|\Psi_i \rangle$ are the only non-vanishing coefficients in any $|\Psi^{ext} \rangle$, then we get the conditions $\langle \eta^i | \eta^j \rangle_{gh} = 0$, $\langle 1 | \eta^i \eta^j \rangle_{gh} = 0$, $\langle \eta^i | \eta^j \eta^k \rangle_{gh} = 0$.

2.6. Quantum BRST observables.

We will consider now various definitions of BRST observables \hat{A}_{ext} . Any \hat{A}_{ext} is required to have ghost number zero and therefore of the form

$$\hat{A}_{ext} = \hat{A} + \hat{A}_i{}^j \hat{\eta}^i \hat{P}_j + \hat{A}_{ij}{}^{kl} \hat{\eta}^i \hat{\eta}^j \hat{P}_k \hat{P}_l + \dots \tag{2.6.1}$$

(O1) Define BRST quantum observables \hat{A}_{ext} by $[\hat{\Omega}, \hat{A}_{ext}] = 0$.

Decomposing into terms with same ghost structure, this yields the conditions (c.f.(2.1.5))

$$\begin{aligned}
i) & \quad [\hat{\Phi}_i, \hat{A}] - i \hbar \hat{A}_i{}^j \hat{\Phi}_j =: \hat{X}_i = 0 \\
ii) & \quad [\hat{\Phi}_{[i}, \hat{A}_{j]}{}^k] - \frac{1}{2} i \hbar C_{[ij]}{}^m \hat{A}_m{}^k + i \hbar C_{m[j}{}^k \hat{A}_{i]}{}^m - 2 i \hbar \hat{A}_{[ij]}{}^{km} \hat{\Phi}_m - \\
& \quad - (i \hbar)^2 C_{mn}{}^k \hat{A}_{[ij]}{}^{mn} =: \hat{X}_{ij}{}^k = 0
\end{aligned} \tag{2.6.2}$$

etc.,

The last term in ii) has no classical analogue. It comes from the factor ordering of higher-order ghost terms. Note also that the quantum constraints $\hat{\Phi}_i$ automatically appear on the right of operators like $\hat{A}_i{}^j$, $\hat{A}_{ij}{}^{km}$, etc.. This is the quantized version of equations (2.1.5), moreover, i) is exactly Dirac's condition on quantum observables \hat{A} . It remains to be seen whether ii) and higher-order equations have any physical content. Classically, for a weakly gauge-invariant

function A , the existence of the higher-order structure functions A_i^j, A_{ij}^{km} etc. is guaranteed [Hen1], i.e. we can successively solve the classical analogues of the equations (2.6.2). The question is: given a classical BRST-invariant observable A_{ext} , when can we find appropriate quantizations for the coefficients $A(s), A_i^j(s), A_{ij}^{km}(s), \dots$, such that the operator equations (2.6.2) hold simultaneously? Since we know that even for the simple case of quantization on a phase space \mathbb{R}^{2n} (without constraints) only a very small number of phase space functions (= observables) can be quantized consistently (namely, the polynomials up to second order in coordinates and momenta), it is a priori not clear whether this will be possible. However, for the special case of constraints arising from a group action we can make use of the fact that any classical Dirac observable A can be written as $A = A' + \lambda^i \Phi_i$, with $\{A', \Phi_i\} = 0$, for some functions λ^i . If we can then find a quantization for A' and Φ_i such that $[\hat{A}', \hat{\Phi}_i] = 0$, a solution to (2.6.2) is given by $\hat{A} = \hat{A}' + \hat{\lambda}^i \hat{\Phi}_i, \hat{A}_i^j = i\hbar C_{ki}^j \lambda^k + [\hat{\lambda}^j, \hat{\Phi}_i], \hat{A}_{ij}^{kl} = 0$ etc.. Hence this condition is equivalent to Dirac's condition (C2) on quantum observables. This will not be true for the general, non-group case where we expect the operator condition $[\hat{\Omega}, \hat{A}_{ext}] = 0$ to be stronger than its classical counterpart, in particular stronger than condition (C2).

The definition (O1) of BRST observable is consistent since the commutator of two such observables is again an observable:

$$[\hat{\Omega}, [\hat{A}_{ext}, \hat{B}_{ext}]] = [[\hat{\Omega}, \hat{A}_{ext}], \hat{B}_{ext}] + [\hat{A}_{ext}, [\hat{\Omega}, \hat{B}_{ext}]] = 0. \tag{2.6.3}$$

The only thing left to check now is the compatibility with the notion of physical BRST states. Since the first condition in (2.6.2) is precisely the (Dirac) condition on physical observables in (C2), the same analysis can be applied here, which means that in general only (B2) is compatible with (O1).

(O2) Define BRST quantum observables \hat{A}_{ext} by $[\hat{\Omega}, \hat{A}_{ext}]|\Psi_{ph}^{ext}\rangle = 0$.
This yields the conditions

$$\begin{aligned} i) \quad & ([\hat{\Phi}_i, \hat{A}] - i\hbar \hat{A}_i^j \hat{\Phi}_j)|\Psi_{ph}\rangle > |\eta^i\rangle_{gh} = 0 \\ ii) \quad & (\hat{X}_i \delta_j^k + i\hbar \hat{X}_{ij}^k)|\Psi_{ph}^k\rangle > |\eta^i \eta^j\rangle_{gh} = 0 \end{aligned} \tag{2.6.4}$$

etc.,

where we have used the abbreviations \hat{X}_i and \hat{X}_{ij}^k introduced in (2.6.2). All ghost wave functions are non-vanishing, and hence i) gives a non-trivial condition on \hat{A} and \hat{A}_i^j which is a mixture of the canonical cases (C2) and (C3).

