3d Lorentzian, Dynamically Triangulated Quantum Gravity

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The model of Lorentzian three-dimensional dynamical triangulations provides a non-perturbative definition of three-dimensional quantum gravity. The theory has two phases: a weak-coupling phase with quantum fluctuations around a "semiclassical" background geometry which is generated dynamically despite the fact that the formulation is explicitly background-independent, and a strong-coupling phase where "classical" space disintegrates into a foam of baby universes.

1. Introduction

Three-dimensional quantum gravity provides a good model for the study of quantum gravity. Locally, the classical solution is just flat space, or in the case of a positive cosmological constant, 3d deSitter space. If one expands around such a classical solution in order to quantize the theory it is non-renormalizable. Nevertheless we know the theory has no dynamical *field degrees of freedom*, but only a finite number number of degrees of freedom. Thus it can be quantized following different procedures, e.g. reduced phase space quantization. However, it remains unclear if anything is "wrong" in an approach where one perform the summation over fluctuating three-geometries and how such an approach deals with the seemingly non-renormalizability of the theory of fluctuating geometries.

In two-dimensional quantum gravity the simplest direct approach to the theory of fluctuating geometries, known as *dynamical triangulations* has been very successful (see e.g. [1]). However, simple generalizations to higher dimensions seem not to work. In four-dimensional space-time the failure could in principle be due to the non-existence of a four-dimensional theory of quantum gravity which is not embedded in a larger theory, but this argument does not apply to the three-dimensional theory of quantum gravity. This led two of us to suggest, following an old idea by Teitelboim, that Euclidean quantum gravity might not be related to the "real" Lorentzian quantum gravity and that one should only include causal geometries in the sum over histories [2]. The geometries which appear in the regularized version of such a theory were called *Lorentzian dynamical triangulations*, and each of these geometries has a well defined rotation to an Euclidean geometry. The opposite is not true however: there are many Euclidean geometries which cannot be rotated to a Lorentzian geometry with a global causal structure. However, it implies that one *can* perform the summation over this restricted class of geometries in the "Euclidean sector", and rotate back after the summation has been done. This is the way we will treat the summation over histories in the following.

In two dimensions one can perform the summation over the class of Lorentzian geometries explicitly and obtain a theory which is *different* from Euclidean two-dimensional quantum grav-

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ity. The difference is best illustrated by considering what is called the proper-time propagator, where one sums over all geometries with the topology $S^1 \times [0,1]$, where the two spatial boundaries are separated by a proper time T. In the Lorentzian theory the spatial slices at a time $T' < T$ are characterized by the fact that the spatial topology always remain a circle. In two-dimensional Euclidean quantum gravity similar spatial slices corresponding to constant proper time split up into many (in the continuum limit infinitely many) disconnected "baby" universes, *each* having the topology S^1.

2. 3d Lorentzian dynamical triangulations

2.1. Theory

As mentioned above the proper-time propagator is a convenient object to study. We choose the simplest possible topology of space-time, $S^2 \times [0,1]$, so that the spatial slices of constant proper time have the topology of a two-sphere. Each spatial slice has an induced two-dimensional Euclidean geometry. In the formalism of Lorentzian dynamical triangulations the space of Euclidean 2d geometries is approximated by the space of 2d dynamical triangulations. In order to obtain a three-dimensional triangulation of space-time we have to fill space-time between two successive time slices. This is done as follows: above (and below) each triangle at proper time $t = n\,a$, n integer, we erect a tetrahedron with its tip at $t + a$, so-called (3,1)-tetrahedra ($t - a$, so-called (1,3)-tetrahedra). Two tetrahedra which share a common spatial link in the t-plane might be glued together along a common time-like triangle. Remaining free time-like triangles are glued together by so-called (2,2)-tetrahedra which have a spatial link both in the t-slice and the $t+a$ slice. (2,2)-tetrahedra can also be glued to each other in all possible ways, the only restriction being that if we cut the triangulation in a constant t-plane between t and $t+a$, the corresponding graph, which consists of triangles and squares (coming from cutting the (2,2)-simplexes) forms a graph with spherical topology.

Summing over all such piecewise linear geometries with the Boltzmann weight given by the Einstein-Hilbert action defines the sum over geometries (see [3] for details). The partition function becomes (up to boundary terms)

$$Z(k_0, k_3, T) = \sum_{T} \mathrm{e}^{k_0 N_0(T) - k_3 N_3(T)}, \qquad (1)$$

where the summation is over the class of triangulations mentioned, and $N_0(T)$ denotes the total number of vertices and $N_3(T)$ the total number of tetrahedra. k_0 is inversely proportional to the bare inverse gravitational coupling constant, while k_3 is linearly related to the cosmological coupling constant.

