

CGPG-93/9-1
Sept 8, 1993

CHROMODYNAMICS AND GRAVITY AS THEORIES ON LOOP SPACE

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Abstract.

We review some attempts of reformulating both gauge theory and general relativity in terms of holonomy-dependent loop variables. The emphasis lies on exhibiting the underlying mathematical structures, which often are not given due attention in physical applications. An extensive list of references is included.

In memoriam M.K.Polivarov

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1. Introduction

This paper is an introduction to the use of loop variables in gauge theory and gravity, and to some of its underlying mathematical structures. It is a considerably enlarged version of a previous review paper on loop approaches to gauge field theory [Lol7]. Its main mathematical ingredient is the holonomy, i.e. the integral along an open or closed path of a gauge potential in space(-time).

Although this review is restricted to the application of loop methods to Yang-Mills theory and general relativity (in the Ashtekar formulation) in four space-time dimensions, many of the mathematical issues discussed in the initial sections (2-9) are relevant to any theory whose basic configuration variable is a connection one-form, as are, for example, some of the so-called topological field theories. The remaining sections (10-15) deal more specifically with the physical applications. I describe and summarize the status of path- and loop-dependent formulations of Yang-Mills theory and gravity, focussing on a few selected topics.

One of my aims is to illustrate that there is not just one, but many different loop spaces, and that the physical properties of a theory should be taken into consideration when making

a specific choice. Another aim is to try and explain why past attempts of using loop variables have often run into problems, in the hope that this may aid further research in this direction. A large number of references is discussed in the text whose selection however reflects my own bias and does not claim to be complete. For additional references and complementary expositions, the reader may consult the review sections of [Gu, Mig2, Bar].

2. Paths and loops

Given a differentiable, simply connected manifold Σ of dimension d , a *path* in Σ (Fig.1a) is a continuous map w from a closed interval of the real line \mathbb{R} into Σ ,

$$\begin{aligned} w : [s_1, s_2] &\rightarrow \Sigma \\ s &\mapsto w^\mu(s). \end{aligned} \tag{2.1}$$

As such it has the properties of a map between two differentiable manifolds, for example, (C^r) differentiability, piecewise differentiability or non-differentiability, and its tangent vector $\frac{dw}{ds}$ may vanish for some or all parameter values s .

A *loop* in Σ (Fig.1b) is a closed path, by which we shall mean a continuous map γ of the unit interval into Σ ,

$$\begin{aligned} \gamma : [0, 1] &\rightarrow \Sigma \\ s &\mapsto \gamma^\mu(s), \\ \gamma(0) &= \gamma(1). \end{aligned} \tag{2.2}$$

We will be using such closed paths in the construction of gauge-invariant quantities in pure gauge theory. Open paths play an important role in gauge theory with fermions, where natural gauge-invariant objects are open flux lines with quarks or Higgs fields “glued to the endpoints” [KogSus, GaGiTr, ForGam, GamSet].

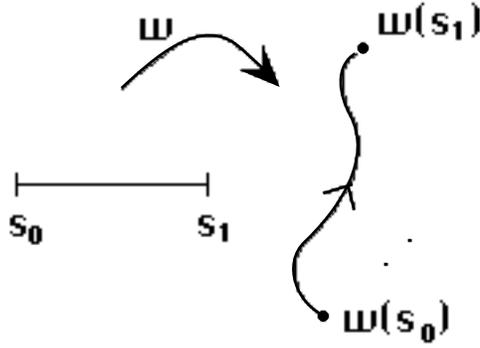


Fig.1a

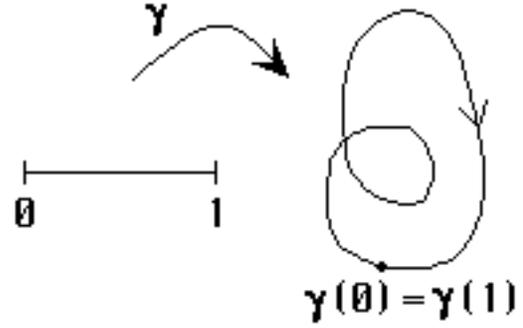


Fig.1b

The manifold Σ may be the real vector space \mathbb{R}^d , possibly with a Euclidean or Minkowskian metric, but may also be non-linear and topologically non-trivial. In this case the possibility arises of having non-contractible loops, i.e. maps γ that cannot be continuously shrunk to a *point loop*

$$p_x(s) = x \in \Sigma, \quad \forall s \in [0, 1]. \quad (2.3)$$

Note that even if a path is closed it has a distinguished image point, namely, its initial and end point, $\gamma(0) = \gamma(1)$, and that each point in the image of γ is labelled by one or more (if there are selfintersections) parameter values s .

3. Holonomy

Suppose in an open neighbourhood V of Σ we are given a configuration $A \in \mathcal{A}$, the set of all *gauge potentials* $\Sigma \rightarrow \Lambda^1 \mathfrak{g}$, i.e. a smooth \mathfrak{g} -valued connection one-form, with \mathfrak{g} denoting the Lie algebra of a finite-dimensional Lie group G . We have

$$A(x) = A_\mu(x) dx^\mu = A_\mu^a(x) X_a dx^\mu, \quad (3.1)$$

where X_a are the algebra generators in the fundamental representation of \mathfrak{g} ($a = 1 \dots \dim G$) and x^μ , $\mu = 1 \dots d$, a set of local coordinates on V .

The *holonomy* U_w of a path $w^\mu(s)$ with initial point s_0 and endpoint s_1 (whose image is completely contained in V) is the solution of the system of differential equations

$$\frac{dU_w(s, s_0)}{ds} = A_\mu(x) \frac{dw^\mu}{ds} U_w(s, s_0), \quad s_0 \leq s \leq s_1, \quad (3.2)$$

with $x = w(s)$, subject to the initial condition

$$U_w(s_0, s_0) = e, \quad (3.3)$$

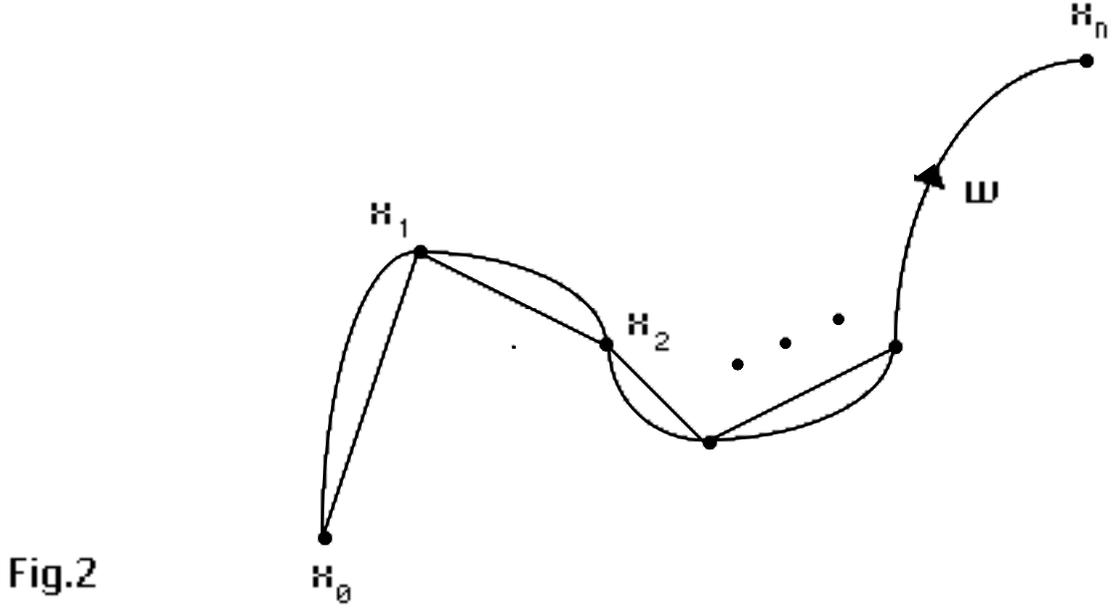
where e denotes the unit element in G . Note that this definition only makes sense for at least piecewise differentiable paths w . The solution of (3.2) is given by the *path-ordered exponential of A along w* ,

$$\begin{aligned} U_w(s_1, s_0) &= \text{P exp } g \int_{s_0}^{s_1} A_\mu(w(t)) \dot{w}^\mu(t) dt := \\ &:= \mathbb{1} + \sum_{n=1}^{\infty} g^n \int_{s_0}^{s_1} dt_1 \int_{t_1}^{s_1} dt_2 \dots \int_{t_{n-1}}^{s_1} dt_n A_{\mu_n}(w(t_n)) \dots A_{\mu_1}(w(t_1)) \dot{w}^{\mu_n}(t_n) \dots \dot{w}^{\mu_1}(t_1). \end{aligned} \quad (3.4)$$

The coupling constant g is necessary to render the argument of the exponential dimensionless. Note that from (3.4) follows the composition law $U_w(s_2, s_0) = U_w(s_2, s_1)U_w(s_1, s_0)$, for $s_0 \leq s_1 \leq s_2$. An alternative definition for U_w that does not need differentiability of the path and employs an approximation of w by n straight line segments ($x_i - x_{i-1}$) is as the limit

$$U_w(s_1, s_0) = \lim_{n \rightarrow \infty} (1 + A(x_n)(x_n - x_{n-1}))(1 + A(x_{n-1})(x_{n-1} - x_{n-2})) \dots (1 + A(x_1)(x_1 - x_0)), \quad (3.5)$$

with $\sup \|x_i - x_{i-1}\| \rightarrow 0$ as n increases [CorHas], and where $x_0 = w(s_0)$, $x_1, \dots, x_n = w(s_1)$ is a set of $n + 1$ points ordered along the path w (Fig.2).



The holonomy U_w takes its values in G and transforms under gauge transformations (smooth functions $g : V \rightarrow G$) according to

$$U_w(s_1, s_0) \xrightarrow{g} g^{-1}(w(s_1))U_w(s_1, s_0)g(w(s_0)). \quad (3.6)$$

Note this would not be true if we had allowed for discontinuities of the path w . The corresponding change of the gauge potential is straightforwardly computed from equation (3.2),

$$A_\mu(x) \xrightarrow{g} g^{-1}(x)A_\mu(x)g(x) - \frac{dg^{-1}(x)}{dx^\mu}g(x) = g^{-1}(x)A_\mu(x)g(x) + g^{-1}(x)\frac{dg(x)}{dx^\mu}. \quad (3.7)$$

Another property of $U_w(s_1, s_0)$ following from (3.2) is its invariance under smooth orientation-preserving reparametrizations f of w , i.e.

$$w(s) = w'(f(s)) \implies U_w(s_1, s_0) = U_{w'}(f(s_1), f(s_0)), \quad (3.8)$$

where $\frac{df}{ds} > 0, \forall s$. The term “non-integrable” (i.e. path-dependent) “phase factor” for U_w was introduced by Yang [Yan], in generalization of its abelian version for $U(1)$ -electromagnetism. It is a well-known result in mathematics that the connection A can (up to gauge transformations) be reconstructed from the knowledge of the holonomies of the *closed* curves based at

a point $x_0 \in \Sigma$ (see [KobNom, Lic] for details on the concept of holonomy group and related issues). The reader may consult the paper by Barrett [Bar] for an account of the mathematical development of these so-called reconstruction theorems, and for more references. The details of such theorems depend on the mathematical setting, for instance whether the underlying fibre bundle is differentiable or just topological. Also generalizations of the concept of connection are possible.

The above description is valid only in a local neighbourhood V of Σ . The appropriate global description is afforded by the mathematical theory of the principal fibre bundles $P : G \rightarrow P \rightarrow \Sigma$ over the base manifold Σ with typical fibre G , with A^P a connection one-form on P . Typically P has no global cross sections (this depends both on the group G and the manifold Σ), and then A^P can be identified with the one-form (3.1) only in a coordinate patch $V \subset \Sigma$, using a local cross section. The global implications of this fact are well known (see, for example, the discussion in [ChoDeW]) and will not be addressed in this paper. Suffice it to remark that the holonomy variables are global in the sense that because of their gauge-invariance they do not “see” non-trivial transition functions between coordinate patches on the manifold Σ . For more comments on global issues, see for example [Bar, Fis, HoScTs, Gro]. However, there are important examples where global cross sections do exist (for instance, $G = SU(2)$ and any three-dimensional Σ) and hence all results described here are valid globally.

Note that for $F = 0$ (also called the case of a “flat connection”), where F is the field strength tensor,

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + ig[A_\mu(x), A_\nu(x)], \quad (3.9)$$

U_w depends only on the homotopy of the path w .

4. Wilson loops

In this section we will be working exclusively with loops, i.e. closed paths γ in Σ , and their associated holonomies,

$$U_\gamma = \text{P exp} \oint_\gamma A_\mu(x) dx^\mu. \quad (4.1)$$

For a loop γ based at a point $x_0 \in \Sigma$ there is a natural way of constructing a gauge-invariant quantity, namely,

$$T_A(\gamma) := \frac{1}{N} \text{Tr } U_\gamma = \frac{1}{N} \text{Tr } \text{P exp } \oint_\gamma A_\mu(x) dx^\mu, \quad (4.2)$$

the *traced holonomy* or so-called *Wilson loop* (introduced by Wilson [Wil] as an indicator of confining behaviour in lattice gauge theory). The trace is taken in a linear representation R of the group G , and the normalization factor in front depends on the dimension N of R . Under a gauge transformation, U_{γ, x_0} transforms homogeneously according to (3.5), but the matrices $g(x_0)$ and $g^{-1}(x_0)$ cancel each other upon taking the trace. Because of the cyclicity of the trace $T_A(\gamma)$ is independent of the choice of base point $x_0 = \gamma(0) = \gamma(1)$ of γ . Note that $T_A(\gamma)$ is invariant under orientation-preserving reparametrizations of γ and is therefore a function on *unparametrized loops*, i.e. reparametrization equivalence classes $\bar{\gamma}$, where any two members of $\bar{\gamma}$ are related by a smooth orientation-preserving bijection $b : S^1 \mapsto S^1$, $\frac{db}{ds} > 0$. Hence an unparametrized loop can be thought of as a continuous ordered set of points in Σ , with a (positive or negative) orientation assigned to each point in a continuous manner (as long as γ does not have any selfintersections or is a point loop (2.3)).