An argument similar to that used in (C2) shows that the condition $([\hat{\Phi}_i, \hat{A}] - i\hbar \hat{A}_i^j \hat{\Phi}_j)|\Psi_{ph}\rangle > |\eta^i\rangle_{gh} = 0$ on physical observables \hat{A} is in general only consistent if we define physical wave functions by (D2) or, equivalently, physical BRST wave functions by (B2). In this case the condition on BRST observables reduces to $\hat{\Omega} \hat{A}_{ext} |\Psi_{ph}^{ext}\rangle = 0$ and equations (2.6.4) become

$$\begin{aligned}
i) \quad & \hat{\eta}^i \hat{\Phi}_i \hat{A} |\Psi_{ph}\rangle = 0 \\
ii) \quad & (\hat{\eta}^i \hat{\Phi}_i - \frac{1}{2} C_{ij}^k \hat{\eta}^i \hat{\eta}^j \hat{P}_k) [\hat{A} \delta_m^n + i\hbar \hat{A}_m^n] |\Psi_{ph,n}\rangle = |\eta^m\rangle_{gh} = 0 \\
iii) \quad & (\hat{\eta}^i \hat{\Phi}_i - \frac{1}{2} C_{ij}^k \hat{\eta}^i \hat{\eta}^j \hat{P}_k) [\hat{A} \delta_m^n \delta_p^q + i\hbar \hat{A}_m^n \delta_p^q + (i\hbar)^2 \hat{A}_{mp}{}^{qn}] \\
& |\Psi_{ph,nq}\rangle = |\eta^m \eta^p\rangle_{gh} = 0
\end{aligned} \tag{2.6.5}$$

etc.

A sufficient condition for i) to hold is the existence of an appropriate factor ordering of $\hat{\Phi}_i$ and \hat{A} such that $[\hat{\Phi}_i, \hat{A}] = \hat{\lambda}_i^j \hat{\Phi}_j$, with some operators $\hat{\lambda}_i^j$ appearing on the *left* of the quantum constraints. This is exactly Dirac's condition on observables \hat{A} . An obvious solution to the remaining equations in (2.6.5) is given by $\hat{A}_i^j = -\frac{1}{i\hbar} \hat{A} \delta_i^j$, and all higher-order operators $\hat{A}_{ij}{}^{km}$ etc. vanishing. Hence, given an operator \hat{A} satisfying i), we can construct a corresponding BRST-invariant quantum observable by

$$\hat{A}_{ext} = \hat{A} - \frac{1}{i\hbar} \hat{A} \hat{\eta}^i \hat{P}_i. \tag{2.6.6}$$

This definition of a BRST observable is consistent since observables map physical (BRST) states into physical ones, and so does the commutator of two such observables. It is weaker than the previous definition (O1).

(O3) Define BRST quantum observables \hat{A}_{ext} by $\langle \Psi'_{ph}{}^{ext} | [\hat{\Omega}, \hat{A}_{ext}] | \Psi_{ph}{}^{ext} \rangle = 0$.

The analysis of $\langle \Psi'_{ph}{}^{ext} | [\hat{\Omega}, \hat{A}_{ext}] | \Psi_{ph}{}^{ext} \rangle = 0$ is similar to that of the weak BRST condition, and the solutions to the resulting set of equations are the same, but now with all wave function coefficients $|\Psi\rangle$, $|\Psi_i\rangle$, $|\Psi_{ij}\rangle$ etc. bearing the suffix "ph". The solutions are:

- 1) $\langle 1 | \eta^i \rangle \neq 0$ and all higher-order coefficients $|\Psi_{ph,i}\rangle$, $|\Psi_{ph,ij}\rangle$ etc. vanish.
- 2) $\langle 1 | \eta^i \rangle \neq 0$ and all products of ghost wave functions except $\langle \eta^i | 1 \rangle_{gh}$ and $\langle 1 | \eta^i \rangle_{gh}$ vanish.
- 3) $\langle 1 | \eta^i \rangle \neq 0$ and some combination of 1) and 2) (c.f. the solutions to (2.5.3)).

This leaves us with just one condition on physical observables \hat{A} ,

$$\langle \Psi'_{ph} | \hat{X}_i | \Psi_{ph} \rangle \equiv \langle \Psi'_{ph} | [\hat{\Phi}_i, \hat{A}] - i\hbar \hat{A}_i^j \hat{\Phi}_j | \Psi_{ph} \rangle = 0, \quad \forall i, \tag{2.6.7}$$

with *no* other conditions on the higher coefficients \hat{A}_i^j , $\hat{A}_{ij}{}^{km}$, ... of \hat{A}_{ext} . Definition (O3) is consistent only if we define physical wave functions by (D2) or, equivalently, use (B2) to determine physical BRST states. Then equation (2.6.7) reduces to

$$\langle \Psi'_{ph} | [\hat{\Phi}_i, \hat{A}] | \Psi_{ph} \rangle = 0, \tag{2.6.8}$$

which is exactly the canonical case (C4) we discussed earlier. Again it turns out that *every* extended operator \hat{A}_{ext} is now a physical observable, and so is of course the commutator of two such objects.

