QUANTIZING CANONICAL GRAVITY IN THE REAL DOMAIN

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We advocate an alternative description of canonical gravity in 3+1 dimensions, obtained by using as the basic variable a *real* variant of the usual Ashtekar connection variables on the spatial three-manifold. With this ansatz, no non-trivial reality conditions have to be solved, and the Hamiltonian constraint, though non-polynomial, can be quantized rigorously in a lattice regularization.

1 A fresh look at the classical structure

There are several possibilities of formulating the classical Hamiltonian theory of pure Einstein gravity. The traditional one, proposed by Arnowitt-Deser-Misner, is in terms of a canonical pair (g_{ab}, π^{ab}) of a Riemannian three-metric and its conjugate momentum. Introducing local, rotational SO(3)-degrees of freedom, one obtains a closely related formulation, based on a variable pair (E^a_i, K^i_a), where the (inverse, densitized) three-metric is expressible as a function of the triad E^a_i, $g^{ab} = E^a_i E^{bi}$, and its conjugate momentum is the extrinsic curvature of the three-manifold Σ, with one of its spatial indices converted to an internal one.

Starting from this latter formulation, one may perform a canonical transformation and end up with yet another canonical pair of variables (A^i_a, E^a_i), where A^i_a is now an SO(3)-valued *connection* variable. To be precise, we will be interested in two variants of this approach. In the first case, the generator of the canonical transformation is $i\int \Gamma^i_a E^a_i$, and one obtains the so-called Ashtekar variables [1] (A^{Ash}, E), with $A^{Ash} = \Gamma + iK$, whereas in the second case one uses the generator $\int \Gamma^i_a E^a_i$, and obtains a real version (A^{Re}, E) of the Ashtekar variables, with $A^{Re} = \Gamma + K$. (The connection variables $\Gamma^i_a = \Gamma^i_a(E)$ are defined by the vanishing of the covariant derivatives, $\mathcal{D}_a(\Gamma)E^b_i = 0$.) These may be understood as two special cases of the canonical transformation [2] $E^a_i = E^a_i$, $A^i_a = \Gamma^i_a + \beta K^i_a$, for a non-vanishing constant β. For the corresponding Poisson brackets, one derives $\{A^i_a(x), E^b_j(y)\} = \beta\ \delta^i_j \delta^b_a \delta^3(x,y)$.

In this new formulation, there are seven first-class constraints,

$$\begin{array}{rcl} G_i &:=& \nabla_a E^a_i \equiv \partial_a E^a_i + \epsilon_{ijk} A^j_a E^{ak} = 0, \\ V_a &:=& F^i_{ab} E^b_i \equiv (\partial_a A^i_b - \partial_b A^i_a + \epsilon^{ijk} A_{aj} A_{bk})\ E^b_i = 0, \\ H &:=& \epsilon^{ijk} E^a_i E^b_j (F_{abk}(A) - (\frac{1}{\beta^2}+1)\ R_{abk}(\Gamma)) = 0, \end{array} \qquad (1)$$

the Gauss law constraints G_i (familiar from Yang-Mills theory), the three spatial diffeomorphism constraints V_a and the Hamiltonian constraint H. As indicated, F denotes the curvature of the connection A, and analogously, R the curvature of the connection Γ. Since Γ_a^i is a rather complicated and non-polynomial function of the triads E_i^a, its appearance in (1) spells trouble for the quantum theory. One may circumvent this by choosing (à la Ashtekar) $\beta = \pm i$, resulting in a polynomial form for H and thus all the constraints.

An unusual feature of this choice is that it makes the fundamental variable A_a^i complex. In order to eliminate the unphysical modes this introduces, one has to impose a sufficient number of conditions on the (a priori arbitrary) SO(3,C)-valued connections A_a^i. One way of doing this is to require that $A_a^i + A_a^{i*} = 2\Gamma_a^i(E)$. Conditions of this type are often called "reality conditions". This sounds simple, but one should keep in mind that they define a highly non-linear subspace of the original space of gauge connections and triads.

One may of course keep the classical formulation real, by setting $\beta = \pm 1$, say. This does not affect the functional form of the Gauss law and spatial diffeomorphism constraints, but the Hamiltonian retains a non-polynomial piece. Thus it seems as if one would run into difficulties similar to those occurring in the canonical quantization based on the metric variable pair (g_{ab}, π^{ab}), where the complicated structure of the Hamiltonian has basically led to an impasse. However, some particular features of the connection representation allow one to take a somewhat different stance, as I will explain in the next section.

2 Quantization

In the quantization of Ashtekar gravity, one roughly speaking proceeds as follows. The local SO(3)-gauge symmetry is eliminated by working with explicitly gauge-invariant wave functions $\Psi(A)$, taking the form of Wilson loop functionals $\Psi(\gamma) = \mathrm{Tr}\ P \exp \int_\gamma A$, where γ denotes a closed curve in Σ. Note that the information about A is encoded in such objects in a rather singular way. Since the wave functions are labelled by the "extended objects" γ, they carry a natural action of the diffeomorphism group (by moving the loop argument), which may be used to construct diffeomorphism-invariant wave functions.