An unparametrized loop possesses certain (differential) topological properties (i.e. invariants under smooth diffeomorphisms of Σ), such as its number of selfintersection points, number of points of non-differentiability (“kinks”), its winding number and its knot class [Kau1,2]. This feature is important for the application of loop methods to gravitational theories, as will be explained in more detail below. In concrete calculations involving an unparametrized loop $\bar{\gamma}$ one usually works with a chosen member γ from the equivalence class $\bar{\gamma}$ and then ensures the end result is independent of this choice. There have also been attempts to set up a loop calculus that is intrinsically reparametrization-invariant [GamTri3,4, Mig2, Gam, Tav].

5. The space \mathcal{A}/\mathcal{G}

There is a more global, geometric way of looking at the setting described in the last section. We have assumed that the configuration space of the theories under consideration is a space \mathcal{A} of \mathfrak{g} -valued connection one-forms, on which the infinite-dimensional group \mathcal{G} of local gauge transformations acts according to (3.7). The physical configuration space therefore consists of the orbits in \mathcal{A} under this gauge group action, where different elements $A(x)$ of the same orbit are gauge-related and hence physically indistinguishable. This quotient space

is usually denoted by \mathcal{A}/\mathcal{G} , but more precisely it may have two different meanings. The first one is the “covariant quotient” where \mathcal{A} consists of space-time connections, and the group \mathcal{G} of local gauge transformations on space-time, the other one is the quotient that comes up in the Hamiltonian formulation (in the $A_0 = 0$ -gauge), where both the connections and the gauge transformations are now purely spatial.

In any case, the quotient \mathcal{A}/\mathcal{G} of Yang-Mills theory is a non-linear space with a non-trivial topology, and many of its properties as an infinite-dimensional differentiable manifold are well known. (Strictly speaking, one has to remove some singular points from \mathcal{A} (those where the connection is reducible), to make the quotient space into a manifold.) One of the main problems of gauge theory has been to find a suitable set of variables to describe this quotient space and the physical dynamics in an efficient way, and which at the same time could serve as a starting point for a non-perturbative quantization scheme. In order to get rid of at least part of the non-trivial transformation behaviour of the gauge potentials A , the non-local holonomy-dependent variables discussed here are a natural choice.

However, it has to be remembered that if one starts from some linear space of loop or path functions, there have to be non-linear constraints if a subset of these functions is to serve as “coordinates” (in the strict mathematical sense) on the space \mathcal{A}/\mathcal{G} . In fact, the essence of these non-local reformulations is to find necessary and sufficient conditions on the elements of such a function space which ensure they are in one-to-one correspondence with the equivalence classes $[A] \in \mathcal{A}/\mathcal{G}$. Usually this goes under the name of “reconstruction theorems”, which were already mentioned earlier, and will be discussed in more detail in Sec.9.

In the application to the Ashtekar formulation of canonical gravity in 2+1 and 3+1 dimensions, it does not suffice to form the quotient of the relevant spaces of connections \mathcal{A} with respect to the groups of local $SO(2,1)$ - and $SU(2)_{\mathfrak{g}}$ -gauge rotations respectively. One still has to factor out by the diffeomorphisms, which also correspond to gauge symmetries, i.e. the gauge equivalence classes $[A]$ do *not* correspond to observables in general relativity. Nevertheless loop-dependent variables $f(\gamma)$ have turned out to be useful in this context, because they carry a non-trivial representation of the diffeomorphism group $Diff(\Sigma)$ of the spatial manifold Σ , namely,

$$(\phi f)(\gamma) := f(\phi \circ \gamma), \quad \forall \phi \in Diff(\Sigma), \quad (5.1)$$

with \circ denoting the composition of maps. Invariance under spatial diffeomorphisms is therefore expressed by the condition

$$\phi f = f, \quad \forall \phi \in \text{Diff}(\Sigma). \quad (5.2)$$

Similar conditions are often adopted to express the diffeomorphism invariance of loop-dependent quantum operators. - There is yet another new feature that occurs in the application to gravity: since the gauge group $SU(2)_{\mathbb{C}} = SL(2, \mathbb{C})$ (and also $SU(1, 1)$) is non-compact, the quotient \mathcal{A}/\mathcal{G} is in fact non-Hausdorff, i.e. its singularities are more ill-behaved than in the compact case. The implications of this result are discussed in [AshLew1].

In case one considers the set of loops based at a point x_0 in a manifold Σ , the gauge dependence of their associated holonomies (4.1) is reduced from the infinite-dimensional group \mathcal{G} to the finite-dimensional group of gauge transformations G at the point x_0 . This is sometimes rephrased by saying that the set of holonomies based at a point x_0 is invariant under the restricted gauge group \mathcal{G}_{x_0} , consisting of those elements of \mathcal{G} which act trivially at the base point. By contrast, the Wilson loops (4.2) are invariant with respect to the entire group \mathcal{G} , and therefore are functions on the quotient space \mathcal{A}/\mathcal{G} .

Alternative descriptions for the space \mathcal{A}/\mathcal{G} or an appropriate generalization thereof are also of interest in the quantum theory, where this space appears as the integration domain in the “sum over all configurations” of the path integral approach and the scalar product of the Hamiltonian approach (see [Lol6] for an introduction). An attempt to use an abelian C^* -algebra generated by the Wilson loop variables for this purpose is described in [AshIsh1], and developed further in [AshLew2, Bae].

6. The space of all loops and its topology

Since it will be of relevance to the field theoretic application later, let us try to give a mathematically meaningful definition of the “space of all loops”. In mathematics, the *loop space* $\Omega_{x_0} X$ associated with some topological space X with distinguished base point x_0 is usually taken to be the function space

$$\Omega_{x_0} X = (X, x_0, x_0)^{(1,0,1)} \quad (6.1)$$

of continuous functions $\gamma : I \mapsto X$ from the unit interval $I = [0, 1]$ to X such that $\gamma(0) = \gamma(1) = x_0$ [Ada]. In most of the following, we will omit the explicit reference to the base point x_0 .

In order to have a notion of convergence for sequences of points in ΩX (i.e. sequences γ_i of functions, $i = 0, 1, 2, \dots$) and a notion of continuity for functions $f : \Omega X \rightarrow Y$ into some space Y , one has to give ΩX a topological structure. A standard choice [Ada, Mic] is the compact-open topology in which open sets are given by subsets of ΩX of the form

$$\Phi(J, O) := \{\gamma \mid \gamma(J) \subset O, J \text{ a closed interval in } [0, 1], O \text{ an open set in } X\}, \quad (6.2)$$

together with their unions and finite intersections [Lip]. It is straightforward to show that with this topology ΩX is Hausdorff (if we assume X to be Hausdorff), i.e. distinct points in ΩX possess disjoint neighbourhoods. However, as was pointed out by Barrett [Bar], holonomy mappings are not in general continuous in the compact-open topology, which therefore seems not suitable for physical applications where holonomies play a central role. He also emphasizes the role that ought to be played by physical measurements in the choice of a topology. Unfortunately, neither for non-abelian gauge theory nor for gravity this is particularly straightforward. It does however seem reasonable to make a choice in which gauge-invariant observables rather than gauge-covariant quantities are continuous. To this end one may employ a well-known construction, namely to induce a topology on loop space by demanding that a given set of functions on it be continuous. Barrett suggests to use the group-valued holonomies as a preferred set of functions. In this context a function $f : \Omega X \rightarrow \mathbb{R}$ is continuous if for every one-parameter family γ_t of loops the composition map $f(\gamma_t)$ is a continuous function of t . The same induced topology is used by Lewandowski [Lew]. Similarly, one may demand that the complex-valued Wilson loop variables (4.2) be continuous.

There is another possibility for defining a topology on loop space, if one uses a Riemannian metric g on the underlying manifold Σ . It is induced by the distance d on loop space, where

$$d(\gamma_1, \gamma_2) = \inf_S Area(S), \quad (6.3)$$

and S runs through all two-surfaces that are bounded by the loop $\gamma_1 \circ \gamma_2^{-1}$. This gives rise to a well-defined metric on loop space (see [DurLei] for more details). It is a natural definition since it takes into account the thin equivalence of loops (cf. Sec.8) in that two thinly equivalent loops have vanishing distance d .

Note that ΩX is not in general a vector space since we cannot add elements of ΩX . However, if X is a linear space and if we choose $x_0 = 0$, we can obtain a linear structure on ΩX by defining addition and scalar multiplication pointwise:

$$\begin{aligned}
(\gamma_1 + \gamma_2)(s) &:= \gamma_1(s) + \gamma_2(s) \\
(a\gamma)(s) &:= a \cdot \gamma(s), \quad a \in \mathbb{R}(\mathbb{C}),
\end{aligned}
\tag{6.4}$$

(the dot denoting scalar multiplication in X), which because of the continuity of these operations makes ΩX into a topological vector space. If X is a differential manifold Σ and we consider $\Omega\Sigma$ as consisting only of differentiable maps γ , $\Omega\Sigma$ can be made into an infinite-dimensional differential manifold (see [Mic] for a discussion of natural topologies on spaces of continuous and differentiable mappings). The finest topology one can impose on a loop space is the discrete topology, in which any element of $\Omega\Sigma$ is an open set. It has been used by Ashtekar and Isham in their treatment of representations of loop algebras [AshIsh1, Ish].

If for some reason one does not want to distinguish a base point in X , a natural loop space to work with is

$$\mathcal{L}X := \bigcup_{x \in X} \Omega_x X,
\tag{6.5}$$

with $\Omega_x X$ as defined in (6.1). - In physical applications one is usually interested in subspaces or quotient spaces of ΩX or $\mathcal{L}X$. Examples of the former are restrictions to the sets of loops without selfintersections, loops without kinks, or contractible loops. Typical quotient spaces are those of loops modulo orientation-preserving reparametrizations, or loops modulo constant translations as employed by Mensky [Men1,2,3] for the special case of $X = \mathbb{R}^n$. In these cases one has to check which properties of the original loop space (and the physical dynamics defined in terms of ΩX or $\mathcal{L}X$) are compatible with the restriction or the projection to the quotient space, respectively.

7. Structures on loop space

Let me emphasize that assigning well-defined mathematical properties to loop space is not a superfluous luxury but necessary if one wants to set up a meaningful differential calculus on ΩX (cf. Sec.12 below). From a physical point of view it is of eminent importance to have additional (topological, algebraic, ...) structure defined on ΩX , which is preserved (or approximately preserved) in the quantum theory.

A loop space ΩX is “better” than an arbitrary infinite-dimensional topological space because its elements can be composed, with the product map given by

$$(\gamma_1 \circ \gamma_2)(s) = \begin{cases} \gamma_1(2s), & 0 \leq s \leq \frac{1}{2} \\ \gamma_2(2s-1) & \frac{1}{2} \leq s \leq 1. \end{cases} \quad (7.1)$$

This product is neither commutative nor associative, and we have neither a unit nor inverse loops in ΩX . (If we work with the loop space $\mathcal{L}X$, then composition is only defined if the endpoint of γ_1 coincides with the initial point of γ_2 .) Taking the inverse γ^{-1} of a loop γ to be

$$\gamma^{-1}(s) := \gamma(1-s), \quad \forall s, \quad (7.2)$$

defines an involution on ΩX since $(\gamma^{-1})^{-1} = \gamma$, and $(\gamma_1 \circ \gamma_2)^{-1} = \gamma_2^{-1} \circ \gamma_1^{-1}$. Note that modifying loop space to

$$\bar{\Omega}_{x_0} X := \bigcup_{\mathbf{I}} (X, x_0, x_0)^{([0,b],0,b)}, \quad (7.3)$$

with the union extending over all closed intervals $\mathbf{I} = [0, b]$, $b \geq 0$, and the product map defined by

$$(\gamma_1 \circ \gamma_2)(s) = \begin{cases} \gamma_1(s) & 0 \leq s \leq b_1 \\ \gamma_2(s) & b_1 \leq s \leq b_1 + b_2, \end{cases} \quad (7.4)$$

which is associative (though not commutative), and with the unit given by the trivial loop $\gamma_0 : [0, 0] \mapsto x_0$, $\bar{\Omega}X$ becomes a topological semigroup (see [Sta] for more comments on associativity, in the context of string theory).

One way ΩX may inherit structure is from X , for example, from a Riemannian metric on X (see below). Another case we will be concerned with is when a quotient space $\Omega X / \sim$ acquires an algebraic structure via certain functions defined on $\Omega \Sigma$, as for example the Wilson loop function (see Sec.8).