In summary, (O1) is the strongest and (O3) the weakest condition on (BRST) observables. For the group case, both definitions (O1) and (O2) are equivalent to Dirac's requirement on quantum observables. In general all three possibilities are compatible only with the strong BRST condition (B2) on physical states. However, as we already mentioned in the canonical approach, in particular cases the weak BRST condition on states may also be sufficient.

2.7. Quantum BRST symmetry and pseudo-selfadjointness.

Independent of the definition we choose for BRST observables, a function of the form $\hat{A}_{ext} = [\hat{\Omega}, \hat{K}_{ext}]$, with any operator \hat{K}_{ext} of ghost number minus one, is always an observable since

$$[\hat{\Omega}, \hat{A}_{ext}] = [\hat{\Omega}, [\hat{\Omega}, \hat{K}_{ext}]] = \hat{\Omega}[\hat{\Omega}, \hat{K}_{ext}] - [\hat{\Omega}, \hat{K}_{ext}]\hat{\Omega} \tag{2.7.1}$$

which vanishes identically because of the nilpotency of the quantum BRST-operator $\hat{\Omega}$. An arbitrary operator \hat{K}_{ext} has the form

$$\hat{K}_{ext} = \hat{K}^i \hat{P}_i + \hat{K}_i{}^{jk} \hat{\eta}^i \hat{P}_j \hat{P}_k + \hat{K}_{ij}{}^{klm} \hat{\eta}^i \hat{\eta}^j \hat{P}_k \hat{P}_l \hat{P}_m + \dots, \tag{2.7.2}$$

hence

$$\begin{aligned} [\hat{K}_{ext}, \hat{\Omega}] = [\hat{\Omega}, \hat{K}_{ext}] = & i\hbar \hat{K}^i \hat{\Phi}_i + \{[\hat{\Phi}_i, \hat{K}^j] + i\hbar C_{ik}{}^j \hat{K}^k + \\ & + 2i\hbar \hat{K}_i{}^{jk} \hat{\Phi}_k + (i\hbar)^2 C_{km}{}^j \hat{K}_i{}^{km}\} \hat{\eta}^i \hat{P}_j + \dots \end{aligned} \tag{2.7.3}$$

Again note that all the quantum constraints appear on the right. The term proportional to $(i\hbar)^2$ has no classical analogue. It arises from the factor re-ordering of higher-order ghost terms. In analogy with the classical case, we require $[\hat{\Omega}, \hat{K}_{ext}]$ to be equivalent to the zero observable, i.e. to have vanishing scalar product when evaluated on physical states,

$$\langle \Psi'_{ph}{}^{ext} | [\hat{\Omega}, \hat{K}_{ext}] | \Psi_{ph}{}^{ext} \rangle_{tot} = 0. \tag{2.7.4}$$

As mentioned earlier, a sufficient condition for this to happen is definition (B2) for physical states together with the (formal) selfadjointness property of the BRST operator, $\hat{\Omega}^\dagger = \hat{\Omega}$. This is meant to mimic the canonical treatment where from $\hat{\Phi}_i | \Psi_{ph} \rangle = 0$ and $\hat{\Phi}_i^\dagger = \hat{\Phi}_i$ it follows immediately that $\langle \Psi'_{ph} | [\hat{K}, \hat{\Phi}_i] | \Psi_{ph} \rangle = 0$, for arbitrary \hat{K} . Analogously, in the case with ghosts we would like all quantities to be defined in such a way that (2.7.4) is fulfilled automatically if we define physical states by (B2).

Let us assume now the existence of a scalar product on \mathcal{H}_{tot} and analyze the selfadjointness condition on $\hat{\Omega}$. From the requirement that

$$\langle \Psi'^{ext} | \hat{\Omega} | \Psi^{ext} \rangle = \langle \Psi'^{ext} | \hat{\Omega}^\dagger | \Psi^{ext} \rangle \quad (2.7.5)$$

should hold for arbitrary wave functions $|\Psi^{ext}\rangle$, $|\Psi'^{ext}\rangle$, it is straightforward to derive a set of conditions on expectation values of canonical and ghost variables, which have as possible solutions (the operators $\hat{\Phi}_i$ are supposed to be self-adjoint with respect to the canonical scalar product on \mathcal{H}):