2.2. Numerical simulations

The model (1) can be studied by Monte Carlo simulations (see [3] for details). There is only one phase[4]. Let us fix the total three-volume of space-time to be N_3, and let us take the total proper time T large. One observes the appearance of a "semiclassical" lump of universe, as shown in Fig. 1. We observe that a typical spatial volume $N_2(t)$

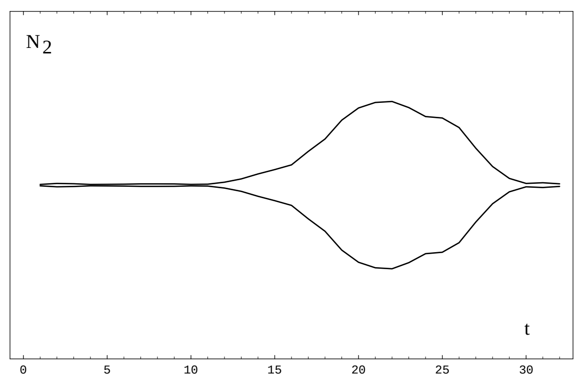

Figure 1. a snapshot of a three-dimensional configuration of space-time. The vertical axis is the spatial volume N_2, the horizontal axis the proper time t

in the lump, and the time-extent ΔT of the lump

[4]In some previous studies we observed a phase transition for large k_0. This was caused by restrictions on the gluing of (2,2)-simplexes. We have now dropped these restrictions.

scale as:

$$N_2 \sim N_3^{2/3}, \quad \Delta T \sim N_3^{1/3}. \tag{2}$$

This justifies the use of the word "semiclassical" for the lump. In the computer simulations the center of mass of the lump moves around randomly and there are fluctuations in the spatial volumes. We have studied the fluctuations of *successive* spatial volumes and the distributions of such spatial volumes is very well described by

$$P(N_2(t), N_s(t+a)) \sim \mathrm{e}^{-c(k_0)\frac{(N_2(t+a)-N_2(t))^2}{N_2(t+a)+N_2(t)}}. \tag{3}$$

The constant $c(k_0)$ decreases as k_0 increases (i.e. the bare gravitational coupling constant decreases). At the same time one can observe that the total number of (2,2)-simplices decreases. The (2,2)-simplices act as glue between successive spatial volumes.

Thus an effective action for the spatial volume of the model seems to be

$$S_{eff}(V_2) = \int dt \,\Big(\frac{\dot{V}_2^2(t)}{V_2(t)} + \Lambda V_2(t)\Big). \tag{4}$$

This is exactly the classical action for the spatial volume in proper-time gauge, thus supporting the semiclassical interpretation of the lump.

3. 3d Lorentzian gravity as a matrix model

If we slice our three-dimensional configurations, not at proper time t, but at proper time $t+a/2$ we will, as mentioned earlier, obtain a spherical graph with from two types of triangles, coming the spatial intersections of (1,3)- and (3,1)-tetrahedra, respectively. In addition the graph will contain squares coming from (2,2)-tetrahedra. This class of graphs can be described by a two-matrix model:

$$Z = \int \mathrm{d}A \mathrm{d}B \, \mathrm{e}^{-N \mathrm{Tr}\,\left((A^2+B^2)-\alpha(A^3+B^3)-\beta ABAB\right)}. \tag{5}$$

A and B are $N \times N$ Hermitean matrices and the special graphs are selected in the large N limit. The coupling constants α, β can be related to the gravitational coupling constants. The phase diagram is shown in Fig. 2 (for a slightly different model, see [5] for details). The matrix model is defined for small values of α, β and has a critical line. This line corresponds to the continuum limit of the 3d quantum gravity theory. Small values of β on the critical line correspond to small values of the bare gravitational coupling constant. Increasing β or the gravitational cou-

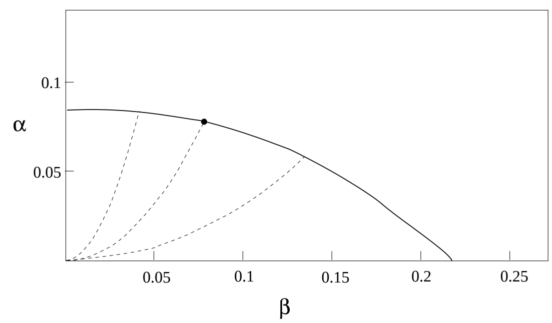

Figure 2. The phase diagram of 3d Lorentzian gravity from matrix models, calculated in [4]

pling constant we meet a phase transition, separating the weak coupling phase we have observed in the computer simulations from a strong coupling phase. The matrix model allows spherical graphs at $t+a/2$ such that the (1,3)- and (3,1)-triangles no longer correspond to triangulations of S^2, but rather a disconnected set of baby universe. One can say that gravity becomes so strong that space is torn apart into many pieces connected by thin wormholes, [5]. In this way 3d Lorentzian dynamical triangulations provide us with an explicit realization of the quantum foam ideas of Hawking and Wheeler.

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