There is a limited amount of rigorous mathematical results on spaces of unparametrized loops, some of which will be summarized in the following. Both Brylinski [Bry] and Schäper [Sch1,2] describe the space of unparametrized loops as the base space of a principal $Diff^+(S^1)$ -bundle. Since the quotient space $\mathcal{L}\Sigma / Diff^+(S^1)$ is singular, they choose as starting point the subset $X = E(S^1, \Sigma) \subset \mathcal{L}\Sigma$ of smooth embeddings of the circle into the Riemannian

manifold Σ . (For $\dim \Sigma \geq 3$, X is a dense open subspace of $C^\infty(S^1, \Sigma)$, consisting of those elements $\gamma \in C^\infty(S^1, \Sigma)$ for which $s \neq t \pmod{1}$ implies $\gamma(s) \neq \gamma(t)$. This excludes all loops with self-intersections and retracings, and the trivial point loops.) Equipping it with the usual C^∞ -topology, one thus obtains a Fréchet principal fibre bundle $X(Y, Diff^+(S^1))$ (Y denoting the quotient manifold $Y = E(S^1, \Sigma)/Diff^+(S^1)$), where X and Y are modelled on the spaces $C^\infty(S^1, \mathbb{R}^d)$ and $C^\infty(S^1, \mathbb{R}^{d-1})$, respectively.

Moreover, this construction extends to the larger manifold \hat{X} of singular loops (of which X is a dense open subspace), defined as the immersions $S^1 \rightarrow \Sigma$ with finitely many self-intersections and tangency points, each of finite order [Bry], and there is an analogous principal fibre bundle $\hat{X}(\hat{Y}, Diff^+(S^1))$. For $\dim \Sigma = 3$, and given a Riemannian metric g on Σ , there is a natural almost-complex structure J on \hat{Y} . If v is a tangent vector to \hat{Y} , and \vec{e} the unit tangent vector to the loop $\gamma \in \hat{X}$, then for every point p of the corresponding $\tilde{\gamma} \in \hat{Y}$, define

$$J \cdot v(p) := \vec{e} \times v(p). \quad (7.5)$$

There is also a natural weak symplectic structure on \hat{Y} , induced from the $Diff^+(S^1)$ -invariant two-form

$$\beta_\gamma(u, v) = \int_0^1 \nu_{\gamma(s)}(\dot{\gamma}(s), u(s), v(s)) ds \quad (7.6)$$

on $\mathcal{L}\Sigma$, where ν denotes a nowhere vanishing volume form on Σ . The corresponding form β on \hat{Y} is non-degenerate in the weak sense that it induces an injection on $T_\gamma \hat{Y} \rightarrow T_\gamma^* \hat{Y}$ with dense image. Hence the Poisson brackets are not defined for all smooth functions, but only on the subset of so-called super-smooth functions. Finally, there is a natural Riemannian metric structure on \hat{Y} , given by

$$G(v, w) = \int_0^1 g(v(s), w(s)) \|\dot{\gamma}(s)\| ds. \quad (7.7)$$

where $\|\dot{\gamma}(s)\|$ denotes the norm of the tangent vector to the loop γ at s . Its relation with the symplectic and complex structures is $G(v, w) = \beta(v, J \cdot w)$.

Schäper calculates the Levi-Civita connection, the Riemannian curvature tensor and the sectional curvature on the Riemannian manifold (X, G) . Furthermore, the metric G gives rise

to a natural connection one-form on the principal bundle $X \rightarrow Y$. Some explicit solutions for horizontal geodesics (corresponding to trajectories of free motion of unparametrized loops) on X are given for the special case of $\Sigma = \mathbb{R}^3$ [Sch1]. Note that this construction is not $Diff(\Sigma)$ -invariant, because it makes explicit use of the metric g on Σ .

Slightly different settings are explored in two papers by Fulp. In the first one [Ful1], a smooth action of the Fréchet Lie group $Diff(I)$ on the space $\mathcal{P}^*\Sigma$ of non-degenerate paths (elements γ of $\mathcal{P}\Sigma = C^\infty(I, \Sigma)$ with nowhere vanishing tangent vector $\dot{\gamma}(s)$) on a manifold Σ is defined such that $\mathcal{P}^*\Sigma/Diff(I)$ inherits a Fréchet manifold structure. Smooth vector fields and one-forms are defined as smooth sections of the bundles $T(\mathcal{P}^*\Sigma) \rightarrow \mathcal{P}^*\Sigma$ and $T^*(\mathcal{P}^*\Sigma) \rightarrow \mathcal{P}^*\Sigma$, respectively. It is then proven that, given a smooth $(k+1)$ -form $\tilde{\alpha}$ on Σ , there is a smooth differential form α on $\mathcal{P}^*\Sigma$ defined by

$$\alpha(\delta_1, \dots, \delta_k) := \int_0^1 \tilde{\alpha}_{\gamma(t)}(\dot{\gamma}(t), \delta_1(t), \dots, \delta_k(t)), \quad (7.8)$$

where $\delta_i \in T_\gamma(\mathcal{P}^*\Sigma)$. These so-called path forms have appeared (in a more formal treatment) in the physics literature, for example, in [CoqPil]. It is not clear how they relate to general differential forms on $\mathcal{P}^*\Sigma$. Explicit formulae for the exterior path space derivative of certain path forms are given, for example, that of the path-ordered exponential $\mathcal{I} : \mathcal{P}^*P \rightarrow G$, $\mathcal{I}(\gamma) = P \exp \int_\gamma \omega$.

Finally, Fulp derives an identity, to be thought of as “the analogue of the Yang-Mills operator on loop space”, essentially equating the integral of the classical Yang-Mills equations along a closed path in P with a certain differential operator on loop space, linear in α , where α is the pullback $\mathcal{I}^*(\theta)$ to \mathcal{P}^*P of the Maurer-Cartan form θ on G . It would be interesting to see if this expression can serve as a starting point of a classical reformulation of Yang-Mills theory on loop space.

In a second paper [Ful2], Fulp investigates natural structures on the principal bundle $\mathcal{P}P(\mathcal{P}\Sigma, C^\infty(I, G))$, the path space of a principal bundle $P(\Sigma, G)$, and various subspaces of $\mathcal{P}P$ in a Fréchet manifold setting. He mainly deals with the submanifold $\mathcal{P}_{u_0}P \subset \mathcal{P}P$ of paths with a fixed base point $u_0 \in P$. It is shown that the corresponding principal bundle $\mathcal{P}_{u_0}P(\mathcal{P}_{\pi(u_0)}\Sigma, C_0^\infty(I, G))$ is trivial. The Fréchet Lie group $C_0^\infty(I, G)$ is the subgroup of all elements $a \in C^\infty(I, G)$ such that $a(0) = e$, with e denoting the unit in G .

Each connection ω on P gives rise to a trivialization $F : \mathcal{P}_{u_0}P \rightarrow \mathcal{P}_{\pi(u_0)}\Sigma \times C_0^\infty(I, G)$ defined by $F(\gamma_P) = (\hat{\pi}(\gamma_P), a)$, where a is the unique element of $C_0^\infty(I, G)$ such that $\gamma_P =$

$s_\omega(\hat{\pi}(\gamma_P)) \cdot a$, with s_ω denoting the unique ω -horizontal lift of a path in $\mathcal{P}_{\pi(u_0)}\Sigma$ to $\mathcal{P}_{u_0}P$. Explicitly, we have

$$a(t) = \text{P exp} \left(\int_0^t \omega(\dot{\gamma}_P(s)) ds \right). \quad (7.9)$$

This path-ordered exponential of the connection form along arbitrary paths in P (not just along the horizontal paths in P usually considered) is an important ingredient in Fulp's work.

Then two types of connections on path space are introduced. Firstly, any connection $\omega : TP \rightarrow \mathfrak{g}$ on P induces a connection $\hat{\omega} : T(\mathcal{P}P)|_{\gamma_P} \rightarrow C^\infty(I, \mathfrak{g})$ "pointwise". Secondly, one can use a construction analogous to that of the canonical flat connection on the trivial bundle $P(\Sigma, G)$ (namely as the pullback to P of the Maurer-Cartan form θ on the fibre, with respect to the fibre projection $P \rightarrow G$ [KobNom]) to obtain flat connections on loop space. The relation with Polyakov's work [Pol2], who also introduced flat connections on path space, remains somewhat elusive, which indicates that it may be hard to make it well-defined in the rigorous mathematical framework presented here. Also the eventual use of the path space $\mathcal{P}P$ in gauge theory may be limited, since its bundle structure is not compatible with parametrization invariance.

Also the paper by Gross [Gro] aims at providing a mathematical framework for some gauge theoretical results obtained by physicists. He chooses to work with the space $\mathcal{P}\mathbb{R}^n$ of piecewise smooth paths in \mathbb{R}^n , based at the origin (the generalization to general manifolds of some of the results is discussed briefly). This path space is a normed vector space with respect to the norm

$$\|\gamma\| = \int_0^1 |\dot{\gamma}(t)| dt \quad (7.10)$$

Next, path forms are introduced, which differ from the ones discussed above in that they are k -forms on \mathbb{R}^n attached to the endpoints $\gamma(1)$ of paths γ from $\mathcal{P}\mathbb{R}^n$. Gross' central point of interest are path two-forms h with values in the Lie algebra \mathfrak{g} , i.e. for two vectors \vec{u} and \vec{v} in \mathbb{R}^n , and $\gamma \in \mathcal{P}\mathbb{R}^n$, $h(\gamma) \langle \vec{u}, \vec{v} \rangle \in \mathfrak{g}$. A particular case of such two-forms are the so-called lasso forms $L(\gamma)$ associated with C^∞ \mathfrak{g} -valued one-forms A on \mathbb{R}^n according to

$$L(\gamma) \langle \vec{u}, \vec{v} \rangle = U_\gamma^{-1} F(\gamma(1)) \langle \vec{u}, \vec{v} \rangle U_\gamma \quad (7.11)$$

where F is the field strength of A . It is argued that these are good variables for Yang-Mills theory, and it is therefore important to characterize the non-linear embedding of the subspace of lasso forms in the space of all smooth path two-forms. Necessary and sufficient conditions are given under which a general $h(\gamma)$ is a lasso form. Furthermore, Gross derives a generalization of Stokes' theorem, i.e. a non-abelian analogue of $\int_{\partial S} A = \int_S F$ in terms of path variables (other references on the non-abelian Stokes' theorem include [Are2, Bra, FiGaKa]). Yang-Mills equations on path space first obtained by Bialynicki-Birula, Mandelstam and others (cf. Sec.11 below) are rederived and discussed. In fact, the lasso forms (7.11) are closely related with the variables (11.3). Gross' paper contains many useful references to both the mathematics and physics literature on the subject.

8. Identities satisfied by the (traced) holonomies

We now give some more properties of the untraced and traced holonomies, which follow from their definitions (3.4) and (4.2). Take G to be the gauge group $GL(N, \mathbb{C})$ or one of its subgroups ($U(N)$, $SU(N)$, $SO(N)$, etc.) in its fundamental representation in terms of $N \times N$ (complex) matrices. For two loops γ_1 and γ_2 based at x , we have

$$U_{\gamma_1 \circ \gamma_2}(A) = U_{\gamma_2}(A) \cdot U_{\gamma_1}(A), \quad \forall A, \quad (8.1)$$

with the loop product as defined in (7.1), the dot denoting matrix multiplication and U_γ defined by (4.1). In other words, the mapping U is compatible with the product structure on loop space. For the inverse loop γ^{-1} of γ , (7.2), we have

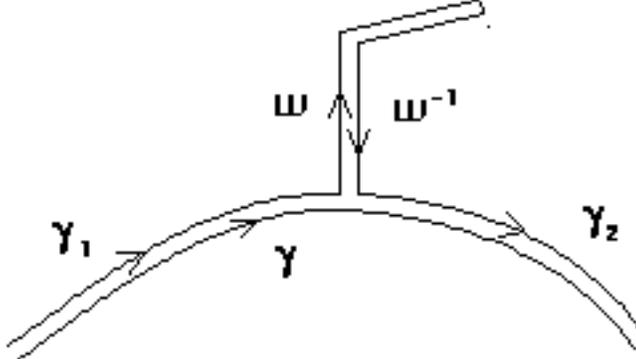
$$U_{\gamma^{-1}} = (U_\gamma)^{-1}. \quad (8.2)$$

Here and in the rest of this section, dependence of U_γ and $T(\gamma)$ on A is understood. Since (8.2) holds also for open paths γ , we derive the important *retracing identity*

$$U_\gamma = U_{\gamma'}, \quad \text{for } \gamma' = ((\gamma_1 \circ w) \circ w^{-1}) \circ \gamma_2, \quad (8.3)$$

where $\gamma_1 \circ \gamma_2 = \gamma$, w is an open path "glued to γ " (see Fig.3) and we have generalized the composition law (7.1) to open paths.

Fig.3



Although the composition of loops is non-associative, it is easily seen that $U_{\gamma'}$ in (8.3) is independent of the order of composition of the loop segments; for example, we could have chosen $\gamma' = \gamma_1 \circ (w \circ (w^{-1} \circ \gamma_2))$ instead. The property (8.3) motivates the introduction of an equivalence relation on paths. A loop γ is called a *thin loop* if it is homotopic to the trivial point loop p by a homotopy of loops whose image lies entirely within the image of γ . Two loops (paths) γ_1 and γ_2 are called *thinly equivalent* if $\gamma_1 \circ \gamma_2^{-1}$ is a thin loop. Note that two loops that differ by an orientation-preserving reparametrization are always thinly equivalent.