- 1) $\hat{\eta}^i = \hat{\eta}^{i\dagger}$ and $\hat{\eta}^i \hat{\eta}^j \hat{P}_k = (\hat{\eta}^i \hat{\eta}^j \hat{P}_k)^\dagger$, implying $\hat{P}_i = -\hat{P}_i^\dagger$
- 2) $\hat{\eta}^i = \hat{\eta}^{i\dagger}$ and $|\Psi_i\rangle$, $|\Psi_{ij}\rangle$, $|\Psi_{ijk}\rangle$ etc. $\equiv 0$
- 3) $\hat{\eta}^i = \hat{\eta}^{i\dagger}$ and all products of ghost wave functions with two or more ghosts vanishing, i.e. $\langle 1 | \eta^i \eta^j \rangle_{gh} = 0$, $\langle 1 | \eta^i \eta^j \eta^k \rangle_{gh} = 0$, $\langle \eta^i | \eta^j \eta^k \rangle_{gh} = 0$ etc.
- 4) $\langle 1 | \eta^i \rangle_{gh} = 0$ (implying $\langle \eta^i | 1 \rangle_{gh} = 0$) and $|\Psi_i\rangle$, $|\Psi_{ij}\rangle$, $|\Psi_{ijk}\rangle$ etc. $\equiv 0$
- 5) $\langle 1 | \eta^i \rangle_{gh} = 0$ and all products of ghost wave functions with two or more ghosts vanishing
- 6) solutions which treat different higher-order coefficients $|\Psi_i\rangle$, $|\Psi_{ij}\rangle$, ... differently. For example, if we set $|\Psi_{ij}\rangle \equiv 0$, $|\Psi_{ijk}\rangle \equiv 0$ etc., one solution is given by $\langle 1 | \eta^i \rangle_{gh} = 0$, $\langle 1 | \eta^i \eta^j \rangle_{gh} = 0$, $\langle \eta^i | \eta^j \rangle_{gh} = 0$ and $\langle \eta^i | \eta^j \eta^k \rangle_{gh} = 0$.

It is easy to check that any of these solutions, together with the condition (B2) on physical wave functions, leads to (2.7.4). Note that the requirement of anti-selfadjointness for \hat{P} leads automatically to the vanishing of $\langle 1 | 1 \rangle_{gh}$ since

$$\langle 1 | \hat{P}_i | \eta^j \rangle_{gh} = - \langle 1 | \hat{P}_i^\dagger | \eta^j \rangle_{gh} = 0, \quad (2.7.6)$$

but on the other hand

$$\langle 1 | \hat{P}_i | \eta^j \rangle_{gh} = i\hbar \delta_i^j \langle 1 | 1 \rangle_{gh}. \quad (2.7.7)$$

A similar argument shows that $\hat{P}_i^\dagger = \pm \hat{P}_i$ leads to the vanishing of any scalar product of ghost wave functions containing less than m ghosts, i.e. the only non-zero scalar products are of the form $\langle \eta^{i_1} \dots | \dots | \eta^{i_m} \rangle$, with $[i_1 \dots i_m]$ being some permutation of $[1 \dots m]$.

As stated earlier, we are interested in systems of type b), where it is essential to relate physical and mathematical structures of the reduced and unreduced quantum theory. Therefore we will insist on the condition $\langle 1 | 1 \rangle_{gh} \neq 0$ since only in this case can the scalar product between two extended wave functions, $\langle \Psi'^{ext} | \Psi^{ext} \rangle_{tot}$, reduce to the ordinary scalar product if $|\Psi^{ext}\rangle$ and $|\Psi'^{ext}\rangle$ do not contain any ghost contributions, i.e. if $|\Psi^{ext}\rangle_{tot} = |\Psi\rangle |1\rangle_{gh}$ and $|\Psi'^{ext}\rangle_{tot} = |\Psi'\rangle |1\rangle_{gh}$. This excludes solution 1) from the above list (the one

normally adopted). This is no particular reason for worry since it only reflects the fact that a ghost scalar product with the ensuing hermiticity assignments for ghost operators was introduced to achieve formal resemblance with the usual canonical quantization, and may turn out to be a not very useful structure to work with.

We will now determine under what conditions $\langle \Psi'_{ph} | \hat{K}^i \hat{\Phi}_i | \Psi_{ph} \rangle = 0$ follows from (2.7.4), without making any assumptions about the hermiticity properties of the ghost operators. We also leave open the definition of physical states for the moment. The conditions are

$$\begin{aligned}
 i) & \langle \Psi'_{ph} | \hat{K}^k \hat{\Phi}_k \delta_i^j + ([\hat{\Phi}_i, \hat{K}^j] + i\hbar C_{ik}^j \hat{K}^k + 2i\hbar \hat{K}_i^{jk} \hat{\Phi}_k + \\
 & \qquad \qquad \qquad (i\hbar)^2 C_{km}^j \hat{K}_i^{km}) | \Psi_{ph,j} \rangle \langle 1 | \eta^i \rangle_{gh} = 0 \\
 ii) & \langle \Psi'_{ph,i} | \hat{K}^j \hat{\Phi}_j | \Psi_{ph} \rangle \langle \eta^i | 1 \rangle_{gh} = 0 \\
 iii) & \langle \Psi'_{ph,ik} | \hat{K}^j \hat{\Phi}_j | \Psi_{ph} \rangle \langle \eta^i \eta^k | 1 \rangle_{gh} = 0 \\
 iv) & \langle \Psi'_{ph,n} | \hat{K}^k \hat{\Phi}_k \delta_i^j + ([\hat{\Phi}_i, \hat{K}^j] + i\hbar C_{ik}^j \hat{K}^k + 2i\hbar \hat{K}_i^{jk} \hat{\Phi}_k + \\
 & \qquad \qquad \qquad (i\hbar)^2 C_{km}^j \hat{K}_i^{km}) | \Psi_{ph,j} \rangle \langle \eta^n | \eta^i \rangle_{gh} = 0 \\
 & \text{etc.}
 \end{aligned} \tag{2.7.8}$$