In the usual Ashtekar approach, the remaining Hamiltonian constraint is given by $H = \epsilon^{ijk} E_i^a E_j^b F_{abk}$. Following Dirac, it is translated into the quantum theory as an operator condition $\hat{H}\Psi = 0$ on physical states Ψ (subject to operator-ordering problems and an appropriate regularization and renormalization). One robust feature of $\hat{H}$ is the existence of a large class of solutions to $\hat{H}\Psi = 0$ in representations where $\hat{A}$ acts by multiplication. Solutions of this type were found already several years ago[3,4], and are labelled by smooth, nonintersecting loops γ. However, there are two problems associated with them: firstly, they are obtained modulo imposition of quantum analogues of the reality conditions (which have not even been formulated in the loop representation); secondly, there are indications that they lie in a physically trivial sector of "states without volume" (see below). A reasonably large class of solutions overcoming both of these difficulties has not yet been found.

One is therefore led to reconsider the real connection approach, corresponding to the choice $\beta = \pm 1$. As a first step, one brings the non-polynomial terms in the Hamiltonian $H = \epsilon^{ijk} E^a_i E^b_j F_{abk} - H^{\rm pot}$ into the form of polynomials modulo powers of the determinant of the metric [5] (recall $\det g = |\det E|$),

$$H^{\rm pot} = \frac{1}{(\det E)^2} \eta_{acd} \eta_{egh} (E^c_k E^d_l E^g_m E^h_n - 2 E^c_m E^d_n E^g_k E^h_l) E^b_k E^f_m (\nabla_b E^a_l)(\nabla_f E^e_n), \tag{2}$$

where the determinant is a cubic function of the triads,

$$\det E = \frac{1}{6} \eta_{abc} \epsilon^{ijk} E^a_i E^b_j E^c_k. \tag{3}$$

What enables us to quantize (2) in spite of its non-polynomiality is the fact that in the loop representation the classical volume function $\mathcal{V} = \int d^3x \sqrt{|\det E|}$ can be quantized self-adjointly, and a basis of eigenstates of $\hat{\mathcal{V}}$ can be constructed in terms of linear combinations of Wilson loop states. (Note that this has no analogue in the metric representation.) This is true both in the continuum [6] and in a lattice-regularized version of the theory [7]. One consequence of this is that – at least on the lattice – any function of $\det E$ can be quantized exactly by going to a basis of eigenstates and defining it in terms of its eigenvalues. In the case of inverse powers of $\det E$, one of course has to take care that zero-eigenstates do not occur.

At this point the reader should be warned that some crucial differences exist between the fixed-lattice approach where the lattice discretizes space itself, and the continuum formalism where the quantum states are labelled by all possible *embedded* loops [4,8]. In the first case, the spatial diffeomorphism symmetry is destroyed by the discretization, and one still has to take a continuum limit (in which the diffeomorphism symmetry may be restored). By contrast, in the continuum the full diffeomorphism group of Σ is still present, and apparently no additional continuum limit is necessary, because one is considering all possible embedded lattices (supporting loop states) simultaneously. At an intermediate stage, the properties of operators acting on loop states can be similar in both approaches (e.g. the spectrum of $\hat{\mathcal{V}}$ is always discrete), but the details of calculations and interpretation tend to be different.

Let us go back to the discussion of the Hamiltonian. Taking into account the anti-symmetrizations in the numerator of $H^{\rm pot}$, one may suspect that the entire expression (2) is not actually as divergent as the $(\det E)^{-2}$-factor suggests. This expectation is in fact correct: consider the continuum expression

$$\frac{1}{2\sqrt{\det E}}\eta_{abc}\epsilon^{ijk}E^b_j E^c_k := e^i_a. \tag{4}$$

Up to a density factor, the e_{ai} are the inverses of the E^a_i, $E^a_i e_{aj} = \sqrt{\det E}\delta_{ij}$. On the other hand, e^i_a may be expressed as the functional derivative of the total volume with respect to E^a_i [9], $e^i_a(x) = 2\{A^i_a(x), \int d^3x\sqrt{\det E}\}$, which is non-singular even for $\det E = 0$. A similar construction can be performed on the lattice, i.e. one can define a lattice link operator that in the continuum limit to lowest order in the lattice spacing a reduces to e_{ai}. Since $H^{\rm pot}$ contains a fourfold product of the e^i_a, this implies that H can in fact be made well-defined on a large number (or possibly all) of the lattice states.

Given this discretization of the real connection approach, one can now try to solve $\hat{H}\Psi = 0$, without having to implement any reality conditions. (Note that one is really trying to find solutions to this set of equations *to lowest order in* a). To what extent this approach is computationally feasible, and what further approximations may be necessary is currently being explored.

We conclude by remarking that the difference between the metric and connection approaches to quantum gravity emerging from the above discussion is not that in the latter one can altogether avoid the appearance of non-polynomial quantities, but rather that one can bring non-polynomiality into a form where it can be controlled.

References

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