We will denote the equivalence class of a loop γ by $\bar{\gamma}$. Under the above equivalence relation, the set of equivalence classes $\Omega_{x_0}\Sigma / \sim$ acquires a group structure, which is characterized by the relations

$$\begin{aligned}
 \bar{\beta} &= \bar{\gamma} && \text{if } \beta = \gamma \text{ mod retracing/reparametrization} \\
 \bar{\gamma} \circ \bar{\gamma}^{-1} &= \bar{p}_{x_0} && (8.4) \\
 (\bar{\alpha} \circ \bar{\beta}) \circ \bar{\gamma} &= \bar{\alpha} \circ (\bar{\beta} \circ \bar{\gamma}) && \text{(associativity)}
 \end{aligned}$$

with the unit \bar{p}_{x_0} , the equivalence class of the constant loop (2.3). This group is usually called the “loop group” or the “group of loops” (not to be confused with the loop groups of maps from S^1 into a Lie group G à la Pressley and Segal [PreSeg]), and will be denoted by $\mathcal{L}\Sigma$. The group $\mathcal{L}\Sigma$ is non-abelian and, as shown by Durhuus and Leinaas [DurLei], not locally compact (in the “area topology” induced by (6.3) in Sec.6), which means it is “very large” even locally. This feature leads to complications when one tries to define a Fourier transform on $\mathcal{L}\Sigma$. The loop group has been discussed by many authors in view of its application to physics [Ana1, AshIsh1, Bar, DurLei, Fis, Frö, GamTri3, Men1]. Its construction has been

around in the mathematics literature for a long time (see the remarks and references in [Bar]). Some toy examples of simpler loop groups in two dimensions are contained in [DurLei]. A simplified version for polygonal loops on \mathbb{R}^4 is discussed in [GamTri3]. - Some authors prefer to induce an equivalence relation on loops by defining $\beta \sim \gamma$ if $U_\beta(A) = U_\gamma(A)$, $\forall A$, which explicitly refers to the space of connections [Lew, AshLew2]. Although sometimes assumed, there is as yet no proof that this definition is equivalent to the thin equivalence introduced above.

The infinitesimal generators of the group $\mathcal{L}\Sigma$ are the “lasso forms” $L(\gamma) \langle \vec{u}, \vec{v} \rangle$ already introduced in (7.11), where γ is a path originating at the base point x_0 , and \vec{u} and \vec{v} are two tangent vectors in the point $\gamma(1)$. More precisely, the linear span of all objects $L(\gamma) \langle \vec{u}, \vec{v} \rangle$ with fixed base point x_0 is the Lie algebra of the restricted holonomy group at x_0 [AmbSin, Gro, Tel]. These objects have also been used in the work by Gambini and Trias [GamTri3,4].

Note that by virtue of (8.1), the holonomy mapping U may be viewed as a group homomorphism $U : \mathcal{L}\Sigma \rightarrow G$. The Wilson loops (4.2) may therefore be regarded as the characters of a representation of the loop group $\mathcal{L}\Sigma$ [Ana1, DurLei, Frö].

There have been several suggestions of embedding the group $\mathcal{L}\Sigma$ into certain large groups with “nice” properties. Di Bartolo et al. introduce a set of distributional functions $X(\gamma)$ on loop space, so-called multi-tangents, defined by the decomposition

$$U_\gamma(A) = \mathbb{1} + \sum_{n=1}^{\infty} \int d^3x_1 \dots d^3x_n A_{a_1}(x_1) \dots A_{a_n}(x_n) X^{a_1x_1 a_2x_2 \dots a_nx_n}(\gamma) \quad (8.5)$$

[DiGaGr]. Then each loop γ gives rise to a set of $X(\gamma)$ satisfying certain algebraic and differential properties. However, there exist more general objects X which satisfy the same properties but do not correspond to any loop γ in $\mathcal{L}\Sigma$. They are identified with elements of an *extended loop group*. This extended group contains elements γ^q (the loop γ traversed q times), where q may be any real number. Tavares [Tav] works with iterated Chen integrals of one-forms ω_i ,

$$\int_\gamma \omega_1 \dots \omega_r := \int_0^1 dt_1 \int_0^{t_1} dt_2 \dots \left(\int_0^{t_{r-1}} dt_r \omega_r(t_r) \right) \dots \omega_1(t_1). \quad (8.6)$$

The set of all such objects gives rise to the so-called shuffle algebra $Sh(\Sigma)$. A generalized loop is now taken to be a continuous complex algebra homomorphism $Sh(\Sigma) \rightarrow \mathbb{C}$ that vanishes on a certain ideal in $Sh(\Sigma)$. The set of all generalized loops can be made into a

group of which $\mathcal{L}\Sigma$ is a subgroup. Note that the holonomy $U_\gamma(A)$ is a special element of $Sh(\Sigma)$ (c.f. expression (3.4)), namely an infinite sum of iterated integrals, with $\omega_1 = \omega_2 = \dots = A$, with A a \mathfrak{g} -valued one-form. In general, iterated integrals do not have a simple behaviour under gauge transformations.

There are analogous identities satisfied by the traced holonomies, characterizing them as a particular subset of complex-valued functions on $\Omega\Sigma \times \mathcal{A}$ (more precisely, $\Omega\Sigma \times \mathcal{A}/\mathcal{G}$ because of their gauge invariance). Independently of the gauge group G , we have the identities

$$T_A(\gamma_1 \circ \gamma_2) = T_A(\gamma_2 \circ \gamma_1) \quad (8.7)$$

because of the cyclicity of the trace, and again a retracing identity,

$$T_A(\gamma) = T_A(\gamma'), \quad \gamma, \gamma' \text{ related as in (8.3)}. \quad (8.8)$$

Another set of identities are the so-called Mandelstam constraints, whose form depends on the dimension N of the group matrices. They can be systematically derived from the identity of N -dimensional δ -functions,

$$\sum_{\pi \in \mathcal{S}_{N+1}} (-1)^{\sigma(\pi)} \delta_{i_1, \pi(j_1)} \dots \delta_{i_{N+1}, \pi(j_{N+1})} = 0, \quad i_k, j_k = 1 \dots N, \quad (8.9)$$

with the sum running over all permutations π of the symmetric group of order $N + 1$, \mathcal{S}_{N+1} , and $\sigma(\pi)$ denoting the parity of the permutation. Contracting $N + 1$ holonomy matrices U_γ with (8.9) results in a trace identity for (combinations of) $N + 1$ loops. For $N = 1$, we have

$$T_A(\alpha)T_A(\beta) - T_A(\alpha \circ \beta) = 0, \quad (8.10)$$

and for $N = 2$,

$$T(\alpha)T(\beta)T(\gamma) - \frac{1}{2} \left(T(\alpha\beta)T(\gamma) + T(\beta\gamma)T(\alpha) + T(\alpha\gamma)T(\beta) \right) + \frac{1}{4} \left(T(\alpha\beta\gamma) + T(\alpha\gamma\beta) \right) = 0, \quad (8.11)$$

etc., where we have omitted the subscript A and the symbol \circ denoting loop composition. Note that the Mandelstam identities are non-linear algebraic equations on the functions T . If we want to consider traced holonomies of specific subgroups of $GL(N, \mathbb{C})$, there will be more identities satisfied by T , for example, deriving from a condition $\det U_\gamma = 1$ (see [GliVir, GamTri6] and the next section for some selected cases). Berenstein and Urrutia [BerUrr] discuss the derivation of Mandelstam identities from the characteristic polynomials of matrices and extend this to the case of supermatrices.

We may use the functions T to induce an equivalence relation on the loop space $\Omega\Sigma$ by defining

$$\beta \sim \gamma \quad \text{if} \quad T_A(\beta) = T_A(\gamma), \quad \forall A. \quad (8.12)$$

The composition law for equivalence classes $\bar{\gamma}$,

$$\bar{\gamma}_1 \circ \bar{\gamma}_2 := \overline{\gamma_1 \circ \gamma_2}, \quad (8.13)$$

induces an abelian group structure on $\Omega\Sigma / \sim$ if in addition the relation $T_A(\alpha) = T_A(\beta) \Rightarrow T_A(\alpha \circ \gamma) = T_A(\beta \circ \gamma)$ is satisfied for all γ [GamTri6, AshIsh1].

9. Equivalence between gauge potentials and holonomies

The importance of the (traced) holonomies lies in the fact that from them one can reconstruct gauge-invariant information about the gauge potential A . The contents of the so-called reconstruction theorems is to specify a set of algebraic and differential conditions on a set of functions on loop space which ensure that from them one can uniquely compute the corresponding equivalence class $[A] \in \mathcal{A}/\mathcal{G}$.

The best-known case is that of holonomies based at a point, for which various versions of the reconstruction theorem are available, depending on the mathematical setting. Although known to mathematicians for a long time, they have been regularly rediscovered by physicists. A very detailed discussion and derivation is contained in the paper by Barrett [Bar]. I quote here his reconstruction theorem for differentiable principal fibre bundles (related treatments may be found in [Ana1, Frö, Gil, GliVir]):

Reconstruction Theorem. Suppose Σ is a connected manifold with basepoint x_0 and the map $H : \Omega_{x_0}\Sigma \rightarrow G$ satisfies the following conditions:

- (i) H is a homomorphism of the composition law of loops, $H(\gamma_1 \circ \gamma_2) = H(\gamma_2)H(\gamma_1)$.
- (ii) H takes the same values on thinly equivalent loops: $\gamma_1 \sim \gamma_2$ if $\gamma_1 \circ \gamma_2^{-1}$ is thin (cf. Sec.8).
- (iii) For any smooth finite-dimensional family of loops $\tilde{\psi} : U \rightarrow \Omega_{x_0}\Sigma$, the composite map $H\psi : U \rightarrow \Omega_{x_0}\Sigma \rightarrow G$ is smooth.

Then there exists a differentiable principal fibre bundle $P(\Sigma, G, \pi)$, a point $p \in \pi^{-1}(x_0)$ and a connection Γ on P such that H is the holonomy mapping of (P, Γ, x_0) . (In (iii) above, U is an open subset of \mathbb{R}^n (parametrizing the family), for any n , and $\tilde{\psi}$ is smooth in the sense that the associated map $\psi : U \times I \rightarrow \Sigma$ is continuous and piecewise C^∞ with respect to the the loop parameter $t \in I$.)

Note that in this formulation the principal bundle P is not fixed a priori, but only the base space Σ and the fibre G are. Fixing P would amount to fixing the homotopy class of the holonomy mapping. The idea of simultaneously considering all possible principal bundles for given (Σ, G) is taken up in [Bar] and [Fis], and the space of PFB isomorphism equivalence classes of triplets (Σ, G, Γ) is called the “grand superspace” \mathcal{S} by Fischer. Recalling the definition of the loop group $\mathcal{L}\Sigma$ introduced in the previous section, one can set up a natural bijection between \mathcal{S} and the space of homomorphisms $\text{Hom}(\mathcal{L}\Sigma, G)/G$ (the quotient by G takes care of the residual gauge freedom of the holonomies at the base point x_0). Fischer observes that one can generalize this construction by taking the quotient of $\mathcal{L}\Sigma$ with respect to the normal subgroup \mathcal{H} of loops homotopic to the trivial loop p_{x_0} , so that $\pi_1(\Sigma, x_0) = \mathcal{L}\Sigma/\mathcal{H}$. One then obtains a one-to-one correspondence between the homomorphisms $\text{Hom}(\pi_1(\Sigma, x_0), G)/G$ and triplets (Σ, G, Γ) where Γ is now a *flat* connection. He suggests to look for some other normal subgroup \mathcal{N} of $\mathcal{L}\Sigma$ such that elements of $\text{Hom}(\mathcal{L}\Sigma/\mathcal{N}, G)/G$ correspond to solutions to the Yang-Mills equations [Fis]. Related ideas are elaborated on by Lewandowski [Lew], who calls $G_{\mathcal{N}} = \mathcal{L}\Sigma/\mathcal{N}$ the generalized gauge group associated with \mathcal{N} and the projection map $\mathcal{L}\Sigma \rightarrow \mathcal{L}\Sigma/\mathcal{N}$ a generalized holonomy map.

Let us go back to the discussion of the reconstruction theorem, now for the case of the traced holonomies. Since we already have a reconstruction theorem for the holonomies, it suffices to show one can recover the holonomies, (4.1), from the Wilson loops (4.2). The main task here is to find algebraic conditions analogous to (i) and (ii) above, characterizing uniquely the traced holonomies as a subset of complex-valued functions on $\Omega\Sigma$. It seems to be much harder to come up with a set of necessary and sufficient conditions, moreover, these

conditions now depend on the gauge group G . To my knowledge, the problem of giving a complete set of such conditions for a general gauge group G has not been solved. The most advanced results in this context are those obtained by Giles [Gil].

The contents of his reconstruction theorem is essentially as follows: Given any complex-valued function $F(\gamma)$ on the loop space $\Omega\Sigma$ satisfying the Mandelstam identities of order N (and possibly some additional identities, characterizing a specific subgroup of $GL(N, \mathbb{C})$), retracing and reparametrization invariance, equation (8.8) and appropriate smoothness conditions, one can construct (modulo a residual gauge freedom) $N \times N$ matrices $U_\gamma \in GL(N, \mathbb{C})$ (or of the subgroup in question) such that the traces of products $U_{\gamma_j} \cdot U_{\gamma_i} \cdot \dots$ are exactly given by $F(\dots \circ \gamma_i \circ \gamma_j)$.

Giles gives an explicit way of reconstructing the holonomies from the traced holonomies, which is very useful in practical applications. However, his results are incomplete in at least two aspects. Firstly, in addition to the usual Mandelstam constraints, there are inequalities restricting the range of the Wilson loops, i.e. given a set of n Wilson loops, they can in general not attain arbitrary complex values, even if all the Mandelstam identities are fulfilled [Lol5]. Secondly, it has not been proven that by running through all admissible $F(\gamma)$'s one recovers indeed *all* possible holonomy configurations U_γ . In general the U_γ 's so obtained will form a subgroup of G . This however can only happen if G is non-compact, a well-studied example being that of $G = SL(2, \mathbb{C})$ [GoLeSt]. This paper explores the degeneracies of the Wilson loops and the momentum loop variables (14.2), as well as their interplay with the dynamics of general relativity.