The possible solutions are

- 1) All higher-order coefficients of physical BRST wave functions vanish: $|\Psi_{ph,i}\rangle = 0$, $|\Psi_{ph,ij}\rangle = 0$ etc..
- 2) All ghost scalar products involving one or more η 's vanish: $\langle 1 | \eta^i \rangle_{gh} = 0$, $\langle 1 | \eta^i \eta^j \rangle_{gh} = 0$, $\langle \eta^i | \eta^j \rangle_{gh} = 0$ etc.
- 3) Other solutions which do not treat all higher-order coefficients of $|\Psi_{ph}^{ext}\rangle$ on an equal footing. For example, if we assume that only $|\Psi_{ph}\rangle$ and $|\Psi_{ph,i}\rangle$ are nonvanishing, we obtain the conditions $\langle 1 | \eta^i \rangle_{gh} = 0$ and $\langle \eta^i | \eta^j \rangle_{gh} = 0$.

All these solutions imply that the extended scalar product of two extended wave functions is proportional to the canonical scalar product of their ghost-free parts, i.e. $\langle \Psi'^{ext} | \Psi^{ext} \rangle_{tot} = \langle \Psi' | \Psi \rangle \langle 1 | 1 \rangle_{gh}$. In particular, since in solution 1) the higher-order coefficients of the *physical* BRST wave function vanish anyway, we can without loss of generality represent all extended wave functions by their ghost-free part. In solution 2) these higher-order coefficients may be non-zero, but they do not contribute to any scalar products. The question of selfadjointness becomes trivial, since both $\langle \Psi'^{ext} | \hat{\eta}^i | \Psi^{ext} \rangle$ and $\langle \Psi'^{ext} | \hat{P}_i | \Psi^{ext} \rangle$ vanish identically. Likewise zero-norm states of the form $\hat{\Omega} | \chi^{ext} \rangle$ vanish identically because they do not contain a ghost-free part,

$$\hat{\Omega}|\chi^{ext} \rangle = \hat{\Phi}_i|\chi \rangle + |\eta^i \rangle_{gh} + [\hat{\Phi}_i|\chi_j \rangle - \dots] |\eta^i \eta^j \rangle_{gh} + \dots \quad (2.7.9)$$

We do not obtain any negative-norm states since there are no wave functions $|\Psi^{ext} \rangle$ with $\langle \Psi^{ext} | \hat{\Omega} \chi^{ext} \rangle \neq 0$. The same analysis applies to the solutions 2), 4), 5) and 6) following equation (2.7.5).

The use of the weak BRST condition on physical states, (B1), is in general not compatible with the assumption that operators $[\hat{K}_{ext}, \hat{\Omega}]$ should correspond to the zero observable, since there is no guarantee that their matrix elements, evaluated on physical (BRST) states vanish. This is equivalent to saying that the weak Dirac condition on states, (D1), does not guarantee the vanishing of expressions of the form $\langle \Psi'_{ph} | \hat{K}^i \hat{\Phi}_i | \Psi_{ph} \rangle$. We could therefore have assumed from the very beginning that physical BRST wave functions are defined by the strong condition (B2). In this case some of the equations in (2.7.5) are fulfilled automatically (the left-hand sides of i), ii) and iii) vanish identically and iv) is the first non-trivial condition). The resulting set of solutions is weaker:

- 1') All higher-order coefficients of physical BRST wave functions vanish: $|\Psi_{ph,i} \rangle = 0, |\Psi_{ph,ij} \rangle = 0$ etc. (same as before).
- 2') All ghost scalar products involving *two* or more η 's vanish.
- 3') Other solutions which do not treat all higher-order coefficients of $|\Psi_{ph}^{ext} \rangle$ on an equal footing. For example, if we assume that only $|\Psi_{ph} \rangle$ and $|\Psi_{ph,i} \rangle$ are nonvanishing, we obtain the condition $\langle \eta^i | \eta^j \rangle_{gh} = 0$.

Cases 2') and 3') allow for a slightly generalized scalar product, with non-vanishing contributions from wave functions with one ghost index (the analysis of the following paragraph applies also to solution 3) following (2.7.5)):

$$\begin{aligned} \langle \Psi'^{ext} | \Psi^{ext} \rangle_{tot} = & \langle \Psi' | \Psi \rangle \langle 1 | 1 \rangle_{gh} + \langle \Psi' | \Psi_i \rangle \langle 1 | \eta^i \rangle_{gh} + \\ & + \langle \Psi'_i | \Psi \rangle \langle \eta^i | 1 \rangle_{gh} . \end{aligned} \quad (2.7.10)$$