For the sake of illustration, and because of their importance in many applications, here are the explicit Mandelstam constraints for $G = SL(2, \mathbb{C})$, in the fundamental representation by complex 2×2 -matrices. One finds

$$\begin{aligned}
(a) \quad & T(\text{point loop } p) = 1 \\
(b) \quad & T(\gamma_1) = T(\gamma_2), \quad \text{if } \gamma_1 \text{ and } \gamma_2 \text{ are thinly equivalent} \\
(c) \quad & T(\gamma_1 \circ \gamma_2) = T(\gamma_2 \circ \gamma_1) \\
(d) \quad & T(\gamma) = T(\gamma^{-1}) \\
(e) \quad & T(\gamma_1)T(\gamma_2) = \frac{1}{2} \left(T(\gamma_1 \circ \lambda \circ \gamma_2 \circ \lambda^{-1}) + T(\gamma_1 \circ \lambda \circ \gamma_2^{-1} \circ \lambda^{-1}) \right).
\end{aligned} \tag{9.1}$$

In (e), λ is a path connecting a point on γ_1 with a point on γ_2 , as illustrated by Fig.4. Note that (e) implies both (a) and (d). However, the conditions have been separated in this way since (a)-(c) hold for any group, whereas (d) and (e) are true just for $SL(2, \mathbb{C})$.

Presumably the set (9.1) exhaust all the Mandelstam constraints for this particular group and representation, although I am not aware of the existence of a formal proof. In any case, it can be shown that there are holonomies U_γ that cannot be reconstructed from the Wilson loops, namely those that lie in the subgroup of so-called null rotations. However, in a sense this incompleteness is negligible in the physical applications considered so far [GoLeSt], and in fact the Wilson loops are “as complete as they could be” [AshLew1].

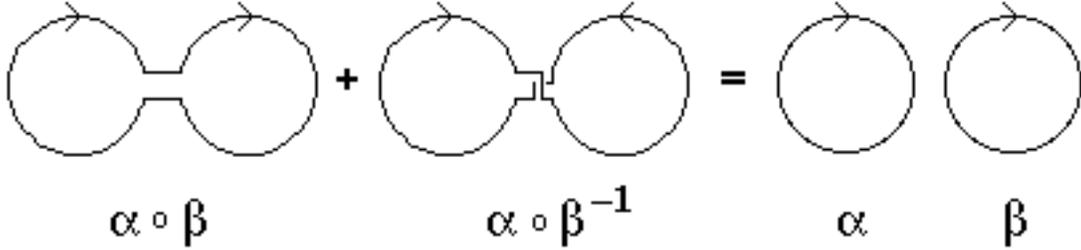


Fig.4

If one wants to restrict the group G to be described by the Wilson loops to a subgroup of $SL(2, \mathbb{C})$, there are further conditions that have to be imposed on the loop variables, including inequalities. For the subgroup $SU(2) \subset SL(2, \mathbb{C})$, one has

$$\begin{aligned}
 (a) \quad & T(\gamma) \text{ real} \\
 (b) \quad & -1 \leq T(\gamma) \leq 1 \\
 (c) \quad & \left(T(\gamma_1 \circ \gamma_2) - T(\gamma_1 \circ \gamma_2^{-1}) \right)^2 \leq 4(1 - T(\gamma_1)^2)(1 - T(\gamma_2)^2) \\
 & \vdots
 \end{aligned} \tag{9.2}$$

whereas the analogous conditions for the subgroup $SU(1, 1) \subset SL(2, \mathbb{C})$ are

$$\begin{aligned}
 (a) \quad & T(\gamma) \text{ real} \\
 (b) \quad & T(\gamma_1)^2 \leq 1 \text{ and } T(\gamma_2)^2 \leq 1 \Rightarrow \\
 & \left(T(\gamma_1 \circ \gamma_2) - T(\gamma_1 \circ \gamma_2^{-1}) \right)^2 \geq 4(1 - T(\gamma_1)^2)(1 - T(\gamma_2)^2) \\
 & \vdots
 \end{aligned} \tag{9.3}$$

As indicated by the dots, there are other inequalities for more complicated loop configurations, involving more than two loops γ_i . Note that all of the above relations hold independently of

A , and hence may be interpreted as *constraints* on loop space functions on $\Omega\Sigma$, rather than as *identities* on the loop-dependent functions on the space of connections, i.e. as functions on $\Omega\Sigma \times \mathcal{A}/\mathcal{G}$. Also at least some of these algebraic conditions can be interpreted as conditions on the underlying loop space. For example, (9.1.d) may be interpreted as a condition that T is a function of unoriented loops.

If one wants to reformulate a theory purely in terms of loop variables, without making any reference to the connection variables A , one has to come up with a complete set of conditions of the type discussed above to ensure equivalence with the usual local formulation. This is a non-trivial task, and not always appreciated in physical applications. The difficulty of proving rigorous reconstruction theorems for the Wilson loops has somewhat hampered this kind of “pure loop approach”. The most progress in isolating the true degrees of freedom in such an approach has been made on the hypercubic lattice [Lol2,3,4]. - Once a reconstruction theorem has been proven, the method is potentially very powerful, since any gauge-invariant statement in terms of the connection A is in principle expressible in terms of the traced holonomies $T(\gamma)$.

Alternatively, one may follow a less radical approach which still incorporates the gauge potentials. For example, one may define a (G -dependent) equivalence relation on loops by identifying two loops if their associated Wilson loops agree for all possible configurations $A(x)$ (8.12). The quotient space of loop space one obtains in this way is of course exactly the one one would like to use as a domain space when constructing a formulation exclusively in terms of loop variables. Another possibility is to postulate the existence of a “loop transform” in the quantum theory (cf. Sec.15) which allows one to translate wave functions and operators from the connection to the loop representation. This automatically takes care of algebraic constraints among the Wilson loop operators, i.e. quantum analogues of the Mandelstam constraints.

Incidentally, there have also been attempts to reformulate gauge theories in terms of the field strengths F instead of the gauge potentials A , because they have a simpler, homogeneous transformation behaviour. However, it is well known that gauge-inequivalent connection configurations may have the same field strength (although this does not happen for “generic” connections), and hence a rigorous reconstruction theorem does not exist. A discussion of this degeneracy and a guide to the literature is contained in the paper by Mostow and Shnider [MosShn].

10. Physical interpretation of the holonomy

It is well known that the existence of a connection on a manifold enables one to define a notion of *parallel transport*. For a field $\Psi(x)$ transforming according to the defining representation of the gauge group G ,

$$\Psi(x) \xrightarrow{g} g^{-1}(x)\Psi(x), \quad (10.1)$$

we can compare fields at different points x and y by parallel-transporting $\Psi(y)$ from y back to x using the holonomy $U_w(s_1, s_0)$ along a path w with $w(s_1) = x$, $w(s_0) = y$, to obtain

$$\Psi_{par,w}(x) := U_w(s_1, s_0)\Psi(y), \quad (10.2)$$

which has now the same behaviour under gauge transformations as $\Psi(x)$. In the limit as the length of w goes to zero, this procedure leads to the definition of the covariant derivative of Ψ [Pol3].

The holonomy of small closed loops measures the *curvature (or field strength) in internal space*. For a small square loop γ of side length ϵ in a coordinate chart around the point $x \in \Sigma$, the base point of γ , the holonomy U_γ can be expanded as

$$U_\gamma = \mathbb{1} + g F_{\mu\nu}^a(x) X_a \epsilon^2 + O(\epsilon^3), \quad (10.3)$$

where γ is defined by its four corners, $(x, x + \epsilon\vec{e}_\mu, x + \epsilon\vec{e}_\mu + \epsilon\vec{e}_\nu, x + \epsilon\vec{e}_\nu)$, with \vec{e}_μ denoting the unit vector in μ -direction. Note that in the non-abelian gauge theory, $F_{\mu\nu}(x) = F_{\mu\nu}^a(x)X_a$ is not an observable, since it transforms non-trivially under gauge transformations. The idea of measuring the eigenvalues of holonomy matrices by non-abelian analogues of interference experiments à la Aharonov and Bohm is discussed by Anandan [Ana2,3] (an earlier treatment can be found in [WuYan]). He also points out differences between the applications to gauge theory and gravity (albeit not in the Ashtekar formulation) [Ana3]. A beautiful geometric description of the modified parallel transport associated with the Ashtekar connection in gravity is given in the review paper by Kuchař [Kuc].

Performing a similar expansion for the traced holonomy of the infinitesimal loop γ , we obtain

$$T_A(\gamma) = \text{Tr } U_\gamma = 1 + \frac{g}{N} \sum_a F_{\mu\nu}^a(x) \text{Tr } X_a \epsilon^2 + \frac{g^2}{N} \sum_{a,b} F_{\mu\nu}^a(x) F_{\mu\nu}^b(x) \text{Tr } X_a X_b \epsilon^4 + O(\epsilon^5), \quad (10.4)$$

(no sum over μ, ν). For a semi-simple Lie algebra \mathfrak{g} we can always find a basis of generators X_a such that $\text{Tr } X_a X_b = \delta_{ab}$. Moreover, for $G = SU(N)$ (more generally, for any subgroup of the special linear group $SL(N, \mathbb{C})$), we have $\text{Tr } X_a = 0$, and the expansion therefore reduces to

$$T_A(\gamma) = 1 + \frac{g^2}{N} \sum_a F_{\mu\nu}^a(x) F_{\mu\nu}^a(x) \epsilon^4 + O(\epsilon^5). \quad (10.5)$$

These expansions illustrate the way local gauge-invariant information about the curvature or field strength F is contained in $T_A(\gamma)$.

Having described the basic mathematical properties of the holonomy and the traced holonomy, there remains the question of the physical interpretation of the underlying loop space itself, and therefore of the way in which a physical theory is to be constructed on it. Formulating a theory in terms of non-local variables depending on loops is potentially very different from the usual local formulations. It makes a difference whether one wants to use loops as convenient auxiliary labels in an otherwise local formulation, or postulates them to be the basic entities of a new, intrinsically non-local description of gauge theory or gravity. There is a variety of ways in which past research has made use of path- and loop-dependent quantities, some examples of which will be described below. It is important to realize that different a-priori physical interpretations of the role of loops suggest different mathematical structures for their description, for example, the initial choice of a loop space or a quotient of a loop space. If one thinks of paths or loops as describing actual trajectories of charged particles [Wil, Bar, Pol3], say, one may work with smooth C^∞ -loops in a (semi-)classical description and nowhere differentiable loops (which are supposed to give the main contribution to the Feynman path integral) in the quantum theory.

Many conceptual issues still remain unanswered. It is not clear whether a non-local description in terms of holonomies is necessarily tied to the *quantum* aspects of the theory. Also it is clearly the non-linearity of Yang-Mills theory and gravity that calls for a non-local loop formulation, since the abelian $U(1)$ -theory does not require such a description, although it is possible [Ash1, AshRov, GamTri2]. One may also ask how big loops are, and whether they have a preferred size in a given theory (which could serve to provide a new fundamental length scale). We are lacking a physical argument to decide whether the (traced) holonomies

should be taken as the basic variables (cf. Polyakov’s “rings of glue” [Pol2]), or whether genuine physical observables are again *composites* of the elementary Wilson loops. - The following sections deal with some issues peculiar to loop formulations of Yang-Mills theory and gravity. They are of an introductory nature, and to be regarded as a commented guide to the literature.

11. Classical loop equations

In this section I give some examples of path-dependent formulations of gauge theory, starting from Mandelstam’s early treatment of electrodynamics coupled to a scalar field, and its subsequent generalization to the non-abelian case. This illustrates how classical equations of motion for path-dependent variables may be obtained. Unfortunately, no analogous classical loop equations have so far been derived for general relativity.

Mandelstam’s description of “QED without potentials” [Man1] is interesting in the present context, because it is the first instance of the use of holonomy-dependent field variables. He demonstrates that a gauge-invariant and Lorentz-covariant formulation of electrodynamics (avoiding the unphysical negative-norm states of the Gupta-Bleuler approach) is possible, *provided* one allows for non-local matter field variables. Instead of using the gauge potential $A_\mu(x)$ and the charged scalar field $\phi(x)$, he works with the field strength $F_{\mu\nu}(x)$ and objects

$$\Phi(\gamma, x) := \phi(x) \exp\left\{-ie \int_{-\infty}^x A_\mu d\gamma^\mu\right\} \quad (11.1)$$

as the basic variables, where γ denotes a space-like path in Minkowski space originating at spatial infinity and ending at the point x . These variables are obviously invariant under the gauge transformations

$$\Phi \rightarrow e^{ies} \Phi, \quad A_\mu \rightarrow A_\mu + \partial_\mu s, \quad (11.2)$$

where s is an arbitrary scalar field satisfying the boundary condition $s(\pm\infty) = 0$. Mandelstam emphasizes that the need for a gauge-invariant formulation arises in the quantum theory, because only commutators between (gauge-invariant) observables are well defined. He argues

that the inherent path dependence of the matter field variable is natural and corroborated by the Aharonov-Bohm effect.

Bialynicki-Birula [Bia] extended these ideas to formulate what one might call “Yang-Mills theory without field strengths”. Since the non-abelian gauge field interacts with itself, there is in a first step no need to couple it to an external matter field in order to make the theory non-trivial. Since the field variables $F_{\mu\nu}(x)$ are themselves not observables (i.e. not gauge-invariant), Bialynicki-Birula introduces in analogy with (11.1) the gauge-invariant fields

$$\mathcal{F}_{\mu\nu}(\gamma, x) := \text{P exp}\left\{-ig \int_{-\infty}^x A_\mu d\gamma^\mu\right\} F_{\mu\nu}(x) \text{P exp}\left\{ig \int_x^{-\infty} A_\mu d\gamma^\mu\right\}, \quad (11.3)$$

which makes use of the holonomies of the space-like paths γ and γ^{-1} , as illustrated by Fig.5.

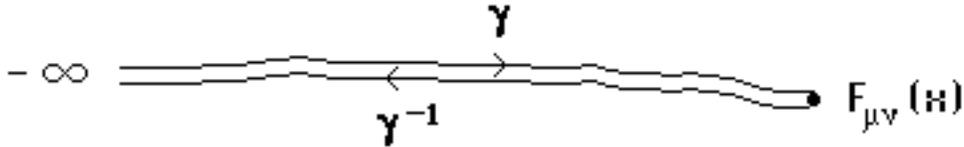


Fig.5

A corresponding path-dependent potential is introduced and discussed by Gambini and Trias [GamTri1]. The usual classical Yang-Mills equations of motion

$$\mathcal{D}_\mu F_a^{\mu\nu}(x) = (\partial_\mu \delta_{ac} - g\epsilon_{abc} A_{b\mu}(x)) F_c^{\mu\nu}(x) = 0, \quad (11.4)$$

with ϵ_{abc} denoting the structure constants of \mathfrak{g} , translate to

$$\partial_\mu(x) \mathcal{F}^{\mu\nu}(\gamma, x) = 0, \quad (11.5)$$

where the differential $\partial_\mu(x)$ acts on the holonomy $U_{\gamma,x}$ as an “endpoint derivative” (see

equation (12.1) for a definition). The Bianchi identities (which are satisfied automatically in the connection formulation) have to be imposed as separate equations,

$$\partial_\lambda(x)\mathcal{F}_{\mu\nu}(\gamma, x) + \partial_\mu(x)\mathcal{F}_{\nu\lambda}(\gamma, x) + \partial_\nu(x)\mathcal{F}_{\lambda\mu}(\gamma, x) = 0. \quad (11.6)$$

It may be somewhat surprising that equations (11.5) and (11.6), unlike equations (11.4), are *linear* in the basic field variables. Mandelstam [Man2] writes: “The field equations are simpler *in appearance* than the Maxwell equations of electrodynamics, since there is no additional current term”. He is obviously aware of the fact that difficulties associated with the non-linear functional form of the Yang-Mills equations must be hidden in (11.5), however, neither spelling out in what sense non-triviality arises, nor making any further use of the equations (11.5) classically (they are used to obtain equations for the corresponding path-dependent Green’s functions in the quantum theory though). A rigorous mathematical derivation of the equations (11.5) and (11.6) is contained in the paper by Gross [Gro], within the mathematical framework discussed already in Sec.7.

Slightly different gauge-covariant field variables are used by Polyakov [Pol1,3], namely,

$$\mathcal{F}_\mu(\gamma, s) := U_\gamma(0, s) F_{\nu\mu}(\gamma(s)) \dot{\gamma}^\nu(s) U_\gamma(s, 0), \quad (11.7)$$

where γ is a closed loop, s some intermediate parameter value, $s \in [0, 1]$, and $U_\gamma(0, s)$ the holonomy along the portion $[0, s]$ of the loop γ (Fig.6)

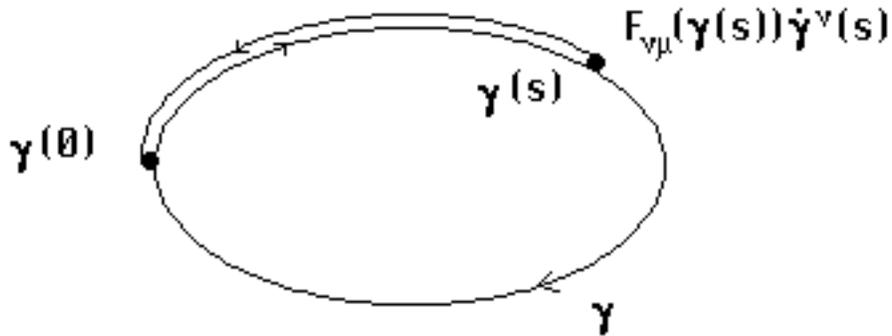


Fig.6

The field equations

$$\frac{\delta}{\delta x^\mu(s)} \mathcal{F}^\mu(\gamma, s) := \dot{\gamma}^\lambda(s) \frac{\delta}{\delta \sigma^{\lambda\mu}(\gamma(s))} \mathcal{F}^\mu(\gamma, s) = 0 \quad (11.8)$$

correspond to the Yang-Mills equations projected onto the tangent vector $\dot{\gamma}^\nu$ of γ at s . The differential operator $\frac{\delta}{\delta \sigma^{\lambda\mu}(\gamma(s))}$ is the projection of the area derivative of definition (12.3), obtained by adding an infinitesimal loop in μ -direction at the point s . Note that \mathcal{F}_μ still depends on the curve parameter s ; parametrization invariance and the (projected) Bianchi identities have to be imposed as separate equations in addition to (11.8). The reason why this reformulation of Yang-Mills theory was thought to be appealing is its close formal resemblance with the two-dimensional non-linear σ -model (see also the work of Aref'eva [Are1,3,4] for a related treatment). It turns out that this simple analogy does not go through, essentially because the theory is not defined on ordinary space, but on loop space. The loop equations (11.8), unlike their analogues for the classical non-linear σ -model, do not contain enough information for constructing a set of conserved currents [GuWan]. In Polyakov's words, the (up-to-date) failure of this approach "has to do with the difficulties we experience in treating equations in loop space, or, what is more or less the same, with string dynamics" [Pol3].

A similar framework is investigated by Hong-Mo, Scharbach and Tsun [HoScTs, HonTsu]. Like Polyakov, they work with parametrized loops and try to exploit the idea that \mathcal{F}_μ (11.7) constitutes a "connection on loop space". The Bianchi identity in loop space then amounts to the statement that the curvature of this connection vanishes, and hence the loop space connection \mathcal{F}_μ is flat ([Pol1,2], see also [Ful2]):

$$\frac{\delta}{\delta x^\nu(s')} \mathcal{F}^\mu(\gamma, s) - \frac{\delta}{\delta x^\mu(s)} \mathcal{F}^\nu(\gamma, s') + [\mathcal{F}_\mu(\gamma, s), \mathcal{F}_\nu(\gamma, s')] = 0. \quad (11.9)$$

According to Hong-Mo et al., the non-flatness of the connection \mathcal{F}_μ , i.e. the non-vanishing of the right-hand side of (11.9), is related to the presence of a non-abelian monopole charge. They also derive a reconstruction theorem for the gauge potential $A_\mu(x)$ from the loop space potential \mathcal{F}_μ , and establish an action principle in loop space, using a formal functional integral over the space of parametrized loops. In order to avoid the explicit parametrization dependence of equations like (11.8) and (11.9), Gambini and Trias in a related work reformulate them in terms of geometric, but distributional quantities [GamTri4].

A further example of loop-dependent equations of motion for Yang-Mills theory, this time in terms of the traced holonomies (4.2), is given by the classical Makeenko-Migdal equations [Mig2]

$$\partial^\mu(x) \frac{\delta}{\delta\sigma^{\mu\nu}(x)} T(\gamma) = 0, \quad (11.10)$$

with $\partial^\mu(x)$ denoting the ordinary differential on space-time and $\frac{\delta}{\delta\sigma^{\mu\nu}(x)}$ Mandelstam's area derivative (see the next section for a definition). What is puzzling about this equation is its linearity in the field variable T ; again the non-linearities seem to have disappeared. In fact, there are spurious solutions to (11.10) [Mig2], which have to be eliminated by other means. This feature is attributed "partly to the presence of the Mandelstam constraints", which have not been taken into account in the derivation of the equation. Migdal concludes that "the classical loop dynamics is quite complicated and implicit, but presumably it is irrelevant as well as the classical colour dynamics".

A similar non-uniqueness for the derivation of equations of motion is present in the quantum theory. Various methods have been proposed to derive an equation for the "Wilson loop average", i.e. the vacuum expectation value $\langle T(\gamma) \rangle$ of the (space-time) Wilson loop, or that of the holonomy operator, the most famous of which is the Makeenko-Migdal equation [MakMig1,2, Mig1, BGNS]. In such derivations, one typically makes use of identities for first and second order functional path derivatives to arrive at expressions that are supposed to serve as quantum Yang-Mills equations. (A question of considerable interest in the late seventies was the relation of such path-dependent equations with the equations of motion for the quantized relativistic string; some references are [CorHas, GamGri, GerNev1, MakMig1,3, Mig1, Nam]). However, no solutions to these equations have been found in three and four space-time dimensions. It has been shown that the $\langle T(\gamma) \rangle$ are multiplicatively renormalizable (for loops γ with a finite number of cusps and self-intersections) [DotVer, Are3,4,5, BrNeSa, BGNS], but no useful equations have been formulated for the renormalized functional $\langle T(\gamma) \rangle_R$.

12. Differential operators

In order to define equations of motion in a path-dependent approach, one needs to have a notion of differentiation. The properties of differential operators are intimately tied to the space they act upon, hence path and loop derivatives assume different meanings in different contexts. Unfortunately those distinctions are often not clearly stated in the literature, nor are the relevant spaces of loop functions and the underlying loop spaces.

Recall that for differentiation on some space X to be well defined, one needs locally at least some topological vector space structure on X [ChoDeW]. If one is lucky, X can

be made into a Banach space (i.e. a complete normed vector space), in which case most of the differential calculus on \mathbb{R}^n can be generalized in a straightforward way to X . Also, the norm induces a translationally invariant metric and a natural topology on X . For more general topological vector spaces one may still be able to define differentiation, but there are in general *no inverse and implicit function theorems and no theorems on the existence and uniqueness of solutions of differential equations*.

In physical applications, X is usually some space of loop functions or functionals, i.e. essentially an infinite-dimensional function space with a vector space structure, but is rarely given any further structure, for example, a topology. A similar statement concerns the loop and path spaces themselves. In some of the previous sections we described the problem of “giving structure” to these spaces, which in turn is an obstruction to defining a meaningful differential calculus on them. There exist several definitions of such operators, which differ in the way they treat the path parametrization and (often implicitly) by which class of functions they are supposed to act on. An explicitly parametrization-invariant geometric loop calculus has been set up and applied by Gambini and Trias (see, for example, [GamTri3,4, Gam]).

In the following I will describe some typical path-dependent differential operators, and outline some of the problems associated with them. For a function $F(w, x)$ depending on an (unparametrized) path w , with x denoting one of its endpoints, we define an “endpoint derivative”

$$\partial_\mu(x) F(w, x) := \lim_{dx_\mu \rightarrow 0} \frac{F(w', x + dx_\mu) - F(w, x)}{dx_\mu}, \quad (12.1)$$

where w' is obtained by adding to w an infinitesimal straight line element dx_μ in μ -direction. It is used, for example, by Bialynicki-Birula [Bia] and Mandelstam [Man2,3] for the special case where F is the holonomy $U_{\gamma, x}$ of a path starting at spatial infinity and going to the point x . (The endpoint derivative of the holonomy taken at the endpoint x of an open path is just the gauge potential $A(x)$.) Gambini and Trias call it “Mandelstam’s covariant derivative” and use it in a generalized context where F is a G -valued function on the set of open paths modulo reparametrizations and retracing [GamTri3].

Note that it is not necessary to use open paths in order to obtain the gauge potential A by differentiation of the holonomy. One may as well stick to loops and use a “triangular derivative” [Gil, Gu, HoScTs]. Within a coordinate patch W on the manifold Σ , choose an origin x_0 and connect each point $x \in W$ in a continuous way to x_0 by an open path, i.e. such that neighbouring points are joined to x_0 by neighbouring paths. For simplicity, we will

use the straight lines (with respect to some auxiliary metric) λ_{x_0x} starting at the origin and going to x (see Fig.7).

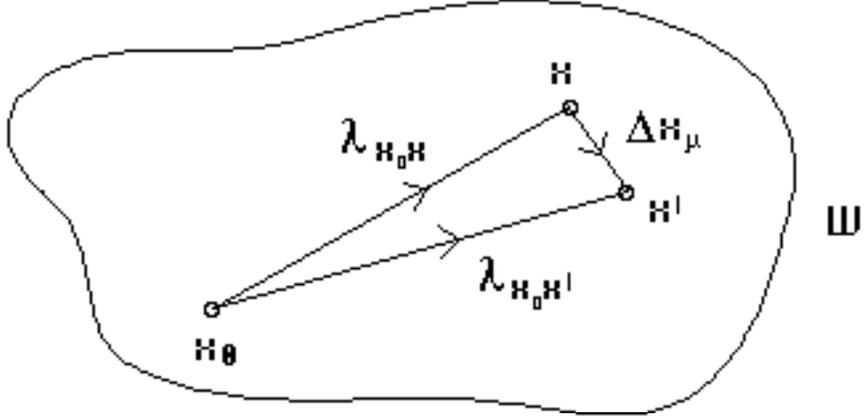


Fig.7

Consider now a point $x' = x + \Delta x_\mu$ close to x and the closed loop $\lambda_{x_0x'}^{-1} \circ \lambda_{xx'} \circ \lambda_{x_0x}$, where $\lambda_{xx'}$ is the infinitesimal straight path linking x and x' . The gauge potential may then be defined as the limit

$$A_\mu(x) = \lim_{\Delta x_\mu \rightarrow 0} \frac{(U_{\lambda_{x_0x'}^{-1} \circ \lambda_{xx'} \circ \lambda_{x_0x}} - \mathbb{1})}{\|\Delta x_\mu\|}. \quad (12.2)$$

This construction is metric-independent. For a different choice of the paths λ_{x_0x} one obtains a gauge potential that is related to (12.2) by a gauge transformation. With respect to a fixed metric, the gauge choice $A_\mu(x)(x^\mu - x_0^\mu) = 0$, $A_\mu(x_0) = 0$ is sometimes called the “central gauge” [Gu].