Hence we may write a general extended wave function as $|\Psi^{ext} \rangle_{tot} = |\Psi \rangle |1 \rangle_{gh} + |\Psi_i \rangle |\eta^i \rangle_{gh}$. Its norm is given by

$$\|\Psi^{ext}\|_{tot}^2 = \|\Psi\|^2 + \langle \Psi | \Psi_i \rangle \langle 1 | \eta^i \rangle_{gh} + \langle \Psi_i | \Psi \rangle \langle \eta^i | 1 \rangle_{gh}, \quad (2.7.11)$$

where we have chosen the normalization $\langle 1 | 1 \rangle_{gh} = 1$. With respect to this scalar product all wave functions of the form $\hat{\Omega}|\chi^{ext} \rangle$ have zero norm, but not necessarily vanishing scalar product with another state $|\Psi^{ext} \rangle$, since

$$\langle \Psi^{ext} | \hat{\Omega} \chi^{ext} \rangle_{tot} = \langle \Psi + \Psi_i \eta^i | \hat{\Phi}_j | \chi \eta^j \rangle_{gh} = \langle \Psi | \hat{\Phi}_j | \chi \rangle \langle 1 | \eta^j \rangle_{gh} . \tag{2.7.12}$$

This vanishes only if $|\Psi\rangle$ is a physical wave function that is annihilated by all the quantum constraints, i.e. the extended ‘‘Hilbert’’ space \mathcal{H}_{tot} contains states with negative norm. However, not only states of the form $\hat{\Omega}|\chi^{ext}\rangle$ have zero norm, but also any state $|\Psi^{ext}\rangle = |\Psi_i \eta^i\rangle$, and their existence leads to negative-norm states as well.

2.8. Negative-norm states.

We must make sure that the final physical Hilbert space does not contain any negative-norm states. In the BRST approach this is usually done in the following way: defining physical wave functions by $\hat{\Omega}|\Psi_{ph}^{ext}\rangle = 0$, all zero-norm states of the form $|\Psi^{ext}\rangle = \hat{\Omega}|\chi^{ext}\rangle$ are physical, but decouple in the sense that their scalar product with any other physical wave function vanishes:

$$\langle \Psi_{ph}^{ext} | \hat{\Omega} | \chi^{ext} \rangle_{tot} = \langle \Psi_{ph}^{ext} | \hat{\Omega}^\dagger | \chi^{ext} \rangle_{tot} = 0 \tag{2.8.1}$$

Even if we ignore the question of hermiticity of the BRST operator for the moment, another problem arises: are all zero-norm states of the form $|\Psi^{ext}\rangle = \hat{\Omega}|\chi^{ext}\rangle$ for some $|\chi^{ext}\rangle$? If they are not, the physical Hilbert space still contains negative-norm states, which is of course unacceptable in any physical theory. In other words, we get into trouble whenever the higher-order BRST quantum cohomology of states is non-trivial (different ghost sectors decouple in this cohomology, hence by ‘‘higher-order’’ we mean wave functions containing at least one ghost, i.e. $|\Psi_i \eta^i\rangle$, $|\Psi_{ij} \eta^i \eta^j\rangle$ etc.).

Exactly the same problem occurs also in the present case, although we have abandoned the condition $\hat{\Omega}^\dagger = \hat{\Omega}$. We can only find a physical interpretation of the theory if the first cohomology of extended wave functions vanishes, i.e. if any m -tuple $\{|\Psi_j\rangle\}$ of wave functions satisfying

$$\hat{\Phi}_{[i} |\Psi_{j]} \rangle = -\frac{1}{2} i\hbar C_{[ij]}^k |\Psi_k \rangle = 0 \tag{2.8.2}$$

can be written as

$$|\Psi_j \rangle = \hat{\Phi}_j |\chi \rangle, \quad j = 1, \dots, m, \tag{2.8.3}$$

for some wave function $|\chi\rangle$. The author has no idea when or whether this happens in the general case. Thus, in the approach we have been following so far, there do not seem to exist compelling reasons for keeping any of the higher-order ghost terms in an extended wave function $|\Psi^{ext}\rangle$.

2.9. Conclusions.

Clearly a quantum observable has to be identified with the ghost-free part \hat{A} of a BRST observable \hat{A}_{ext} . For the group case, defining BRST observables by $[\hat{\Omega}, \hat{A}_{ext}] = 0$ or $[\hat{\Omega}, \hat{A}_{ext}]|\Psi_{ph}^{ext}\rangle = 0$ is equivalent to Dirac’s condition (C2).

However, for more general types of constraints we expect $[\hat{\Omega}, \hat{A}_{ext}] = 0$ to impose stronger restrictions on the \hat{A}_{ext} because the whole series of equations (3.1.4) for the higher-order coefficients A_i^j, A_{ij}^{km} etc. has to be satisfied as operator equations. In this case not every quantum observable \hat{A} according to Dirac's definition (c.f. after (C2)) would have a corresponding BRST-invariant observable \hat{A}_{ext} such that $\hat{A}_{ext} = \hat{A} + more$.

None of the definitions of BRST observable is in general consistent with the weak BRST condition on BRST wave functions. In (O2), for any operator that is a quantum observable in the sense of Dirac, we can construct a BRST-invariant observable according to (2.6.6), in (O3) we can consistently set all higher ghost coefficients to zero, i.e. choose $\hat{A}_{ext} = \hat{A}$. Working with definition (O1), the operators $\hat{A}_i^j, \hat{A}_{ij}^{km}$ etc. are the quantized versions of the classical higher-order coefficients, in (O2) and (O3) they do not have any obvious physical interpretation.

We have seen earlier that in the classical case any linear combination $K^i \Phi_i$ of the constraints is equivalent to the zero observable and that this statement was encoded into the classical BRST symmetry $A_{ext} \rightarrow A_{ext} + \{\Omega, K_{ext}\}$. In order to preserve this "symmetry" in the quantum theory, we had to require that $[\hat{\Omega}, \hat{K}_{ext}]$ vanish between physical states, which imposes severe restrictions on BRST wave functions and the extended scalar product. Also, we must necessarily define physical BRST states by the strong BRST condition, $\hat{\Omega}|\Psi_{ph}^{ext}\rangle = 0$.

Furthermore we showed that the condition of pseudo-selfadjointness for $\hat{\Omega}$ is incompatible with the requirement that for wave functions with ghost number zero the extended scalar product should reduce to the canonical one. This is not surprising since the condition $\hat{\Omega}^\dagger = \hat{\Omega}$ was only introduced to achieve formal resemblance with expressions in the canonical approach. We therefore dropped the selfadjointness condition and are now left with essentially two possibilities of defining the extended scalar product. The first one is given by

$$\langle \Psi'^{ext} | \Psi^{ext} \rangle_{tot} = \langle \Psi' | \Psi \rangle ; \tag{2.9.1}$$

in this case the higher-order coefficients in the extended wave functions have no physical significance whatsoever, and there are zero-norm, but no negative-norm states. In calculating matrix elements of BRST observables \hat{A}_{ext} , (2.6.1), only the ghost-free part \hat{A} gives a contribution to the scalar product,

$$\langle \Psi'^{ext} | \hat{A}_{ext} | \Psi^{ext} \rangle_{tot} = \langle \Psi' | \hat{A} | \Psi \rangle . \tag{2.9.2}$$

The alternative scalar product is

$$\langle \Psi'^{ext} | \Psi^{ext} \rangle_{tot} = \langle \Psi' | \Psi \rangle + \langle \Psi' | \Psi_i \rangle \langle 1 | \eta^i \rangle_{gh} + \langle \Psi'_i | \Psi \rangle \langle \eta^i | 1 \rangle_{gh} . \tag{2.7.10}$$

where we assume that neither $\langle \eta^i | 1 \rangle_{gh}$ nor $\langle 1 | \eta^i \rangle_{gh}$ vanish. For matrix elements of BRST observables (2.6.1) we find

$$\begin{aligned} \langle \Psi'^{ext} | \hat{A}_{ext} | \Psi^{ext} \rangle_{tot} = & \langle \Psi' | \hat{A} | \Psi \rangle + \langle \Psi'_i | \hat{A} | \Psi \rangle \langle \eta^i | 1 \rangle_{gh} + \\ & + \langle \Psi' | \hat{A} | \Psi_i \rangle \langle 1 | \eta^i \rangle_{gh} + i\hbar \langle \Psi' | \hat{A}_i^j | \Psi_j \rangle \langle 1 | \eta^i \rangle_{gh} . \end{aligned} \tag{2.9.3}$$

Since we know that \hat{A}_{ext} represents the canonical quantum operator \hat{A} , we must require that the last three terms on the right-hand side of (2.9.3) vanish, at least on physical wave functions $|\Psi_{ph}^{ext}\rangle$. We analyze the implications for each definition of BRST observable separately:

- 1) Using definition (O1), we cannot set $\hat{A}_i^j \equiv 0$, hence we have to assume that for *physical* wave functions the first-order coefficients $|\Psi_{ph,i}\rangle$ vanish, which effectively leads us back to the scalar product (2.9.1).
- 2) Using definition (O2), the last two terms on the right-hand side of (2.9.3) cancel each other due to $\hat{A}_i^j = -\frac{1}{i\hbar} \hat{A} \delta_i^j$, but we still need $|\Psi_{ph,i}\rangle \equiv 0$, leading back to (2.9.1).
- 3) Using definition (O3), we are free to set \hat{A}_i^j to zero, but again need the condition $|\Psi_{ph,i}\rangle \equiv 0$ to cancel the second and third term in (2.9.3), and the conclusion is the same as before.

We conclude that under the assumptions we have made, without loss of generality the only scalar product one can adopt on the extended ‘‘Hilbert’’ space \mathcal{H}_{tot} is the one given by (2.9.1), which completely ignores the ghost structure of BRST wave functions and observables. In this sense problems concerning negative-norm states and pseudo-hermiticity of operators appear here as mere artefacts of the BRST formalism.

Let us at this point comment on the scalar product proposed by Henneaux in [Hen1]. The discussion there essentially concerns the case of abelian constraints, where classically the constraints are supposed to be given by the first m canonical momenta p_i , $i = 1, \dots, m$, of the original phase space. The extended scalar product involves an integration over the m ghost variables η^i , and the ghost and antighost operators $\hat{\eta}^i$ and \hat{P}_i are supposed to be selfadjoint and anti-selfadjoint respectively. As we have seen above, this leads to the vanishing of the scalar product of two ghost-free functions. It is suggested in [Hen1] that the zeros coming from the ghost integration $\int d\eta^i$ should cancel against the infinities arising from the integration over the first m canonical coordinates, $\int dq^i$, to give unity. However, this assumption cannot be correct in a more general context, since we could equally well have chosen a compact gauge group, in which case the integration over the analogues of the canonical coordinates q would have been finite, but the scalar product of states with zero ghost number would still have vanished. Of course one can always *locally* abelianize the constraints in

the classical theory, but in general this is not true globally and therefore of not much use in the quantum theory [Ish]. Note also that nowhere in our previous discussion have we resorted to the possibility of abelianizing the constraints.