Another frequently used differential operator is the so-called area derivative. For a path-dependent function $F(w)$ it is usually defined as

$$\frac{\delta}{\delta \sigma^{\mu\nu}(x)} F(w) := \lim_{d\sigma^{\mu\nu} \rightarrow 0} \frac{F(w \circ_x \gamma_{\mu\nu}) - F(w)}{d\sigma^{\mu\nu}}, \quad (12.3)$$

where $\gamma_{\mu\nu}$ is an infinitesimal planar loop in the μ - ν -plane attached (by path composition) to the path w at the point x on w . The area derivative of a loop function is therefore a function of loops with a marked point. In a local coordinate chart $\{x_\mu\}$, the area of the small loop is given by

$$d\sigma^{\mu\nu} = \frac{1}{2} \int_{\gamma_{\mu\nu}} x^\mu dx^\nu, \quad (12.4)$$

which is antisymmetric in μ and ν . For the special case where $F(w) = U_w$, the holonomy of a path w with initial point x_0 and endpoint x_1 , we have

$$\frac{\delta}{\delta\sigma^{\mu\nu}(x)} U_{w;x_0,x_1} = U_{w;x_0,x} F_{\mu\nu}(x) U_{w;x,x_1}, \quad (12.5)$$

and hence the area derivative is automatically antisymmetric in μ and ν . For more general cases of loop functions F , one may have to introduce an explicit antisymmetrization in (12.3) in order for the definition to make sense. Another derivative that is sometimes used is obtained by displacing an entire curve $\gamma(s)$ by an infinitesimal amount $\delta\gamma(s)$ (i.e. each point of the loop γ may be shifted) [CorHas, MakMig2]. A way of defining a rigorous path derivative by introducing a generalized Fourier decomposition for functions on the unit interval has been described by Gervais and Neveu [GerNev2].

Further discussions about path and area derivatives, and their relations to various functional derivatives can be found in [Ble, BGNS, BrüPul, GamGri, Gro, Mig2, Pol2, Tav]. One way of defining an “anti-area derivative” (which does not coincide with a simple surface integration) is discussed by Durhuus and Olesen [DurOle]. For the application to gravity one wants to avoid the explicit use of the metric in the definition of those derivatives. Special attention to this demand is given in [BrüPul, Tav].

Let me again emphasize that for general loop functions there is no reason for the limits in (12.1-3) to be well defined and exist. Furthermore, if we talk about “infinitesimal loops”, this implies we have chosen some topology on loop space which tells us about small variations, i.e. what it means for two loops to be infinitesimally close to each other. Our “intuitive” notion of closeness of two paths is that coming from viewing them as embedded in the manifold Σ and using the Euclidean metric of \mathbb{R}^n in local charts of Σ . However, this may not be the appropriate thing to do. For example, in the context of the “group of loops” introduced in Sec.7 above, two paths are considered equivalent if they are in the same class with respect to retracing. This leads to a generalized and non-local (with respect to the metric topology on

\mathbb{R}^n) notion of closeness, as has been explained elsewhere in these notes. Since the issue of giving topological structures to infinite-dimensional spaces is mathematically involved, there is a genuine need for physical arguments to restrict the possible choices, and thus obtain meaningful equations of motion in a path-dependent approach.

13. Lattice gauge theory

The only way of obtaining quantitative results about the behaviour of non-abelian gauge theory, such as the values of hadron masses, and testing the hypothesis of quark confinement, are in a regularized version of the theory, where continuous space-time is approximated by a finite hypercubic lattice. The basic gauge field variables in this case are the link holonomies U_l , i.e. gauge potentials integrated over elementary lattice links.

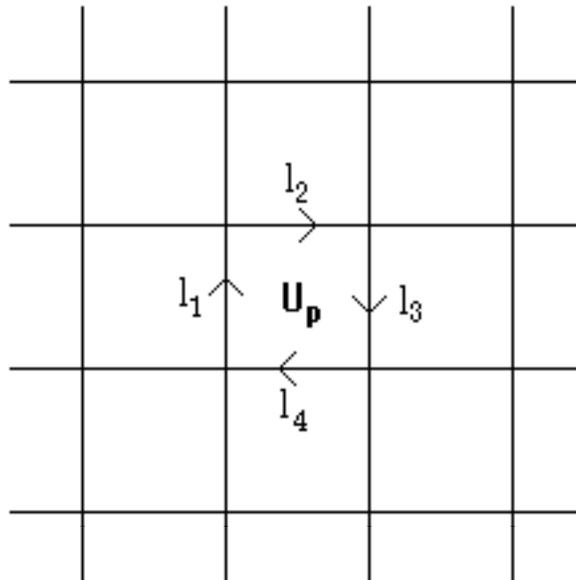


Fig.8

Wilson, who first proposed this approach to gauge theory [Wil], identified closed flux lines on the lattice as natural gauge-invariant objects, and introduced a discretized, Euclidean form of the Yang-Mills action, which in the continuum limit (as the length a of lattice links goes to zero) reduces to the ordinary one. For gauge group $SU(N)$, it is given essentially as the

sum over all lattice plaquettes (elementary square loops made up of four contiguous oriented links) P of the traces of the corresponding holonomies $U(P)$,

$$S_W(U_i) = -\frac{1}{Ng^2} \sum_P (N - \text{Tr } U_P). \quad (13.1)$$

The plaquette holonomy U_P is to be thought of as the product of the four link holonomies U_l associated with P (see Fig.8, where we have $U_P = U_{l_1}U_{l_2}U_{l_3}U_{l_4}$). The underlying physical interpretation for holonomies of closed paths is that of weight factors in the Feynman path integral, associated with classical quark trajectories. More precisely, in order to compute the current-current propagator for quark fields between two points 0 and x in space-time, one has to average over all possible classical quark trajectories and classical gauge field configurations. The relevant quark configurations are pairs of quarks created at the origin 0 and annihilated at x (Fig.9a),

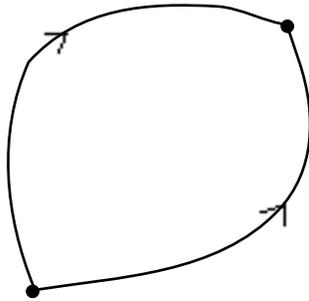


Fig.9a

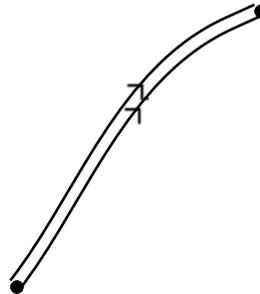


Fig.9b

the weight for such a pair being exactly the holonomy U_γ along the loop formed by the pair of quark trajectories. In the strong coupling limit of the lattice gauge theory ($g \rightarrow \infty$), one can produce arguments that the gauge field average of U_γ for a fixed lattice loop γ behaves as $\exp icA(\gamma)$, with $A(\gamma)$ denoting the area enclosed by the loop γ , and constant c . This suggests confining behaviour for quarks, because large areas $A(\gamma)$ (corresponding to the quarks being far apart) are strongly suppressed in the sum over all paths, whereas narrow “flux tubes” (Fig.9b) are favoured [Wil]. Note this argument was made within the pure gauge theory, without explicitly including fermionic degrees of freedom.

In the Hamiltonian formulation of the lattice theory due to Kogut and Susskind, the

(spatial) link holonomies play again an important role [KogSus]. Together with an appropriate set of conjugate momentum variables they form a closed Poisson-bracket algebra, which is of the form of a semi-direct product. This algebra is then quantized, leading to a representation where the quantized link holonomies \hat{U}_l are diagonal. The wave functions in the Hilbert space are not gauge-invariant, and the physical subspace has to be projected out by imposing the Gauss law constraint as a Dirac condition on quantum states. It turns out that gauge-invariant states can be labelled by closed paths of links on the lattice.

However, with growing lattice size, selecting the physical subspace and calculating the action of the Hamiltonian on it become quickly involved. Furthermore this set of variables does not seem particularly suited for treating the weak coupling limit. For that reason, there have been various alternative proposals to formulate the Hamiltonian lattice theory in an explicitly gauge-invariant manner [FurKol, GaLeTr, RovSmo3, Brü]. They involve Hilbert spaces of gauge-invariant states labelled by loops, and the action of the Hamiltonian operator typically results in geometric deformations or rearrangements (fusion, fission etc.) of these loop arguments. (A lattice analogue of the equivalence theorems discussed in Sec.9 above is investigated by Durhuus [Dur], more precisely, he derives a necessary and sufficient condition on the gauge group G such that the linear span of products of Wilson loop variables is dense in the space of continuous gauge-invariant observables.)

Unfortunately there is now another drawback, since such bases of loop states are vastly overcomplete (due to the Mandelstam constraints), which renders the physical interpretation of the geometric picture of “loops interacting through the action of the Hamiltonian” rather obscure. The problem is how to isolate in an efficient way a set of independent loop states, since the number of loops on the lattice grows very fast with growing lattice size. A frequently used approximation employs a truncated loop basis, which only considers states labelled by loops that are shorter than a given number n of links [FurKol, Brü], and calculate the spectrum of the Hamiltonian in this restricted context. This seems to work reasonably well, at least for some sectors of the theory. However, what one ideally would like to have is a formulation directly in terms of the physical degrees of freedom, which is gauge-invariant and has a minimal redundancy with respect to the Mandelstam constraints. Such a set of variables has been given in [Lol2], and recent results, obtained in a Lagrangian context, show that it is indeed possible to perform calculations directly in terms of these independent loop variables [Lol3].

Another advantage of the loop representation is that it can be naturally extended to include fermions. Gauge-invariant variables may be obtained by “gluing” fermion fields Ψ to the ends of an open path w , leading to the holonomy-dependent variables $\Psi^\dagger(y)U_w(y,x)\Psi(x)$. The canonical loop algebra of the pure gauge theory (cf. the next section) can be enlarged

by these path variables, and quantized. The corresponding dynamics of paths and loops on the lattice, together with some small scale calculations is described by Gambini and Setaro [GamSet] (see also for references to the same method applied to $U(1)$ -gauge theory).

The lattice methods described above cannot be readily applied to the case of general relativity, because the way in which the lattice approximation has been set up explicitly violates diffeomorphism invariance. This is a serious obstacle since the diffeomorphism group acts as a gauge group (not just as a symmetry group like the Lorentz group, say), and a major problem in lattice formulations of gravity has been to recover diffeomorphism invariance in the continuum limit. Some problems that occur in translating the $SL(2, \mathbb{C})$ -loop approach à la Rovelli and Smolin to the lattice have been discussed by Renteln [Ren]. - In spite of the difficulties of applying standard lattice methods to the connection formulation of gravity, the discrete structures appearing in its canonical quantization (cf. Sec.15) indicate that lattice-like constructions may still play a role. However, they would be defined through their intrinsic topology and connectivity (i.e. diffeomorphism-invariant properties) rather than through an imbedding as hypercubic lattices into a metric manifold. For example, there are ways of defining diffeomorphism-invariant measures on the space \mathcal{A}/\mathcal{G} of connections modulo gauge transformations introduced in Sec.5. They allow for the integration of connection configurations that are supported on one-dimensional lattices or graphs [AshLew2, Bae].

14. Loop algebras

In this section we will discuss some algebras of loop- and path-dependent functions. The algebras we shall be concerned with involve holonomies and are defined either on a configuration space \mathcal{A} or a phase space $T^*\mathcal{A}$. The first such algebra was written down by Mandelstam in a Lagrangian framework [Man2]. Splitting $\mathcal{F}_{\mu\nu}(\gamma, x) = \mathcal{F}_{\mu\nu}^a(\gamma, x)X_a$ (c.f. (11.3)) into its components \mathcal{F}_{ij} and \mathcal{F}_{0i} , $i = 1, 2, 3$, he finds for the equal-time commutators of an $SU(2)$ Yang-Mills theory

$$\begin{aligned}
[\mathcal{F}_{ij}^a(\gamma, x), \mathcal{F}_{kl}^b(\gamma', x')] &= 0, \\
[\mathcal{F}_{0i}^a(\gamma, x), \mathcal{F}_{jk}^b(\gamma', x')] &= -i\delta_{ab}(\delta_{ik}\partial_j - \delta_{ij}\partial_k)\delta^3(x - x') + i\epsilon_{abc} \int_{\gamma'} d\xi_i \delta^3(x - \xi) \mathcal{F}_{jk}^c(\gamma', x'), \\
[\mathcal{F}_{0i}^a(\gamma, x), \mathcal{F}_{0j}^b(\gamma', x')] &= i\epsilon_{abc} \int_{\gamma'} d\xi_i \delta^3(x - \xi) \mathcal{F}_{0j}^c(\gamma', x') + i\epsilon_{abc} \int_{\gamma} d\xi_j \delta^3(x' - \xi) \mathcal{F}_{0i}^c(\gamma, x).
\end{aligned} \tag{14.1}$$

The relations (14.1) define a closing algebra (if we include the unit generator appearing on the

right-hand side of the second equation), i.e. the commutator of two \mathcal{F} 's is again proportional to an algebra generator. A characteristic feature is the fact that the structure constants of the algebra are distributional and depend on the path configurations appearing as arguments of the \mathcal{F} -variables.