A similar comment concerns the attempt to accommodate a rigged Hilbert space structure in the BRST approach, as has been proposed by Thomi [Tho]. This deals with the problem that for the case of pure momentum constraints the eigenvectors corresponding to the eigenvalue zero of the quantum constraints are not elements of a Hilbert space, but are so-called generalized eigenvectors which are not square-integrable. The main idea put forward in [Tho] is to consider wave function coefficients of all ghost orders and assign them to different parts of the rigged Hilbert space (consisting of test functions, square-integrable functions and distributions) in such a way that all pairings between these coefficients are well defined.

However, it seems to be rather unnatural to treat different “ghost sectors” of extended wave functions in this way. The condition $\hat{\Omega}|\Psi^{ext}\rangle = 0$ never leads to any condition on the highest-order coefficient of $|\Psi^{ext}\rangle$, i.e. the wave functions with m ghost indices, hence their interpretation is unclear. For the case of a non-abelian gauge group, the higher-order coefficients $|\Psi_{ph,i}\rangle$, $|\Psi_{ph,ij}\rangle$ etc. of a physical BRST state obey equivariance rather than invariance conditions, which are of the type of generalized Dirac conditions we discussed in §1. Unfortunately, in our discussion no natural meaning emerged for these higher-order coefficients, and hence for the higher-order BRST cohomology of physical states. On the contrary, when in §2.8 we assumed the first-order coefficients $|\Psi_i\rangle$ to be non-zero, we could only find a physical interpretation for the theory if the first BRST cohomology was trivial, i.e. if every wave function $|\Psi_{ph,i}\rangle = |\eta^i\rangle_{gh}$ was of the form of a BRST coboundary, $\hat{\Omega}|\chi^{ext}\rangle$ (c.f. (2.8.2), (2.8.3)). Therefore it seems sensible to adopt from the outset a zero-ghost number condition on BRST wave functions, $\hat{N}|\Psi^{ext}\rangle = 0$.

The BRST “gauge” symmetry $A_{ext} \rightarrow A_{ext} + \{\Omega, K_{ext}\}$ at the quantum level boils down to the assertion that we can add to any quantum observable \hat{A} an operator of the form $\hat{K}^i \hat{\Phi}_i$, for arbitrary operators \hat{K}^i , as long as we define physical states by the strong Dirac (or, equivalently, BRST) condition. The usefulness of this symmetry and the significance of the BRST quantum cohomology of physical observables is unclear, at least in the present context. Moreover, it excludes the possibility of defining physical states by a weak Dirac (or BRST) condition.

We started by asking how much of the classical canonical BRST symmetry and algebraic structure can be preserved in the quantum theory; the investigations described in this paper suggest the answer: “little”. This final conclusion is in fact quite similar to that reached in [Tuy2], where the BRST quantum theory is discussed within the framework of an extended geometric quantization procedure, and with a much heavier mathematical machinery. Our results also illustrate the observation made in [DET] about the general absence of a quantization of (subsets of) the super-Poisson algebra of extended phase space

functions $C^\infty(\mathcal{S}_{ext}) \simeq C^\infty(\mathcal{S}) \otimes \Lambda(\mathfrak{g} \oplus \mathfrak{g}^*)$, because of the form of the classical super-Poisson bracket,

$$\begin{aligned} \{A(s) \otimes F(\eta, P), B(s) \otimes G(\eta, P)\} = \\ = \{A(s), B(s)\} \otimes F(\eta, P) \wedge G(\eta, P) + A(s)B(s) \otimes \{F(\eta, P), G(\eta, P)\}, \end{aligned} \quad (2.9.4)$$

which makes essential use of the (anti-)commutativity of elements of both $C^\infty(\mathcal{S})$ and $\Lambda(\mathfrak{g} \oplus \mathfrak{g}^*)$. However, these authors (implicitly) are mainly concerned with systems of type a); we have seen that for systems of type b) these observations become even more significant. We would like to emphasize that apart from quantizing the classical BRST and ghost number operator, one wants to quantize as many physical observables as possible. If in the quantum theory only very few quantum observables can be found, one should get seriously worried about the status of the quantization procedure used. The canonical BRST formulation does not seem to simplify the search for observables and does not solve operator ordering problems associated with the Grassmann-even variables. On the contrary, depending on the condition we impose on BRST observables, it may be more restrictive than the usual canonical approach.

These (in a sense negative) conclusions we have reached for the simplest and most regular case of a finite-dimensional Lie algebra of first-class constraints arising from a free symplectic action of a connected Lie group G on phase space, which (at least for systems of type b)) does not provide much indication that the canonical BRST approach will advance our physical understanding of the quantization of more general and physically interesting cases like, for instance, that of an open gauge algebra of constraints.

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