In the Hamiltonian formulation in terms of traced holonomies, one finds a similar algebra structure after introducing conjugate holonomy variables on the Yang-Mills phase space (in the $A_0 = 0$ -gauge) coordinatized by the pairs $(A_i^a(x), E_a^i(x))$, with canonical Poisson brackets $\{A_i^a(x), E_b^j(y)\} = \delta_b^a \delta_i^j \delta^3(x - y)$. They depend on a loop γ , a marked point $\gamma(s)$ on γ , and both the gauge potential and the generalized electric field and are defined as ([GamTri6, RovSmo2])

$$T_{A,E}^i(\gamma, s) := \text{Tr } U_\gamma(s, s) E^i(\gamma(s)). \quad (14.2)$$

For the special case of $SU(2)$, computing the Poisson brackets on phase space of these loop variables leads to

$$\begin{aligned} \{T(\gamma), T(\gamma')\} &= 0, \\ \{T^i(\gamma, s), T(\gamma')\} &= -\Delta^i(\gamma', \gamma(s))(T(\gamma \circ_s \gamma') - T(\gamma \circ_s \gamma'^{-1})), \\ \{T^i(\gamma, s), T^j(\gamma', t)\} &= -\Delta^i(\gamma', \gamma(s))(T^j(\gamma \circ_s \gamma', u(t)) + T^j(\gamma \circ_s \gamma'^{-1}, u(t))) + \\ &\quad + \Delta^j(\gamma, \gamma'(t))(T^i(\gamma' \circ_t \gamma, v(s)) - T^i(\gamma' \circ_t \gamma^{-1}, v(s))). \end{aligned} \quad (14.3)$$

The structure constants Δ are again distributional:

$$\Delta^i(\gamma, x) = \oint_\gamma dt \delta^3(\gamma(t), x) \dot{\gamma}^i(t). \quad (14.4)$$

In its general form for gauge group $SU(N)$, the algebra (14.3) was first introduced by Gambini and Trias [GamTri6]. For the case of $G = \text{SL}(2, \mathbb{C})$ (for which the algebra coincides with (14.3)), it was later rediscovered by Rovelli and Smolin in a loop approach to canonical quantum gravity [RovSmo2]. The same algebra is also relevant for 2 + 1-dimensional gravity, which may be formulated on a space \mathcal{A} of $su(1, 1)$ -valued connection one-forms [AHRSS, Smo1]. A generalization to a Hamiltonian algebra of fields depending on open paths in the context of Higgs fields is described in [GaGiTr]. Related loop-dependent algebras on the

phase space of general relativity in terms of the metric variables and that of scalar field theory are discussed by Rayner [Ray1].

Note that the algebra (14.3) has the form of a semi-direct product, with the abelian subalgebra of the traced holonomies $\{T\}$ acted upon by the non-abelian algebra of the T^i -variables, and is similar in structure to the Lagrangian algebra (14.1). This is not true for $N \neq 2$, for which the right-hand sides of the Poisson brackets in (14.3) are *not* linear in T . In order to make the algebra (14.3) non-distributional, one has to “smear out” the loop variables appropriately. One way of doing this is to integrate (14.2) over a ribbon or “strip” R [Rov2, AshIsh1], i.e. a non-degenerate one-parameter family $\gamma_t(s) =: R(s, t)$ of loops, $t \in [0, 1]$, according to

$$T(R) := \int_0^1 dt \int_0^1 ds R'^i(s, t) \dot{R}^j(s, t) \epsilon_{ijk} T^k(\gamma_t(s)). \quad (14.5)$$

The dot and the prime denote differentiation with respect to s and t respectively, and ϵ_{ijk} is the totally antisymmetric ϵ -tensor in three dimensions. The resulting algebra of the loop and ribbon variables, $T(\gamma)$ and $T(R)$, has real structure constants that can be expressed in terms of the intersection numbers of the loops and ribbons appearing in their arguments. Although the algebraic structure of the relations (14.3) can be neatly visualized by “cutting and gluing” of diagrams of loops and ribbons, a satisfactory physical interpretation has not been given so far. Part of the problem is again the overcompleteness of these non-local variables, which affects also the ribbon variables $T(R)$. There have been attempts of integrating the algebra (14.3) with the help of a formal group law expansion [Lol1], and thus possibly to obtain a new infinite-dimensional group structure on the space of three-dimensional loops. Unfortunately it is difficult to complete this program, again due to the lack of a suitable topological and differentiable structure on loop space.

As has already been mentioned, the Wilson loop variables occurring in the algebra (14.3) are not observables for general relativity, because they are not invariant under diffeomorphisms. However, it is formally easy to implement the three-dimensional spatial diffeomorphism invariance by requiring the (generalized) Wilson loops T to be constant on the orbits of the group $Diff(\Sigma)$, as in (5.2). Since the canonical loop algebra depends entirely on the topology of loop configurations, it projects down to the quotient space $T^*\mathcal{A}/Diff(\Sigma)$.

Although it is not the subject of this article, let me point out that the $2 + 1$ -dimensional model for general relativity is an ideal testing ground for some of the loop methods, since it has a non-abelian gauge group, but still only a finite number of degrees of freedom. Loop

algebras similar to (14.3) have been used by Martin [Mar] and Nelson et al. [NelReg, NeReZe] to describe and subsequently quantize 2 + 1 gravity.

The main utilization of the loop algebras described above is their importance for canonical non-standard quantization schemes, which will be the subject of the next section. (There are also straightforward lattice analogues of the algebra (14.4), which have been employed in [RovSmo3, Brü].) Another example of a non-trivial loop algebra, involving non-intersecting loops, is due to ‘t Hooft [tHo1,2]. He supplements the Wilson loops $T(\gamma)$ by a set of dual loop operators $\bar{T}(\gamma)$, corresponding to magnetic flux lines. Both the T - and the \bar{T} -operators commute among themselves, but the commutation relation between a $T(\gamma)$ and a $T(\gamma')$ depends on the linking number of γ and γ' . This algebra is used to extract qualitative behaviour of different phases of Yang-Mills theory (see also the discussions by Mandelstam [Man3] and Gambini and Trias [GamTri5] (who construct a quantum representation of ‘t Hooft’s algebra on a space of loop states), and the related work by Hosoya and Shigemoto [Hos, HosShi] on the idea of duality between electric and magnetic flux lines).

15. Canonical quantization

The only non-perturbative quantization schemes put forward in the loop formulation are Hamiltonian and operator-based. “Non-perturbative” in this context means “not resorting to a perturbation expansion in terms of the gauge potentials A_μ ”. One might argue that such attempts defeat the purpose of the loop approach. Indeed, the hope in the non-local formulations we have been discussing is often for an *inequivalence* of the quantum theory and the usual local field-theoretic quantization. Unfortunately, a corresponding, alternative “loop perturbation theory” has not yet been developed. One also has to decide about how to treat the Mandelstam constraints in the quantization, for example, whether to solve them before or after the quantization, and different choices may well lead to inequivalent quantum theories.

All existing canonical quantizations for Yang-Mills theory and gravity in the loop formulation postulate the existence of (self-adjoint) operator analogues of a set of basic loop variables (such as the traced holonomies and appropriate momentum variables), defined on some “Hilbert space” of loop functionals, such that their commutation relations are preserved in the quantum theory. All of them are defined at a formal level, in the sense that there is no proper Hilbert space structure, and the wave functions are just elements of some linear function space, depending on loops. Some of them have in common that the action of the operator analogue $\hat{T}(\gamma)$ of the traced holonomy in this function space is given by multiplication by $T(\gamma)$. Note that we cannot employ a Schrödinger-type quantization, because the

algebra relations of the basic variables of the theory (for example, (14.3)) are not of the form of canonical commutation relations.

Gambini and Trias were the first ones to write down an algebra of quantum operators, realizing the algebra (14.3) [GamTri6]. Wave functions in their approach are labelled by individual loops and sets of loops, and there is a “vacuum state”, the no-loop state $|0\rangle$, which is annihilated by the momentum operators \hat{T}^i . In this aspect their representation is similar to the highest-weight representation of the loop algebra proposed by Aldaya and Navarro-Salas [AldNav].

For reasons inherent in the loop formulation of canonical gravity, Rovelli and Smolin [RovSmo2] quantize an extended set of $\text{SL}(2, \mathbb{C})$ -loop variables $T^{i_1 \dots i_n}$, which depend on n “electric field” variables inserted into the traced holonomy, and are a straightforward generalization of the expression (14.2). It turns out that in order to obtain a closed Poisson algebra one has to include the infinite tower of these generalized holonomy variables, for any n . The resulting algebra has a graded structure, schematically given by

$$\begin{aligned} \{T^0, T^0\} &= 0, \\ \{T^m, T^n\} &\sim T^{m+n-1}, \quad m+n > 0 \end{aligned} \tag{15.1}$$

where m and n denote the numbers of electric field insertions. This algebra contains the algebra (14.3) as a closed subalgebra. The corresponding quantum algebra obtained in [RovSmo2] is isomorphic to (15.1), with the exception of higher-order correction terms proportional to \hbar^k , $k \geq 2$, appearing on the right-hand sides of the commutators. (See also the work by Rayner [Ray2] on this particular quantum representation.)

These higher-order terms do not appear in the quantization proposed in [Lol1], where the semi-direct product structure of the algebra (14.3) is exploited, using methods from the theory of unitary irreducible representations of semi-direct product algebras. This theory is very powerful in finite dimensions, giving a complete classification and construction of such representations. Some of the formalism can be applied to the infinite-dimensional case too, although there is no reason to expect that similarly strong results will hold.

Note at this stage that in spite of many similarities, “solving” quantum gauge theory and quantum gravity requires entirely different strategies. In case of the former one needs some (generalized) measure on a suitable space of loops (the domain space for wave functions Ψ), and then looks for solutions Ψ to the eigenvector equation $\hat{H}_{YM}\Psi = E\Psi$, where \hat{H}_{YM} is the quantized Yang-Mills Hamiltonian. Up to now one does not know how to tackle this

problem non-perturbatively in the continuum theory, but progress has been made in the lattice-regularized theory, as explained in Sec.13.

For the case of gravity, the quantum dynamics is contained in the constraint equation $\hat{H}_{GR}\Psi = 0$, i.e. physical wave functions Ψ are annihilated by the gravitational quantum Hamiltonian (they also have to be annihilated by the quantum constraints corresponding to spatial diffeomorphisms). Then, a suitable measure has to be found on the space $\{\Psi\}$ of these solutions. It is not my intention to describe in any detail the progress that has been made in finding such “non-perturbative solutions to quantum gravity” with the help of loop methods, because a number of excellent reviews on the subject is already available [Rov1,2, Pul, Smo2, Ash2].

There are many intriguing aspects to this line of research, and the development of appropriate mathematical structures is not keeping pace with the many heuristic ideas that are being put forward. Examples are the relation of the quantum solutions to knot and link invariants [BrGaPu, DiGaGr, AshLew2], the construction of diffeomorphism-invariant observables [Smo3], and the introduction of a “weave” of loops to model space at small scales [AsRoSm, Ash2]. It is clear however that discrete mathematical objects (like the generalized knot classes [RovSmo1]) play a prominent role.

One interesting mathematical tool that has been introduced to relate the quantum representations on spaces of connection wave functions $\Psi([A])$ and loop wave functions $\tilde{\Psi}(\gamma)$ is the so-called loop transform. The idea of exploiting the (non-linear) duality between those two spaces appeared first in [GamTri5]. In the form introduced by Rovelli and Smolin [RovSmo2], the loop transform reads

$$\tilde{\Psi}(\gamma) = \int_{\mathcal{A}/\mathcal{G}} [dA]_{\mathcal{G}} T_A(\gamma)\Psi([A]), \quad (15.2)$$

where the Wilson loops $T_A(\gamma)$ serve as kernel of the transform. Also self-adjoint operators may be translated from the connection to the loop representation with the help of (15.2). For Yang-Mills theory, this construction is of limited practical use, in lack of a well-defined gauge-invariant measure $[dA]_{\mathcal{G}}$. Nevertheless many of its properties have been explored in both gauge and gravitational theories (see, for example, [BrüPul, AshIsh1, AshRov, Lol6, AshLew2, AshLol]).

The approach that leads closest to the construction of a rigorous representation theory of the loop algebra is that of Ashtekar and Isham [AshIsh1], subsequently extended by Ashtekar and Lewandowski [AshLew2]. They start from the abelian subalgebra of the traced

holonomies $T(\gamma)$, i.e. the first line of the algebra (14.3), endow it with the structure of a C^* -algebra, and then look for its cyclic representations, using Gel'fand spectral theory. The Hilbert spaces involved are given by spaces of square-integrable functions on the space of ideals of the C^* -algebra, a certain completion $\overline{\mathcal{A}/\mathcal{G}}$ of the space \mathcal{A}/\mathcal{G} , allowing also for distributional connections A . The precise mathematical structure of the space $\overline{\mathcal{A}/\mathcal{G}}$ and the inclusion of appropriate conjugate momentum variables are currently under investigation.

Beyond these mainly kinematical considerations of how to quantize algebras of basic phase space variables, little is known about a proper formulation of the Hamiltonian dynamics, i.e. about how to express the quantum Hamiltonians of Yang-Mills theory and gravity as well-defined (self-adjoint) operators in terms of the non-local (quantized) loop variables. One main ingredient is the choice of an appropriate regularization and renormalization prescription (which in the case of gravity has to respect diffeomorphism invariance). On the other hand, we know from finite-dimensional examples that the quantization of non-canonical commutation relations at a kinematic level usually is non-unique, and we expect the situation to be much worse in the present, field-theoretic case (see also [AshIsh2] on the ambiguity of field-theoretic quantizations). Whichever representation theory we come up with for the loop algebra, further physical criteria will be needed to decide which of the multitude of possible representations is physically relevant, for example, by selecting those in which the Hamiltonian assumes a simple form, and by finding ways to relate them to physical observations.

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